and by our assumptions, there must be an A_{ij} such that $B_{ij}\delta\lambda_j = A_{ij}\lambda_j$. Then $|B_{ij}\delta\lambda_j|^2 = \delta\lambda_j B_{ji} A_{ik}\lambda_k$ becomes, with (6), $|B_{ij}\delta\lambda_j|^2 = \delta\lambda_j [B,A]_{j,k}\lambda_k$. Since the matrices form a Lie algebra, $[B,A]$ is also in $\mathfrak{A}$, and the right-hand side vanishes by (4). This establishes (7) and completes the proof.

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[6]This condition is actually necessary. Sidney Coleman and Erick Weinberg (unpublished) have considered massless scalar electrodynamics in the one-loop approximation and found symmetry-breaking solutions which go continuously to the symmetric solution as the coupling constants go to zero.

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Constraints on Anomalies*

Stephen L. Adler and Curtis G. Callan, Jr.†
Institute for Advanced Study, Princeton, New Jersey 08540

and

David J. Gross
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

and

Roman Jackiw
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 16 June 1972)

The various coupling-constant-dependent numbers describing anomalous commutators are constrained by the nonrenormalization of the axial-vector–current anomaly. The axial-vector current continues to behave anomalously even if the underlying unrenormalized field theory is finite due to the vanishing of the Gell-Mann–Low eigenvalue function.

I. INTRODUCTION

It has now been established that the canonical formalism of quantum field theory frequently yields results that are not verified in perturbation theory.[1] These "anomalies" are of two distinct kinds. Firstly there are failures of the Bjorken-Johnson-Low (BJL)[2] limit: Equal-time commutators between operators, when evaluated by the BJL technique in perturbation theory, usually do not agree with the canonical determination of these commutators. A well-known consequence is the failure of the Callan-Gross sum rule for electroproduction.[3] Secondly there are violations of Ward identities associated with exact or partial symmetries; the two known examples being the triangle anomaly of the axial-vector current and the trace anomaly of the new improved energy-momentum tensor.[1] (When a Ward identity is anomalous, there is also a corresponding BJL anomaly.) The Sutherland-Veltman low-energy theorem for neutral pion decay is falsified as a consequence.[4] Both categories of anomalies arise from the divergences of unrenormalized perturbation theory, which require the introduction of regulators to define the theory. The BJL anomaly reflects the noncommutativity of the BJL high-energy limit with the infinite regulator limit which must be taken to define renormalized, physical amplitudes. Failures of Ward identities arise when no regulator exists which preserves the relevant symmetry.

Although the common cause for both classes of anomalies is evident, it has not been appreciated that an intimate relationship exists between the BJL anomalies and the failures of Ward identities. In this paper we demonstrate that the very interesting analysis by Crewther[5] of the triangle anomaly in terms of Wilson's short-distance expansion[6] can be extended to exhibit this relationship. Further we show that in lowest nontrivial order of perturbation theory the q-number anomaly in the equal-time commutator of space-components of currents can be completely determined in terms of the c-number anomaly in the equal-time commutator between the time component and space component of the current, i.e., the ordinary Schwinger term. Finally we inquire to what extent the canonical formalism can be reestablished if the unrenormalized theory becomes finite due to the vanishing of the Gell-Mann–Low[7] eigenvalue function. Our conclusion, at least for the axial-vector current, is that naive manipulations continue to lead to error.

II. THE CREWTHER ANALYSIS

Assume that one is dealing with a theory which is conformally invariant at short distances. Consider the vector–vector–axial-vector current amplitude

$$T^{\mu\nu\alpha}_{abc}(x,y,z)=\langle 0|T(V^\mu_a(x)V^\nu_b(y)A^\alpha_c(z))|0\rangle\,. \qquad (2.1)$$

Schreier[8] has shown that a conformally invariant three-index pseudotensor of dimension 9 must be proportional to the fermion triangle graph constructed from massless fermions in free field theory. Hence (2.1) is given by[9]

$$T^{\mu\nu\alpha}_{abc}(x,y,z)=N\Delta^{\mu\nu\alpha}_{abc}(x,y,z)+\cdots\,, \qquad (2.2\text{a})$$

$$\Delta^{\mu\nu\alpha}_{abc}(x,y,z)=\frac{d_{abc}}{16\pi^6}\,\frac{\mathrm{Tr}\gamma^5\gamma^\mu\gamma^\delta\gamma^\nu\gamma^\epsilon\gamma^\alpha\gamma^\varphi(x-y)_\delta(y-z)_\epsilon(z-x)_\varphi}{[(x-y)^2-i\epsilon]^2[(y-z)^2-i\epsilon]^2[(z-x)^2-i\epsilon]^2}\,. \qquad (2.2\text{b})$$

Here N is a number and the dots in (2.2a) represent less singular, non-scale-invariant contributions to $T^{\mu\nu\alpha}_{abc}(x,y,z)$, which vanish in the scale-invariant (= conformally invariant) limit. The precise assumption about these subdominant terms is that they can be identified and separated from $\Delta^{\mu\nu\alpha}_{abc}(x,y,z)$ in *sequential* short-distance limits,

$$\lim_{x_i\to x_k}\ \lim_{x_j\to x_k} T^{\mu\nu\alpha}_{abc}(x,y,z)=\lim_{x_i\to x_k}\ \lim_{x_j\to x_k} N\Delta^{\mu\nu\alpha}_{abc}(x,y,z)+\text{less singular terms}\,, \qquad (2.3)$$

where $\{x_i, x_j, x_k\}$ are any of $\{x,y,z\}$. Thus we know that

$$\lim_{x\to 0}\lim_{y\to 0} T^{\mu\nu\alpha}_{abc}(x,y,0)=\frac{Nd_{abc}}{4\pi^6}\,\frac{\epsilon^{\nu\alpha\delta\epsilon}y_\epsilon(2x^\mu x_\delta-g^\mu_\delta x^2)}{(y^2-i\epsilon)^2(x^2-i\epsilon)^4}+\text{less singular terms}\,, \qquad (2.4\text{a})$$

$$\lim_{z\to 0}\lim_{x\to 0} T^{\mu\nu\alpha}_{abc}(x,0,z)=\frac{Nd_{abc}}{4\pi^6}\,\frac{\epsilon^{\mu\nu\delta\epsilon}x_\epsilon(2z^\alpha z_\delta-g^\alpha_\delta z^2)}{(x^2-i\epsilon)^2(z^2-i\epsilon)^4}+\text{less singular terms}\,. \qquad (2.4\text{b})$$

Next a scale-invariant short-distance expansion for current commutators is postulated,

$$[V^\mu_a(x),V^\nu_b(0)]\underset{x\sim 0}{=}-iS_{VV}\delta_{ab}(g^{\mu\nu}x^2-2x^\mu x^\nu)\,\frac{\epsilon(x^0)\delta'''(x^2)}{6\pi^3}+iK_{VV}d_{abc}\epsilon^{\mu\nu}{}_{\alpha\beta}\,A^\alpha_c(0)x^\beta\,\frac{\epsilon(x^0)\delta'(x^2)}{\pi}+\cdots\,, \qquad (2.5\text{a})$$

$$[A^\mu_a(x),A^\nu_b(0)]\underset{x\sim 0}{=}-iS_{AA}\delta_{ab}(g^{\mu\nu}x^2-2x^\mu x^\nu)\,\frac{\epsilon(x^0)\delta'''(x^2)}{6\pi^3}+iK_{AA}d_{abc}\epsilon^{\mu\nu}{}_{\alpha\beta}\,A^\alpha_c(0)x^\beta\,\frac{\epsilon(x^0)\delta'(x^2)}{\pi}+\cdots\,, \qquad (2.5\text{b})$$

$$[V^\mu_a(x),A^\nu_b(0)]\underset{x\sim 0}{=}iK_{VA}d_{abc}\epsilon^{\mu\nu}{}_{\alpha\beta}V^\alpha_c(0)x^\beta\,\frac{\epsilon(x^0)\delta'(x^2)}{\pi}+\cdots\,. \qquad (2.5\text{c})$$

The dots indicate less singular contributions, or operators with quantum numbers and symmetries different from the exhibited terms. S and K are constants which appear in the following equal-time commutators,

$$[V^0_a(x),V^i_b(0)]|_{x^0=0}=i\delta_{ab}S_{VV}\Lambda\partial^i\delta^3(\vec{x})-\frac{i}{24\pi^2}\delta_{ab}S_{VV}\partial^i\partial_k\partial^k\delta^3(\vec{x})+\cdots\,, \qquad (2.6\text{a})$$

$$[V^i_a(x),V^j_b(0)]|_{x^0=0}=iK_{VV}d_{abc}\epsilon^{ijk}A^k_c(0)\delta^3(\vec{x})+\cdots\,, \qquad (2.6\text{b})$$

etc.

Λ is a quadratically divergent constant, and the omitted terms have different quantum numbers. An expansion similar to (2.5) is written for T products,

$$T(V_a^\mu(x)V_b^\nu(0)) \underset{x\sim 0}{=} S_{VV}\frac{\delta_{ab}(g^{\mu\nu}x^2-2x^\mu x^\nu)}{2\pi^4(x^2-i\epsilon)^4}-K_{VV}\frac{d_{abc}\epsilon^{\mu\nu}{}_{\alpha\beta}A_c^\alpha(0)x^\beta}{2\pi^2[x^2-i\epsilon]^2}+\cdots, \tag{2.7a}$$

$$T(A_a^\mu(x)A_b^\nu(0)) \underset{x\sim 0}{=} S_{AA}\frac{\delta_{ab}(g^{\mu\nu}x^2-2x^\mu x^\nu)}{2\pi^4(x^2-i\epsilon)^4}-K_{AA}\frac{d_{abc}\epsilon^{\mu\nu}{}_{\alpha\beta}A_c^\alpha(0)x^\beta}{2\pi^2[x^2-i\epsilon]^2}+\cdots, \tag{2.7b}$$

$$T(V_a^\mu(x)A_b^\nu(0)) \underset{x\sim 0}{=} -K_{VA}\frac{d_{abc}\epsilon^{\mu\nu}{}_{\alpha\beta}V_c^\alpha(0)x^\beta}{2\pi^2(x^2-i\epsilon)^2}+\cdots. \tag{2.7c}$$

Crewther's observation is that the constants N, S, and K are not independent.[5] From (2.1) and (2.7c) it follows that

$$\lim_{y\to 0}T_{abc}^{\mu\nu\alpha}(x,y,0)=-K_{VA}\frac{d_{bcd}\epsilon^{\nu\alpha}{}_{\omega\varphi}y^\varphi}{2\pi^2(y^2-i\epsilon)^2}\langle 0|T(V_a^\mu(x)V_d^\omega(0))|0\rangle \tag{2.8a}$$

while (2.6a) implies that

$$\lim_{x\to 0}\lim_{y\to 0}T_{abc}^{\mu\nu\alpha}(x,y,0)=S_{VV}K_{VA}\frac{d_{abc}\epsilon^{\nu\alpha\delta\epsilon}y_\epsilon(2x^\mu x_\delta-g_\delta^\mu x^2)}{4\pi^6(y^2-i\epsilon)^2(x^2-i\epsilon)^4}. \tag{2.8b}$$

Hence upon comparing (2.8b) and (2.4a), one finds

$$N=S_{VV}K_{VA}. \tag{2.9a}$$

Additionally, from (2.1), (2.4b), (2.7a), and (2.7b) it follows that

$$N=S_{AA}K_{VV}. \tag{2.9b}$$

If a similar analysis is performed on the axial-vector–axial-vector–axial-vector current amplitude one gets

$$N'=S_{AA}K_{AA}, \tag{2.10}$$

where N' is the proportionality constant defined analogously to (2.2a).[10] As emphasized by Crewther, the interest in relations (2.9) and (2.10) derives from the fact that S and K are measurable (in principle) in various deep-inelastic processes – thus the anomaly in low-energy processes, at an unphysical point, is directly determined by experimental high-energy behavior.

III. CONSTRAINTS ON ANOMALIES

We shall use (2.9) and (2.10) to probe the structure of anomalies in various models. Consider first a free massless field theory with

$$V_a^\mu(x)=:\bar\psi(x)\gamma^\mu\tfrac{1}{2}\lambda_a\psi(x):,$$

$$A_a^\mu(x)=:\bar\psi(x)i\gamma^\mu\gamma_5\tfrac{1}{2}\lambda_a\psi(x):.$$

It is trivial to verify that, as already stated by Crewther,[5] (2.9) and (2.10) are satisfied with $N=K=S=1$. The nonvanishing of S and N is conventionally described as anomalous. A naive evaluation of the equal-time commutator (2.6a) yields a vanishing result; the nonvanishing of S measures the famous Schwinger-term anomaly. Similarly a naive evaluation of $\partial_\mu^x T_{abc}^{\mu\nu\alpha}(x,y,z)$, $\partial_\nu^y T_{abc}^{\mu\nu\alpha}(x,y,z)$, and $\partial_\alpha^z T_{abc}^{\mu\nu\alpha}(x,y,z)$ yields zero since the currents are conserved. Nevertheless, as Schreier has shown,[8] one cannot consistently set all divergences of $\Delta_{abc}^{\mu\nu\alpha}(x,y,z)$ to zero, because this quantity is singular when all three points coincide. In momentum space this corresponds to the well-known violation of Ward-Takahashi identities of the fermion, axial-vector triangle graph. Hence N measures the axial-vector–current anomaly.[1] Since a naive determination of K from the equal-time commutator (2.6b) also gives $K=1$, we have a connection between the anomalies: $N=S$; the triangle anomaly is a consequence of the Schwinger-term anomaly.

Next consider a fermion theory with an SU(3)-invariant Yukawa interaction of strength g involving neutral vector gluons. (Spin-zero gluons render the axial-vector current infinite; hence we do not consider them.) In order to apply Crewther's analysis,[5] it is necessary to satisfy his hypotheses: (1) the existence of finite currents; (2) the existence of a scale-invariant expansion for products of currents, (2.5) and (2.7); and (3) the existence of a conformally invariant short-distance limit for $T_{abc}^{\mu\nu\alpha}(x,y,z)$, (2.2). No complete calculation of $T_{abc}^{\mu\nu\alpha}(x,y,z)$ in higher order has been performed which can be used to check the third hypothesis. We shall nonetheless *assume* that this result is valid, provided the other two are satisfied. This assumption is motivated by the fact that the triangle anomaly has no higher-order corrections,[11] and is almost certainly true for the class of graphs, discussed in detail below, which contain only a single fermion loop. (See the Appendix.) Therefore we set $N=N'=1$, *even in the presence of interactions*. In order to satisfy the first hypothesis, we must not consider SU(3) singlet vector currents, since these are not well defined in perturbation theory. The interaction with the vector gluon gives rise to infinite vacuum polarization, which modifies the singlet current.

In lowest-order perturbation theory in this model, a scale-invariant expansion for current products exists. This can be seen as follows. A BJL-limit determination of the equal-time commutator (2.6b) yields a finite expression.[1] Hence the q-number portion of the expansion (2.5) and (2.7) exists. That the c-number part also exists follows from the Jost-Luttinger calculation of the proper vacuum polarization tensor.[12] Their result is that in second order of perturbation theory this object is no more singular than in the free-field model. [In momentum space both the free-field graph and the lowest order graphs of Fig. 1 go as a single power of $\ln(-k^2)$ for large k.] However, S and K depart from their free-field values. The Jost-Luttinger formula[12] for S is $1+3g^2/16\pi^2$. Because $N=1$, we must have $K=(1+3g^2/16\pi^2)^{-1}\approx 1-3g^2/16\pi^2$; and the BJL calculation of K gives indeed this answer.[1] Hence we see that the BJL anomaly in the commutator of two spatial components of currents is determined by the higher-order terms in the c-number Schwinger term.

Beyond lowest order, perturbation theory no longer satisfies the hypotheses (1) and (2). The axial-vector current ceases to be well defined, since the graph of Fig. 2 is not rendered finite by external wave-function renormalization factors.[1] Also the c-number portion of the expansion (2.5) and (2.7) is not of the assumed scale-invariant form, since the proper vacuum polarization tensor acquires quadratic and higher powers of $\ln(-k^2)$. (No calculations have been performed on the q-number part of the expansion; but we expect that it too is no longer scale-invariant.) However, subsets of graphs can be chosen which probably continue to satisfy the hypotheses. For example if fermion creation and annihilation is ignored, then the vacuum expectation value of the currents is given by the one-fermion-loop graphs. In this approximation we have[13]

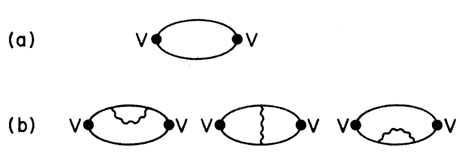

FIG. 1. Contributions to $\int d^4x\, e^{ikx}\,\langle 0|T(V_a^\mu(x)V_b^\nu(0))|0\rangle$ which go as $\ln(-k^2)$ for large k. (a) Free-field theory graph. (b) Lowest-order perturbation theory graphs.

$$\int \frac{d^4x}{(2\pi)^4} e^{iqx}\langle 0|T(V_a^\mu(x)V_b^\nu(0))|0\rangle^{\text{one-fermion-loop}}$$
$$=(g^{\mu\nu}q^2-q^\mu q^\nu)\frac{i}{24\pi^2}\delta_{ab}F(g^2)\ln(-q^2/m^2)$$
$$+\text{less singular terms}. \tag{3.1a}$$

Here $F(g^2)$ is the Baker-Johnson function,[13] whose first three terms in a power series expansion are known:

$$F(g^2)=1+\frac{3g^2}{16\pi^2}-\frac{3}{512}\frac{g^4}{\pi^4}+\cdots. \tag{3.1b}$$

Also the axial-vector current is no longer infinite, since the graph of Fig. 2 is absent. Evidently the c-number term in the expansion (2.5) and (2.7) is scale invariant with $S=F(g^2)$, and it is likely that so also is the q-number term. Hence we conjecture that if the current commutator were computed in the BJL limit, without including fermion creation or annihilation processes, one would find

$$K(g^2)=\frac{1}{F(g^2)}=1-\frac{3g^2}{16\pi^2}+\frac{21}{512}\frac{g^4}{\pi^4}+\cdots. \tag{3.2}$$

IV. ANOMALIES IN THE GELL-MANN–LOW LIMIT

We consider quantum electrodynamics, and assume that the Gell-Mann–Low eigenvalue function possesses a zero, so that Z_3 is finite.[7] Now one *can* discuss currents, since the vacuum polarization no longer diverges. In this limit it should be possible to set the electron mass m to zero and scale invariance becomes exact.[14] (Z_2, the electron wave-function renormalization constant, can be made finite by appropriate choice of gauge.) We examine this (hypothetical) theory in the context of the ideas developed in Secs. II and III. It will be seen that singular behavior survives even in this finite theory and that naive canonical reasoning continues to be inapplicable.[15] [All our previous formulas hold with SU(3) indices suppressed, and the following replacements: $V_a^\mu\to J^\mu$, $A_a^\mu\to J_5^\mu$, $d_{abc}\to 2$, $\delta_{ab}\to 2$.] Observe first that, since the triangle anomaly has no radiative corrections, N continues to be equal to unity. However, because Z_3 is *finite*, the quadratically divergent Schwinger term is ab-

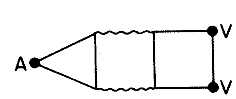

FIG. 2. Graph which renders the axial-vector current infinite.

sent; i.e., $S=0$. Since $K=1/S$, we see that the coefficient of the axial-vector current in the short-distance expansion of the product of two currents is infinite. In other words the c-number singularity (2.5) is weaker than x^{-6}, but the q-number singularity is stronger than x^{-3}, so that their product remains singular as x^{-9}. Consequently the BJL limit, which naively gives (2.5), is anomalous, and this is true regardless whether or not the electron mass is set to zero.

Further difficulties emerge if we set the electron mass to zero. In that limit all vacuum matrix elements of current products vanish by the Federbush-Johnson theorem,[14]

$$\langle 0|J^{\mu_1}(x_1)\cdots J^{\mu_n}(x_n)|0\rangle=0\,. \tag{4.1}$$

Nevertheless we now show that one cannot conclude the strong statement that $J^\mu(x)|0\rangle=0$. For if this were true then

$$T^{\mu\nu\alpha}(x,y,z)=\langle 0|T(J^\mu(x)J^\nu(y)J_5^\alpha(z))|0\rangle \tag{4.2}$$

must be purely a seagull, since no matter what the values of x^0, y^0, and z^0 are, there is always an electromagnetic current adjacent to the vacuum. However, a seagull cannot give rise to the anomalous divergence,[1] which survives even in the Gell-Mann–Low scale-invariant limit, since it is mass independent and is not renormalized. *Consequently the equation* $J^\mu(x)|0\rangle=0$ *is false.* Evidently only a weaker statement can be true

$$\langle 0|OJ^\mu(x)|0\rangle=0\,, \tag{4.3}$$

where O stands for some, *but not all,* operators. In particular, products of electromagnetic currents can comprise O, but O cannot be

$$J_5^\alpha(z)J^\nu(y) \quad\text{or}\quad J^\nu(y)J_5^\alpha(z)\,.$$

A further problem appears if we combine the Federbush-Johnson theorem with the results which we obtained above from Crewther's analysis when fermion creation and annihilation were neglected. As Baker and Johnson[14] have shown, when the coupling g^2 is equal to the value $g_0^{\,2}$ which makes Z_3 finite, and the electron mass is zero, (4.1) holds even in the one-fermion-loop approximation. In particular, $g_0^{\,2}$ is a zero of the function $F(g^2)$ defined in (3.1a) and the four-point function satisfies

$$T^{\mu\nu\lambda\sigma}(x0yz)=\langle 0|T(J^\mu(x)J^\nu(0)J^\lambda(y)J^\sigma(z))|0\rangle^{\text{one-fermion-loop}}_{g^2=g_0^2,\,m=0}=0\,. \tag{4.4}$$

But now let us take the limit $x\to 0$ in (4.4) and substitute the short-distance expansion of (2.7a). In the one-loop approximation $S_{VV}=F(g_0^{\,2})=0$, so the leading contribution comes from the second term in (2.7a) and is given by

$$T^{\mu\nu\lambda\sigma}(x0yz)\underset{x\sim 0}{\propto}K(g_0^{\,2})\frac{\epsilon^{\mu\nu}{}_{\alpha\beta}x^\beta}{(x^2-i\epsilon)^2}\langle 0|T(J_5^\alpha(0)J^\lambda(y)J^\sigma(z))|0\rangle^{\text{one-fermion-loop}}_{g^2=g_0^2,\,m=0}\,. \tag{4.5}$$

This is *infinite*, since according to (3.2) the coefficient $K(g_0^{\,2})$ is equal to $F^{-1}(g_0^{\,2})=\infty$, while we have seen that the three-point function appearing in (4.5) cannot vanish. So we have reached the impossible conclusion that $0=\infty$! Evidently, if the theory has an eigenvalue g_0 which makes Z_3 finite, the naive short-distance expansion is invalid *at the eigenvalue*, even though it may be true order-by-order in perturbation theory. In particular, the limiting operations $g\to g_0$ and $x\to 0$ do not commute. One can easily write down simple examples which have this property, e.g.,

$$\frac{F^{-1}(g^2)}{1+f(x)F^{-2}(g^2)}\,, \tag{4.6}$$

where $f\neq 0$ for $x\neq 0$ but $f(0)=0$. For all nonzero x, (4.6) vanishes as $F(g^2)$ in the limit $g^2\to g_0^{\,2}$, but for $x=0$, (4.6) diverges as $F^{-1}(g^2)$ in the same limit. Whether such behavior can actually emerge from field theory, when all the constraints imposed by current conservation and conformal invariance are taken into account, remains an open question, as indeed does the question of whether an eigenvalue $g_0^{\,2}$ exists in the first place.

V. CONCLUSION

We have shown that the coupling-constant-dependent numbers, describing various BJL anomalies, are constrained by the nonrenormalization of the triangle anomaly. Furthermore the axial-vector current continues to behave in a singular fashion even in the finite theory of Gell-Mann and Low. In particular the following three phenomena are incompatible:

(1) The triangle anomaly is unrenormalized.

(2) There is an eigenvalue $g^2=g_0^{\,2}$ which makes Z_3 finite.

(3) Naive scale-invariant short-distance expansions involving the axial-vector current are valid at the eigenvalue.

Crewther[5] also applies Wilson's method to anomalies of scale invariance.[1] Unfortunately there does not seem to be a "no renormalization theorem" for these anomalies since *all* regulators violate scale invariance. (Chiral invariance is not violated by boson regulators; these render finite all graphs but the basic fermion triangle.) Therefore results analogous to the above cannot be deduced for scale invariance anomalies.[16]

We have benefited from conversations with R. Crewther, K. Johnson and K. Wilson, which we are happy to acknowledge. SLA and CGC, Jr. wish to acknowledge the hospitality of the National Accelerator Laboratory, where part of this work was done.

APPENDIX

In this Appendix we shall give arguments for our assertion that the vacuum-polarization-free triangle is asymptotically conformal invariant. Our starting point is the Ward identities for scale and conformal invariance. At the naive canonical level, these have the form

$$\int dz \langle 0|T(\Theta(z)\phi^{(1)}(x_1)\cdots\phi^{(n)}(x_n))|0\rangle = i\sum_{i=1}^{n}\left(x_i\cdot\frac{\partial}{\partial x_i}+d_i\right)\langle 0|T(\phi^{(1)}(x_1)\cdots\phi^{(n)}(x_n))|0\rangle, \tag{A1}$$

$$\int dz\, z_\mu \langle 0|T(\Theta(z)\phi^{(1)}(x_1)\cdots\phi^{(n)}(x_n))|0\rangle$$
$$= i\sum_{i=1}^{n}\left(2x^i_\mu x^i\cdot\frac{\partial}{\partial x^i}-x_i{}^2\frac{\partial}{\partial x_i^\mu}+2x_i^\nu(d_i g_{\mu\nu}+\Sigma^{(i)}_{\mu\nu})\right)\langle 0|T(\phi^{(1)}(x_1)\cdots\phi^{(n)}(x_n))|0\rangle, \tag{A2}$$

where Θ is the trace of the "improved" energy-momentum tensor (hence containing only mass terms and other soft operators), d_i is the canonical dimension of the field $\phi^{(i)}$ and $\Sigma^{(i)}_{\mu\nu}$ is the corresponding intrinsic spin matrix.

The basis for the naive argument for asymptotic conformal invariance is the observation that since Θ must contain explicit factors of mass the left-hand sides of (A1) and (A2) must, on dimensional grounds alone, be less singular at short distances than the corresponding right-hand side.

The work of Zimmermann,[17] Lowenstein,[18] and Schroer[19] indicates that when the unavoidable divergences of perturbation theory are properly taken into account, the above Ward identities are modified by the addition to Θ of operator contributions of dimension four (nonsoft). These new terms have no explicit dimensional factors and need not vanish relative to the right-hand side in the short-distance limit. As a result, asymptotic scale and conformal invariance are not realized in renormalized perturbation theory, except in special cases.

The nonsoft contributions to Θ are associated with the various wave-function and coupling-constant renormalization subtractions needed to make the theory finite. The pieces associated with wave-function renormalization can in fact be absorbed in (A1) and (A2) by replacing the canonical dimensions d_i by coupling-constant-dependent "anomalous dimensions" $\bar{d}_i$. The pieces associated with coupling-constant renormalization are proportional to the various interaction terms in the Lagrangian and simplify only in (A1): The insertion at zero four-momentum of an interaction term is equivalent to differentiation with respect to the corresponding coupling constant.

In the body of the paper we considered a theory of an SU_3 singlet vector-meson coupling via a conserved current to a fermion. The *octet* vector and axial-vector currents in such a theory require no renormalization subtractions, since they cannot be coupled to the singlet vector meson by vacuum polarization bubbles of the type illustrated in Fig. 1. Thus, the $SU_3 \times SU_3$ currents will, according to the preceding paragraph, act like fields with canonical dimensions. The same statement applies to both the electromagnetic and axial-vector currents in quantum electrodynamics with vacuum-polarization insertions omitted. Since the vector meson couples via a conserved current, the usual Ward-identity argument guarantees that coupling-constant infinities arise only from vacuum polarization graphs. If such graphs are excluded – either by fiat, or by looking at a sufficiently low order in perturbation theory – no coupling-constant renormalization is needed, and Θ in (A1) and (A2) may be treated as a soft operator. Further, if we consider a Green's function involving only nonrenormalized currents, so that the relevant dimensions are all canonical, the scale and conformal Ward identities assume their naive form and the argument for asymptotic scale and conformal invariance becomes correct.

Let us apply these remarks to the *VVA* triangle. To $O(g^0)$ (g being the coupling constant of the gluon), we obtain the bare triangle, which is trivially conformally invariant in the short-distance limit. To $O(g^2)$ we obtain the triangle decorated in all possible ways with one gluon. At this level, no vacuum polarization is possible and the above argument indicates that asymptotic conformal invariance still holds. But there is only one possible form for a conformal-invariant *VVA* amplitude. Therefore, in the short-distance limit, $\Gamma_{VVA} \to (1+cg^2)\Gamma^{(0)}_{VAA}$, where $\Gamma^{(0)}_{VVA}$ stands for the asymptotic limit of the bare triangle. On the other hand, the PCAC (partially conserved axial-vector current) anomaly is determined precisely by the short-distance limit of Γ_{VVA} and is also known to be coupling-constant independent. This is possible only if $C=0$, which is to say that the $O(g^2)$ graphs succeed in being conformal invariant by vanishing. Now consider the $O(g^4)$ contributions to Γ_{VVA}. At this level there *are* vacuum-polarization graphs and the argument for conformal invariance breaks down. Nonetheless scale invariance survives. We argued that when coupling-constant renormalization is needed, (A1) is modified by adding a term

$$\beta(g)\frac{\partial}{\partial g}\langle 0|T(\phi^{(1)}(x_1)\cdots\phi^{(n)}(x_n))|0\rangle$$

to the left-hand side. It turns out that β is $O(g^3)$, so that if we need $\beta(\partial/\partial g)\Gamma_{VVA}$ to $O(g^4)$ it suffices to know Γ_{VVA} to $O(g^2)$. We have just argued that the $O(g^2)$ contribution to Γ_{VVA} vanishes more rapidly in the asymptotic limit than naive power counting would suggest. Therefore, the left-hand side of (A1), computed to $O(g^4)$, still vanishes relative to the right-hand side in the short-distance limit, leading to asymptotic scale invariance. In higher orders, scale invariance presumably breaks down as well.

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[9]Our conventions are the following: $\{\gamma^\mu, \gamma^\nu\}=2g^{\mu\nu}$; $g^{\mu\nu}=0$, $\mu\neq\nu$, $g^{00}=-g^{11}=-g^{22}=-g^{33}=1$; $\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3$. We assume SU(3) symmetry realized by triplet quarks, hence the occurrence of $d_{abc}=\frac{1}{4}\operatorname{Tr}\{\lambda_a, \lambda_b\}\lambda_c$.

[10]Henceforth we take $N=N'$; $S_{VV}=S_{AA}=S$; $K_{VV}=K_{VA}=K_{AA}$.

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