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Higher-Order Contributions to μ Decay in a Spontaneously Broken Gauge Model

T. W. Appelquist,* J. R. Primack,† and H. R. Quinn*

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

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The one-loop contributions to μ decay are calculated in Weinberg's model of the weak and electromagnetic interactions of leptons. The higher-order weak contributions are shown to be finite in this model, despite the high order of divergence of individual Feynman graphs. Thus this calculation is practical evidence in favor of the renormalizability of spontaneously broken gauge theories. The electromagnetic corrections to μ decay are finite after renormalization in the Weinberg model. The renormalized higher-order weak corrections do not affect the shape of the final electron spectrum. The effect on the μ decay rate is of the same order of magnitude as the electromagnetic correction, $\sim\frac{1}{2}\%$. Since this is effectively just a small "renormalization" of the weak coupling constant, it appears to be impossible to detect this effect experimentally unless other weak processes can be calculated and measured with comparable precision. The calculation is done by evaluating Feynman graphs dispersively. This calculational technique may be of interest in further work with spontaneously broken gauge theories.

I. INTRODUCTION

An important recent development in the theory of weak interactions is the possibility that a certain class of intermediate-vector-boson models may provide a renormalizable theory of weak and electromagnetic processes.[1,2] In such models the vector bosons acquire their masses through a spontaneous breakdown of local gauge invariance.[3] This idea was first proposed by Weinberg[4] in 1967 in the context of an SU(2)×U(1) model of leptons. Recent work has demonstrated the cancellation of leading divergences for several processes in the Weinberg model.[5] The cancellation of all divergences on the one-loop level has also been shown in a simplified (Abelian) version of the model for a number of processes.[6]

In this paper we discuss techniques to calculate one-loop contributions for models with spontaneously broken gauge symmetry. Calculating such contributions dispersively, we obtain a finite result despite the fact that the individual Feynman graphs give highly divergent contributions. We present an explicit calculation of the one-loop contributions to μ decay in the Weinberg model.

We find that in this model the electromagnetic corrections to μ decay have no ultraviolet divergence after renormalization. We reserve for a subsequent paper a detailed discussion of the radiative corrections including infrared effects, questions of μ-e universality, and a comparison with previous calculations.[7,8] Weak corrections to μ decay, in which the neutral intermediate vector boson Z plays a role analogous to that of the photon in the electromagnetic corrections, are also finite and contribute new terms. These are of the same order of magnitude as the hard-photon part of the electromagnetic corrections and have essentially the same momentum-transfer dependence as the Born term. Thus their only effect is a correction of order α to the decay rate of the muon.

To test the universality of the weak coupling one would like to be able to calculate the neutron β-decay rate also. Since no attractive spontaneously broken gauge models yet exist which include strongly interacting particles, we can only comment briefly on some constraints placed on such models by the requirement that the β decay rate be finite.

Our explicit results are thus rather insensitive to experimental test. Nevertheless the fact that μ decay is finite in second order in this theory is a significant further indication of the renormalizability of spontaneously broken gauge theories. The calculational techniques used here may also be useful for further calculations in such models.

The organization of the paper is as follows: Sec. II discusses some general properties of spontaneously broken gauge models in relation to our approach in this calculation. Section III contains introductory remarks about our calculational techniques. Section IV presents the details of the calculation. This section is divided as follows: A. lowest-order μ decay; B. canceling divergences; C. vector-vector cut contribution; D. electromagnetic corrections; E. other contributions; F. summary of results for the Weinberg model. Section V contains a brief discussion of neutron β decay, and Sec. VI presents some concluding comments.

II. BACKGROUND

The Weinberg model of leptons belongs to a large class of field-theoretic models[9] in which the vector mesons have obtained their masses through the Higgs mechanism[3]; that is, by spontaneous breakdown of a gauge symmetry of the original Lagrangian. In such models one begins with a Lagrangian where all vector mesons are massless and minimally coupled to exactly conserved fermion currents and to a set of scalar fields ϕ. Some of the scalar fields develop nonzero vacuum expectation values and thereby introduce vector-meson and fermion masses into the Lagrangian; the other scalar fields are massless but uncoupled gauge degrees of freedom. The spontaneous symmetry breaking also establishes relationships between different couplings in the resulting theory.

The Weinberg model has a spontaneously broken $SU(2)\times U(1)$ symmetry. That is, it begins with a triplet plus a singlet of vector mesons minimally coupled to left-handed doublet (νe) and a right-handed singlet (e) of leptons, and to a scalar-field doublet. Spontaneous symmetry breaking generates masses for three vector-meson states and for the electrons, leaving one linear combination of neutral vector-meson fields, the photon, massless and coupled to electric charge. The charged vector mesons couple to the usual $V-A$ weak currents and the remaining heavy neutral vector combination, the Z, couples both to electrons and neutrinos. In the model adopted for calculation here, muons and their neutrinos are introduced on an equal footing with electrons and their neutrinos. The only change is that the coupling of the scalar field to the muons is larger by a factor of m_μ/m_e, so that the correct mass ratio will obtain from the spontaneous symmetry breaking.

By choosing different sets of field variables one can cast the resulting Lagrangian in one of two forms. One choice, discussed in detail by Lee[1] for an Abelian model, and Lee and Zinn-Justin[2] for an SU(2) model, yields a Lagrangian which is renormalizable by the usual power-counting criterion, but in which unitarity is by no means obvious, since the Lagrangian contains wrong-metric fields. Another choice of field variables yields a "manifestly" unitary S matrix. However, with this choice one has a theory which is highly divergent and apparently unrenormalizable. This lack of convergence results from the form of the vector propagator, which behaves as a constant for large momentum. Feynman rules[10] in this formalism are given in Fig. 1 for Weinberg's model.

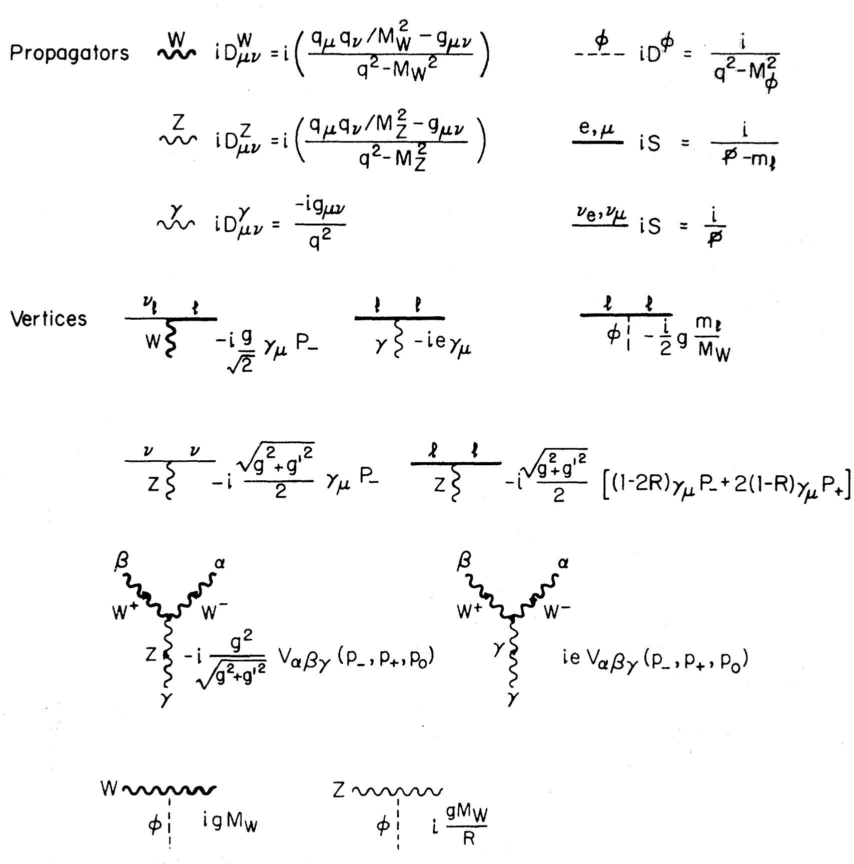

FIG. 1. Feynman rules for the Weinberg model. Here l denotes either an electron or a muon, e is the charge of the electron, $g^2=e^2/(1-R)$, $R=M_W{}^2/M_Z{}^2$, and $V_{\alpha\beta\gamma}(p_-,p_+,p_0)=(p_+-p_0)_\alpha g_{\beta\gamma}+(p_0-p_-)_\beta g_{\gamma\alpha}+(p_--p_+)_\gamma g_{\alpha\beta}$.

Following Lee and Zinn-Justin[2] we call the former choice of field variables the "R" (renormalizable) formalism, and the latter the "U" (unitary) formalism. Formal arguments have been given[2] to show that the two are equivalent and that, therefore, all S-matrix elements are both finite and unitary.

In the R formalism, the Green's functions are not unitary at off-mass-shell points. It is a nontrivial matter to demonstrate the cancellation of the wrong-metric contributions for any S-matrix element. In the U formalism this difficulty is replaced by the divergence of off-shell Green's functions (even after all renormalization subtractions are performed), and therefore of the amplitude corresponding to individual Feynman graphs. Finiteness of S-matrix elements cannot be shown by naive power counting, since it requires cancellation of divergences from different graphs. If such a theory yields finite results for all S-matrix elements it may be called cryptorenormalizable,[11] since its renormalizability is certainly not evident without considerable calculation.

In this paper, we have chosen to calculate in the U formalism. The normal procedure would be to regularize each of the Feynman diagrams in order to avoid ambiguities while performing the renormalization subtractions and demonstrating the cancellation between various graphs of the remaining divergences. While a regularization procedure has been developed by Lee and Zinn-Justin[2] for the R formalism, regulating the U formalism is more difficult.[12]

We have avoided the difficulties of regularization by calculating dispersively. Absorptive parts from one-loop graphs are of course finite and unambiguous. We show that when absorptive parts from sets of graphs are added together, the most rapidly growing contributions cancel. After the renormalization subtractions, the dispersive integrals converge.

As in a conventional renormalization program,[13] the subtractions are limited to those corresponding to Lagrangian counterterms generated by rescaling the fields and parameters of the original Lagrangian.[14] The way in which the necessary subtractions are incorporated in our calculation is discussed in further detail in Sec. III A for the vector propagator and Sec. IV E for the vector-lepton vertices. There is a slight complication in the renormalization because of the instability of the W boson; this is discussed in Sec. IV E.

The coupling constants which appear in our dispersive calculation are the physical (renormalized) coupling constants. Actually, since we are calculating only to order g^4, the only coupling constants which need to be discussed in connection with μ decay are those at the $e\nu_e W$ and $\mu\nu_\mu W$ vertices. The unrenormalized coupling constants at these vertices are identical. However, since the spontaneous symmetry breaking is arranged to give $m_\mu \neq m_e$, the corresponding renormalized coupling constants $g = g_{e\nu W}$ and $g_\mu = g_{\mu\nu W}$ will differ in perturbation theory. (We define these coupling constants to be the values of their respective vertices with all three particles on-mass-shell – a situation which would occur, for example, in W decay.)

It is obviously important to determine how much g_μ/g differs from unity. Individual Feynman graphs give quadratically divergent terms independent of masses, and logarithmically divergent terms some of which are proportional to m_e^2/M_W^2. We have verified, however, that when *all* wave-function and vertex renormalizations are included, including those involving the scalar meson ϕ, the *divergent* terms in the $e\nu_e W$ and $\mu\nu_\mu W$ renormalizations are equal, so that the ratio g_μ/g is finite. We have furthermore found that the finite contributions to g_μ/g from weak effects (graphs containing the massive vector mesons W, Z, or the massive scalar meson ϕ) are very small (of order m_μ^2/M_W^2 or smaller). However, the graphs containing photons introduce finite deviations from μ-e universality, contributing to g_μ/g terms of order $\alpha \ln(m_\mu/m_e)$. These graphs also contain infrared divergences, which as usual must be treated to-

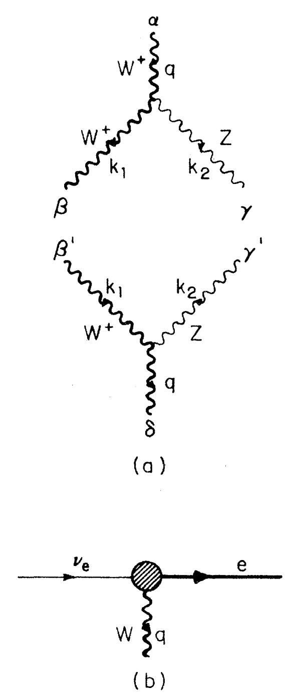

FIG. 2. (a) Vector self-energy contribution, showing notation for momenta and indices. (b) Weak-interaction vertex.

gether with the infrared divergences from the graphs for soft-photon emission. This subject is discussed briefly in Sec. IV D. The infrared problem and the renormalization program will be discussed in more detail in a future publication.

There is an inherent ambiguity which is always present in a dispersive calculation: The result of evaluating the dispersive integral could differ from the equivalent Feynman-integral calculation by a real polynomial. The only way to be certain that this does not occur in our calculation is to compare it with a conventional regularized-Feynman-integral calculation, which we have not done. We note, however, that after the addition of the absorptive parts from sets of graphs, the dispersive integrals in the Weinberg theory behave no differently from those in quantum electrodynamics. If all of the usual renormalization subtractions are performed, it is well known that dispersive calculations in quantum electrodynamics give the same results as Feynman integral calculations.[15]

III. PRELIMINARIES

A. Self-Energy Graph

In order to explain our calculational procedure, we will consider in some detail the Feynman graph drawn in Fig. 2(a), the second-order contribution to $\mu\nu_e \to \nu_\mu e$ (or μ decay) from the exchange of a charged (W^+) boson with a neutral (Z) boson self-energy bubble. The absorptive part of the W^+ vacuum-polarization tensor is

$$-\mathrm{Disc}\Pi_{\alpha\beta}(q)=g^2R\int d\tau\, V_{\alpha\beta\gamma}(q,-k_1,-k_2)V_{\delta\beta'\gamma'}(q,-k_1,-k_2)i(k_1^\beta k_1^{\beta'}/M_W^2-g^{\beta\beta'})i(k_2^\gamma k_2^{\gamma'}/M_Z^2-g^{\gamma\gamma'}), \tag{1}$$

where $V_{\alpha\beta\gamma}$ is the Yang-Mills 3-vertex (see Fig. 1) and $d\tau$ is the usual 2-particle phase space

$$\begin{aligned} d\tau &= \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^4(k_1+k_2-q) \\ &\quad\times(2\pi i)\delta_+(k_1^2-M_W^2)(2\pi i)\delta_+(k_2^2-M_Z^2)i^2 \\ &= \frac{|\vec{k}|}{4q_0}\frac{d\Omega_k}{(2\pi)^2}. \end{aligned} \tag{2}$$

The second expression follows in the center-of-mass frame, where

$$\begin{aligned} q &= (q_0,\vec{0}), \\ k_1 &= \left(\frac{q^2+M_W^2-M_Z^2}{2q_0}, +\vec{k}\right), \\ k_2 &= \left(\frac{q^2-M_W^2+M_Z^2}{2q_0}, -\vec{k}\right), \\ \vec{k}^2 &= \frac{q^2}{4}-\frac{M_W^2+M_Z^2}{2}+\frac{(M_W^2-M_Z^2)^2}{4q^2}. \end{aligned} \tag{3}$$

For our present purposes we do not need to give the explicit expression for $\mathrm{Disc}\Pi_{\alpha\beta}$, but merely note that it has the form

$$(-i)\,\mathrm{Disc}\Pi_{\alpha\beta}(q)=g_{\alpha\beta}b(q^2)+q_\alpha q_\beta c(q^2). \tag{4}$$

$\Pi_{\alpha\beta}$ will be transverse [i.e., $b(q^2)=-q^2c(q^2)$] in the special case where both particles in the self-energy loop have the same mass.

In calculating $\Pi_{\alpha\beta}$ in the usual Feynman way – i.e., nondispersively – one would subtract it twice:

$$\tilde{\Pi}_{\alpha\beta}(q)=\Pi_{\alpha\beta}(q)-c_1g_{\alpha\beta}-c_2(q^2g_{\alpha\beta}-q_\alpha q_\beta). \tag{5}$$

The first subtraction corresponds to rescaling the spontaneously induced $W_\mu W^\mu$ term in the Lagrangian (mass renormalization) and the second subtraction corresponds to rescaling the W_μ kinetic-energy term (wave-function renormalization).[14] The real coefficients c_1 and c_2 are chosen so that in the expression

$$\tilde{\Pi}_{\alpha\beta}(q)=g_{\alpha\beta}\tilde{B}(q^2)+q_\alpha q_\beta\tilde{C}(q^2), \tag{6}$$

$\mathrm{Re}\,\tilde{B}(M_W^2)=\mathrm{Re}\,\tilde{B}'(M_W^2)=0$.[16] ($M_W$ is of course the physical W mass.)

Correspondingly, we subtract the dispersion relation for $\tilde{B}(q^2)$ twice, that for $\tilde{C}(q^2)$ thereby receives a subtraction, and the polarization tensor is

$$\begin{aligned} \Pi_{\alpha\beta}(q) &= g_{\alpha\beta}(t-M_W^2)^2\frac{1}{2\pi}\int\frac{dt'}{t'-t}\frac{b(t')}{(t'-M_W^2)^2} \\ &\quad+q_\alpha q_\beta\left[\frac{1}{2\pi}\int\frac{dt'}{t'-t}c(t')+\frac{1}{2\pi}\int\frac{dt'b(t')}{(t'-M_W^2)^2}\right], \end{aligned} \tag{7}$$

with $t\equiv q^2$. Let us now use this expression to calculate the amplitude corresponding to the $\mu\nu_e\to\nu_\mu e$ graph of Fig. 3(a). Suppressing inessential factors, the result is

$$\begin{aligned} A &= J_\alpha^{(e)}J_\beta^{(\mu)\dagger}g^{\alpha\beta}\frac{1}{2\pi}\int\frac{dt'}{t'-t}\frac{b(t')}{(t'-M_W^2)^2} \\ &\quad+J^{(e)}J^{(\mu)\dagger}\frac{m_em_\mu}{M_W^4} \\ &\quad\times\frac{1}{2\pi}\int\frac{dt'}{t'-t}\left[c(t')+b(t')\frac{t'-2M_W^2}{(t'-M_W^2)^2}\right], \end{aligned} \tag{8}$$

where $J_\alpha^{(l)}\equiv\bar{l}\gamma_\alpha P_-\nu_l$, $J^{(l)}\equiv\bar{l}P_-\nu_l$, $P_\pm\equiv\frac{1}{2}(1\pm\gamma_5)$, $l=e$ or μ.

This same result is obtained if, instead of calculating the subtracted W^+ polarization tensor in Fig.

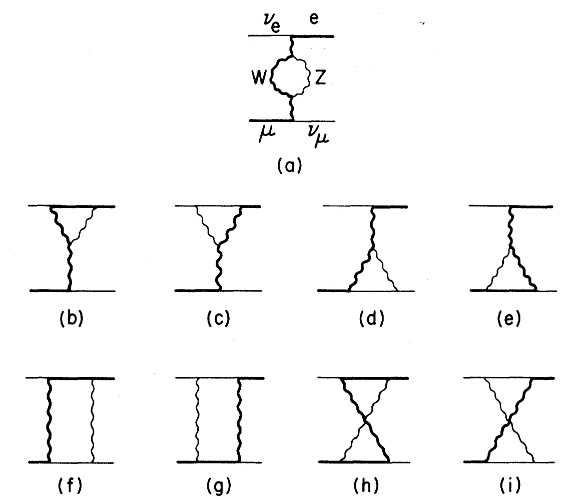

FIG. 3. Feynman graphs which have a W-Z cut.

2(a) dispersively and inserting it into the graph [Fig. 3(a)], we instead disperse the entire graph without subtraction.[17] The absorptive part of this graph, again suppressing inessential factors, is

$$\begin{aligned}\mathrm{Disc}A &= J_\rho^{(e)} J_\sigma^{(\mu)\dagger} D^{W\rho\alpha}(q)[g_{\alpha\beta}b(t) + q_\alpha q_\beta c(t)] D^{W\beta\sigma}(q) \\ &= J_\rho^{(e)} J_\sigma^{(\mu)\dagger} \frac{g^{\rho\sigma} b(t)}{(t - M_W{}^2)^2} \\ &\quad + J^{(e)} J^{(\mu)\dagger} \frac{m_e m_\mu}{M_W{}^4}\left[c(t) + b(t)\,\frac{t - 2M_W{}^2}{(t - M_W{}^2)^2}\right]. \qquad (9)\end{aligned}$$

To obtain the second expression we have used momentum conservation ($q = p_{\nu_e} - p_e = p_{\nu_\mu} - p_\mu$) and the Dirac equation. It is evident that we will recover Eq. (8) if we insert Eq. (9) into an unsubtracted dispersion relation.[18]

This fact will considerably simplify our calculations below, by enabling us to combine the contribution to the absorptive part from Fig. 2(a) with that from other graphs (see, e.g., Fig. 3). We will thus be able to exhibit cancellations of terms in the absorptive part which, if they did not cancel, would lead to divergences in the scattering amplitude. We have, in fact, been dealing with divergent dispersion relations for the graph in Fig. 2, since explicit calculation shows that $b(t) \sim t^3$ and $c(t) \sim t^2$. Thus $\Pi_{\alpha\beta}$ actually requires four subtractions to render it finite, although there is physical justification only for two; similarly, the amplitude A appearing in Eq. (8) requires two subtractions. Thus in this theory, as in a nonrenormalizable theory, the Green's functions are not made finite by the subtractions which correspond to multiplicative renormalization. However, as we have already mentioned and will demonstrate below, when we add together sets of graphs corresponding to a given S-matrix element, the nonrenormalizable divergences cancel. This is the characteristic behavior of a cryptorenormalizable theory. Until we actually demonstrate this cancellation, we can provisionally regard the dispersive integrals as being defined by means of an upper cutoff.

B. Vertex Graphs

In the previous section we showed that the subtractions corresponding to the W mass and kinetic-energy Lagrangian counterterms are correctly performed, if we calculate a self-energy graph of the form of Fig. 3(a) by dispersing the entire graph. There is no counterterm corresponding to lepton-lepton scattering, so the box graphs, Figs. 3(f)–3(i) receive no subtraction. In this section we will discuss the treatment of vertex-type graphs.

Consider a graph of the form of Fig. 2(b). This vertex part has the general form

$$\mathrm{Disc}\Lambda_\rho(q) = J_\rho^{(e)} f_1(q^2) + J_\rho^{(2)} f_2(q^2) + \sum_{i=3}^{n} J_\rho^{(i)} f_i(q^2), \qquad (10)$$

where $J_\rho^{(e)} \equiv (\bar{e}\gamma_\rho P_- \nu_e)$, $J_\rho^{(2)} = J_\sigma^{(e)} q^\sigma q_\rho = -m_e J^{(e)} q_\rho$, and $J_\rho^{(i)}$ are other possible vector expressions. There is a counterterm in the Lagrangian corresponding to $J_\rho^{(e)}$, so f_1 is to be subtracted once, but the other terms in Λ_ρ are not to be subtracted. Thus

$$\tilde{\Lambda}_\rho(q) = J_\rho^{(e)}(t - M_W{}^2)\frac{1}{2\pi i}\int \frac{dt'}{t' - t}\,\frac{f_1(t')}{t' - M_W{}^2} + \sum_{i=2}^{n} J_\rho^{(i)} \frac{1}{2\pi i}\int \frac{dt'}{t' - t} f_i(t'), \qquad (11)$$

where, as usual, we define $t \equiv q^2$ for convenience. The amplitude corresponding to a graph of the form of Fig. 3(b) is thus

$$\begin{aligned}A_{3b} &= \tilde{\Lambda}_\rho(q) D^W_{\rho\sigma}(q) J_\sigma^{(\mu)\dagger} \\ &= \frac{1}{2\pi i}\int \frac{dt'}{t' - t}\left[\frac{f_1(t')}{t' - M_W{}^2} J_\rho^{(e)} + \sum_{i=2}^{n} \frac{f_i(t')}{t - M_W{}^2} J_\rho^{(i)}\right]\left(\frac{q^\rho q^\sigma}{M_W{}^2} - g^{\rho\sigma}\right) J_\sigma^{(\mu)\dagger}. \qquad (12)\end{aligned}$$

If we were to disperse the whole graph, we would instead obtain

$$A'_{3b}=\frac{1}{2\pi i}\int\frac{dt'}{t'-t}\left\{\left[\frac{f_1(t')}{t'-M_W{}^2}J_\rho^{(e)}+\sum_{i=2}^{n}\frac{f_i(t')}{t'-M_W{}^2}J_\rho^{(i)}\right]\left(\frac{q^\rho q^\sigma}{M_W{}^2}-g^{\rho\sigma}\right)J_\sigma^{(\mu)\dagger}\right\}. \tag{13}$$

Note that here $J_\rho^{(2)}q^\rho=t'J_\alpha^{(e)}q^\alpha=-m_e t' J^{(e)}$. Since

$$\frac{1}{t-M_W{}^2}m_e J^{(e)}\left(\frac{t}{M_W{}^2}-1\right)m_\mu J^{(\mu)\dagger}=\frac{1}{t'-M_W{}^2}m_e J^{(e)}\left(\frac{t'}{M_W{}^2}-1\right)m_\mu J^{(\mu)\dagger}, \tag{14}$$

the f_1 and f_2 terms in A_{3b} and A'_{3b} are equal. This equality does not necessarily hold for other vertex functions f_i; for example, it would be invalid for $J_\rho^{(i)}=J^{(e)}(p_{\nu_e}+p_e)_\rho$. Although such additional vertex functions are actually present, it is not difficult to show that in our calculation the terms containing them are not divergent, and are furthermore suppressed by factors of lepton masses divided by intermediate vector-boson masses. We neglect such small terms ($m_\mu{}^2/M_W{}^2<10^{-6}$). Thus by "dispersing" each full graph we will have correctly treated both the self-energy insertion and all significant contributions from the vertex insertions.

IV. CALCULATIONS

A. Lowest-Order μ Decay

The Born amplitude for μ decay in the Weinberg model is

$$\begin{aligned}A_{\text{Born}}&=\tfrac{1}{2}gg_\mu J_\alpha^{(\mu)\dagger}J_\beta^{(e)}D^{W\alpha\beta}(q)\\&=\frac{gg_\mu}{8M_W{}^2}[\nu_\mu\gamma_\alpha(1-\gamma_5)\mu][e\gamma^\alpha(1-\gamma_5)\nu_e]\left(1+\frac{q^2}{M_W{}^2-q^2}\right)-\frac{gg_\mu}{8M_W{}^2}[\nu_\mu(1-\gamma_5)\mu][e(1-\gamma_5)\nu_e]\frac{m_e m_\mu}{M_W{}^2-q^2}.\end{aligned} \tag{15}$$

In the $V-A$ current-current theory of weak interactions,

$$A_{\text{Born}}^{(V-A)}=\frac{G}{\sqrt{2}}[\nu_\mu(1-\gamma_5)\mu][e(1-\gamma_5)\nu_e]. \tag{16}$$

The relation between the coupling constant g in Weinberg's model and the weak coupling constant G is

$$\frac{G}{\sqrt{2}}=\frac{gg_\mu}{8M_W{}^2}\approx\frac{g^2}{8M_W{}^2} \tag{17}$$

to lowest order in g^2 (cf. Sec. II). Also to this order,

$$g^2=e^2/(1-R),$$

where

$$R=M_W{}^2/M_Z{}^2.$$

Thus

$$M_W{}^2=\frac{\pi}{\sqrt{2}}\frac{\alpha}{1-R}\frac{m_p{}^2}{1.02\times10^{-5}}. \tag{18}$$

For $R=0$, $M_W=37.3$ GeV, Lee's value[19]; for any other value of R, M_W is larger than this, and $M_Z\geqslant M_W$. (Actually, as we shall see below, both $R=0$ and $R=1$ are singular limits of the Weinberg theory.)[20] Since $m_\mu{}^2\geqslant q^2\geqslant m_e{}^2$ in μ decay, $q^2/M_W{}^2\leqslant10^{-6}$. Thus the difference between the μ-decay Born amplitude in the current-current theory and the Born amplitude in an intermediate-vector-meson theory such as Weinberg's is microscopic.

The higher-order weak contributions to the μ-decay amplitude (or any other amplitude) in the current-current theory of weak interactions are divergent. The amplitude can be schematically represented as

$$A_{\mu\ \text{decay}}\cong A_{\text{Born}}(1+G\Lambda^2+G^2\Lambda^4+\cdots), \tag{19}$$

where Λ^2 is an ultraviolet cutoff. Simple arguments lead one to expect that the intermediate vector-boson masses in the Weinberg model will effectively take the place of the cutoff Λ^2, and render the amplitude finite as long as $M_W{}^2$, $M_Z{}^2<\infty$. We shall see below that this is exactly what happens. Thus the expansion parameter is of the same order as α, the fine-structure constant, for reasonable values of $M_W{}^2$, $M_Z{}^2$, and the higher-order weak contributions are of the same order of magnitude as the electromagnetic corrections.

B. Sets of One-Loop Graphs with Canceling Divergences

All the graphs which contribute to the S-matrix element for $\mu\nu_e\to\nu_\mu e$ (or μ decay) in second order in g^2 in the Weinberg model are displayed in Fig. 4. Here row (a) represents the graphs drawn explicitly in Fig. 3, (b) represents the electromagnetic corrections, and (c) and (d) are additional weak corrections. We calculate these graphs by dispersing in the t channel ($\mu\nu_\mu\to\nu_e e$), as we have already illustrated for the self-energy graph Fig. 3(a). We begin by discussing the canceling of nonrenormalizable divergences.[21]

Let us, for a moment, imagine that the muon and electron have the same mass (but nevertheless re-

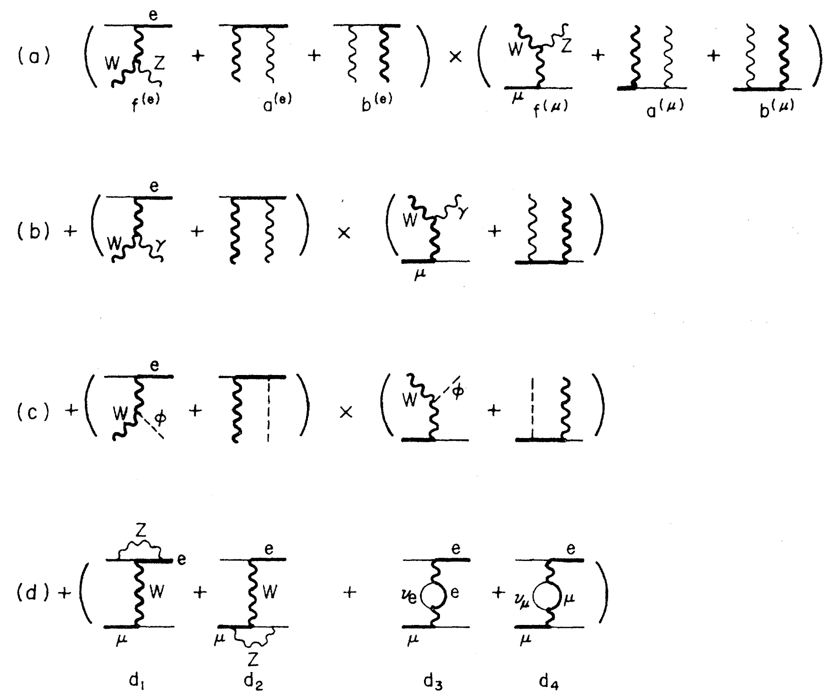

FIG. 4. Cut contributions to μ decay: (a) W-Z cut, (b) W-photon cut, (c) W-scalar cut, (d) lepton cuts.

main distinguishable particles). Let us furthermore consider forward scattering ($p_\mu = p_e$, $p_{\nu_\mu} = p_{\nu_e}$). Then the contributions to the total absorptive part from rows (a), (b), and (c) in Fig. 4 are each of the form

$$\text{Disc}A = i\int d\tau_n A^\dagger(n|p_e, p_{\nu_e})A(n|p_e, p_{\nu_e});$$

that is, each is a perfect square. This is not the case for the first two graphs of row (d). However, each of the graphs of row (d) is finite after renormalization.[22] Consequently, the divergences in the remaining graphs must cancel among themselves. Furthermore, because rows (a), (b), and (c) are each positive definite, the divergences must cancel in each row separately.

Returning to the actual kinematics, with unequal masses and nonforward scattering, the leading (quadratic) divergences are independent of masses and external momenta, and cancel as before. The nonleading (logarithmic) divergences depend in a simple way on the lepton masses and are independent of external momenta, as will be apparent when we calculate them explicitly below. The pattern of cancellation of logarithmic divergences is therefore the same as in our idealized example. Thus we can consider each row of Fig. 4 separately.

C. Graphs with W-Z Intermediate States

In this section we will discuss in some detail the graphs of Fig. 3, which correspond to Fig. 4(a). As before, we denote the $W(Z)$ momentum by k_1 (k_2) and Lorentz index by β (γ). Defining $Q = k_1 - k_2$, we note the following kinematic identities:

$$k_1^2 = M_W^2, \quad k_2^2 = M_Z^2,$$

$$q = k_1 + k_2, \quad q\cdot Q = M_W^2 - M_Z^2,$$

$$2k_1\cdot q = q^2 + M_W^2 - M_Z^2, \quad 2k_2\cdot q = q^2 - M_W^2 + M_Z^2,$$

$$2k_1\cdot k_2 = q^2 - M_W^2 - M_Z^2,$$

The phase-space integral is simplest in the c.m. system where $q = (q_0, \vec{0})$; see Eq. (2). The amplitudes f, a, and b defined in Fig. 4(a) are

$$f^{(e)}_{\beta\gamma} = \frac{-ig^3}{\sqrt{2}\,(g^2+g'^2)^{1/2}}\, J^{(e)\rho} D^W_{\rho\alpha}(q)\, V^{\alpha\beta\gamma}(q, -k_1, -k_2), \tag{20}$$

$$a^{(e)}_{\beta\gamma} = \frac{-ig(g^2+g'^2)^{1/2}}{2\sqrt{2}}\; \frac{\bar{e}\gamma_\gamma[(1-2R)P_- + 2(1-R)P_+](\not p_{\nu_e} - \not k_1 + m_e)\gamma_\beta P_- \nu_e}{(p_{\nu_e} - k_1)^2 - m_e^2}$$

$$=\frac{-ig(g^2+g'^2)^{1/2}}{2\sqrt{2}}\;\frac{\bar{e}\gamma_\gamma[(1-2R)(\not{p}_e+\not{k}_2)+2(1-R)m_e]\gamma_\beta P_-\nu_e}{(p_e+k_2)^2-m_e^2}, \tag{21}$$

$$b^{(e)}_{\beta\gamma}=\frac{-ig(g^2+g'^2)^{1/2}}{2\sqrt{2}}\;\frac{\bar{e}\gamma_\beta(\not{p}_e+\not{k}_1)\gamma_\gamma P_-\nu_e}{(p_e+k_1)^2}, \tag{22}$$

and similarly for the μ graphs. The discontinuity corresponding to the entire amplitude represented by Fig. 4(a) is thus

$$\mathrm{Disc}\,A_{(a)}=\int d\tau\,(a^{(\mu)}+b^{(\mu)}+f^{(\mu)})_{\beta'\gamma'}(a^{(e)}+b^{(e)}+f^{(e)})_{\beta\gamma}\,i\left(\frac{k_1^\beta k_1^{\beta'}}{M_W^2}-g^{\beta\beta'}\right)i\left(\frac{k_2^\gamma k_2^{\gamma'}}{M_Z^2}-g^{\gamma\gamma'}\right), \tag{23}$$

where the phase-space integral $d\tau$ is defined in Eq. (2). This is the discontinuity across the t cut at fixed s for graphs 3(f) and 3(g), and at fixed u for graphs 3(h) and 3(i). This sum is particularly convenient since in each case the dispersive integral is over the same interval, i.e., t from $(M_W+M_Z)^2$ to ∞.

Let us begin by considering

$$(a+b+f)^{(e)}_{\beta\gamma}k_1^\beta k_2^\gamma=\frac{-ig(g^2+g'^2)^{1/2}}{2\sqrt{2}}\left[J^{(e)}_\mu Q^\mu\left(\frac{q^2R}{q^2-M_W^2}-R-\frac{1/2}{(p_e+k_2)^2-m_e^2}\right)\right.$$
$$\left.+m_e J^{(e)}\left(\frac{q^2}{q^2-M_W^2}(R-1)-\frac{1/2}{(p_e+k_2)^2-m_e^2}\right)\right]. \tag{24}$$

The first two terms in the first parenthesis, multiplied by the similar terms in $(a+b+f)^{(\mu)}_{\beta'\gamma'}k_1^{\beta'}k_2^{\gamma'}$, would each produce quadratically divergent terms in the dispersive integral. The cancellation of the leading q^2 dependence between these terms – which is a cancellation between the Yang-Mills-coupling f graph and the a and b graphs – removes this divergence. The only divergent term remaining after this cancellation comes from the product of the first term in the second parenthesis with the similar term from $(a+b+f)^{(\mu)}_{\beta'\gamma'}k_1^{\beta'}k_2^{\gamma'}$. This term is proportional to the small quantity $m_\mu m_e/M_W^2\leq 10^{-8}$, but we keep it because it is divergent. Thus

$$(a+b+f)^{(e)}_{\beta\gamma}k_1^\beta k_2^\gamma(a+b+f)^{(\mu)}_{\beta'\gamma'}k_1^{\beta'}k_2^{\gamma'}$$
$$=\frac{-g^4}{8}J^{(e)}_\rho J^{(\mu)\dagger}_\sigma\left[\frac{Q^\rho Q^\sigma}{(q^2-M_W^2)^2}R^2+J^{(e)}J^{(\mu)\dagger}\frac{m_e m_\mu}{M_W^4}(R-1)^2+\text{finite and small terms (of order } m_e m_\mu/M_W^4)\right]. \tag{25}$$

Computing similarly, we find the following additional contributions:

$$(a+b+f)^{(e)}_{\beta\gamma}k_2^\gamma(a+b+f)^{(\mu)}_{\beta'\gamma'}k_2^{\gamma'}g^{\beta\beta'}=\frac{-g^4}{8}\left[J^{(e)}_R J^{(\mu)\dagger}_\sigma\frac{Q^\rho Q^\sigma}{(q^2-M_W^2)^2}R^2+J^{(e)}J^{(\mu)\dagger}\frac{m_e m_\mu}{M_W^4}(R^2-2R)+\text{finite and small}\right], \tag{26}$$

$$(a+b+f)^{(e)}_{\beta\gamma}k_1^\beta(a+b+f)^{(\mu)}_{\beta'\gamma'}k_1^{\beta'}g^{\gamma\gamma'}$$
$$=-\frac{g^4}{8}\left\{J^{(e)}_\rho J^{(\mu)\dagger}_\sigma\frac{1}{(q^2-M_W^2)^2}[Q^\rho Q^\sigma(1-2R)-4M_Z^2(R-1)^2g^{\rho\sigma}]-J^{(e)}J^{(\mu)\dagger}\frac{m_e m_\mu}{M_W^4}(1-2R)+\text{finite and small}\right\}. \tag{27}$$

In the remaining term the only divergent (i.e., constant as $q^2\to\infty$) contribution comes from the self-energy diagram Fig. 3(a):

$$(a+b+f)^{(e)}_{\beta\gamma}(a+b+f)^{(\mu)}_{\beta'\gamma'}g^{\beta\beta'}g^{\gamma\gamma'}\big|_{\text{divergent}}=+\frac{g^4}{8}J^{(e)}J^{(\mu)\dagger}\frac{m_e m_\mu}{M_W^4}2R. \tag{28}$$

When this is added to the similar terms obtained above, they all cancel exactly. This proves our assertion that there is no divergence when all the graphs of Fig. 3, or equivalently of Fig. 4(a), are added together. The finite contributions from the final term (the $g^{\beta\beta'}g^{\gamma\gamma'}$ term) of Eq. (23) are as follows. The graph of

Fig. 3(a) gives

$$-\frac{g^4}{2}J_\mu^{(e)}J_\nu^{(\mu)\dagger}\,\frac{R}{(q^2-M_W{}^2)^2}\,[4\,g^{\mu\nu}(q^2-M_W{}^2)+g^{\mu\nu}(M_W{}^2+M_Z{}^2)+\tfrac{5}{2}\,Q^\mu Q^\nu];$$

the vertex graphs, Figs. 3(b)–3(c), give

$$-\frac{g^4}{2}J_\mu^{(e)}J_\nu^{(\mu)\dagger}\,\frac{1}{q^2-M_W{}^2}\left\{\left[g^{\mu\nu}(M_W{}^2+M_Z{}^2)+\tfrac{1}{2}\,Q^\mu Q^\nu\right]\left[\frac{1}{d_1^e}+\frac{1}{d_1^\mu}-(1-2R)\left(\frac{1}{d_2^e}+\frac{1}{d_2^\mu}\right)\right]-4R\,g^{\mu\nu}\right\};$$

and the box graphs, Figs. 3(f)–3(i), give[23]

$$-\frac{g^4}{2}J_\mu^{(e)}J_\nu^{(\mu)\dagger}\,\frac{1}{R}\left\{\left[\frac{k_1^\mu k_1^\nu}{d_1^e\,d_1^\mu}+(1-2R)\,\frac{k_2^\mu k_2^\nu}{d_2^e\,d_2^\mu}\right]-\tfrac{1}{2}\,g^{\mu\nu}(1-2R)\left(\frac{1}{d_1^e}+\frac{1}{d_1^\mu}+\frac{1}{d_2^e}+\frac{1}{d_2^\mu}\right)\right\},$$

where we define

$$d_1^{e,\mu}=(p_{e,\mu}+k_1)^2,\quad d_2^{e,\mu}=(p_{e,\mu}+k_2)^2-m_{e,\mu}{}^2.$$

Having verified that the divergences cancel we drop all terms proportional to lepton masses. Performing the phase-space integration yields

$$-i\,\mathrm{Disc}\,A_{(a)}=\frac{-g^4}{2}J_\alpha^{(e)}J^{(\mu)\dagger\alpha}\frac{|\vec{\mathrm{k}}|}{4\sqrt{q^2}}\,\frac{1}{4\pi}\,\chi_{(a)}(t),\tag{29}$$

$$\chi_{(a)}(t)=\frac{4}{(t-M_W{}^2)^2}\left[M_W{}^2\left(4R+3-\frac{1}{R}\right)-\tfrac{1}{3}\vec{\mathrm{k}}^2(8R+1)\right]+\frac{8}{t-M_W{}^2}\left[M_W{}^2(1+R)L-\frac{1}{t}R(2l-M_W{}^2M_Z{}^2L)\right]+\eta_{(a)}(t),\tag{30}$$

$$\eta_{(a)}(t)=\frac{4}{R}\left[(2R-1)L+\frac{1}{t}(2R^2-2R+1)(2-lL)\right],$$

where

$$L=\frac{1}{n}\ln\left|\frac{l+n}{l-n}\right|,$$

$$l=\tfrac{1}{2}(M_W{}^2+M_Z{}^2-t),$$

$$n=\sqrt{t}\;|\vec{\mathrm{k}}|=\tfrac{1}{2}[t^2-2t(M_W{}^2+M_Z{}^2)+(M_Z{}^2-M_W{}^2)^2]^{1/2}.$$

The amplitude corresponding to the graphs of Fig. 3 [or Fig. 4(a)] is then given by an unsubtracted dispersion relation:

$$A_{(a)}(t)=\frac{1}{2\pi i}\int_{(M_W+M_Z)^2}^\infty\frac{dt'}{t'-t}\,\mathrm{Disc}\,A_{(a)}\,(t').$$

Because of the large value of the t threshold compared to lepton masses, the t dependence of $A_{(a)}$ is negligible. Since $A_{(a)}$ is proportional to the Born amplitude, Eq. (15), these weak corrections effectively just change the relationship between g^2 and the weak-interaction coupling constant G measured in μ decay from Eq. (17) to

$$\frac{G}{\sqrt{2}}=\frac{g^2}{8M_W{}^2}[1+\tfrac{1}{2}\,g^2\xi(R)+O(g^4)].\tag{31}$$

The contribution of $A_{(a)}$ to $\xi(R)$ which we denote by $\xi_{(a)}(R)$, can be evaluated by straightforward quadrature after the substitution $n=\sqrt{R}\,M_Z{}^2\sinh\theta$. The resulting expressions are rather complicated, except for the contribution from Figs. 3(f)–3(i), the box and crossed-box graphs, obtained by keeping only the $g_{\mu\nu}$ parts of the W and Z propagators. (The only contributions from the $k_\mu k_\nu$ terms in the vector propagators in these graphs, after the cancellation described above, are terms $\sim m_\mu{}^2/M_W{}^2$, which we drop.) We denote this contribution by $\eta_{(a)}(t)$ in Eq. (30). Quadrature gives

$$A_{(\eta)}=\frac{g^4}{64\pi^2M_W{}^2}J_\alpha^{(e)}J^{(\mu)\dagger\alpha}\,\frac{2R^2+6R-3}{1-R}\ln R$$

(evaluating at $t=0$, consistent with neglect of lepton masses). The same answer is also obtained by evaluating these graphs by momentum integration in the usual way, thus checking this part of our dispersive calculation.

We have evaluated the entire contribution $\xi_{(a)}(R)$ for several values of R; see Table I. As we have advertised, $g^2\xi_{(a)}\sim\alpha$ for most values of $R=M_W{}^2/M_Z{}^2$. Note that the limits $R\to0$ and $R\to\infty$ are singular.

D. Electromagnetic Corrections

The second row of Fig. 4 represents electromagnetic corrections to μ decay. By calculations

TABLE I. Contributions of the W-Z cut and lepton-cut graphs [Fig. 4(a) and 4(d)] for various values of R. The W-ϕ cut contribution [Fig. 4(c)] is given explicitly in Eq. (38). The W-γ cut contributions [Fig. 4(b)] are discussed briefly in Sec. IV D and will be given explicitly in paper II.

R	M_W (GeV)	M_Z (GeV)	$g^2=4\pi\alpha/(1-R)$	$g^2\xi(R)\times 10^3$ $W-Z$ cut	Lepton cuts
10^{-4}	37.3	3.73×10^3	0.091	−16	1.9
0.1	39.3	124	0.102	− 3.9	0.50
0.3	44.6	81.5	0.131	− 1.3	− 0.48
0.5	52.7	74.7	0.183	1.2	− 1.5
0.7	68.1	81.5	0.306	6.1	− 3.6
0.9	118	124	0.917	28	−13
0.95	167	171	1.83	64	−30

analogous to those already discussed we have verified that there are no nonrenormalizable divergences in the sum of these graphs. This result depends only on the electromagnetic properties of the charged vector boson. In a Yang-Mills model of the sort considered here, these properties are gyromagnetic ratio $g=2$ and quadrupole moment $Q=-e/M_W^2$.

The graphs of Fig. 4(b) give an infrared divergence for the differential decay rate as a function of electron energy. As in the classic $V-A$ current-current theory calculations,[7] the cross terms between these graphs and the Born amplitude must be combined with the contribution to the differential decay rate from soft-photon emission and folded with experimental apertures to give a finite prediction for any measurement.

Let us review briefly the results of the $V-A$ calculations.[7] The soft-photon effects cause a large change of about 7% in the final electron spectrum, as measured by the Michel effective-shape parameter ρ. The electromagnetic correction to the decay rate is not enhanced by soft-photon effects, however, and is about $\frac{1}{2}$%. In determining the weak-coupling constant from μ decay this correction is traditionally included:

$$\frac{G}{\sqrt{2}}=\left(\frac{G}{\sqrt{2}}\right)_{\text{measured}}\left[1-\frac{\alpha}{4\pi}\left(\tfrac{25}{4}-\pi^2\right)\right].$$

The weak corrections calculated in the preceding sections are of the same order as this electromagnetic correction.

It has been found in previous calculations in intermediate vector-boson theories that the electromagnetic corrections to μ decay are the same as in the $V-A$ current-current theory, up to negligible terms (of order m_μ^2/M_W^2).[8] In the dispersive formalism employed in this paper, the evaluation of the electromagnetic corrections contains a number of novel features. Since this problem is not strongly connected with other considerations of this paper, we reserve a complete treatment for a following paper and here comment only on some of the main points. As we have noted in Sec. II, the on-mass-shell subtraction procedure implicit in the dispersive calculation introduces both infrared divergences and dependence on the lepton masses into the renormalized $e\nu_e W$ and $\mu\nu_\mu W$ couplings. (The presence of such infrared divergences is not entirely unreasonable since the $W\to e\nu_e$ decay rate, for example, will not be infrared convergent without inclusion of soft-photon emission.) Physical quantities are free of infrared divergences only when all of these terms are included. We have also noted above the presence of a term of order $\alpha\ln(m_\mu/m_e)$ in the ratio of the renormalized $e\nu_e W$ and $\mu\nu_\mu W$ couplings. Terms of the same order are also found in the subtracted second-order amplitude corresponding to the graphs of Fig. 4(b). A complete examination of all such terms will be given in paper II.

E. Other Contributions

The divergent contributions from the graphs of Fig. 4(c) cancel. The only significant finite contribution from these graphs is from the W self-energy insertion. It is weakly dependent on the mass M_ϕ, of the scalar meson. The contribution is given by

$$\chi_{(c)}=\frac{4}{(t^2-M_W^2)^2}\left(M_W^2+\tfrac{1}{3}|\vec{k}|^2\right) \tag{32}$$

where

$$|\vec{k}|=\frac{1}{2\sqrt{t}}\left[t^2-2t(M_W^2+M_\phi^2)+(M_W^2-M_\phi^2)^2\right]^{1/2}$$

and χ is defined as in Eq. (29). Performing the dispersion integral yields

$$g^2\xi_{(c)}=\frac{\alpha}{4\pi(1-R)}\left[\ln\left|\frac{M_W}{M_\phi}\right|-\tfrac{47}{24}+O(M_\phi/M_W)\right] \tag{33}$$

for the effect of these graphs on the decay rate as expressed in Eq. (31). [We have assumed $M_\phi\leq M_W$; the case $M_\phi>M_W$ can also be evaluated using Eq. (32) and is not essentially different.]

The graphs of Fig. 4(d) are each finite after renormalization, and they can be calculated by the methods used above. As usual, the amplitude is obtained from an unsubtracted dispersion relation. However, there is a new complication in considering these graphs that we have not encountered before: The instability of the W. The dispersive integral starts at $t=m_{\mu(e)}^2$ which of course lies far below M_W^2. The usual procedure for renormalizing vertex parts, discussed in Sec. III B, leads us to make a subtraction at $q^2 = M_W{}^2$ so that, for example, the $\nu e W$ vertex

$$\Gamma_\mu^{(e)}(q,\, p,\, p') = \frac{1}{\sqrt{2}}\, g\gamma_\mu P_- + g^3 \Lambda_\mu^{(e)}(q,\, p,\, p')$$

goes to its lowest-order value on the mass shell:

$$\bar{u}_e(p')[\Gamma_\mu^{(e)}]_{q^2 = M_W{}^2} \quad u_{\nu_e}(p) = \frac{1}{\sqrt{2}}\, g\, J_\mu^{(e)}\,,$$

where g is the physical (renormalized) coupling constant. Here we cannot satisfy this condition, because the contribution to $\Lambda_\mu^{(e)}$ from the first two graphs of Fig. 4(d) has a nonvanishing imaginary part at $q^2 = M_W{}^2$. Subtractions correspond to Hermitian counterterms in the Lagrangian and hence must be real. Actually, all we should demand physically is that the real part of $\Lambda_\mu^{(e)}$ should vanish on mass shell. For the vertex-type graphs which we consider here, the subtraction will be done correctly if we take the principal part of the dispersive integral.

Defining χ as in Eq. (29), we find that graph (d_1) gives

$$\chi_{(d_1)} = \left[\frac{2}{q^2 - M_W{}^2} - \left(1 + \frac{M_Z{}^2 + M_W{}^2}{q^2 - M_W{}^2}\right) L' + \frac{1}{q^2 - M_W{}^2}\,\frac{1}{2n'}\,\left[2l' - (l'^2 - n'^2)L'\right]\right]\frac{2R-1}{R}\,, \tag{34}$$

where we have defined

$$l' = M_Z{}^2 + \frac{(q^2 + m_e{}^2)^2}{2q^2} - 2m_e{}^2 \cong M_Z{}^2 + \frac{q^2}{2}\,,$$

$$n' = \frac{(q^2 - m_e{}^2)^2}{2q^2} \cong \frac{q^2}{2}\,,$$

$$L' = \frac{1}{n'} \ln \left|\frac{l' + n'}{l' - n'}\right| \,.$$

Note that here $\vec{k}^2 = (q^2 - m_e{}^2)^2/4q^2$. The second graph (d_2) gives rise to identical expressions, but with $m_e \to m_\mu$.

We now turn our attention to the second pair of graphs in Fig. 4 row (d). The amplitude corresponding to (d_3) is

$$A_{(d_3)} = \tfrac{1}{2} g^2 J_\alpha^{(e)} J_\beta^{(\mu)\dagger} D^{W\alpha\mu}(q)\, D^{W\beta\nu}(q) \tilde{\Pi}_{\mu\nu}^{(e)}(q)\,. \tag{35}$$

Let us write the $(e\nu)$-loop contribution to the W vacuum-polarization tensor in the form $\Pi_{\mu\nu}^{(e)}(q) = B(q^2)\, g_{\mu\nu} + C(q^2) q_\mu q_\nu$. It is evident that the second (longitudinal) term makes a contribution to $A_{(d_3)}$ smaller by a factor $m_e m_\mu / M_W{}^2$ than that of the first term, so we ignore it here. The absorptive part of the first term is then calculated to be

$$-i\, \mathrm{Disc}\, B(q^2) = \frac{g^2}{2}\, \frac{|\vec{k}|}{4\sqrt{q^2}}\, \frac{1}{\pi}(q^2 - m_e{}^2 - \tfrac{4}{3}\vec{k}^2)\,.$$

One can verify that setting $m_e = 0$ results in an error of only $\sim m_e{}^2/M_W{}^2$ in calculating $A_{(d_3)}$. Making this usual approximation, we have

$$-i\, \mathrm{Disc}\, B(t) = \frac{g^2}{24\pi}\, t$$

and we immediately recognize that the corresponding (unrenormalized) $B(t)$ is

$$B(t) = -\frac{g^2}{48\pi^2}\, t \ln\left(-\frac{t}{\Lambda^2}\right)\,.$$

The renormalized $\tilde{B}(t)$ is then given by

$$\tilde{B}(t) = B(t) - \mathrm{Re}B(M_W{}^2) - (t - M_W{}^2)\frac{d}{dt}\mathrm{Re}B(t)\Big|_{t = M_W{}^2} = -\frac{g^2}{48\pi^2}\left[t \ln\left(-\frac{t}{M_W{}^2}\right) - (t - M_W{}^2)\right]\,.$$

Substituting this into Eq. (35) gives

$$A_{(d_3)} = -\frac{g^4}{96\pi^2} J_\alpha^{(e)} J_\beta^{(\mu)\dagger} g^{\alpha\beta} \times \frac{1}{(t - M_W{}^2)^2}\left[t \ln\left(-\frac{t}{M_W{}^2}\right) - (t - M_W{}^2)\right] \cong -\frac{g^4}{96\pi^2} J_\alpha^{(e)} J_\beta^{(\mu)\dagger} g^{\alpha\beta}\, \frac{1}{M_W{}^2}\,. \tag{36}$$

(For μ decay, $m_e{}^2 \ll M_\mu{}^2$.) Graph (d_4) gives the same result.

Neglecting terms $\sim m_e{}^2/M_W{}^2$, the graphs of Fig. 4(d) give the same dependence on the energy of the final electron in μ decay as the Fermi-theory Born term. Thus these graphs, like the graphs in Fig. 4(a) affect only the normalization of the weak coupling constant. The contributions of all of these graphs are summarized in Table I.

F. Results for Weinberg Model

The most significant feature of this calculation is that the result is finite. Electromagnetic corrections are finite and will be discussed in paper II. There are additional corrections to the decay rate of the same order as the hard-photon electromagnetic contribution. Table I summarizes the calculation by presenting the results of a computer evaluation of the dispersion integral. One readily sees that these contributions are insignificant; that is, they represent only a fraction of a percent change in the decay rate except for $R = 0$ and

$R > 0.9$. (The contributions diverge logarithmically for $R \to 0$, and quadratically as $R \to 1$ for fixed G and α.) In the numerical calculation we set

$$\frac{g^2}{8m_W^2} = \frac{G}{\sqrt{2}};$$

that is, we use the lowest-order contribution to define the relationship between Weinberg's parameters g and M_W. Since $g^2 = e^2/(1-R)$, this leaves only one free parameter, which we choose to be R. Table I displays the values of the various quantities for a range of values of R. Since $g^2\xi(R)$ is at most a few percent the above approximation is valid for any reasonable value of R. Values of $R < 0.65$ have been shown by Chen and Lee[24] to be inconsistent with present experimental data for $\bar{\nu}_e + e \to \bar{\nu}_e + e$. Values of R very close to one are obviously excluded in the sense of this calculation, since for such values the coupling g becomes large and perturbation expansion is no longer possible.

V. β DECAY

It would be interesting to perform a similar calculation for neutron β decay in order to discuss the universality of the weak coupling. Unfortunately, no satisfactory spontaneously broken gauge model exists for strongly interacting particles. Furthermore, the effects of the strong interactions would make it very difficult to calculate β decay with sufficient precision. In this section we nevertheless make some general observations about this process.

It is clear that the most naive possible extension of Weinberg's lepton model, in which one neglects all strong-interaction effects and the existence of strange particles, would yield the same results for neutron β decay as for μ decay. Only the kinematics would change, and the neglected terms would be of order m_N^2/M_W^2 where m_N = nucleon mass. (This is still a small quantity.) We remark that the equality between electromagnetic corrections to μ decay, and those in a bare-nucleon calculation, is a property of any intermediate-vector-boson theory, but not of the $V-A$ current-current theory. In that theory the $(\mu\nu_\mu)$ $(e\nu_e)$ vertex may be written in charge-retention form by a Fierz transformation, and thus the electromagnetic corrections to it are finite; but the electromagnetic corrections to β decay are infinite.

One can abstract the essential features of the bare-nucleon calculation in the form of current-algebraic requirements which must be satisfied by any strongly interacting model in order to obtain a finite result for β decay. Similar arguments for other processes would give further requirements.

Let us imagine a model in which the Lagrangian contains the usual vector and scalar particles of the gauge model which will always be treated perturbatively, and also some further strongly interacting particles such as a vector gluon[25] the effects of which we wish to include to all orders in the expressions for the currents. We recognize that the freedom to introduce such particles is considerably restricted by the requirement that the original Lagrangian must respect certain local gauge symmetries.

Writing J_μ^- for the hadronic current which couples to the charged vector meson and J_μ^Z for that which couples to the neutral vector meson, we can express the sum of diagrams $(a+b)$ of Fig. 4(a) (reading $n \to p$ for $\mu \to \nu$) as[26]

$$(a+b)^{np}_{\beta\gamma} = i \int d^4x\, e^{-ik_2\cdot x} \langle n |\, T^*(J_\beta^-(0) J_\gamma^Z(x)) | p\rangle . \tag{37}$$

The f term will have the same form as previously, with the leptonic current $J_\rho^{(\mu)\dagger}$ replaced by the hadronic $\langle n | J_\rho^-(0) | p\rangle$. The cancellation of leading divergences then occurs if

$$\begin{aligned}\langle n | [J_\alpha^-(0),\ Q^Z(0)] | p\rangle &= \langle n | [Q^-(0),\ J_\alpha^Z(0)]\, | p\rangle \\ &= 2R \langle n | J_\alpha^-(0) | p\rangle ,\end{aligned} \tag{38}$$

where

$$Q^Z(t) = \int d^3x\, J_0^Z(t, \vec{x}),$$

$$Q^-(t) = \int d^3x\, J_0^-(t, \vec{x}).$$

[Note that these charges are not constants, since the currents are not conserved.] Equation (38) is a necessary but not sufficient condition for the finiteness of β decay, and is a feature that one would wish to require of any weak-interaction model including hadrons. One can formulate further constraints in the form of limits on the allowable growth as $q^2 \to \infty$ of

$$R^1_\eta = \int d^4x\, e^{-ik_1\cdot x} \langle n |\, T^*(\partial^\mu J_\mu^-(x) J_\eta^Z(0))\, | p\rangle$$

and

$$R^2_\eta = \int d^4x\, e^{-ik_2\cdot x} \langle n |\, T^*(\partial^\mu J_\mu^Z(x) J_\eta^-(0)) | p\rangle .$$

These conditions can be derived by making the substitution (37) in the expressions of Eq. (23) and following the algebraic steps laid out in Sec. IV C. They are sufficiently complicated that we do not feel it is worthwhile to give them here.

VI. COMMENTS

The intimate relationship of weak and electromagnetic interactions is a fundamental property of spontaneously broken gauge models. It has been demonstrated explicitly that the good high-energy behavior and renormalizability of such models generally obtains as a result of an interplay and cancellation between these interactions.[5] However, in the process $\mu\nu_e \to \nu_\mu e$ (or μ decay), the divergences in one-loop graphs with t-channel W-Z, W-ϕ, and W-γ cuts cancel among themselves, as we have explained in Sec. III. Thus the first-order electromagnetic corrections to μ decay are ultraviolet finite all by themselves, after renormalization. We note that this finiteness of the electromagnetic corrections depends only upon the electromagnetic properties of the W, and not upon the details of our gauge model of weak interactions.

The importance of the gauge model for this calculation becomes evident when one examines the universality questions discussed in Sec. II. It is only after *all* vertex and wave-function renormalizations, including graphs involving the photon, the vector bosons, and the scalar ϕ, that g_μ/g does not contain logarithmically divergent terms proportional to m_e^2/M_W^2 or m_μ^2/M_W^2. The finite contributions to g_μ/g from graphs containing the W, Z, and ϕ are very small (of order m_μ^2/M_W^2 or smaller). The graphs containing photons contribute terms of order $\alpha \ln m_\mu/m_e$ to g_μ/g, as we stated in Sec. IV D. A detailed discussion of these terms is reserved for paper II.

The main result of our explicit calculation of μ decay in the Weinberg model of leptons is the demonstration that it is finite and that the weak corrections to the rate in this model are the same size as the electromagnetic corrections. It is perhaps unfortunate that the weak corrections have the same dependence on the final electron's energy as the Born term, except for negligible corrections of order m_μ^2/M_W^2. As a result, the effect of these weak corrections is just effectively to "renormalize" the weak interaction coupling constant by $\sim\frac{1}{2}\%$. It thus appears impossible to detect this effect experimentally until other weak processes can be calculated and measured with comparable precision.

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†Junior Fellow, Society of Fellows.

[1] G. 't Hooft, Nucl. Phys. B35, 167 (1971); B. W. Lee, Phys. Rev. D 5, 823 (1972).

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[3] P. W. Higgs, Phys. Rev. 145, 1156 (1966); T. W. B. Kibble, *ibid.* 155, 1554 (1967), and references therein.

[4] S. Weinberg, Phys. Rev. Letters 19, 1264 (1967); A. Salam, in *Proceedings of the Eighth Nobel Symposium* (Almqvist and Wicksel, Stockholm, 1968).

[5] S. Weinberg, Phys. Rev. Letters 27, 1688 (1972); S. Weinberg and R. Jackiw, Phys. Rev. D 5, 2396 (1972).

[6] T. W. Appelquist and H. R. Quinn, Phys. Letters 39B, 229 (1972).

[7] S. M. Berman, Phys. Rev. 112, 267 (1958); T. Kinoshita and A. Sirlin, *ibid.* 113, 1652 (1959); L. Durand, III, L. F. Landovitz, and R. B. Marr, *ibid.* 130, 1188 (1963), present a dispersive calculation.

[8] Electromagnetic corrections to μ decay have been calculated in nonrenormalizable intermediate-vector-boson theories by T. D. Lee, Phys. Rev. 128, 899 (1962), and D. Bailin, *ibid.* 135, 166 (1964). These calculations contain divergences for intermediate vector bosons with $g \neq 2$.

[9] For another example, see H. Georgi and S. L. Glashow Phys. Rev. Letters 28, 1494 (1972); the model proposed in this paper does not possess neutral currents.

[10] Throughout the paper we use the metric and other conventions given in J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[11] This term was suggested by S. Weinberg (private communication); "crypto" denotes "hidden," which accurately describes the situation.

[12] It is worthwhile emphasizing that a regulator procedure for the U formalism is not a direct by-product of a regulated R formalism. The functional transformation from the R formalism to the U formalism is nonlinear. Both the renormalization constants and the Green's functions can be much more divergent in the latter. It is only the renormalized S matrix that is expected to survive this transformation unscathed.

[13] Some references on the Lagrangian counterterm formulation of renormalization are: S. N. Gupta, Proc. Phys. Soc. (London) A64, 426 (1951); G. Takeda, Prog. Theor. Phys. (Kyoto) 7, 359 (1952); P. T. Matthews and A. Salam, Phys. Rev. 94, 185 (1954); N. N. Bogoliubov and O. S. Parasiuk, Acta Math. 97, 227 (1957); K. Hepp, Commun. Math. Phys. 2, 301 (1966).

[14] Because the model was obtained by spontaneous symmetry breaking, there are fewer renormalization constants than there are vertices and propagators; thus certain relationships must be satisfied between the various subtraction constants. Cf. Ref. 2. The situation is similar to that in the σ model, discussed by B. W.

Lee, Nucl. Phys. B9, 649 (1969), and by J.-L. Gervais and B. W. Lee, Nucl. Phys. B12, 627 (1969).

[15]V. B. Berestetskii, Sov. Phys.—Usp. 5, 7 (1962); T.-T. Chou and Max Dresden, Rev. Mod. Phys. 39, 143 (1967).

[16]The contribution to the W vacuum-polarization tensor from the graph under consideration here is real on the mass shell; but this is not so for the (νe)-loop contribution, for example. We discuss this case in more detail in Sec. IV E.

[17]One can verify that in the ϕ^3 theory all one-loop graphs are correctly evaluated by dispersing the whole graph in the manner we explain. We discuss the validity of the procedure here because the complications of spin are not entirely trivial.

[18]It is the coefficients of $J_\rho^{(e)} J_\sigma^{(\mu)\dagger} g^{\rho\sigma}$ and $J^{(e)} J^{(\mu)\dagger}$ which are actually dispersed.

[19]T. D. Lee, Phys. Rev. Letters 26, 801 (1971).

[20]We note that the singularity as $R \to 0$ ($m_Z \to \infty$) implies that models such as that of S. Schechter and Y. Ueda, Phys. Rev. D 2, 736 (1970), which attempt to remove the neutral currents without introducing additional leptons, are nonrenormalizable. Although for μ decay the singularity is only logarithmic, one-loop contributions to non-charge-exchange processes will diverge quadratically in this limit.

[21]It is important to emphasize that we are calculating in terms of renormalized coupling constants. Thus we do not explicitly include external line corrections. This is to be contrasted with earlier calculations (cf. T. Kinoshita and A. Sirlin, Ref. 7) of the radiative corrections to the Fermi theory where the answer is expressed in terms of the unrenormalized Fermi constant G.

[22]Because of the vanishing neutrino mass the divergent part of the first two graphs of Fig. 4(d) has exactly the right structure to be removed by renormalization. This is not true for the analogous graphs for the process $e\mu \to e\mu$, for example. There one must combine graphs such as these with graphs involving scalar particles in order to cancel a divergent contribution which does not correspond to renormalization subtraction. Cf. Ref. 6.

[23]The identities

$$(\bar{e}\gamma_\gamma\gamma_\rho\gamma_\beta P_- \nu_e)(\bar{\nu}_\mu P_+\gamma^\gamma\gamma_\sigma\gamma^\beta\mu) = 4J_\eta^{(e)} J^{(\mu)\dagger\eta} g_{\rho\sigma}$$

and

$$(\bar{e}\gamma_\gamma\gamma_\rho\gamma_\beta P_- \nu_e)(\bar{\nu}_\mu P_+\gamma^\beta\gamma_\sigma\gamma^\gamma\mu) = 4J_\sigma^{(e)} J_\rho^{(\mu)\dagger}$$

are helpful in calculating these contributions. Also note that we hold s fixed in dispersing graphs 3(f) and 3(g), and u fixed in dispersing graphs 3(h) and 3(i), as noted following Eq. (23). We consistently neglect lepton masses in obtaining the expressions given; note that $m_e^2 < s,\ u < m_\mu^2$.

[24]H. H. Chen and B. W. Lee, Phys. Rev. D 5, 1874 (1972).

[25]One problem which must be considered in introducing such particles is the absence of triangle anomalies. Actually, the Weinberg model has such anomalies and is consequently not renormalizable unless the anomalies are canceled by the introduction of heavy leptons or hadrons. We have ignored the anomaly problem in this paper since it does not arise until one goes to higher orders in perturbation theory. For further discussion see C. Bouchiat, J. Iliopoulos, and P. Meyer, Phys. Letters 36B, 519 (1972); D. Gross and R. Jackiw, Phys. Rev. D 6, 477 (1972), and H. Georgi and S. L. Glashow, *ibid.* 6, 429 (1972).

[26]The T^* product is defined by subtracting a c-number part:

$$T^*(AB) = T(AB) - \langle T(AB)\rangle_\Omega\,.$$

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Reciprocal Bootstrap on the Light Cone

A. Zee*

Institute for Advanced Study, Princeton, New Jersey 08540

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We introduce dynamical considerations onto the light cone in the form of the static bootstrap. We obtain (1) the prediction that the asymmetry in the deep-inelastic electron scattering on polarized deuteron targets is small, and (2) a relation between $F_2^{\nu p} + F_2^{\nu n}$, $F_2^{\gamma p}$, and $F_2^{\gamma n}$. The "physical origin" of these results is discussed. The result (2) also follows as a "chiral-limit theorem."

I. INTRODUCTION

Just as our understanding of electromagnetic and weak interaction allows the measurement of the matrix elements of the local currents,[1] the assumption of light-cone dominance[2] enables the matrix elements of an infinite collection of local operators to be measured in deep-inelastic scattering experiments. These are the local operators contained in the so-called bilocal operators defined on the light cone. Unhappily, what one can say about these operators has so far been limited. Their Lorentz tensor property is presumed known.[3] They are also believed to transform like SU(3) sin-