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Spontaneous Breakings of Chiral Symmetries. II. Mass Relations and Particle Mixings

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A detailed analysis of SU(3)×SU(3) symmetry breaking by a $(3,\bar{3})+(\bar{3},3)$ linear term is given, with no reference to a Lagrangian model and by carefully taking into account the SU(3) noninvariance of the vacuum. Sum rules are derived and a detailed discussion of the κ mass and the η-η' mixing angle is given. By carefully handling the delicate problem of approximations, no inconsistency is found with the experimental data, in contrast with results that have appeared in the literature. The over-all picture is instead in very good agreement with experiments, thus showing that SU(3)×SU(3) symmetry breaking is a very precise framework for discussion of elementary-particle symmetries. Corrections to SU(3) results, arising from the SU(3) noninvariance of the vacuum, are discussed. It is found that, in general, neglecting $\lambda_8 \equiv \langle 0|u_8|0\rangle$ is a dangerous approximation. In particular, the value of the η-η' mixing angle is shown to depend very strongly on λ_8, a point which has not been realized in the literature. A generalized Gell-Mann–Okubo mass formula taking into account the SU(3) noninvariance of the vacuum is derived and it is found to be in extremely good agreement with experiment, thus showing that SU(3)×SU(3) symmetry breaking may be even more precise than SU(3) symmetry breaking. The high degree of accuracy of the sum rules resolves the ambiguity in the identification of the ninth pseudoscalar meson and provides a very precise determination of the η-η' mixing angle.

I. INTRODUCTION

The aim of the present paper is to analyze in detail the consequences of SU(3)×SU(3) symmetry breaking, such as mass splittings, relations between coupling constants, mixing angles, etc.[1] The literature on the subject is very rich.[2] Still, a detailed discussion of the results which follow simply from the assumption of a $(3,\bar{3})+(\bar{3},3)$ symmetry breaking,[3] without reference to any Lagrangian model and without assuming SU(3) invariance of the vacuum, is lacking. As a matter of fact, what will emerge from our analysis is that SU(3) ×SU(3) symmetry breaking is a much better framework for discussion of strong-interaction symmetries than the conventional SU(3) symmetry breaking. The relations we will obtain by studying SU(3) ×SU(3) symmetry breaking appear in much better agreement with the experimental data than the analogous SU(3) relations, obtained by neglecting the vacuum noninvariance under SU(3). A typical example is the Gell-Mann–Okubo formula for mesons, the chiral version of which seems to work extremely well. *In this respect, SU(3)×SU(3) symmetry breaking appears much more precise than conventional SU(3) symmetry breaking.*

Another impressive feature of the analysis is the internal consistency of the set of sum rules obtained. Independently of the input parameters one chooses they all work with a high degree of accuracy. This may appear in contrast with some results discussed in the literature. The point is that the problem of the approximations is a very delicate one. For example, neglecting the η-η' mixing angle θ and/or the vacuum noninvariance under SU(3) (both parameters are actually small) may in some cases lead to bad results. This is a phenomenon typical of the spontaneous breaking of the symmetry. Small variations of the parameters may lead to rather different results. Therefore, very mild looking approximations (like neglecting θ or $\lambda_8 \equiv \langle 0|u_8|0\rangle$) may considerably change the physical consistency of a sum rule. An example of this is the determination of the κ mass, which appears to be rather critical and whose inaccurate treatment may yield misleading conclusions.

In order to avoid difficulties of the type described above, we will try to be as general as possible. In particular, we will make no reference to a Lagrangian model, like the σ model, or to a semiclassical approximation. It will turn out that some of the results obtained in the literature by using a specific model are actually model-independent and follow only from the assumption of a $(3,\bar{3})+(\bar{3},3)$ symmetry breaking. To get these results we will use the functional approach to quantum field theory, which does not require any perturbative expansion or first-order approximations with respect to the breaking parameters. A general discussion of this approach as well as its connection to the semiclassical approximation are given in Secs. III and IV.

A particular emphasis is put on the group-theoretical content of the relations obtained. It will appear from our discussion that the Ward-like identities are nothing but the group-theoretical characterization of the spontaneous symmetry breaking. In this respect they are the model-independent features of the spontaneous symmetry breaking, in the same way as the consequences of the Wigner-Eckart theorem do not depend on any specific model.

The breaking of SU(3)×SU(3) symmetry is analyzed in both the cases of isospin invariance ($\epsilon_3=0$) and of isospin breaking ($\epsilon_3\neq 0$) in Secs. VI and VII. By using the third-order Ward identities we may obtain relations involving only measurable quantities like F_π, F_K, and masses (but not the breaking parameters ϵ_0, ϵ_8). This allows us to determine the vacuum expectation values $\lambda_0 \equiv\langle 0|u_0|0\rangle$ and $\lambda_8\equiv\langle 0|u_8|0\rangle$, which characterize the noninvariance of the vacuum in two independent ways. The agreement is very good, showing the remarkable consistency of the equations and of the whole theoretical scheme (Sec. VI). Besides relations which have already appeared in the literature, we find two interesting sum rules involving the η-η' mixing angle and the κ-meson mass [Eqs. (53) and (54)]. A careful discussion of these two parameters is given in Sec. VI B. If proper care is exerted, no inconsistency arises with the presently available experimental data, in contrast to what has been claimed in the literature. Actually the agreement seems rather good with an angle θ being rather small.

An interesting feature steadily emerging from the analysis is that SU(3) noninvariance of the vacuum is small but not negligible with respect to the chiral noninvariance ($\lambda_8/\lambda_0\simeq 20-25\%$). This raises the question of how good are the conventional SU(3) results obtained by neglecting λ_8. This point is discussed in Sec. VI D, where corrections to SU(3) arising from λ_8 are analyzed. An interesting result is a generalized Gell-Mann–Okubo (GMO) formula

$$4F_K m_K{}^2=3f_8 m_\eta{}^2-F_\pi m_\pi{}^2+3(m_{\eta'}{}^2-m_\eta{}^2)\sin\theta(f_8\sin\theta-f_8'\cos\theta)$$

(see Sec. VI D for details). The angle θ may be carefully determined from an independent equation (which connects it to F_K/F_π), giving $\theta\simeq 2°58'$. It may be surprising that the angle θ is so small with respect to the value obtained from the conventional GMO formula. The reason is that the corrections to the GMO formula mainly arise from the $\lambda_8\neq 0$ effect and not from the mixing angle θ [in contrast with the conventional SU(3) scheme, where θ is introduced *ad hoc* to adjust the GMO formula]. The generalized GMO formula with this small value for θ is in very good agreement (about 1%) with the experimental data. The previous equations can be also used to obtain sum rules involving η'. Their high degree of accuracy resolves the ambiguity about the ninth pseudoscalar meson by giving $\eta'=X^0(958)$. The choice $E(1422)$ would make the sum rules work rather badly.

The implications of SU(3)×SU(3) symmetry breaking about the coupling constants are discussed in Sec. VI E.

The case $\epsilon_3\neq 0$ is analyzed in Sec. VII. Sum rules are obtained but the check with experimental data is difficult because some of the parameters involved are not known. The general features which emerge are that the mixing between π^0 and the η-η' system is very small, and the results of Sec. VI are essentially unchanged. This is not a trivial result because, from what is known about spontaneous symmetry breaking, even a small perturbation like $\epsilon_3 u_3$ could cause reasonable changes especially in the mixing angles.

II. BROKEN SYMMETRIES

The importance of broken internal symmetries in elementary-particle physics has been realized for a long time. Only recently, however, spontaneously broken symmetries[4] have been suggested to play a fundamental role in the classification of elementary particles.[5] In order to clarify the discussion we will distinguish between two cases of broken symmetries:

(i) Wigner-type symmetries. The simplest case of broken internal symmetries is realized when (a) the Hamiltonian has definite and simple transformation properties under the group G describing the given symmetry:

$$H = H_{\text{inv}} + gH_{\text{break}}; \quad (1)$$

(b) the "basic" fields of the theory transform in the first approximation as a representation of the group G:

$$[\underline{Q}^\alpha, \phi_i(x)] = \underline{D}_{ij}\, \phi_j(x), \quad (2)$$

where $\underline{Q}^\alpha$ are the generators of the group[6] (sum over repeated indices is always understood, unless explicitly stated otherwise); (c) the vacuum state is approximately invariant[7,8] under the group G:

$$\underline{Q}^\alpha |0\rangle \simeq 0. \quad (3)$$

A situation of this kind is realized, e.g., for the isospin symmetry. Mass splittings, relations between coupling constants and transition rates, etc., are easily computed in this case by simply using the Wigner-Eckart theorem. For definiteness, these broken symmetries will be called the Wigner-type. They will not be discussed in detail in this paper.

(ii) Nambu-Goldstone-type symmetries. A more involved situation is realized when the vacuum is not invariant, *not even approximately*, under G. In contrast to the previous case (i), now one does not expect to recover a fully symmetric theory in the limit $g \to 0$. Rather, the limit of symmetric Hamiltonian corresponds to the spontaneous symmetry breaking, as discussed by Nambu and Goldstone.[4] For simplicity, we will denote by Nambu-Goldstone broken symmetry the more general case $g \neq 0$.[9]

Even if the generators $\underline{Q}^\alpha$ fail to exist as well-defined operators as a consequence of the spontaneous symmetry breaking, it is reasonable to assume that the basic fields $\phi_i(x)$ transform *locally* as a representation of the group G, in the first approximation.[10]

It is not difficult to realize that in the Nambu-Goldstone type of symmetry breaking the Wigner-Eckart theorem is not as helpful as in case (i) and one has to find a new method for computing mass splitting, relations between coupling constants, etc. An attempt in this direction is discussed in this paper.

III. LAGRANGIAN FORMULATION

In order to simplify the discussion we will consider the "semiclassical" approximation first. The general features and the equations so obtained remain unchanged in the quantum-field-theory case, as we will prove in Sec. IV. The advantage of the semiclassical approximation is that here the method is not obscured by the problems connected with the functional formulation of quantum field theory and by the singular functions appearing there.

In the semiclassical approximation the theory is described by a Lagrangian L, which is a function of the basic classical fields $\phi_i(x)$, $i = 1, \ldots, n$:

$$L = L(\phi_i).$$

The spontaneous breaking of the symmetry is realized by the existence of one extremal point for the function $L(\phi_i)$, i.e., a point $\phi_i = \text{constant} = \tilde{\phi}_i$, where[11]

$$\left(\frac{\partial L}{\partial \phi_i}\right)_{\phi = \tilde{\phi}} = 0. \quad (4)$$

When expanded around the extremal point, the Lagrangian shows explicitly the symmetry breaking:

$$L = L(\tilde{\phi}) + \frac{1}{2}\left(\frac{\partial^2 L}{\partial \phi_i \partial \phi_j}\right)_{\phi = \tilde{\phi}} (\phi_i - \tilde{\phi}_i)(\phi_j - \tilde{\phi}_j) + \cdots, \quad (5)$$

with a clear interpretation of

$$-\left(\frac{\partial^2 L}{\partial \phi_i \partial \phi_j}\right)_{\phi = \tilde{\phi}} \equiv M^2{}_{ij} = M^2{}_{ji}$$

as the mass-squared matrix,

$$\left(\frac{\partial^3 L}{\partial \phi_i \partial \phi_j \partial \phi_k}\right)_{\phi = \tilde{\phi}} \equiv g_{ijk}$$

as the coupling constant between the i, j, k modes, etc.

A better understanding of the symmetry breaking is obtained by discussing its group-theoretical content. To this purpose, it is useful to choose the following representation for the generators $\underline{G}^\alpha$:

$$\underline{G}^\alpha \to i\underline{g}^\alpha_{ij}\phi_j \frac{\partial}{\partial \phi_i}, \quad (6)$$

where $\underline{g}^\alpha_{ij}$ is the matrix representation of the generator $\underline{G}^\alpha$, in the vector space spanned by the basic fields ϕ_i. Under an infinitesimal transformation the fields undergo the following variation:

$$\phi_i \to \phi_i + \epsilon \delta^\alpha \phi_i, \qquad \delta^\alpha \phi_i = -i[\underline{G}^\alpha, \phi_i] = \underline{g}^\alpha_{ij}\phi_j, \quad (7)$$

or, in a compact form,

$$\delta^\alpha \phi = -i[\underline{G}^\alpha, \phi] = \underline{g}^\alpha \phi.$$

Similarly, the Lagrangian function transforms in the following way:

$$L \to L + \epsilon \delta^\alpha L, \qquad \delta^\alpha L = -i[\underline{G}^\alpha, L] = (\underline{g}^\alpha \phi)_k \frac{\partial}{\partial \phi_k} L. \quad (8)$$

The above equation is a trivial consequence of Eq. (7) and it does not seem to have deep implications at first sight. However, if $\delta^\alpha L$ is specified *independently*, then *Eq. (8) fixes the transformation properties of L under* $\underline{G}^\alpha$, and therefore it contains the basic information about the symmetry. Our philosophy will be to exploit as much as possible Eq. (8): As we will see, all the physical information like mass splitting, relations between coupling constants, etc., can be obtained from Eq. (8).

In the following, we will regard $\delta^\alpha L$ as a given function of the fields ϕ_i, so that Eq. (8) can be considered as a group-theoretical statement on the transformation properties of L.

We are interested in the derivatives of L at the point $\phi=\tilde{\phi}$, since they have a physical interpretation as mass matrix, coupling constants, etc. Since the left-hand side of Eq. (8) is regarded as known, Eq. (8) and its derivatives provide the relevant information about the mass-squared matrix, the coupling constants, etc.

First of all, the condition of spontaneous symmetry breaking, Eq. (4), gives a constraint on $\delta^\alpha L$, at the point $\phi=\tilde{\phi}$:

$$\left[(\underline{g}^\alpha\phi)_k \frac{\partial L}{\partial\phi_k}\right]_{\phi=\tilde{\phi}} = (\delta^\alpha L)_{\phi=\tilde{\phi}} = 0 . \tag{9}$$

Furthermore, the derivatives of Eq. (8), at the point $\phi=\tilde{\phi}$, give information about the mass-squared matrix,

$$\left[\frac{\partial^2 L}{\partial\phi_i\partial\phi_k}(\underline{g}^\alpha\phi)_k + \underline{g}^\alpha_{ki}\frac{\partial L}{\partial\phi_k}\right]_{\phi=\tilde{\phi}} = \left(\frac{\partial\delta^\alpha L}{\partial\phi_i}\right)_{\phi=\tilde{\phi}} , \tag{10}$$

i.e.,

$$[M^2{}_{ik}(\underline{g}^\alpha\phi)_k]_{\phi=\tilde{\phi}} = -\left(\frac{\partial\delta^\alpha L}{\partial\phi_i}\right)_{\phi=\tilde{\phi}} , \tag{11}$$

where again

$$M^2{}_{ik} \equiv -\left(\frac{\partial^2 L}{\partial\phi_i\partial\phi_k}\right)_{\phi=\tilde{\phi}} .$$

In a similar way, the second derivatives of Eq. (8) give

$$g_{ijk}(\underline{g}^\alpha_{kl}\phi_l)_{\phi=\tilde{\phi}} \equiv \left(\frac{\partial^3 L}{\partial\phi_i\partial\phi_j\partial\phi_k}\,\underline{g}^\alpha_{kl}\phi_l\right)_{\phi=\tilde{\phi}}$$
$$= [M^2,\underline{g}^\alpha]_{ij} + \left(\frac{\partial^2\delta^\alpha L}{\partial\phi_i\partial\phi_j}\right)_{\phi=\tilde{\phi}} , \tag{12}$$

where

$$[M^2,\underline{g}^\alpha]_{ij} \equiv M^2{}_{ik}\underline{g}^\alpha_{kj} - \underline{g}^\alpha_{ik}M^2{}_{kj} . \tag{12'}$$

Equation (12) yields nontrivial relations between the coupling constants and the symmetry properties of the mass-squared matrix.

The above procedure of evaluating the derivatives of Eq. (8) at the point $\phi=\tilde{\phi}$ can be further pursued. In this way, one gets a chain of equations, each of which relates the n-point coupling constants to lower-order coupling constants. For example, four-point couplings are related to three-point couplings by the following equation:

$$-g_{ijkl}(\underline{g}^\alpha_{lp}\phi_p)_{\phi=\tilde{\phi}} = g_{jkp}\underline{g}^\alpha_{pi} + g_{ikp}\underline{g}^\alpha_{pj}$$
$$+ g_{ijp}\underline{g}^\alpha_{pk} - \left(\frac{\partial^3\delta L}{\partial\phi_i\partial\phi_j\partial\phi_k}\right)_{\phi=\tilde{\phi}} . \tag{13}$$

As we will prove in Sec. IV, the chain of Eqs. (9), (11), (12), (13), etc., corresponds to the chain of Ward identities one gets in the quantum-field-theory case as a consequence of the spontaneous symmetry breaking. *In both the classical and the quantum-field-theory case, the above structure of Ward-like identities appears as the group-theoretical characterization of a spontaneous symmetry breaking.* The relevant information contained in these equations as well as their physical interpretation will become clear in Secs. V, VI, and VII, where the above method will be applied to concrete physical examples.

IV. BROKEN SYMMETRIES IN QUANTUM FIELD THEORY: FUNCTIONAL METHOD

The natural objection to the results of Sec. III is that they may be confined to the semiclassical approximation and that large corrections may be expected in the quantum-field-theory case.[12] Another difficulty of the semiclassical approximation is that it is essentially based on Eq. (5), and a nonanalytic behavior of the theory would raise serious doubts about the validity of this expansion. There are in fact indications that a nonanalytic behavior of the theory occurs whenever the symmetry is spontaneously broken.[13]

As a matter of fact, if one tries to interpret the results of Sec. III as a tree approximation of the quantum-field-theory case, one is faced with the difficulty of proving that the tree approximation is a very good approximation and that the corrections are very small, at least in the low-energy case. An alternative and more convincing way of discussing the quantum-field-theory case is to look at the problem from a completely different point of view. Instead of attempting to justify the semiclassical approximation as a tree approximation, we will prove that the equations obtained in Sec. III, if suitably interpreted, remain valid in the quantum-field-theory case, with no approximation. This is not surprising; as stressed before, the chain of Eqs. (9)–(13), etc., are essentially the group-theoretical statement of spontaneous symmetry breaking. Their deep group-theoretical content does not depend on whether the treatment is semiclassical or quantum-field-theoretical, just as the results of the Wigner-Eckart theorem are independent of the formulation adopted.

The discussion of the spontaneous breakdown of symmetries in quantum field theory is suitably done by using the functional method.[12] To simplify the discussion we will consider the case in which the field theory is described by a Lagrangian $L(x)$ of the form[14,15]

$$L(x)=L_{\text{free}}(x)+L_{\text{int}}(x)+\epsilon_i\phi_i(x)\,, \tag{14}$$

where $L_{\text{free}}(x)+L_{\text{int}}(x)\equiv L_{\text{inv}}(x)$ is invariant under the given group G, ϵ_i are constants, and $\phi_i(x)$ are the basic local fields in terms of which the Lagrangian is constructed and which transform locally as a representation of the group G:

$$\phi_i(x)\rightarrow \underline{g}^{\alpha}_{ij}\phi_j(x)\,. \tag{15}$$

Following the standard procedure[12] one introduces the functional

$$\begin{aligned}Z(\epsilon)&=-i\ln\left\langle\underline{\psi}_0\middle|T\exp\left[i\int d^4x[\underline{L}_{\text{int}}(x)+\epsilon_i(x)\underline{\phi}_i(x)]\right]\middle|\underline{\psi}_0\right\rangle\\&=-i\ln\left\langle\underline{\psi}_0\middle|T\exp\left[i\int d^4x\{\underline{L}_{\text{int}}(x)+\epsilon_i\underline{\phi}_i(x)+[\epsilon_i(x)-\epsilon_i]\underline{\phi}_i(x)\}\right]\middle|\underline{\psi}_0\right\rangle\,,\end{aligned} \tag{16}$$

corresponding to the modified Lagrangian

$$L'(x)=L_{\text{free}}(x)+L_{\text{int}}(x)+\epsilon_i(x)\phi_i(x)\,,$$

$\epsilon_i(x)$ describing c-number external sources. [In Eq. (16), $\underline{\psi}_0$, $\underline{L}_{\text{int}}(x)$, and $\underline{\phi}_i(x)$ denote ψ_0, $L_{\text{int}}(x)$, and $\phi_i(x)$ in the interaction picture.] The limit $\epsilon_i(x)\rightarrow$ constant $=\epsilon_i$ will yield the original Lagrangian (14) and will be performed at the end. By performing a Legendre transformation one may introduce the c-number functions

$$\lambda_i(x)\equiv\frac{\delta Z}{\delta\epsilon_i(x)}=\langle\psi_0|\phi_i(x)|\psi_0\rangle \tag{17}$$

and define the action integral[16]

$$\begin{aligned}A(\lambda)&\equiv Z(\lambda)-\int d^4x[\epsilon_i(x)-\epsilon_i]\lambda_i(x)\\&\equiv W(\lambda)+\epsilon_i\int\lambda_i(x)d^4x\,.\end{aligned} \tag{18}$$

Here $Z(\lambda)$ is obtained from $Z(\epsilon)$ by expressing the $\epsilon_i(x)$'s as functionals of the $\lambda_i(x)$'s through Eq. (17) (Legendre transformation).

It is easy to prove the following equations:

$$\frac{\delta A}{\delta\lambda_i(x)}=-[\epsilon_i(x)-\epsilon_i]\,, \tag{19}$$

$$\frac{\delta^2 A}{\delta\lambda_i(x)\delta\lambda_j(y)}=\frac{\delta^2 W}{\delta\lambda_i(x)\delta\lambda_j(y)}=\Delta^{-1}{}_{ij}(x,y)\,, \tag{20}$$

$$\frac{\delta^3 A}{\delta\lambda_i(x)\delta\lambda_j(y)\delta\lambda_k(z)}=\frac{\delta^3 W}{\delta\lambda_i(x)\delta\lambda_j(y)\delta\lambda_k(z)}=\Gamma_{ijk}(x,y,z)\,, \tag{21}$$

etc., where $\Delta^{-1}{}_{ij}(x,y)$ is the inverse of the "propagator"

$$\Delta_{ij}(x,y)=\langle\psi_0|T(\phi_i(x)\phi_j(y))|\psi_0\rangle-\lambda_i(x)\lambda_j(y) \tag{22}$$

and $\Gamma(x,y,z)$ is the amputated three-point function.

The above equations (19)–(22) suggest that the action functional $A(\lambda)$ can be interpreted as a classical action integral for the classical fields $\lambda_i(x)$. As we will see, the analogy with the classical case is very strong.

In order to get the group-theoretical transformation properties of $A(\lambda)$ we consider the following infinitesimal transformation of the "classical fields" $\lambda_i(x)$:

$$\lambda_i(x)\rightarrow\lambda_i(x)+\epsilon\delta^{\alpha}\lambda_i(x)\,, \tag{23}$$

$$\delta^{\alpha}\lambda_i(x)=\underline{g}^{\alpha}_{ij}\lambda_j(x)\,. \tag{24}$$

To the above transformation of the $\lambda_i(x)$'s will correspond a variation of the functional $A(\lambda)$ given by

$$\delta^{\alpha}A=\int d^4x\,\underline{g}^{\alpha}_{ij}\lambda_j(x)\frac{\delta A}{\delta\lambda_i(x)}\,. \tag{25}$$

The above equation may look like a trivial identity if A is a known functional of the $\lambda_i(x)$'s. However, if $\delta^{\alpha}A$ is assigned independently, the above equation has a deep group-theoretical content because it becomes a statement about the transformation properties of A under the group G. The nontrivial question which arises at this point is how can one assign $\delta^{\alpha}A$ by the knowledge of the transformation properties of the Lagrangian. In the semiclassical approximation, Eq. (25) was written for the Lagrangian itself and there was no problem in making assumptions about $\delta^\alpha L$. Here, the group-theoretical behavior of $\delta^\alpha A$ looks less direct, and one would rather like to make assumptions about the transformation properties of the Lagrangian L or of the Hamiltonian density H. These last quantities have in fact a more direct physical meaning[17] than the functional $A(\lambda)$. The connection between $\delta^\alpha L$ and $\delta^\alpha A$ is given by the following:

Statement. Under a transformation $\delta^\alpha \lambda_i = \underline{g}^\alpha_{ij}\lambda_j$ the change $\delta^\alpha A$ in the functional A is given by the vacuum expectation value of the change $\delta^\alpha L$ of the Lagrangian under the transformation $\delta^\alpha \phi_i(x) = \underline{g}^\alpha_{ij}\phi_j(x)$:

$$\delta^\alpha A(\lambda) = \langle \psi_0 | \delta^\alpha L | \psi_0\rangle \ . \tag{26}$$

Proof. Since we will be concerned only with the case in which $\delta^\alpha L$ is linear in the fields, we will give the proof for that case. The proof then becomes very simple. Since $W(\lambda)$ is an invariant functional of λ,[12] one has

$$\begin{aligned}\delta^\alpha A &= \delta^\alpha W + \epsilon_i \underline{g}^\alpha_{ik}\lambda_k(x)\\ &= \epsilon_i \underline{g}^\alpha_{ik}\lambda_k(x)\\ &= \langle \psi_0 | \delta^\alpha L|\psi_0\rangle \ .\end{aligned}$$

As a consequence of the above equation, the group transformation properties of $A(\lambda)$ are tightly bound to the transformation properties of $L(x)$ and one may make assumptions either on $L(x)$ or on $A(\lambda)$. Equation (25) has the same form as Eq. (8) of Sec. III, and assigning the function $\delta^\alpha A$ is equivalent to assigning $\langle\psi_0|\delta^\alpha L|\psi_0\rangle$, in close analogy with the semiclassical approximation.

Now in the limit $\epsilon_i(x) \to \epsilon_i$, Eq. (19) gives

$$\frac{\delta A}{\delta \lambda_i} = 0 \, . \tag{27}$$

This is the analog of the extremal condition, Eq. (4), of the semiclassical approximation. If the explicit form of A is known, the above equation determines the values of $\lambda_i(x) \to \text{constant} = \lambda_i$, for which the symmetry is broken. However, even if A is not known, Eq. (27) is very useful: The group transformation properties of A fix the "directions" of the vector $\lambda_i = \text{constant}$ for which spontaneous symmetry breaking occurs. We do not insist on this point, which has been discussed in detail elsewhere.[18]

In conclusion, the treatment of Sec. III applies step by step and one gets the following equations [in the limit $\epsilon_i(x) \to \text{constant} = \epsilon_i$]:

$$\epsilon_i \underline{g}^\alpha_{ij}\lambda_j = 0\,, \tag{28}$$

$$\tilde\Delta^{-1}{}_{ij}(0)\underline{g}^\alpha_{jk}\lambda_k = \underline{g}^\alpha_{ik}\epsilon_k\,, \tag{29}$$

$$\tilde\Gamma_{ijk}(0,0)\underline{g}^\alpha_{kl}\lambda_l = -[\tilde\Delta^{-1}(0), \underline{g}^\alpha]_{ij}\,, \tag{30}$$

etc., where $\tilde\Delta^{-1}(0)$ is the Fourier transform, at zero momentum, of Δ^{-1}, and $\tilde\Gamma(0,0)$ is the Fourier transform of the amputated vertex function, with external legs at zero four-momentum.

It is easy to recognize in the above equations the Ward identities obtained by Glashow and Weinberg,[3] with a completely different method. The advantage of the present formulation is that *the chain of Ward identities emerges as the result of successive derivations of a single equation (25), which contains the basic information about the group properties of A without any reference to a specific model.*[19]

Equations (28)–(30) become identical to Eqs. (9), (11), and (12) in the following case: (i) the renormalization constants of the fields $\phi_i(x)$ may be considered equal within a very good approximation, and then put equal to 1, by a redefinition of the fields; (ii) the two-point functions Δ_{ij} are dominated by poles, so that at zero four-momentum one may neglect the continuum and obtain

$$\lim_{p^2\to 0}\tilde\Delta^{-1}{}_{ij}(p^2) = \lim_{p^2\to 0}(p^2\delta_{ij} + M^2{}_{ij}) = M^2{}_{ij}\,,$$

where $M^2{}_{ij}$ has the physical meaning of a mass-squared matrix.

In conclusion, if we introduce the abridged notation

$$\lambda = \lambda_i\,\hat\phi_i\,, \quad \epsilon = \epsilon_i\hat\phi_i\,,$$

where $\hat\phi_i$ is the unit vector in the direction i, the previous equations can be written, in the basis where the mass matrix is diagonal, in the following more convenient form:

$$\epsilon_i[\underline{G}^\alpha, \lambda]_i = 0\,, \tag{31}$$

$$m_{(i)}{}^2[\underline{G}^\alpha, \lambda]_i = [\underline{G}^\alpha, \epsilon]_i\,, \tag{32}$$

$$g_{ijk}[\underline{G}^\alpha, \lambda]_k = (m^2_{(i)} - m^2_{(j)})[\underline{G}^\alpha, \hat\phi_j]_{\bar i} \quad \text{(no sum over } i \text{ and } j)\,, \tag{33}$$

$$-g_{ijkl}[\underline{G}^\alpha, \lambda]_l = g_{ijp}[\underline{G}^\alpha, \hat\phi_k]_p + g_{jkp}[\underline{G}^\alpha, \hat\phi_i]_p + g_{ikp}[\underline{G}^\alpha, \hat\phi_j]_p\,. \tag{34}$$

Here $\bar i$ denotes the charge conjugate of the particle i and

$$-i[\underline{G}^\alpha, \hat\phi_l]_p = g^\alpha_{lp}$$

are the matrix elements of the generator $\underline{G}^\alpha$ in the representation spanned by the particles under consideration. The general Ward identity is

$$-g^{(n)}_{i_1 i_2 \cdots i_n}[\underline{G}^\alpha, \lambda]_{i_n} = g^{(n-1)}_{[i_1] i_2 \cdots i_{n-1} p}[\underline{G}^\alpha, \hat{\phi}_{i_1}]_p + g^{(n-1)}_{i_1 [i_2] \cdots i_{n-1} p}[\underline{G}^\alpha, \hat{\phi}_{i_2}]_p + \cdots + g^{(n-1)}_{i_1 i_2 \cdots [i_{n-1}] p}[\underline{G}^\alpha, \hat{\phi}_{i_{n-1}}]_p, \tag{35}$$

where $[i]$ means that the index i is to be omitted and $g^{(n)}_{i_1 i_2 \cdots i_n}$ is the nth-order coupling constant between the particles $i_1, i_2, \ldots, i_n$.

The application of the above treatment to specific physical examples, in particular SU(3)×SU(3), will be done in Secs. V, VI, and VII.

V. BREAKDOWN OF CHIRAL SU(3)×SU(3)

In the following we want to apply the previous general results to the case of chiral SU(3)×SU(3) symmetry.

In this case the physical situation indicates that the only fields with nonvanishing vacuum expectation value can be u_0, u_8, and u_3 [with the standard notation, where u_i and v_i, $i = 0, 1, \ldots, 8$, represent the scalar and pseudoscalar $0^\pm$ mesons, belonging to the $(3, \bar{3}) + (\bar{3}, 3)$ representation of SU(3)×SU(3)]. In addition, the most commonly accepted idea is that the strong-interaction Lagrangian is SU(3)×SU(3)-invariant except for a linear term transforming according to the $(3, \bar{3}) + (\bar{3}, 3)$ representation, namely,[3]

$$L = L_0 + \epsilon_0 u_0 + \epsilon_8 u_8 + \epsilon_3 u_3 .$$

The use of the functional method enables us to obtain results which hold for any quantum-field-theory model with a $(3, \bar{3}) + (\bar{3}, 3)$ linear breaking. No assumption is needed about the form of the quantum-field-theory Lagrangian and no reference is made to a specific Lagrangian model (like the σ model). In this way our results are *model-independent* and may be regarded as a check of SU(3)×SU(3) symmetry with the only assumption of $(3, \bar{3}) + (\bar{3}, 3)$ symmetry breaking. This has to be compared with the results of the many papers dealing with SU(3) σ model, where it is not clear which are the consequences of SU(3)×SU(3) alone and which are the results depending on the structure of the σ model. In addition, Eqs. (31)–(35) are not the consequence of a first-order approximation in the breaking parameters ϵ_i, which has repeatedly been questioned.[20] Therefore they are not affected by the difficulties arising from nonanalytic behavior of the theory with respect to the breaking parameters ϵ_i.[19]

In this case the physical limit of terms appearing in the functional equations is the following:

$$\epsilon_i(x) \to \begin{cases} \epsilon_i \neq 0 & \text{when } i \text{ corresponds to } u_0, u_8, u_3 \\ = 0 & \text{otherwise,} \end{cases}$$

$$\lambda_i(x) = \langle 0|\phi_i(x)|0\rangle \to \begin{cases} \lambda_i \neq 0 & \text{when } i \text{ corresponds to } u_0, u_8, u_3 \\ = 0 & \text{otherwise.} \end{cases}$$

A few comments concerning the above chain of Ward-like identities may be useful. Equation (32) generalizes the content of Goldstone's theorem. In the limit $\epsilon \to 0$, the "modes" i, for which $[\underline{G}^\alpha, \lambda]_i$ remain different from zero when $\epsilon \to 0$, correspond to Goldstone bosons:

$$m_{(i)}{}^2(\epsilon \to 0) = 0 \quad \text{if } [\underline{G}^\alpha, \lambda]_i \neq 0 .$$

To identify these would-be Goldstone bosons, one should know the behavior of λ as $\epsilon \to 0$, and this is in general a very delicate limit. As far as SU(3)×SU(3) symmetry breaking is concerned, one expects that the physical solution of Eq. (27) has the following properties:

$$\lim_{\epsilon \to 0} \lambda_0 \neq 0, \qquad \lim_{\epsilon \to 0} \lambda_8 \neq 0, \tag{36}$$

so that when $\epsilon \to 0$, all the pseudoscalar octet becomes massless. In general, however, there are solutions[18] for which Eq. (36) holds. For convenience, we will call "possible" Goldstone bosons all the particles i for which $[\underline{G}^\alpha, \lambda]_i \neq 0$, λ being taken at its physical value. With this terminology the chain of Ward identities give the nth-order coupling constants *involving at least one "possible" Goldstone boson* in terms of lower-order ones. Thus, by recurrence relations one may express coupling constants of arbitrary order in terms of the masses. The spontaneous breaking of the symmetry gives therefore strong restrictions on the coupling constants of arbitrary order with a "possible" Goldstone boson. Unfortunately, the coupling constants appearing in Eqs. (31)–(35) correspond to zero four-momentum and the extrapolation to the physical value is not trivial. We will discuss this point later. For the moment, we would like to remark that the chain of the Ward identities provides also a simple method for obtaining relations between masses (or in general between coupling constants of the same order). This is the case when a coupling constant $g^{(n)}$ of order n is expressed in several independent ways in terms of the coupling constants $g^{(n-1)}$. Then, by a direct comparison, some relations between these $g^{(n-1)}$ constants are obtained. For instance, by comparing the third-order constants appearing in the chain, we can obtain some relations between masses, mixing angles, and the parameters λ_i, as we shall discuss and exploit in detail in the following sections.

In Sec. VI we will consider the case

$$\epsilon_3 = 0 \text{ and } \lambda_3 = \langle u_3 \rangle_0 = 0 ,$$

i.e., the case of conserved isospin, for both $0^\pm$ and $1^\pm$ multiplets of mesons.

The more general case

$$\epsilon_3 \neq 0 \text{ and } \lambda_3 \neq 0 ,$$

where isospin is also broken, will be discussed in Sec. VII. In particular we shall discuss carefully the effect of these nonvanishing terms on the previous results. We will show that the corrections are actually negligible because ϵ_3 and λ_3 are rather small. We emphasize that this result is by no means trivial; it is known in fact that, in the case of the spontaneously broken symmetries, the presence of a very little term might, in principle, produce very large effects, in particular as far as mixing angles are concerned.

VI. BREAKING OF CHIRAL SU(3)×SU(3) DOWN TO SU(2)

A. Mass Splitting and Breaking Parameters

We consider now the case when ϵ_0, ϵ_8, and $\lambda_0 = \langle u_0 \rangle_0$, $\lambda_8 = \langle u_8 \rangle_0$ can be different from zero, whereas $\epsilon_3 = 0$ and $\lambda_3 = 0$.

The first Ward identity (31) shows simply the consistency of this choice of nonvanishing parameters. The second-order identity (32) gives

$$\pi = \frac{\sqrt{2}\,\epsilon_0 + \epsilon_8}{\sqrt{2}\,\lambda_0 + \lambda_8} , \tag{37}$$

$$K = \frac{2\sqrt{2}\,\epsilon_0 - \epsilon_8}{2\sqrt{2}\,\lambda_0 - \lambda_8} , \tag{38}$$

$$K_S = \frac{\epsilon_8}{\lambda_8} , \tag{39}$$

$$\eta = \frac{\sqrt{2}\,\epsilon_8 \sin\theta + (\sqrt{2}\,\epsilon_0 - \epsilon_8)\cos\theta}{\sqrt{2}\,\lambda_8 \sin\theta + (\sqrt{2}\,\lambda_0 - \lambda_8)\cos\theta} , \tag{40}$$

$$\eta' = \frac{\sqrt{2}\,\epsilon_8 \cos\theta - (\sqrt{2}\,\epsilon_0 - \epsilon_8)\sin\theta}{\sqrt{2}\,\lambda_8 \cos\theta - (\sqrt{2}\,\lambda_0 - \lambda_8)\sin\theta} , \tag{41}$$

where π means $(\text{mass})^2$ of the π, etc., K_S denotes the scalar partner of the K (sometimes denoted by κ in the literature), and the angle θ describes the η-η' mixing:

$$\begin{aligned} \hat{\eta} &= \hat{v}_8 \cos\theta + \hat{v}_0 \sin\theta , \\ \hat{\eta}' &= -\hat{v}_8 \sin\theta + \hat{v}_0 \cos\theta . \end{aligned} \tag{42}$$

There is a certain arbitrariness in choosing the input parameters in order to determine the remaining ones. A detailed discussion will be done in Secs. VI B–VI D. For the moment, we note that the PCAC (partial conservation of axial-vector current) hypothesis provides an independent way for evaluating the parameters λ_0 and λ_8 [and then ϵ_0 and ϵ_8, through Eqs. (37) and (38)]:

$$\begin{aligned} F_\pi &= (\tfrac{2}{3})^{1/2}(\sqrt{2}\,\lambda_0 + \lambda_8) , \\ F_K &= (\tfrac{2}{3})^{1/2}(\sqrt{2}\,\lambda_0 - \tfrac{1}{2}\lambda_8) . \end{aligned} \tag{43}$$

Thus, by using the experimental estimates $F_\pi = 0.96 m_\pi$, $F_K/F_\pi = 1.28$, one obtains[21] from the above equations

$$\begin{aligned} &\lambda_0 = 0.99 m_\pi , \quad \lambda_8 = -0.22 m_\pi , \\ &\epsilon_0 = 9.92 m_\pi{}^3 , \quad \epsilon_8 = -12.86 m_\pi{}^3 . \end{aligned} \tag{44}$$

Let us now examine the third-order Ward identities (33). In the case where the generator $\underline{G}^\alpha$ is one of the generators $(F_6 + iF_7)/\sqrt{2}$, F_3^5, or $(F_6^5 + iF_7^5)/\sqrt{2}$, Eq. (33) takes the simple form

$$g_{ijk} = (m_{(i)}{}^2 - m_{(j)}{}^2) C_{ijk} \quad (\text{no sum over } i \text{ and } j) . \tag{45}$$

In Table I we have collected the values of C_{ijk} for this case. Taking as $\underline{G}^\alpha$ the generator F_8^5, and putting

$$\begin{aligned} F_\eta &\equiv \frac{2}{\sqrt{3}}\lambda_8 \sin\theta + \left(\frac{2}{\sqrt{3}}\lambda_0 - (\tfrac{2}{3})^{1/2}\lambda_8\right)\cos\theta , \\ F_{\eta'} &\equiv \frac{2}{\sqrt{3}}\lambda_8 \cos\theta - \left(\frac{2}{\sqrt{3}}\lambda_0 - (\tfrac{2}{3})^{1/2}\lambda_8\right)\sin\theta , \end{aligned} \tag{46}$$

we obtain from the same Eq. (33) the following relations:

$$g_{\pi^0\pi_S^0\eta} F_\eta + g_{\pi^0\pi_S^0\eta'} F_{\eta'} = (\tfrac{2}{3})^{1/2}(\pi - \pi_S) , \tag{47}$$

$$g_{\overline{K}^0K_S^0\eta} F_\eta + g_{\overline{K}^0K_S^0\eta'} F_{\eta'} = -\tfrac{1}{2}(\tfrac{2}{3})^{1/2}(K - K_S) , \tag{48}$$

and

$$\begin{aligned} &g_{\eta\eta_S\eta} F_\eta + g_{\eta\eta_S\eta'} F_{\eta'} \\ &\quad = \sqrt{2}\,(\eta - \eta_S)[\cos\theta_S \sin(\theta - \bar{\theta}) + (\tfrac{2}{3})^{1/2}\cos\theta \sin\theta_S] , \\ &g_{\eta'\eta_S\eta} F_\eta + g_{\eta'\eta_S\eta'} F_{\eta'} \\ &\quad = \sqrt{2}\,(\eta' - \eta_S)[\cos\theta_S \cos(\theta - \bar{\theta}) - (\tfrac{2}{3})^{1/2}\sin\theta \sin\theta_S] , \\ &g_{\eta\eta_S'\eta} F_\eta + g_{\eta\eta_S'\eta'} F_{\eta'} \\ &\quad = \sqrt{2}\,(\eta - \eta_S')[-\sin\theta_S \sin(\theta - \bar{\theta}) + (\tfrac{2}{3})^{1/2}\cos\theta \cos\theta_S] , \\ &g_{\eta'\eta_S'\eta} F_\eta + g_{\eta'\eta_S'\eta'} F_{\eta'} \\ &\quad = -\sqrt{2}\,(\eta' - \eta_S')[\sin\theta_S \cos(\theta - \bar{\theta}) + (\tfrac{2}{3})^{1/2}\sin\theta \cos\theta_S] , \end{aligned} \tag{49}$$

where θ_S is the η_S-η_S' mixing angle defined as in Eq. (42), and

$$\cos\bar{\theta} = (\tfrac{2}{3})^{1/2} , \quad \sin\bar{\theta} = \frac{1}{\sqrt{3}} . \tag{50}$$

TABLE I. $g_{ijk}=(m_{(i)}^2-m_{(j)}^2)C_{ijk}$.

$\underline{G}^\alpha$	i	j	k	C_{ijk}	
$(F_6+iF_7)2^{-1/2}$	π^0	$\bar{K}^0$	K_S^0	$2^{-1/2}(F_\pi-F_K)^{-1}$	(I.1)
	η	$\bar{K}^0$	K_S^0	$-(\frac{3}{2})^{1/2}(F_\pi-F_K)^{-1}\cos\theta$	(I.2)
	η'	$\bar{K}^0$	K_S^0	$(\frac{3}{2})^{1/2}(F_\pi-F_K)^{-1}\sin\theta$	(I.3)
	π_S^0	$\bar{K}_S^0$	K_S^0	$2^{-1/2}(F_\pi-F_K)^{-1}$	(I.4)
	η_S	$\bar{K}_S^0$	K_S^0	$-(\frac{3}{2})^{1/2}(F_\pi-F_K)^{-1}\cos\theta_S$	(I.5)
	η_S'	$\bar{K}_S^0$	K_S^0	$(\frac{3}{2})^{1/2}(F_\pi-F_K)^{-1}\sin\theta_S$	(I.6)
	ρ^0	$\bar{K}^{*0}$	K_S^0	$2^{-1/2}(F_\pi-F_K)^{-1}$	(I.7)
	ϕ	$\bar{K}^{*0}$	K_S^0	$-(\frac{3}{2})^{1/2}(F_\pi-F_K)^{-1}\cos\theta_V$	(I.8)
	ω	$\bar{K}^{*0}$	K_S^0	$(\frac{3}{2})^{1/2}(F_\pi-F_K)^{-1}\sin\theta_V$	(I.9)
	A^0	$\bar{K}_A^0$	K_S^0	$2^{-1/2}(F_\pi-F_K)^{-1}$	(I.10)
	ϕ_A	$\bar{K}_A^0$	K_S^0	$-(\frac{3}{2})^{1/2}(F_\pi-F_K)^{-1}\cos\theta_A$	(I.11)
	ω_A	$\bar{K}_A^0$	K_S^0	$(\frac{3}{2})^{1/2}(F_\pi-F_K)^{-1}\sin\theta_A$	(I.12)
F_3^5	K^0	$\bar{K}_S^0$	π^0	$-2^{-1/2}F_\pi^{-1}$	(I.13)
	η	π_S^0	π^0	$2^{1/2}F_\pi^{-1}\sin(\theta+\bar{\theta})$	(I.14)
	η'	π_S^0	π^0	$2^{1/2}F_\pi^{-1}\cos(\theta+\bar{\theta})$	(I.15)
	π^0	η_S	π^0	$2^{1/2}F_\pi^{-1}\sin(\theta_S+\bar{\theta})$	(I.16)
	π^0	η_S'	π^0	$2^{1/2}F_\pi^{-1}\cos(\theta_S+\bar{\theta})$	(I.17)
	A^+	ρ^-	π^0	$-2^{1/2}iF_\pi^{-1}$	(I.18)
	$\bar{K}_A^0$	K^{*0}	π^0	$-2^{-1/2}iF_\pi^{-1}$	(I.19)
$(F_6^5+iF_7^5)2^{-1/2}$	π^0	$\bar{K}_S^0$	K^0	$-2^{-1/2}F_K^{-1}$	(I.20)
	η	$\bar{K}_S^0$	K^0	$6^{-1/2}F_K^{-1}(2^{3/2}\sin\theta-\cos\theta)$	(I.21)
	η'	$\bar{K}_S^0$	K^0	$6^{-1/2}F_K^{-1}(2^{3/2}\cos\theta+\sin\theta)$	(I.22)
	$\bar{K}^0$	π_S^0	K^0	$-2^{-1/2}F_K^{-1}$	(I.23)
	$\bar{K}^0$	η_S	K^0	$6^{-1/2}F_K^{-1}(2^{3/2}\sin\theta_S-\cos\theta_S)$	(I.24)
	$\bar{K}^0$	η_S'	K^0	$6^{-1/2}F_K^{-1}(2^{3/2}\cos\theta_S+\sin\theta_S)$	(I.25)
	A^0	$\bar{K}^{*0}$	K^0	$-2^{-1/2}iF_K^{-1}$	(I.26)
	$\bar{K}_A^0$	ρ^0	K^0	$2^{-1/2}iF_K^{-1}$	(I.27)
	$\bar{K}_A^0$	ϕ	K^0	$-(\frac{3}{2})^{1/2}iF_K^{-1}\cos\theta_V$	(I.28)
	$\bar{K}_A^0$	ω	K^0	$(\frac{3}{2})^{1/2}iF_K^{-1}\sin\theta_V$	(I.29)
	ϕ_A	$\bar{K}^{*0}$	K^0	$(\frac{3}{2})^{1/2}iF_K^{-1}\cos\theta_A$	(I.30)
	ω_A	$\bar{K}^{*0}$	K^0	$-(\frac{3}{2})^{1/2}iF_K^{-1}\sin\theta_A$	(I.31)

For the sake of completeness, we add some simple comments. First, it is easy to see that the use of other generators does not give further information. For example, starting from the generator $(F_6-iF_7)/\sqrt{2}$ instead of $(F_6+iF_7)/\sqrt{2}$, we would obtain the "charge conjugate" relations.

Similarly, starting from the "charged" generators $(F_4+iF_5)/\sqrt{2}$, etc., we would obtain the coupling constants between charged particles, e.g.,

$$g_{\pi^+K^-K_S^0}=g_{\pi^+K^0K_S^-}=\sqrt{2}\,g_{\pi^0K^-K_S^+}=-\sqrt{2}\,g_{\pi^0\bar{K}^0K_S^0}\,,$$

$$g_{\eta K^-K_S^+}=g_{\eta\bar{K}^0K_S^0}\,.$$

These relations are clearly a consequence of the residual SU(2) symmetry. All the other independent coupling constants not listed in Table I and involving at least one *possible* Goldstone boson are zero. The present method does not give any information about coupling constants involving only π_S, η_S, η'_S.

We note that Eq. (33) holds also if the fields labeled i and j do *not* belong to the same multiplet of the fields which are "responsible" for the breaking. In our case, e.g., the indices i and j can label the multiplet of the vector ($1^{\pm}$) mesons ρ, K^*, ϕ, ω and A, K_A, ϕ_A, ω_A which can be accommodated in the $(8+1,1)+(1,8+1)$ representation of SU(3) $\times$SU(3). In Table I we give also the coupling constants we obtain in this case (θ_V and θ_A are the mixing angles between ω and ϕ, and their axial-vector partners ω_A and ϕ_A, respectively). The Ward identity generated by F_8^5 is instead

$$g_{\overline{K}{}^0_A K^{*0}\eta}F_\eta + g_{\overline{K}{}^0_A K^{*0}\eta'}F_{\eta'} = -i(\tfrac{3}{2})^{1/2}(K_A - K^*) \,. \quad (51)$$

As in the case of the $0^{\pm}$ mesons, other coupling constants can be obtained from those appearing in Table I and Eq. (51), by simply taking into account the residual invariance SU(2) and charge conjugation.

B. η-η' Mixing Angle and K_S Mass

There is a first immediate use of the third-order Ward identities. As already noted, a given coupling constant can appear several times in the set of these identities, being related to different generators $\underline{G}^\alpha$. This is the case, e.g., of the constant $g_{\pi^0\overline{K}{}^0K_S^0}$ which appears in Eqs. (I.1), (I.13), and (I.20) of Table I. Then, an obvious comparison of the respective right-hand sides gives mass relations. In this way one gets

$$\frac{K-\pi}{F_\pi - F_K} = \frac{K-K_S}{F_\pi} = \frac{\pi - K_S}{F_K} \,. \quad (52)$$

Actually, the second identity is a consequence of the first one, and vice versa. This is a quite general result, i.e., only one relation can be obtained in these cases, since one of the three generators involved can be obtained from the commutator of the other two, and therefore "depends" on them. The previous relation (52) can be also obtained directly from Eqs. (37)–(39) and (43) and has been given also by other authors. Using the same trick for the coupling constants $g_{\eta\overline{K}{}^0K_S^0}$ [Eqs. (I.2), (I.21), and (48)] and $g_{\eta'\overline{K}{}^0K_S^0}$ [Eqs. (I.3), (I.22), and (48)], we obtain the two following independent relations:

$$2\sqrt{2}\,\tan\theta - 1 = 3\frac{(K_S-\pi)(\eta-K)}{(K-\pi)(\eta-K_S)} \,, \quad (53)$$

$$2\sqrt{2}\,\cot\theta + 1 = -3\frac{(K_S-\pi)(\eta'-K)}{(K-\pi)(\eta'-K_S)} \,, \quad (54)$$

having used Eq. (52). A very nice feature of these two relations is that they *do not involve the parameters* ϵ *and* λ.[22] The above equations look suitable for discussing the K_S mass and the mixing angle θ (for both of which the experimental values are not yet well established). In particular, the equation for K_S resulting from (53) and (54) is

$$(K_S-\pi)^2\left[8+3\frac{(\eta-\pi)(\eta'-\pi)}{(K-\pi)^2} - 4\frac{\eta+\eta'-2\pi}{K-\pi}\right] + 2(K_S-\pi)\left[\frac{(\eta-\pi)(\eta'-\pi)}{K-\pi} - 2(\eta+\eta'-2\pi)\right] + 3(\eta-\pi)(\eta'-\pi) = 0 \,. \quad (55)$$

Using the masses of $\pi(134.97)$, $K(497.76)$, $\eta(548.8)$, and $\eta'(957.7)$, we obtain the following solutions for the K_S mass:

$$m_{K_S} = 938 \text{ MeV}\,, \quad m_{K_S} = 698 \text{ MeV}\,,$$

and correspondingly

$$\theta = -48'\,, \quad \theta = -15^\circ \,.$$

These results are very near to those obtained by Glashow[23]; the values of mass of K_S, however, do not appear to be supported by the physical situation, which seems to suggest a K_S mass in the region 1080–1260 MeV. Actually, the *previous Eq. (55) is somewhat "unstable" in the sense that very small variations of some masses yield a relatively large change in the* K_S *mass*. In fact, the variation δK_S around the previous value $K_S = 938$ MeV may be expressed in terms of the variations of the other masses appearing in Eq. (55) by the following relation:

$$\delta K_S = 33.0\,\delta K - 32.4\,\delta\eta + 1.3\,\delta\eta' + 0.7\,\delta\pi \,.$$

This shows that K_S is relatively unaffected by variations of η' and π; instead, a variation of only 1% in the masses of η and K yields a change of about 22% in the mass of K_S. For instance, putting in Eq. (55) $K = 503$ MeV and $\eta = 544$ MeV, we obtain

$$m_{K_S} = 1148 \text{ MeV} \,.$$

With these values, the angle θ turns out to be 5° 50′. A better determination of θ will be given in Sec. VI D.

C. PCAC Coupling Constants

In Sec. VI A we have used the experimental estimates of F_π and F_K to determine the breaking parameters ϵ_0, ϵ_8. Now we can check the consistency of the theory. Since we have relations not involving ϵ_0 and ϵ_8 we may determine F_π and F_K from the values of the masses. In particular we can evaluate, by means of Eq. (52), the ratio F_K/F_π. By varying the mass of K_S between the values 938 and 1148 MeV, we obtain

$$F_K/F_\pi = 1.36 - 1.22\,,$$

very near to the experimental value and the PCAC hypothesis. Equivalently, one has

$$\lambda_8/\lambda_0 = -(0.28 - 0.18)\,.$$

This, in turn, enables us to determine the symmetry-breaking parameters ϵ_0, ϵ_8 by using Eqs. (37) and (38). We get

$$c = \epsilon_8/\epsilon_0 = -(1.30 - 1.29)\,,$$

$$\epsilon_0/\lambda_0 = (10.16 - 9.94) m_\pi^{\,2}\,.$$

It is worthwhile to stress the "stability" of the Gell-Mann parameter c under variations of K_S mass in the range we have considered. In addition, we can see that increasing the K_S mass reduces the breaking of SU(3) symmetry. Finally we want to emphasize at this point that these values are obtained in a way which is completely self-contained in our approach.

D. Generalized GMO Mass Formula

The GMO mass formula for pseudoscalar mesons is usually derived under the approximation in which particle states may be classified according to SU(3) multiplets (apart from the well-known singlet-octet mixing). This means that the noninvariance of the vacuum under SU(3) is not taken into account and it implies the approximation $\lambda_8 = 0$. This kind of approximation has also been used by Gell-Mann, Oakes, and Renner.[3] According to our previous analysis, however, it turns out that λ_8, though small, is not negligible with respect to λ_0 ($\lambda_8/\lambda_0 \simeq 20\%$), and one might in principle expect reasonable corrections to the GMO formula. It seems therefore interesting to investigate how the GMO formula gets corrected by the nonvanishing of λ_8.

A generalized GMO formula taking into account the nonvanishing of λ_8 may be obtained by using our Ward identities (52)–(54). Putting

$$F_{K_S} = (\tfrac{3}{2})^{1/2}\lambda_8 = F_\pi - F_K\,,$$

we get in fact

$$F_\pi \pi = F_K K + F_{K_S} K_S\,, \tag{56}$$

$$F_\eta \eta = F_K K \cos\theta - F_{K_S} K_S \cos(\theta + 2\bar{\theta})\,, \tag{57}$$

$$F_{\eta'} \eta' = -F_K K \sin\theta + F_{K_S} K_S \sin(\theta + 2\bar{\theta})\,, \tag{58}$$

which are essentially equivalent to Eqs. (52)–(54) given before.

By taking suitable combinations of the above equations, we get

$$4F_K K = 3\eta f_8 + F_\pi \pi + 3(\eta' - \eta)\sin\theta(f_8 \sin\theta - f_8' \cos\theta)\,, \tag{59}$$

$$2\sqrt{2}\,(F_K K - F_\pi \pi) + 3\eta f_8' = 3(\eta' - \eta)\cos\theta(f_8 \sin\theta - f_8' \cos\theta)\,, \tag{60}$$

where

$$f_8 = (\tfrac{2}{3})^{1/2}(\sqrt{2}\,\lambda_0 - \lambda_8)\,,$$

$$f_8' = \frac{2}{\sqrt{3}}\lambda_8\,.$$

Equation (59) is the generalized GMO formula and shows very well why the GMO formula works. The corrections are in fact proportional to $\sin^2\theta$ or $\lambda_8 \sin\theta$, which are both very small. Moreover, in the limit $\lambda_8 \simeq 0$, $F_\pi = F_K = f_8$ and one obtains

$$4K - 3\eta - \pi = 3(\eta' - \eta)\sin^2\theta \simeq 0\,,$$

which is the original GMO formula and works reasonably well. We will discuss later the validity of the approximation $\lambda_8 \simeq 0$.

The approximation of neglecting λ_8 does *not* work, however, in Eq. (60), where the terms on the right-hand side are proportional to $\sin\theta$ and λ_8. They are not negligible with respect to the left-hand side: otherwise one would get

$$F_K K = F_\pi \pi\,,$$

in bad agreement with the present data. One may show that if the renormalization constants Z_K, Z_π are properly taken into account the above equation is the relation found by Khuri[24]:

$$Z_K^{\;-1} F_K K = Z_\pi^{\;-1} F_\pi \pi\,.$$

This shows that very mild looking assumptions, $\lambda_8 \simeq 0$, $\theta \simeq 0$, may lead to very questionable results. This point does not seem to have been fully realized in many of the papers about SU(3)×SU(3) symmetry breaking. In particular *the fact that the GMO formula works so well does not imply that the SU(3) noninvariance of the vacuum may always be neglected.*

We are now in the position of checking the validity of Eq. (59), since we may determine θ from Eq. (60). It is worthwhile to stress that the situation is quite different from the SU(3) case. There, the mixing angle θ is a parameter extraneous to the theory and it is introduced just to make the GMO formula work exactly. Here, θ may be determined by an independent equation [Eq. (60)] and the degree of accuracy of the generalized GMO formula may be checked consistently. In addition by using

Eqs. (59) and (60) the ambiguity in identifying the ninth pseudoscalar meson may be resolved. In SU(3) this is not possible because θ and η' are not unambiguously determined by the theory. Their "determination" is done in an indirect way through the GMO formula and since this is only one equation one cannot determine both θ and η'.

(i) Determination of θ. This is easily done by writing Eq. (60) in the form

$$2\sqrt{2}\,(F_K K - F_\pi \pi) + 3\eta' f_8' = 3(\eta' - \eta)\sin\theta(f_8\cos\theta + f_8'\sin\theta)\,.$$

Since θ is small one gets the approximate[25] formula

$$\sin\theta = \frac{2\sqrt{2}\,(F_K K - F_\pi\pi) + 3f_8'\eta'}{3(\eta' - \eta)f_8} \qquad (61)$$

yielding

$$\sin\theta \simeq 0.052 \pm 0.019\,,$$

i.e.,

$$\theta \simeq 2°58' \pm 1°08\,.$$

These values are obtained by using as input parameters $K(0.2458\ \text{GeV}^2 \pm 0.8\%)$, $\pi(0.0188\ \text{GeV}^2 \pm 3.3\%)$, $\eta(0.30118\ \text{GeV}^2)$, $\eta'(0.9168\ \text{GeV}^2)$, and $F_K/F_\pi = 1.27 \pm 0.02$.

One might be surprised of such a small value of θ to be compared with the SU(3) "determination": $\theta \simeq 11°$. The reason is that in SU(3), θ is determined through the formula

$$\sin^2\theta = \frac{4K - 3\eta - \pi}{3(\eta' - \eta)}\,,$$

obtained under the assumption of SU(3) invariance of the vacuum.[26] The corrections due to the nonvanishing of λ_8 are however not negligible if one wants to determine θ. This is easily seen by writing Eq. (59) in the form

$$\sin^2\theta - \frac{f_8'}{f_8}\sin\theta\cos\theta = \frac{4F_K K - 3f_8\eta - F_\pi\pi}{3(\eta' - \eta)f_8}\,.$$

(ii) Accuracy of the generalized GMO formula. We may now check the validity of Eq. (59). By using $\sin\theta = 0.032$, $\pi = 0.01885\ \text{GeV}^2$, $F_K/F_\pi = 1.29$, we get $K = 0.2500\ \text{GeV}^2$. The agreement is as good as it could be. Deviations of the order 1–2% are in fact of the same order as the electromagnetic corrections which are not taken into account in this formulation. In this respect it seems reasonable to conclude that *the generalized GMO formula is a much more precise relation than the original GMO formula.* Of course, the smallness of θ enables us to write formula (59) in the simplified form

$$4F_K K = 3f_8\eta + F_\pi\pi\,.$$

(iii) Identification of η'. One actually may ask whether the particle η' in the above formulas is the $X^0(958)$ or the $E(1422)$ meson. To resolve this ambiguity, one could try to fit with the $E(1422)$ the Eqs. (59) and (60) or, more directly, the following equation:

$$\left(1 - 2\frac{F_K}{F_\pi}\,\frac{\eta - K}{\eta - \pi}\right)\left(1 - 2\frac{F_K}{F_\pi}\,\frac{\eta' - K}{\eta' - \pi}\right) = -\frac{1}{2}\,, \qquad (62)$$

obtained by eliminating θ from (59) and (60). It is easily seen that, with this choice, these equations would exhibit an imprecision of about 5–6%. This is very much outside the precision 1–2% according to which Eqs. (59) and (60) seem to work. Therefore if one takes this precision seriously, η' cannot be identified with the $E(1422)$.[27] As a matter of fact, the *whole* scheme seems to exhibit such a high precision (1–2%) that one is tempted to conclude that in fact SU(3)×SU(3) symmetry breaking is more precise than SU(3) symmetry breaking. In order to accommodate the $E(1422)$ one should give up this very striking consistence of the set of sum rules derived and admit deviations of the order of 5–6%.

In conclusion, in spite of the uncertainty due to the K_S mass, the whole picture emerging from the analysis of this section strongly supports the SU(3)×SU(3) symmetry breaking according to the $(3,\bar{3}) + (\bar{3},3)$ representation. The whole theoretical scheme seems in fact to exhibit a very good internal consistency and the agreement with the experimental data is always good.

E. Branching Ratios

A full analysis of the coupling constants derived by the Ward identities and in particular their extrapolation to the physical mass is outside the scope of the present paper. In general we cannot expect to obtain the actual decay rates using naively these coupling constants. For instance, we would have $\Gamma(\rho \to \pi\pi) = 0$, in contrast with experiments. A better agreement can possibly appear if one takes the ratio of "similar" decay rates, e.g., $R = \Gamma(\pi_S \to KK)/\Gamma(\pi_S \to \eta\pi)$ or $\Gamma(\eta_S \to \pi\pi)/\Gamma(\eta_S \to KK)$. In the first case we obtain $R \approx 0.18$ whereas experiments suggest $R \lesssim 0.2$. In the second case, the mixing angle θ_S is involved: We obtain for the ratio a value near to the experimental data if we take $\theta_S \approx -20°$, a value which is not in contrast with the Gell-Mann–Okubo mass formula for the 0^+ nonet (with $m_{\eta_S} = 1070$ MeV and $m_{\eta_S'} = 750$ MeV).

VII. BREAKING OF SU(3)×SU(3) SYMMETRY DOWN TO U(1)

In this section we want to consider the case in which the strong-interaction Lagrangian and the vacuum state are not isospin invariant.

This amounts to saying that, in addition to the parameters ϵ_0, ϵ_8, λ_0, λ_8 previously considered, the other nonzero quantities ϵ_3 and λ_3 should be introduced. The simultaneous nonvanishing of these six parameters is consistent with the first-order Ward identity (31). According to the second Ward identity (32), an extended mixing between π^0, η, and η' (and between π^0_S, η_S, η'_S) is to be expected. For describing these mixings, let us put

$$\hat{v}_8 = a_{88}\hat{\eta} + a_{80}\hat{\eta}' + a_{83}\hat{\pi}_0, \quad \hat{u}_8 = b_{88}\hat{\eta}_S + b_{80}\hat{\eta}'_S + b_{83}\hat{\pi}^0_S,$$

$$\hat{v}_0 = a_{08}\hat{\eta} + a_{00}\hat{\eta}' + a_{03}\hat{\pi}_0, \quad \hat{u}_0 = b_{08}\hat{\eta}_S + b_{00}\hat{\eta}'_S + b_{03}\hat{\pi}^0_S,$$

$$\hat{v}_3 = a_{38}\hat{\eta} + a_{30}\hat{\eta}' + a_{33}\hat{\pi}_0, \quad \hat{u}_3 = b_{38}\hat{\eta}_S + b_{30}\hat{\eta}'_S + b_{33}\hat{\pi}^0_S, \tag{63}$$

where

$$\begin{pmatrix} a_{88} & a_{80} & a_{83} \\ a_{08} & a_{00} & a_{03} \\ a_{38} & a_{30} & a_{33} \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\beta & -\cos\alpha\sin\beta\cos\gamma - \sin\alpha\cos\beta & \cos\alpha\sin\gamma \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta & -\sin\alpha\sin\beta\cos\gamma + \cos\alpha\cos\beta & \sin\alpha\sin\gamma \\ -\cos\beta\sin\gamma & \sin\beta\sin\gamma & \cos\gamma \end{pmatrix}. \tag{64}$$

Similarly, one introduces a mixing matrix b_{ij} between the scalar mesons.

The second-order Ward identity gives now

$$\pi^+_S = \frac{\epsilon_3}{\lambda_3} = \pi^-_S,$$

$$K^+_S = \frac{\sqrt{3}\,\epsilon_8 + \epsilon_3}{\sqrt{3}\,\lambda_8 + \lambda_3} = K^-_S,$$

$$K^0_S = \frac{\sqrt{3}\,\epsilon_8 - \epsilon_3}{\sqrt{3}\,\lambda_8 - \lambda_3} = \bar{K}^0_S,$$

$$\pi^+ = \frac{\sqrt{2}\,\epsilon_0 + \epsilon_8}{\sqrt{2}\,\lambda_0 + \lambda_8} = \pi^-,$$

$$K^+ = \frac{2\sqrt{2}\,\epsilon_0 + \sqrt{3}\,\epsilon_3 - \epsilon_8}{2\sqrt{2}\,\lambda_0 + \sqrt{3}\,\lambda_3 - \lambda_8} = K^-,$$

$$K^0 = \frac{2\sqrt{2}\,\epsilon_0 - \sqrt{3}\,\epsilon_3 - \epsilon_8}{2\sqrt{2}\,\lambda_0 - \sqrt{3}\,\lambda_3 - \lambda_8} = \bar{K}^0, \tag{65}$$

$$\pi^0 = \frac{(\sqrt{2}\,\epsilon_0 + \epsilon_8)a_{33} + \epsilon_3(a_{83} + \sqrt{2}\,a_{03})}{(\sqrt{2}\,\lambda_0 + \lambda_8)a_{33} + \lambda_3(a_{83} + \sqrt{2}\,a_{03})},$$

$$\eta = \frac{(\sqrt{2}\,\epsilon_0 + \epsilon_8)a_{38} + \epsilon_3(a_{88} + \sqrt{2}\,a_{08})}{(\sqrt{2}\,\lambda_0 + \lambda_8)a_{38} + \lambda_3(a_{88} + \sqrt{2}\,a_{08})},$$

$$\eta' = \frac{(\sqrt{2}\,\epsilon_0 + \epsilon_8)a_{30} + \epsilon_3(a_{80} + \sqrt{2}\,a_{00})}{(\sqrt{2}\,\lambda_0 + \lambda_8)a_{30} + \lambda_3(a_{80} + \sqrt{2}\,a_{00})},$$

$$\pi^0 = \frac{\sqrt{2}\,\epsilon_8 a_{03} + (\sqrt{2}\,\epsilon_0 - \epsilon_8)a_{83} + \epsilon_3 a_{33}}{\sqrt{2}\,\lambda_8 a_{03} + (\sqrt{2}\,\lambda_0 - \lambda_8)a_{83} + \lambda_3 a_{33}},$$

$$\eta = \frac{\sqrt{2}\,\epsilon_8 a_{08} + (\sqrt{2}\,\epsilon_0 - \epsilon_8)a_{88} + \epsilon_3 a_{38}}{\sqrt{2}\,\lambda_8 a_{08} + (\sqrt{2}\,\lambda_0 - \lambda_8)a_{88} + \lambda_3 a_{38}},$$

$$\eta' = \frac{\sqrt{2}\,\epsilon_8 a_{00} + (\sqrt{2}\,\epsilon_0 - \epsilon_8)a_{80} + \epsilon_3 a_{30}}{\sqrt{2}\,\lambda_8 a_{00} + (\sqrt{2}\,\lambda_0 - \lambda_8)a_{80} + \lambda_3 a_{30}}.$$

As discussed in Sec. VI, the third-order Ward identities provide a simpler way of getting relations involving only physically measurable quantities.

In this way one gets the following relations (see the Appendix for details):

$$\frac{K^0 - K^-_S}{F_\pi} = \frac{K^0 - \pi^+}{F_\pi - F_K + (1/\sqrt{2})\lambda_3}, \tag{66}$$

$$\frac{K^- - K^0_S}{F_\pi} = \frac{K^- - \pi^+}{F_\pi - F_K - (1/\sqrt{2})\lambda_3}, \tag{67}$$

$$\frac{K^0 - \pi^+_S}{F_K + (1/\sqrt{2})\lambda_3} = \frac{K^0 - K^-}{\sqrt{2}\,\lambda_3}. \tag{68}$$

By eliminating the parameters F_π, F_K, and λ_3 from (66)–(68) one may get the following mass relation:

$$(K^- - \pi^+)(K^0 - K^-_S)(K^0_S - \pi^+_S) = (K^0 - \pi^+)(K^- - K^0_S)(K^-_S - \pi^+_S). \tag{69}$$

Similarly, one obtains relations involving π^0, η, η'. They involve the mixing angles and are listed in the Appendix [Eqs. (A6)–(A10)].

We may now discuss in detail the implications of the isospin-breaking parameter λ_3. From Eqs. (66)–(68) we obtain

$$\frac{\sqrt{2}\,\lambda_3}{F_\pi} = (K^- - K^0)\frac{(K^0_S - \pi^+)}{(K^0_S - K^-)(\pi^+_S - K^0)}, \tag{70}$$

which expresses λ_3 in terms of the *strong* mass difference $K^- - K^0$. Even if this difference at this level is not known (and, in addition, some uncertainty exists about the masses of K_S and π_S), one can roughly expect a value of the order $10^{-2}F_\pi$ for λ_3. A more refined evaluation of λ_3 may be given in the following way. Using Eqs. (A7) and (A8), we obtain

TABLE II. $g_{ijk}=(m_{(i)}^2-m_{(j)}^2)C_{ijk}$.

G^α	i	j	k	C_{ijk}	
$(F_1+iF_2)2^{-1/2}$	K^0	K^-	π_S^+	$2^{-1/2}\lambda_3^{-1}$	(II.1)
	π^0	π^-	π_S^+	$-\lambda_3^{-1}a_{33}$	(II.2)
	η	π^-	π_S^+	$-\lambda_3^{-1}a_{38}$	(II.3)
	η'	π^-	π_S^+	$-\lambda_3^{-1}a_{30}$	(II.4)
	K_S^0	K_S^-	π_S^+	$2^{-1/2}\lambda_3^{-1}$	(II.5)
	π_S^0	π_S^-	π_S^+	$-\lambda_3^{-1}b_{33}$	(II.6)
	η_S	π_S^-	π_S^+	$-\lambda_3^{-1}b_{38}$	(II.7)
	η_S'	π_S^-	π_S^+	$-\lambda_3^{-1}b_{30}$	(II.8)
$(F_4+iF_5)2^{-1/2}$	π^-	$\bar{K}^0$	K_S^+	$-(F_\pi-F_K+2^{-1/2}\lambda_3)^{-1}$	(II.9)
	π^0	K^-	K_S^+	$-2^{-1/2}(F_\pi-F_K+2^{-1/2}\lambda_3)^{-1}(a_{33}+3^{1/2}a_{83})$	(II.10)
	η	K^-	K_S^+	$-2^{-1/2}(F_\pi-F_K+2^{-1/2}\lambda_3)^{-1}(a_{38}+3^{1/2}a_{88})$	(II.11)
	η'	K^-	K_S^+	$-2^{-1/2}(F_\pi-F_K+2^{-1/2}\lambda_3)^{-1}(a_{30}+3^{1/2}a_{80})$	(II.12)
	π_S^-	$\bar{K}_S^0$	K_S^+	$-(F_\pi-F_K+2^{-1/2}\lambda_3)^{-1}$	(II.13)
	π_S^0	K_S^-	K_S^+	$-2^{-1/2}(F_\pi-F_K+2^{-1/2}\lambda_3)^{-1}(b_{33}+3^{1/2}b_{83})$	(II.14)
	η_S	K_S^-	K_S^+	$-2^{-1/2}(F_\pi-F_K+2^{-1/2}\lambda_3)^{-1}(b_{38}+3^{1/2}b_{88})$	(II.15)
	η_S'	K_S^-	K_S^+	$-2^{-1/2}(F_\pi-F_K+2^{-1/2}\lambda_3)^{-1}(b_{30}+3^{1/2}b_{80})$	(II.16)
$(F_6+iF_7)2^{-1/2}$	π^+	K^-	K_S^0	$-(F_\pi-F_K-2^{-1/2}\lambda_3)^{-1}$	(II.17)
	π^0	$\bar{K}^0$	K_S^0	$2^{-1/2}(F_\pi-F_K-2^{-1/2}\lambda_3)^{-1}(a_{33}-3^{1/2}a_{83})$	(II.18)
	η	$\bar{K}^0$	K_S^0	$2^{-1/2}(F_\pi-F_K-2^{-1/2}\lambda_3)^{-1}(a_{38}-3^{1/2}a_{88})$	(II.19)
	η'	$\bar{K}^0$	K_S^0	$2^{-1/2}(F_\pi-F_K-2^{-1/2}\lambda_3)^{-1}(a_{30}-3^{1/2}a_{80})$	(II.20)
	π_S^+	K_S^-	K_S^0	$-(F_\pi-F_K-2^{-1/2}\lambda_3)^{-1}$	(II.21)
	π_S^0	$\bar{K}_S^0$	K_S^0	$2^{-1/2}(F_\pi-F_K-2^{-1/2}\lambda_3)^{-1}(b_{33}-3^{1/2}b_{83})$	(II.22)
	η_S	$\bar{K}_S^0$	K_S^0	$2^{-1/2}(F_\pi-F_K-2^{-1/2}\lambda_3)^{-1}(b_{38}-3^{1/2}b_{88})$	(II.23)
	η_S'	$\bar{K}_S^0$	K_S^0	$2^{-1/2}(F_\pi-F_K-2^{-1/2}\lambda_3)^{-1}(b_{30}-3^{1/2}b_{80})$	(II.24)

$$|a_{38}|\leq\left|\frac{\eta-\pi_S^-}{\eta-\pi^-}\,\frac{\sqrt{2}\,\lambda_3}{F_\pi}\right|=2.60\left|\frac{\sqrt{2}\,\lambda_3}{F_\pi}\right|, \tag{71}$$

$$|a_{30}|\leq\left|\frac{\eta'-\pi_S^-}{\eta'-\pi^-}\,\frac{\sqrt{2}\,\lambda_3}{F_\pi}\right|=0.13\left|\frac{\sqrt{2}\,\lambda_3}{F_\pi}\right|, \tag{72}$$

where we have put $m_{\pi_S}=1016$ MeV [the right-hand side of both inequalities slightly decreases if we use instead $\pi_S=\delta(962\text{ MeV})$]. The smallness of both matrix elements a_{38} and a_{30} implies immediately that the angle γ between π^0 and the direction $\hat{v}_3$ is very small (and therefore there is a very little mixing of π^0). Moreover, by using Eqs. (71) and (72), one gets

$$\tan\gamma\leq 2.6\left|\frac{\sqrt{2}\,\lambda_3}{F_\pi}\right|,\qquad \left|\frac{a_{83}+\sqrt{2}\,a_{03}}{\sqrt{3}\,a_{33}}\right|\leq 2.6\left|\frac{\sqrt{2}\,\lambda_3}{F_\pi}\right|. \tag{73}$$

Finally, from Eqs. (70), (73), and (A6), we get

$$\left|\frac{\pi^0-\pi^-}{\pi^-}\right|\leq 21.7\left|\frac{K^--K^0}{K^-}\right|^2, \tag{74}$$

where we have put $m_{K_S}=1100$ MeV. For any reasonable value of the strong mass difference K^0-K^-, this relation indicates that the strong mass difference $\pi^0-\pi^-$ is extremely smaller than its physically observed value (it is of the *second* order with respect to the difference K^0-K^-, or, equivalently, with respect to λ_3/F_π). Then by appealing to Dashen's theorem

$$|K^0-K^-|_{\text{e.m.}}=|\pi^0-\pi^-|_{\text{e.m.}},$$

one may put

$$|K^0-K^-|_{\text{strong}}=|K^0-K^-|_{\text{phys}}+|\pi^--\pi^0|_{\text{phys}},$$

TABLE II. *(continued)*

G^α	i	j	k	C_{ijk}	
$(F_1^5+iF_2^5)2^{-1/2}$	K^0	K_S^-	π^+	F_π^{-1}	(II.25)
	K^-	K_S^0	π^+	F_π^{-1}	(II.26)
	π^0	π_S^-	π^+	$(\frac{2}{3})^{1/2}F_\pi^{-1}(a_{83}+2^{1/2}a_{03})$	(II.27)
	η	π_S^-	π^+	$(\frac{2}{3})^{1/2}F_\pi^{-1}(a_{88}+2^{1/2}a_{08})$	(II.28)
	η'	π_S^-	π^+	$(\frac{2}{3})^{1/2}F_\pi^{-1}(a_{80}+2^{1/2}a_{00})$	(II.29)
	π^-	π_S^0	π^+	$(\frac{2}{3})^{1/2}F_\pi^{-1}(b_{83}+2^{1/2}b_{03})$	(II.30)
	π^-	η_S	π^+	$(\frac{2}{3})^{1/2}F_\pi^{-1}(b_{88}+2^{1/2}b_{08})$	(II.31)
	π^-	η'_S	π^+	$(\frac{2}{3})^{1/2}F_\pi^{-1}(b_{80}+2^{1/2}b_{00})$	(II.32)
$(F_4^5+iF_5^5)2^{-1/2}$	$\bar{K}_0$	π_S^-	K^+	$(F_K+2^{-1/2}\lambda_3)^{-1}$	(II.33)
	π^-	$\bar{K}_S^0$	K^+	$(F_K+2^{-1/2}\lambda_3)^{-1}$	(II.34)
	π^0	K_S^-	K^+	$6^{-1/2}(F_K+2^{-1/2}\lambda_3)^{-1}(2^{3/2}a_{03}+3^{1/2}a_{33}-a_{83})$	(II.35)
	η	K_S^-	K^+	$6^{-1/2}(F_K+2^{-1/2}\lambda_3)^{-1}(2^{3/2}a_{08}+3^{1/2}a_{38}-a_{88})$	(II.36)
	η'	K_S^-	K^+	$6^{-1/2}(F_K+2^{-1/2}\lambda_3)^{-1}(2^{3/2}a_{00}+3^{1/2}a_{30}-a_{80})$	(II.37)
	K^-	π_S^0	K^+	$6^{-1/2}(F_K+2^{-1/2}\lambda_3)^{-1}(2^{3/2}b_{03}+3^{1/2}b_{33}-b_{83})$	(II.38)
	K^-	η_S	K^+	$6^{-1/2}(F_K+2^{-1/2}\lambda_3)^{-1}(2^{3/2}b_{08}+3^{1/2}b_{38}-b_{88})$	(II.39)
	K^-	η'_S	K^+	$6^{-1/2}(F_K+2^{-1/2}\lambda_3)^{-1}(2^{3/2}b_{00}+3^{1/2}b_{30}-b_{80})$	(II.40)
$(F_6^5+iF_7^5)2^{-1/2}$	K^-	π_S^+	K^0	$(F_K-2^{-1/2}\lambda_3)^{-1}$	(II.41)
	π^+	K_S^-	K^0	$(F_K-2^{-1/2}\lambda_3)^{-1}$	(II.42)
	π^0	$\bar{K}_S^0$	K^0	$6^{-1/2}(F_K-2^{-1/2}\lambda_3)^{-1}(2^{3/2}a_{03}-3^{1/2}a_{33}-a_{83})$	(II.43)
	η	$\bar{K}_S^0$	K^0	$6^{-1/2}(F_K-2^{-1/2}\lambda_3)^{-1}(2^{3/2}a_{08}-3^{1/2}a_{38}-a_{88})$	(II.44)
	η'	$\bar{K}_S^0$	K^0	$6^{-1/2}(F_K-2^{-1/2}\lambda_3)^{-1}(2^{3/2}a_{00}-3^{1/2}a_{30}-a_{80})$	(II.45)
	$\bar{K}^0$	π_S^0	K^0	$6^{-1/2}(F_K-2^{-1/2}\lambda_3)^{-1}(2^{3/2}b_{03}-3^{1/2}b_{33}-b_{83})$	(II.46)
	$\bar{K}^0$	η_S	K^0	$6^{-1/2}(F_K-2^{-1/2}\lambda_3)^{-1}(2^{3/2}b_{08}-3^{1/2}b_{38}-b_{88})$	(II.47)
	$\bar{K}^0$	η'_S	K^0	$6^{-1/2}(F_K-2^{-1/2}\lambda_3)^{-1}(2^{3/2}b_{00}-3^{1/2}b_{30}-b_{80})$	(II.48)

where the subscript "strong" denotes the contribution due to $\epsilon_3 u_3$. With this assumption, we get from Eq. (70)

$$\lambda_3/F_\pi = -(5.3-7.1)\times 10^{-3},$$

respectively, when $K_S = 1200$ MeV, $\pi_S = 1016$ MeV and $K_S = 938$ MeV, $\pi_S = 962$ MeV. The corresponding values of ϵ_3, from Eq. (65), are

$$\epsilon_3 = -(0.28-0.33)m_\pi^{\ 3}.$$

It is immediately seen that, due to the smallness of the parameter λ_3, the quantities F_π and F_K (or λ_0 and λ_8) and ϵ_0 and ϵ_8, as evaluated in Sec. VI, are practically unaffected by the presence of the new term λ_3. We can obtain also a more accurate estimation for the mixing angle γ between π^0 and the direction $\hat{v}_3$:

$$|\gamma| \leq 1°30'.$$

Now at the limit $\gamma = 0$, one has from Eq. (64) $\alpha+\beta = \theta$, where θ is the angle defined in Eq. (42) for the η-η' mixing. Thus one may easily show that also this angle is not appreciably changed by the term λ_3.

APPENDIX

We give here some details concerning the case $\epsilon_3 \neq 0$, $\lambda_3 \neq 0$, considered in Sec. VII.

If $\underline{G}^\alpha$ is one of the generators $(F_1+iF_2)/\sqrt{2}$, $(F_4+iF_5)/\sqrt{2}$, $(F_6+iF_7)/\sqrt{2}$, $(F_1^5+iF_2^5)/\sqrt{2}$, $(F_4^5+iF_5^5)/\sqrt{2}$, $(F_6^5+iF_7^5)/\sqrt{2}$, the third-order Ward identities [Eq. (33)] acquire the simple form of Eq. (45), and the corresponding values of C_{ijk} can be found in Table II. In this table we have used the definition of F_π and F_K already given (43) in Sec. VI A. Actually, one could put, for instance,

$$F_{K^+}=F_K+\frac{1}{\sqrt{2}}\lambda_3\,,\qquad F_{K^0}=F_K-\frac{1}{\sqrt{2}}\lambda_3\,; \tag{A1}$$

however, the discussion of the limit where λ_3 is very small is simplified if the parameters F_π and F_K are used. The Ward identities generated by F_3^5 and F_8^5 are of the following type:

$$g_{ij\pi^0}\mathfrak{F}_{\pi^0}^{(3,8)}+g_{ij\eta}\mathfrak{F}_{\eta}^{(3,8)}+g_{ij\eta'}\mathfrak{F}_{\eta'}^{(3,8)}=(m_{(i)}{}^2-m_{(j)}{}^2)C_{ij}\,, \tag{A2}$$

where

$$\mathfrak{F}_{\pi^0}^{(3)}=\frac{1}{\sqrt{2}}F_\pi a_{33}+\frac{1}{\sqrt{3}}\lambda_3(a_{83}+\sqrt{2}\,a_{03})\,,$$

$$\mathfrak{F}_{\eta}^{(3)}=\frac{1}{\sqrt{2}}F_\pi a_{38}+\frac{1}{\sqrt{3}}\lambda_3(a_{88}+\sqrt{2}\,a_{08})\,, \tag{A3}$$

$$\mathfrak{F}_{\eta'}^{(3)}=\frac{1}{\sqrt{2}}F_\pi a_{30}+\frac{1}{\sqrt{3}}\lambda_3(a_{80}+\sqrt{2}\,a_{00})$$

and

$$\mathfrak{F}_{\pi^0}^{(8)}=\frac{2}{\sqrt{3}}\lambda_8 a_{03}+(\tfrac{2}{3})^{1/2}(\sqrt{2}\,\lambda_0-\lambda_8)a_{83}+(\tfrac{2}{3})^{1/2}\lambda_3 a_{33}\,,$$

$$\mathfrak{F}_{\eta}^{(8)}=\frac{2}{\sqrt{3}}\lambda_8 a_{08}+(\tfrac{2}{3})^{1/2}(\sqrt{2}\,\lambda_0-\lambda_8)a_{88}+(\tfrac{2}{3})^{1/2}\lambda_3 a_{38}\,, \tag{A4}$$

$$\mathfrak{F}_{\eta'}^{(8)}=\frac{2}{\sqrt{3}}\lambda_8 a_{00}+(\tfrac{2}{3})^{1/2}(\sqrt{2}\,\lambda_0-\lambda_8)a_{80}+(\tfrac{2}{3})^{1/2}\lambda_3 a_{30}\,,$$

corresponding to the generators F_3^5 and F_8^5, respectively. The values of C_{ij} are tabulated in Table III for both cases.

By direct comparison of the third-order coupling constants, one easily obtains, using Eqs. (II.25) and (II.9), (II.26) and (II.17), (II.33) and (II.1) of Table II, the relations (66)–(68). Similarly, from Eqs. (II.13) and (II.5) one has

$$\frac{K_S^0-\pi_S^+}{F_\pi-F_K+(1/\sqrt{2})\lambda_3}=\frac{K_S^0-K_S^-}{\sqrt{2}\,\lambda_3}\,. \tag{A5}$$

This relation, and also those obtained from Eqs.

TABLE III. Values of C_{ij}. [See Eqs. (A2).]

$\underline{G}^\alpha$	i	j	C_{ij}	
F_3^5	K^-	K_S^+	2^{-1}	(III.1)
	$\bar{K}^0$	K_S^0	-2^{-1}	(III.2)
	π^0	π_S^0	$3^{-1/2}[b_{33}(a_{83}+2^{1/2}a_{03})+a_{33}(b_{83}+2^{1/2}b_{03})]$	(III.3)
	η	π_S^0	$3^{-1/2}[b_{33}(a_{88}+2^{1/2}a_{08})+a_{38}(b_{83}+2^{1/2}b_{03})]$	(III.4)
	η'	π_S^0	$3^{-1/2}[b_{33}(a_{80}+2^{1/2}a_{00})+a_{30}(b_{83}+2^{1/2}b_{03})]$	(III.5)
	π^0	η_S	$3^{-1/2}[b_{38}(a_{83}+2^{1/2}a_{03})+a_{33}(b_{88}+2^{1/2}b_{08})]$	(III.6)
	η	η_S	$3^{-1/2}[b_{38}(a_{88}+2^{1/2}a_{08})+a_{38}(b_{88}+2^{1/2}b_{08})]$	(III.7)
	η'	η_S	$3^{-1/2}[b_{38}(a_{80}+2^{1/2}a_{00})+a_{30}(b_{88}+2^{1/2}b_{08})]$	(III.8)
	π^0	η_S'	$3^{-1/2}[b_{30}(a_{83}+2^{1/2}a_{03})+a_{33}(b_{80}+2^{1/2}b_{00})]$	(III.9)
	η	η_S'	$3^{-1/2}[b_{30}(a_{88}+2^{1/2}a_{08})+a_{38}(b_{80}+2^{1/2}b_{00})]$	(III.10)
	η'	η_S'	$3^{-1/2}[b_{30}(a_{80}+2^{1/2}a_{00})+a_{30}(b_{80}+2^{1/2}b_{00})]$	(III.11)
F_8^5	π^-	π_S^+	$(\frac{2}{3})^{1/2}$	(III.12)
	K^-	K_S^+	$-6^{-1/2}$	(III.13)
	K^0	$\bar{K}_S^0$	$-6^{-1/2}$	(III.14)
	π^0	π_S^0	$(\frac{2}{3})^{1/2}[a_{33}b_{33}+2^{1/2}a_{03}b_{83}+a_{83}(2^{1/2}b_{03}-b_{83})]$	(III.15)
	η	π_S^0	$(\frac{2}{3})^{1/2}[a_{38}b_{33}+2^{1/2}a_{08}b_{83}+a_{88}(2^{1/2}b_{03}-b_{83})]$	(III.16)
	η'	π_S^0	$(\frac{2}{3})^{1/2}[a_{30}b_{33}+2^{1/2}a_{00}b_{83}+a_{80}(2^{1/2}b_{03}-b_{83})]$	(III.17)
	π^0	η_S	$(\frac{2}{3})^{1/2}[a_{33}b_{38}+2^{1/2}a_{03}b_{88}+a_{83}(2^{1/2}b_{08}-b_{88})]$	(III.18)
	η	η_S	$(\frac{2}{3})^{1/2}[a_{38}b_{38}+2^{1/2}a_{08}b_{88}+a_{88}(2^{1/2}b_{08}-b_{88})]$	(III.19)
	η'	η_S	$(\frac{2}{3})^{1/2}[a_{30}b_{38}+2^{1/2}a_{00}b_{88}+a_{80}(2^{1/2}b_{08}-b_{88})]$	(III.20)
	π^0	η_S'	$(\frac{2}{3})^{1/2}[a_{33}b_{30}+2^{1/2}a_{03}b_{80}+a_{83}(2^{1/2}b_{00}-b_{80})]$	(III.21)
	η	η_S'	$(\frac{2}{3})^{1/2}[a_{38}b_{30}+2^{1/2}a_{08}b_{80}+a_{88}(2^{1/2}b_{00}-b_{80})]$	(III.22)
	η'	η_S'	$(\frac{2}{3})^{1/2}[a_{30}b_{30}+2^{1/2}a_{00}b_{80}+a_{80}(2^{1/2}b_{00}-b_{80})]$	(III.23)

(II.42), (II.34), (II.41), (II.21), are actually not independent of Eqs. (66)–(68).

Finally, comparing Eqs. (II.2) and (II.27), (II.3) and (II.28), (II.4) and (II.29), (II.11) and (II.36), (II.12) and (II.37) of Table II we obtain the following relations:

$$\frac{\pi^0-\pi^-}{\pi^0-\pi_S^-}=-\frac{a_{83}+\sqrt{2}\,a_{03}}{\sqrt{3}\,a_{33}}\,\frac{\sqrt{2}\,\lambda_3}{F_\pi}, \tag{A6}$$

$$\frac{\eta-\pi^-}{\eta-\pi_S^-}=-\frac{a_{88}+\sqrt{2}\,a_{08}}{\sqrt{3}\,a_{38}}\,\frac{\sqrt{2}\,\lambda_3}{F_\pi}, \tag{A7}$$

$$\frac{\eta'-\pi^-}{\eta'-\pi_S^-}=-\frac{a_{80}+\sqrt{2}\,a_{00}}{\sqrt{3}\,a_{30}}\,\frac{\sqrt{2}\,\lambda_3}{F_\pi}, \tag{A8}$$

$$\frac{K^--\eta}{K_S^--\eta}=-\frac{2\sqrt{2}\,a_{08}+\sqrt{3}\,a_{38}-a_{88}}{\sqrt{3}\,(a_{38}+\sqrt{3}\,a_{88})}\,\frac{F_\pi-F_K+\lambda_3/\sqrt{2}}{F_K+\lambda_3/\sqrt{2}}, \tag{A9}$$

$$\frac{K^--\eta'}{K_S^--\eta'}=-\frac{2\sqrt{2}\,a_{00}+\sqrt{3}\,a_{30}-a_{80}}{\sqrt{3}\,(a_{30}+\sqrt{3}\,a_{80})}\,\frac{F_\pi-F_K+\lambda_3/\sqrt{2}}{F_K+\lambda_3/\sqrt{2}}, \tag{A10}$$

which, together with the Eqs. (66)–(68), provide the whole set of independent relations we can derive in this way.

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[1]This is a slightly expanded version of a paper with the same title submitted to the Kiev Conference on Elementary Particles, 1970. Since then, other papers have appeared dealing with the same subject (Ref. 2). In our opinion, however, *model-independent* features of SU(3) × SU(3) summetry have not been discussed in detail and a critical over-all analysis of the sum rules arising from SU(3) × SU(3) symmetry breaking has not been given. In particular, a careful model-independent discussion about the role played by the κ meson, the determination of the η-η' mixing angle, as well as the corrections to the Gell-Mann–Okubo formula seems to be lacking. For these and other reasons explained in the Introduction, we believe that the present paper may still be of some interest.

[2]The number of papers about SU(3) × SU(3) symmetry is very large and it is impossible to list all of them in any reasonable fair way. In the following we will quote only those papers which have a direct connection with our results.

[3]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968); S. L. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968).

[4]Y. Nambu, Phys. Rev. 117, 648 (1960); J. Goldstone, Nuovo Cimento 19, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. 127, 965 (1962).

[5]Y. Nambu, Phys. Rev. Letters 4, 380 (1960); S. L. Glashow and S. Weinberg, *ibid.* 20, 224 (1968); Gell-Mann, Oakes, and Renner, Ref. 3; M. Lévy, Nuovo Cimento 52A, 23 (1967); G. Cicogna, F. Strocchi, and R. Vergara Caffarelli, Phys. Rev. Letters 22, 497 (1969); Phys. Rev. D 1, 1197 (1970).

[6]For the sake of simplicity, we do not enter here into the problem of giving a precise meaning to Eq. (2). See, for example, D. W. Robinson, in *Symmetry Principles and Fundamental Particles*, edited by B. Kurşunoğlu and A. Perlmutter (Freeman, San Francisco, 1967), p. 457.

[7]One cannot expect exact invariance of the vacuum under $\underline{G}$; by Coleman's theorem (Ref. 8), this would imply strict invariance of the theory and no breaking at all. However, one may consider the situation in which the symmetry breaking is "small" ($g \ll 1$), and the vacuum noninvariance disappears in the limit $g \to 0$, yielding a fully invariant theory. In this case one may neglect the noninvariance of the vacuum as a first approximation.

[8]S. Coleman, J. Math. Phys. 7, 787 (1966).

[9]It is, however, worthwhile to remark that the case $g \neq 0$ has not been as fully discussed and understood in the literature as the case $g = 0$. For example, it is not obvious how to give a precise meaning to Eq. (2), provided that something similar holds, as a first approximation.

[10]This assumption plays the same role as assumption (b) of case (i). For a more precise definition of the local transformation properties of the fields in the case of spontaneous symmetry breakdown, see, e.g., Robinson, Ref. 6.

[11]For the general ideas about the semiclassical approximation, see S. L. Glashow, in *Particles Currents Symmetries*, Proceedings of VII Internationale Universitätswochen für Kernphysik, Schladming, 1968, edited by P. Urban (Springer, New York, 1968), p. 245; P. de Mottoni and E. Fabri, Nuovo Cimento 54A, 42 (1968). Some recent papers [J. Schechter and Y. Ueda, Phys. Rev. D 3, 168 (1971); 3, 176 (1971); 3, 2874 (1971); 4, 733 (1971)] have appeared dealing with the semiclassical approximation of the σ model and having some overlap with our paper. Their point of view is, however, different. The group-theoretical content has not been emphasized; in addition, their analysis does not include the careful discussion of the η-η' mixing and K_S mass sum rules, the generalized GMO formula, and the other sum rules obtained in Secs. VI and VII.

[12]B. Zumino, in *Brandeis University Summer Institute in Theoretical Physics*, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass, 1971). The main ideas go back to J. Schwinger, Proc. Natl. Acad. Sci. (U.S.) 37, 452 (1951); 37, 455 (1951); G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964); see also G. Parisi and M. Testa, Nuovo Cimento 67A, 13 (1970). In the last reference one may also find a discussion of SU(3) × SU(3) symmetry breaking.

[13]The above remarks apply, for example, to the papers of Ref. 11.

[14]This case will cover all the physical examples we will discuss in Secs. V, VI, and VII in connection with chiral symmetry breaking. A linear breaking in the Lagrangian or Hamiltonian density is in fact the basic assumption of the Gell-Mann–Oakes–Renner and of the Glashow-Weinberg model of SU(3) × SU(3) breaking (Ref. 3). The modifications required when the breaking is not linear in the basic fields, as it occurs in the case of

U(3)×U(3) breaking (Ref. 15), will be discussed elsewhere.

[15]F. Strocchi and R. Vergara Caffarelli, Phys. Letters 35B, 595 (1971).

[16]The advantage of using $A(\lambda)$ instead of $W(\lambda)$ is that $A(\lambda)$ will likely have a well-defined limit as $\epsilon_i(x) \to \epsilon_i$ = constant, whereas $W(\lambda)$ does not (Ref. 12). Moreover, the connections with the semiclassical approximation will be more apparent in terms of $A(\lambda)$ rather than $W(\lambda)$, as we will see below.

[17]In particular, the group-theoretical properties of $H(x)$ govern the divergences of the local currents and are connected with the PCAC equations.

[18]G. Cicogna, F. Strocchi, and R. Vergara Caffarelli, Phys. Rev. Letters 22, 497 (1969); Phys. Rev. D 1, 1197 (1970).

[19]The functional method proves to be very useful also in discussing the analyticity properties of the vacuum expectation values with respect to the breaking parameters ϵ_i. This will be discussed in a subsequent paper.

[20]R. Dashen, Phys. Rev. 183, 1245 (1969); L.-F. Li and H. Pagels, Phys. Rev. Letters 26, 1204 (1971); 27, 1089 (1971).

[21]Similar values have been obtained by Parisi and Testa, Ref. 12.

[22]These relations could be obtained also from the second-order equations (37)–(41). From a practical point of view, however, the present method automatically gives these formulas, whereas extracting them by direct elimination of ϵ and λ from Eqs. (37)–(41) can be a somewhat tedious task. This remark will appear even more relevant in the case $\epsilon_3 \neq 0$ where the second-order Ward identities are complicated by the occurrence of three mixing angles.

[23]S. L. Glashow, in *Hadrons and Their Interactions*, edited by A. Zichichi (Academic, New York, 1968), p. 83.

[24]N. N. Khuri, Phys. Rev. Letters 16, 75 (1966); 16, 601(E) (1966).

[25]Alternatively, one may use the following formula:

$$\tan\theta = -2\sqrt{2}\left(1 - \frac{F_K}{F_\pi}\frac{\eta' - K}{\eta' - \pi}\right) \bigg/ \left(1 - 4\frac{F_K}{F_\pi}\frac{\eta' - K}{\eta' - \pi}\right),$$

which can be derived without making any approximation from Eqs. (59) and (60).

[26]This equation follows easily from Eq. (59) by putting $\lambda_8 = 0$, and is the standard GMO formula including mixing.

[27]Putting $m_{\eta'} = 1422$ MeV would lead to an angle of the order $-5°$. A further possibility, actually, cannot be excluded, viz., the occurrence of a mixing between X^0 and E. Such a mixing would in fact be consistent with our Eqs. (59) and (60).

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Quarks, Sum Rules, and Low-Energy Parameters in πN Scattering

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Finite-energy sum rules and current-algebra sum rules are shown to work at the quark level. Making use of these rules, and of a factorization assumption for the basic meson-quark amplitudes above threshold, some well-known SU(6) results are derived. Low-energy parameters in πQ and πN scattering are also evaluated. Using our model for the πN $a_{1\pm}^{(-)}$ p-wave scattering lengths, an inconsistency is found between the usual PCAC (partially conserved axial-vector current) or ρ-exchange-model treatments and dispersion relations. It can be removed if double counting of resonances and ρ-exchange terms is avoided. This provides good agreement with experiment.

I. INTRODUCTION

The quark model is usually seen as an easy way of applying unitary symmetries to hadronic interactions. The determination of coupling constants and widths of resonances at low energies and relations between cross sections at high energies are classic examples in which the quark model and the SU(6) symmetry scheme give the same results. However the quark model is not identical to SU(6), and the physics of hadrons is, on the other hand, much more complex than SU(6) or any other simple symmetry scheme. In this paper we exploit possible non-SU(6) [or SU(6)$_W$] aspects of the quark model.

Inverting the normal procedure, we start by going from hadrons to quarks rather than the other way round. The reason is that if we want more than SU(6), we have to use experimental information where it exists and where it has motivated and justified a large variety of theoretical approaches. That means that we have to incorporate the knowledge gained in hadron physics. We thus apply to quarks the well-established