PHYSICAL REVIEW D VOLUME 6, NUMBER 10 15 NOVEMBER 1972

Rigorous Parametric Dispersion Representation with Three-Channel Symmetry*

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(Received 30 June 1972)

Starting with an analyticity domain in the two Mandelstam variables which is contained in the domain obtained by Martin, we derive a parametric dispersion representation for scattering amplitudes in the equal-mass case. For pion-pion scattering this representation is a rigorous consequence of the axioms of local field theory; it displays in a symmetric and explicit way the contributions of all three channels, and it has only "physical" absorptive parts. This representation is useful for deriving sum rules involving only absorptive parts and relating all three channels. Some of these sum rules are given in this paper, the most important of which form a set of independent physical relations that lead to necessary and sufficient conditions ensuring full crossing symmetry.

I. INTRODUCTION

In a recent paper,[1] one of us derived parametric dispersion relations for the off-shell Compton amplitude to obtain new sum rules for the electroproduction form factors. The main feature of Ref. 1 was the use of analyticity in the two variables ν and q^2 to get sum rules by integrating along contours that lie on analytic hypersurfaces in the complex ν-q^2 space. The selection of allowed contours was carried out by parametrizing ν and q^2 by analytic functions, $\nu=\nu(z)$ and $q^2=q^2(z)$, regular for $\mathrm{Im}z>0$, with the added restriction that the points $(\nu(z),q^2(z))$ be inside the analyticity domain of the Compton amplitude for all $\mathrm{Im}z>0$. The question immediately arises as to whether similar ideas can be applied to the usual on-shell strong amplitudes.

The present paper is devoted to deriving rigorous parametric dispersion relations for elastic on-shell amplitudes. These again are functions of two variables s and t, the usual Mandelstam variables. For the case of pion-pion scattering it is also known that, as a consequence of the axioms of local field theory, these amplitudes are analytic functions of s and t in domains first derived by Mandelstam[2] and later, using more powerful techniques, extended by Martin.[3] We consider domains E defined by inequalities of the form $|(s-a)(t-a)(u-a)|<C_a$, where a is a real parameter. We shall show that E can be imbedded in the Martin domains. Any point (s,t) that satisfies this inequality, and does not lie on a physical s-, t-, or u-channel cut, is inside the analyticity domain.

The main aim of this paper is to find a convenient parametrization of (s,t), i.e., a rational mapping $z\to(s,t)$ such that the following conditions are fulfilled: (a) For all complex z the points $(s(z),t(z))$ are in E, except for those values of z which correspond to the image of the three cuts of the s, t, and u channels; (b) the mapping exhibits the full 3-fold symmetry of the s, t, and u channels and the symmetry of the domains; and (c) along the image of the cuts in the z plane, the absorptive parts are either physical or obtainable from physical partial-wave amplitudes via convergent Legendre expansions.

The mappings $s=s(z,a)$, $t=t(z,a)$ that we obtain also depend on the real parameter a which appears in the definition of the domains. For certain admissible values of a, the amplitude $F(s(z,a),t(z,a))$ defines a function $\overline{F}(z,a)$ which is analytic in the z plane except for the three cuts. By writing "dispersion relations" in z for fixed a we get rigorous representations of the amplitude that have explicit contributions from all the three channels appearing simultaneously. For fixed a this representation is somewhat reminiscent of the old heuristic Cini-Fubini representation.[4] The kernels we have are more complicated than a simple Cauchy kernel, but our result is an exact consequence of local field theory and not an approximation. These dispersion relations are used first in the case of $\pi^0\pi^0\to\pi^0\pi^0$ scattering to give a class of sum rules that restrict the absorptive part. This class of sum rules is shown to provide a physical set of *independent* necessary and sufficient conditions on the absorptive part of a single channel that guarantee crossing symmetry. Using partial-wave projections, we translate these sum rules into a set of necessary and sufficient conditions on the imaginary parts of the partial-wave amplitudes, $\mathrm{Im}f_l(s)$, that guarantee crossing. Although a similar set of conditions has been derived before[5] via different techniques, our conditions have the advantage of being independent, while those of Ref. 5 are dependent and overdetermined. Obviously, the same method can be applied to charged pion-pion scattering.

In Sec. II, we review briefly some of the results of Martin[5] on the analyticity domains and define the domains that we shall use. In Sec. III, we use the cubic equation which defines the domain E to lead us in a fairly clear way to a rational mapping $z \to (s, t)$ that has the properties (a), (b), and (c) mentioned above. The dispersion relations in z are derived in Sec. IV. There we also calculate the domain of admissible values for the parameter a. The case of $\pi^0\pi^0$ scattering is taken up in Sec. V, and the main result is an infinite set of physical sum rules on $\text{Im}f_l(s)$ which give a set of necessary and sufficient conditions that guarantee full crossing. A procedure is then defined to go from a set $\{\text{Im}f_l(s)\}$ respecting unitarity to an amplitude $F(s,t)$ that is not only crossing-symmetric but also analytic in the Martin domains, and respects the Froissart bound. Finally, in Sec. VI, we discuss briefly the case of charged pion-pion scattering and give two examples of similar sum rules.

II. ANALYTICITY DOMAIN IN TWO VARIABLES

Our starting point is the analyticity domain of the elastic scattering amplitude, as deduced from axiomatic field theory. More precisely, we want to use, *in all three channels* connected by the crossing relations, enlarged domains of the type derived by Martin.[3] The main ingredients of Martin's derivation are the validity of fixed-transfer dispersion relations and the positivity condition. Since we have to require these properties to be true in all three channels, we are forced to restrict ourselves to the scattering of the lightest hadronic particles, as already pointed out by Mandelstam.[2] Therefore, if one insists on working exclusively with rigorous results from field theory, our approach can only be applied to pion-pion scattering. However, the approach can be of some use in other processes also if one postulates similar analyticity properties (for example, if one assumes the full Mandelstam analyticity without assuming the validity of a Mandelstam representation with a finite number of subtractions). In this paper, for the sake of definiteness and simplicity, we shall concentrate on the pion-pion scattering amplitudes and use only analyticity domains that follow from axiomatic field theory.

The mass of the pion will be set equal to unity, and we will consider the scattering amplitude, for any charge or isospin combination, as an analytic function of the usual variables $s,t,u=4-s-t$. We shall also use the alternative notation

$$s_1 = s, \quad s_2 = t, \quad s_3 = u\,. \tag{2.1}$$

Our aim is to write "parametric dispersion relations" on some analytic hypersurfaces, the real sections of which are curves approaching the forward and the backward pieces of the physical region in all three channels. Obviously, these curves have to be asymptotically parallel to the s, t, and u axes. Hence a natural choice for these hypersurfaces is the family of cubics

$$\mathcal{C}:\ (s-a_1)(t-a_2)(u-a_3)=K\,. \tag{2.2}$$

Let D be the axiomatic domain referred to above (and which we do not characterize explicitly). By definition the amplitude is a holomorphic function of s and t in D minus the physical cuts:

$$D' = D \cap C\{s,t\,|\,s \geq 4,\ t \geq 4,\ u \geq 4\}\,. \tag{2.3}$$

In order to be able to write simple parametric dispersion relations on the path (2.2) in the (s,t) space, we have to choose the real parameters a_i and K in such a way that the cubic $\mathcal{C}$ satisfies the following two conditions:

(i) $\mathcal{C} \subset D$.

(ii) There must exist a (complex) coordinate z on $\mathcal{C}$ such that the mapping $z \to (s,t)$ is a rational one.

In other words we seek parametrizations of (2.2), $s=s(z)$ and $t=t(z)$, which are rational functions of z. From condition (i) it is clear that the pion-pion amplitudes will be analytic functions of z except for those values of z that correspond to the physical cuts.

The restrictions on the parameters a_i and K which are necessary and sufficient to ensure condition (ii) will be worked out in the next section. In order to satisfy condition (i) we can use the methods and results of Ref. 3.

What we need is that, for some values of the a_i's, there exist an $A>0$ such that the domain

$$E=\{s,t\,|\,|(s-a_1)(t-a_2)(u-a_3)|<A\} \tag{2.4}$$

is contained in D. If we have such a domain then it suffices to choose $K<A$ to guarantee that $\mathcal{C} \subset D$. Now from the work of Martin[3] we know that there exist domains $F \subset D$ of the form

$$F=\{s,t\,|\,|(s-a_1)(t-a_2)|<A_1\} \cup \{s,t\,|\,|(t-a_2)(u-a_3)|<A_2\} \cup \{s,t\,|\,|(u-a_3)(s-a_1)|<A_3\}\,. \tag{2.5}$$

Comparing Eqs. (2.4) and (2.5), we see that $E \subset D$ if one takes $A^{2/3} \leq \min A_i$ for $i=1,2,3$.

A method to compute the allowed range of the parameters a_i and A_i [and hence the admissible cubics of the form (2.2)] has also been indicated in Ref. 3. However, an explicit computation is not really required for the limited application we have in mind. As a matter of fact, we will concentrate mainly on the symmetric case $a_1 = a_2 = a_3 \equiv a$. We will start by choosing a and K in small neighborhoods of $\frac{4}{3}$ and 0, respectively. For such values the argument above easily applies, since using a result of Martin[3] it can be shown that the domains $F \subset D$ of the form (2.5) include the family

$$F_a = \{s, t \mid |s-a|\,|t-a| < [8 + ((4-a)(16-a))^{1/2}]^2\} \cup \{s \to t, t \to u\} \cup \{s \to u, t \to s\}, \tag{2.6}$$

where $0 \leqslant a < 4$.

Eventually, the extension to larger values of $|a|$ will be carried out on our final expressions by a simple argument.

III. CUBIC SECTIONS AND CHOICE OF VARIABLES

As we stated in the previous section, we want to parametrize the cubic $\mathcal{C}$ by means of a rational mapping $z \to (s, t)$. We insist on rational mappings in order to avoid spurious branch points and to be able to use contour integrals in z without worrying about Riemann sheets. Of course, this requirement is not of a fundamental nature, but merely a matter of practical convenience.

According to Lüroth's theorem,[6] an algebraic curve has a parametric representation by rational functions only if its genus is zero. The genus of a cubic is 1 − (number of nodes) − (number of cusps). A simple inspection of Eq. (2.2) shows that $\mathcal{C}$ has no cusps. We then conclude that one node (double point) is needed. The ensuing constraint on the parameters of $\mathcal{C}$ is easily derived. Solving Eq. (2.2) for s, we get

$$s = -\tfrac{1}{2}(t + a_3 - a_1 - 4) \pm \frac{1}{2}\left(\frac{[(t + a_3 + a_1 - 4)^2(t - a_2) - 4K]}{t - a_2}\right)^{1/2}. \tag{3.1}$$

A double point clearly occurs if and only if the cubic polynomial inside the bracket has a double zero. Other than the trivial degenerate case $K = 0$, simple algebra shows that a double root of the polynomial in Eq. (3.1) occurs when

$$K = \tfrac{1}{27}(4 - a_1 - a_2 - a_3)^3. \tag{3.2}$$

Thus our curves belong to the three-parameter family

$$(s - a_1)(t - a_2)(u - a_3) = [\tfrac{1}{3}(4 - a_1 - a_2 - a_3)]^3, \tag{3.3}$$

or equivalently

$$s = -\tfrac{1}{2}(t + a_3 - a_1 - 4) \pm \tfrac{1}{2}[t - \tfrac{1}{3}(4 - a_1 + 2a_2 - a_3)] \times \left(\frac{t - \frac{1}{3}(16 - 4a_1 - a_2 - 4a_3)}{t - a_2}\right)^{1/2}. \tag{3.4}$$

Equation (3.3) represents a rather restricted class of cubics. The double point is also the center of rotational symmetry of order 3. Its coordinates are free, and there is one parameter left which fixes the scale of the curve. Furthermore, the double point is an *isolated* point of the real section of the curve.

One should remark that these features are not necessary restrictions. A class of cubics larger than (2.2) could also be considered containing curves with only axial symmetry and with a double point connecting a real infinite branch to a real loop. In that case, however, the condition (i) of Sec. II would be harder to check, and such a possibility will not be discussed here.

The form (3.4) immediately suggests various rational parametrizations of our cubics. Note that Eq. (3.4) has the form

$$s = L_1(t) \pm \left(\frac{L_2(t)}{L_3(t)}\right)[Q(t)]^{1/2},$$

where the L_i's are linear and $Q(t)$ is quadratic in t. It is a trivial matter to find a rational parametrization $t = t(\lambda)$ such that $[Q(t(\lambda))]^{1/2}$ is also rational in λ.

A particularly convenient choice, which respects the symmetry of order 3, is given by the following:

$$s_k = a_k + \tfrac{1}{3}(4 - a_1 - a_2 - a_3)\frac{(z - z_k)^3}{z^3 - 1}, \quad k = 1, 2, 3 \tag{3.5}$$

where the z_k's are the cube roots of unity; $z_k = e^{(k-1)2i\pi/3}$. It is easily checked that (3.5) satisfies (3.2) with $s = s_1$, $t = s_2$, $u = s_3$. The mapping $z \to (s_1, s_2)$ is one-to-one, except for the double point which is the image of $z = 0$ and $z = \infty$. The three asymptotic points $s_k = a_k$ correspond to $z = z_k$, $k = 1, 2, 3$.

From this point on we restrict ourselves to the case $a_1 = a_2 = a_3 \equiv a$. The double point then coincides with the center of the Mandelstam triangle, $s_k = \frac{4}{3}$, $k = 1, 2, 3$. The three branches of our cubic will be symmetric over the three channels. We have from Eq. (3.5)

$$s_k = a + (\tfrac{4}{3} - a)\frac{(z - z_k)^3}{z^3 - 1}, \quad k = 1, 2, 3. \tag{3.6}$$

It is trivial to check that for real a in the interval $0 \leqslant a < 4$, and for all complex z,

$$|(s_1(z) - a)(s_2(z) - a)(s_3(z) - a)| = |(\tfrac{4}{3} - a)^3| < 8^3. \tag{3.7}$$

A simple comparison with Eq. (2.6) shows that, at least for $0 \leq a < 4$, we have $(s_1(z), s_2(z)) \in D$ for all complex z. Thus the amplitude $F(s_1(z), s_2(z))$ can be considered as an analytic function of z and the parameter a:

$$\overline{F}(z,a) \equiv F(s_1(z), s_2(z)). \tag{3.8}$$

For admissible values of a, the only singularities of $\overline{F}(z,a)$ in the z plane are given by the image $V(a)$ of the physical cuts $s_k \geq 4$, $k=1,2,3$. We have shown above that the values $0 \leq a < 4$ are admissible. Actually we shall show in the next section that the set of admissible values is much larger.

Using Eq. (3.6) one can carry out a simple calculation to obtain $V(a)$, the image of the three physical cuts in the z plane. The result is

$$V(a) = V_1(a) \cup V_2(a) \cup V_3(a), \tag{3.9}$$

where

$$V_1(a) = \begin{cases} \{z \,|\, |z| = 1,\ \tfrac{2}{3}\pi \leq |\arg z| \leq \phi_0(a)\} & \text{if } \tfrac{4}{9} < a < \tfrac{4}{3} \quad \text{(I)}, \\ \{z \,|\, |z| = 1,\ \phi_0(a) \leq |\arg z| \leq \tfrac{2}{3}\pi\} & \text{if } \tfrac{4}{3} < a < 4 \quad \text{(II)}, \\ \{z \,|\, |z| = 1,\ \tfrac{2}{3}\pi \leq |\arg z| \leq \pi\} \cup \{z \,|\, \rho_-(a) \leq |z| \leq \rho_+(a),\ \arg z = \pi\} & \text{if } a \leq \tfrac{4}{9} \quad \text{(III)}, \end{cases} \tag{3.10}$$

and

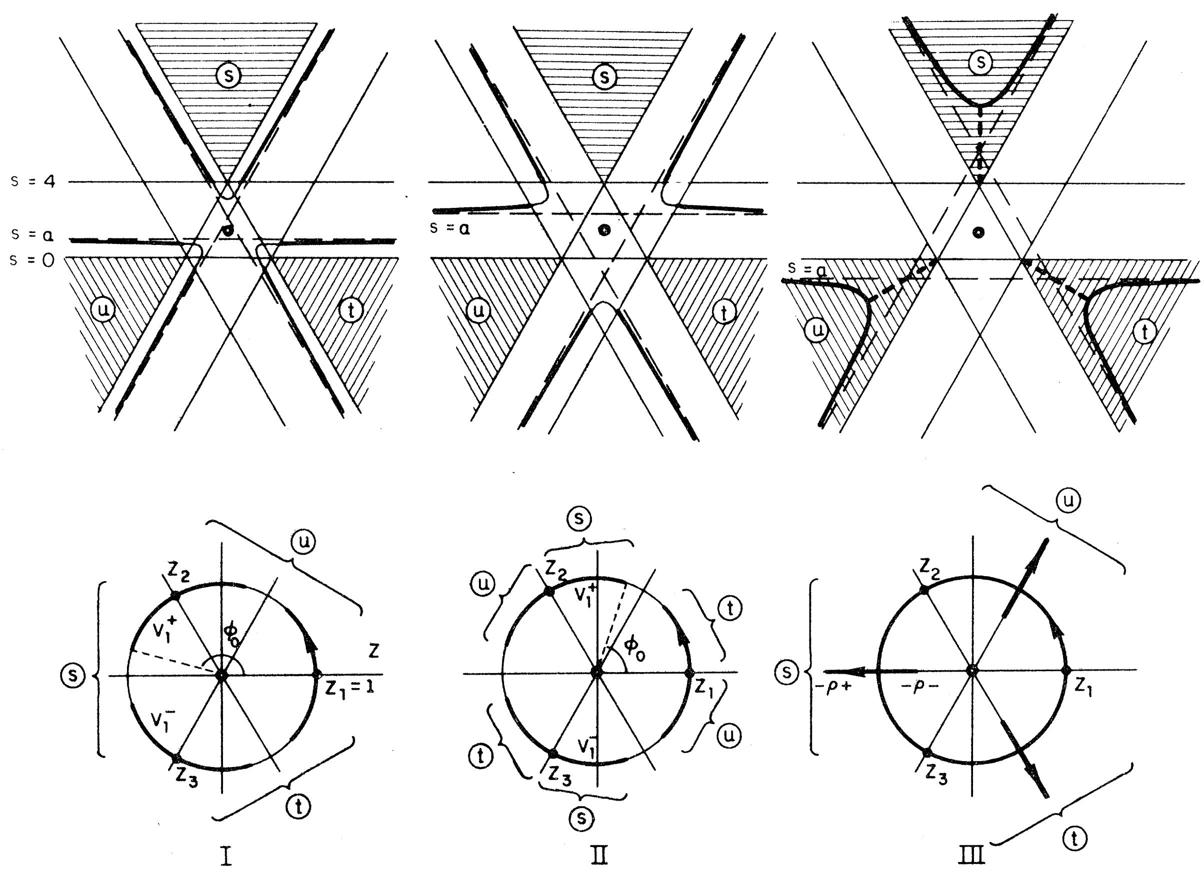

FIG. 1. Integration paths for the dispersion relations Eqs. (4.12) and (4.13) are represented both in the Mandelstam plane and in the z plane (thick line). The three cases correspond to three qualitatively different situations encountered when the parameter a is varied: $\frac{4}{9} < a < \frac{4}{3}$ (I); $\frac{4}{3} < a < 4$ (II); $a \leq \frac{4}{9}$ (III). In case (III), the thick dotted line in the (s,t) plane is the real projection of the complex piece of the path. The image of that piece in the z plane is formed by the three straight segments.

$$V_2(a) = e^{2i\pi/3} V_1(a)\,,$$

$$V_3(a) = e^{4i\pi/3} V_1(a)\,.$$

The functions $\phi_0(a)$ and $\rho_\pm(a)$ are given by

$$\phi_0(a) = \tan^{-1}\left\{\frac{[(4-a)(a-\frac{4}{9})]^{1/2}}{a-\frac{20}{9}}\right\}, \quad 0<\phi_0\leq\pi$$
$$\rho_\pm(a) = \tfrac{9}{16}\{(\tfrac{20}{9}-a)\pm[(4-a)(\tfrac{4}{9}-a)]^{1/2}\}\,. \tag{3.11}$$

The three different cases encountered in Eq. (3.10) are clearly displayed in Fig. 1, where the real section of the (s,t) space and the z plane are both shown. Note that in cases I and II the image of the cuts lies on the unit circle in the z plane but it does not fully cover the unit circle. In these cases there are gaps on the unit circle free of singularities and one can find analytic continuations from $|z|<1$ to $|z|>1$. The case $a=\frac{4}{3}$ corresponds to the degeneracy of the cubic into three straight lines and has to be excluded.

Finally, we should stress the fact that along the cuts $V(a)$ we have by construction guaranteed the fact that the amplitude will either be physical or obtainable from physical partial-wave expansions which are convergent. For example along the s cut, $V_1(a)$, i.e., $s\geq 4$, the corresponding values of $t=t(s,a)$ will always be inside the s-channel Martin-Lehmann ellipse. Each of the three domains whose union forms F_a in (2.6) contains all three channel Martin-Lehmann ellipses [see discussion below Eq. (2.5) in Ref. 3].

IV. PARAMETRIC DISPERSION RELATIONS AND ANALYTIC CONTINUATION IN a

In this section we derive dispersion relations in the variable z that have explicit contributions from all three channels. The absorptive parts that appear in these representations will be physical or else determined by physical partial waves. We shall also use these representations to calculate the domain of admissible values of the parameter a.

In order to write dispersion relations in the z plane, we have first to identify the discontinuity of $\overline{F}(z,a)$ on $V(a)$ with the absorptive parts of the amplitude in the three channels. The latter will be denoted by

$$A_1(s,t) \equiv \lim_{\epsilon\downarrow 0}\frac{1}{2i}[F(s+i\epsilon,t)-F(s-i\epsilon,t)]\,, \quad s\geq 4 \tag{4.1}$$

with similar definitions for $A_2(s,t)$ and $A_3(s,t)$ in the t and u channels, respectively. We note that A_k is not necessarily real, because t is not required to be real in Eq. (4.1). In any case A_k has to be understood as the analytic continuation, via the appropriate partial-wave expansion, of the physical absorptive part in the k channel.

Using Eq. (3.6), the reality condition $F(s,t) = F^*(s^*,t^*)$ becomes

$$\overline{F}(z,a) = \overline{F}^*\left(\frac{1}{z^*},a\right)\,. \tag{4.2}$$

We define the "discontinuity" of $\overline{F}(z,a)$ across $V(a)$ by

$$\overline{A}(z,a) \equiv \begin{cases} \lim\limits_{\epsilon\downarrow 0}\dfrac{1}{2i}[\overline{F}((1+\epsilon)z,a)-\overline{F}((1-\epsilon)z,a)] & \text{for } |z|=1\,,\\[2ex] \lim\limits_{\epsilon\downarrow 0}\dfrac{1}{2i}[\overline{F}(ze^{-i\epsilon},a)-\overline{F}(ze^{i\epsilon},a)] & \text{for } \arg z=\pi \ (\mathrm{mod}\,\tfrac{2}{3}\pi)\,.\end{cases} \tag{4.3}$$

The second part of the definition is only needed in case III, where the gap disappears. From Eq. (4.2) we have

$$\overline{A}(z,a) = \overline{A}^*(z,a) \qquad \text{for } |z|=1\,,$$

but

$$\overline{A}(z,a) = -\overline{A}^*\left(\frac{1}{z^*},a\right) \qquad \text{for } \arg z=\pi\ (\mathrm{mod}\,\tfrac{2}{3}\pi)\,. \tag{4.4}$$

Finally we need to know the sign of $\mathrm{Im}s_k$ on each side of $V(a)$. A simple calculation using Eq. (3.6) gives us

$$\mathrm{Im}s = 3\epsilon(\tfrac{4}{3}-a)\begin{cases}\dfrac{2\sin\phi}{(1+2\cos\phi)^2}\,, & z=(1+\epsilon)e^{i\phi}\\[2ex] \dfrac{\rho(\rho+1)^3}{(\rho^3+1)^2}(\rho-1)\,, & z=-\rho e^{-i\epsilon}\,.\end{cases} \tag{4.5}$$

In order to relate $\overline{A}(z,a)$ to $A_k(s,t)$, it is convenient to define $V_k^\pm(a)$ by (see Fig. 1)

$$V_k(a)=V_k^+(a)\cup V_k^-(a)\,,\quad \text{with}\quad \operatorname{Im}(z/z_k)\quad\text{or}\quad (|z|-1)\begin{cases}\geqslant 0 \text{ in } V_k^+(a)\\[4pt] \leqslant 0 \text{ in } V_k^-(a)\,.\end{cases} \tag{4.6}$$

It then follows from Eq. (4.5) that

$$\begin{aligned} \overline{A}(z,a)&=\begin{cases}A_k(s,t)\,, & z\in V_k^+(a)\\ -A_k(s,t)\,, & z\in V_k^-(a)\,,\end{cases} \qquad a<\tfrac{4}{3}\ \text{(cases I and III)}\,,\\ \overline{A}(z,a)&=\begin{cases}-A_k(s,t)\,, & z\in V_k^+(a)\\ A_k(s,t)\,, & z\in V_k^-(a)\,,\end{cases} \qquad a>\tfrac{4}{3}\ \text{(case II)}\,.\end{aligned} \tag{4.7}$$

To settle the question of subtractions we have to know the behavior of $\overline{F}(z,a)$ in the neighborhood of $z=z_k$. These points are the images of the asymptotic points, for Eq. (3.5) gives us as $z\to z_k$

$$s_k\to a\,,\qquad s_j\simeq \text{const}\,\frac{a-\frac{4}{3}}{z-z_k}\qquad (j\neq k)\,. \tag{4.8}$$

From the work of Jin and Martin,[7] we know that

$$F(s,t)=o(s^2)\quad \text{for } |s|\to\infty\,,\quad t \text{ fixed},\quad (s,t)\in D\,, \tag{4.9}$$

and similar properties at fixed s and u. Hence we see that

$$\overline{F}(z,a)=o(1/(z-z_k)^2)\,,\quad \text{as}\quad z\to z_k\,,\quad a \text{ fixed}\,. \tag{4.10}$$

Now if we write contour integrals involving the function $(z^3-1)\overline{F}(z,a)$ instead of $\overline{F}(z,a)$, it follows from Eq. (4.10) that the part coming from the integration over small circles around $z=z_k$, $k=1,2,3$, vanishes when the radii of these circles tend to zero. This procedure is equivalent to the introduction of two subtractions in the usual fixed-t dispersion relations. In our case, the subtraction constants will be related to the coefficients of the Taylor expansion

$$\overline{F}(z,a)=\sum_{n=0}^{\infty} f_n(a)z^n\,, \tag{4.11}$$

which is convergent for $|z|<\rho_-(a)$ (or $|z|<1$ for cases I and II). Note that f_0 is real and independent of a since

$$f_0=\overline{F}(0,a)=F(\tfrac{4}{3},\tfrac{4}{3})\,.$$

Moreover, Eq. (3.6) shows that

$$\overline{F}(\infty,a)=\overline{F}^*(0,a)=f_0\,.$$

Thus, using the Cauchy formula for two simple circuits C_i and C_e, interior and exterior to $V(a)$, respectively, we get for $z\notin V(a)$, $|z|<1$,

$$\frac{1}{2\pi i}\oint_{C_i} dz'\frac{z'^3-1}{z'^3(z'-z)}\,\overline{F}(z',a)=\frac{z^3-1}{z^3}\,\overline{F}(z,a)+\frac{f_0}{z^3}+\frac{f_1(a)}{z^2}+\frac{f_2(a)}{z}\,,$$

$$\frac{1}{2\pi i}\oint_{C_e} dz'\frac{z'^3-1}{z'^3(z'-z)}\,\overline{F}(z',a)=\overline{F}(\infty,a)=f_0\,.$$

By subtracting, letting C_e and C_i go onto $V(a)$, and using Eq. (4.3), we obtain our dispersion relations on the cubic $\mathcal{C}$:

$$\overline{F}(z,a)=f_0+f_1(a)\frac{z}{1-z^3}+f_2(a)\frac{z^2}{1-z^3}+\frac{z^3}{(1-z^3)}\,\frac{1}{\pi}\int_{V(a)} dz'\frac{z'^3-1}{z'^3(z'-z)}\,\overline{A}(z',a)\,. \tag{4.12}$$

At this stage, it would be instructive to rewrite Eq. (4.12) in terms of the familiar s, t, and u variables. To avoid unessential discussions, we do this only for the case I, i.e., $\frac{4}{9}<a<\frac{4}{3}$. For any fixed a in this interval we use the mapping $z\to(s(z,a),t(z,a))$ given in Eq. (3.6) to obtain from Eq. (4.12) the following representation:

$$F(s,t,u)=f_0+\frac{f_1(a)}{9(a-\frac{4}{3})}(z_1s+z_2t+z_3u)-\frac{f_2(a)}{9(a-\frac{4}{3})}\left(\frac{s}{z_1}+\frac{t}{z_2}+\frac{u}{z_3}\right)$$
$$-\frac{(a-\frac{4}{3})(s/z_1+t/z_2+u/z_3)}{(z_1s+z_2t+z_3u)\pi}\left\{\int_4^\infty ds'[K_+(s';s,t,u)A_s(s',t_+(s',a))+K_-(s';s,t,u)A_s(s',t_-(s',a))]\right.$$
$$\left.+\int_4^\infty dt'[s'\to t';(stu)\to(tus)]+\int_4^\infty du'[s'\to u';(stu)\to(ust)]\right\}, \quad (4.13)$$

where

$$t_\pm(s',a)=-\frac{1}{2}\left[(s'-4)\mp(s'-\tfrac{4}{3})\left(\frac{s'+3a-\frac{16}{3}}{s'-a}\right)^{1/2}\right]. \quad (4.14)$$

The kernels $K_\pm$ are given by

$$K_\pm(s';s,t,u)=\frac{2s'-3a+\frac{4}{3}}{2(s'-\frac{4}{3})^2}\left[1\pm i\frac{3\sqrt{3}\,(a-\frac{4}{3})}{2s'-3a+\frac{4}{3}}\left(\frac{s'-a}{s'+3a-\frac{16}{3}}\right)^{1/2}\right]$$
$$\times(z_1s+z_2t+z_3u)\left[z_1s+z_2t+z_3u+\left(\frac{s}{z_1}+\frac{t}{z_2}+\frac{u}{z_3}\right)\frac{1}{z'_\pm(s',a)}\right]^{-1}, \quad (4.15)$$

with

$$z'_\pm(s',a)=-\frac{1}{2(s'-\frac{4}{3})}\{(s'+\tfrac{8}{3}-3a)\mp i[3(s'-a)(s'+3a-\tfrac{16}{3})]^{1/2}\}. \quad (4.16)$$

In both Eqs. (4.13) and (4.15), *it has to be understood that the variables* s, t, *and* $u=4-s-t$ *are related through the equation of the cubic,* $(s-a)(t-a)\times(u-a)=(\frac{4}{3}-a)^3$. The complex numbers z_k, $k=1,2,3$, are just the cube roots of unity defined below Eq. (3.5). The algebra involved in the derivation of Eq. (4.13) is quite simplified by noting the relations

$$9(a-\tfrac{4}{3})z(1-z^3)^{-1}=\sum_{k=1}^{3}z_k s_k$$

and

$$9(a-\tfrac{4}{3})z^2(1-z^3)^{-1}=-\sum_{k=1}^{3}z_k^{-1}s_k,$$

which follow from Eq. (3.6).

A few remarks about the new representation (4.13) are in order. First, it is a completely rigorous consequence of local field theory, at least for pion-pion scattering. Second, it displays the contributions of all three channels in an explicit and symmetric manner. Third, it is physical in the sense that the $A_{s,t,u}$ in Eq. (4.13) are either physical absorptive parts or determined by convergent Legendre expansions from the physical partial-wave absorptive parts. Finally, (4.13) undergoes a tremendous simplification in the symmetric case, $\pi^0\pi^0\to\pi^0\pi^0$ (see *Note added in proof*).

One should also note the strong similarity between our approach and the method used by Wanders[3] to tackle the same kind of problems by means of homogeneous variables. This latter method assumes the validity of the Mandelstam representation, but this is probably not an essential ingredient.[9] Actually, for the totally symmetric case $\pi^0\pi^0\to\pi^0\pi^0$, our dispersion relations on the cubics (3.6) are completely equivalent to dispersion relations written in a particular bundle of straight lines in the homogeneous-variables plane. Two important differences between the two methods have to be stressed, however: (i) The homogeneous-variables method can only be used in the totally symmetric case, i.e., $\pi^0\pi^0\to\pi^0\pi^0$, while without particular choice of cubics we do not have to require any symmetry properties of the amplitude, as it is clear in the preceding discussion of this section.[10] (ii) For an arbitrary straight line in the homogeneous-variable plane, the dispersion relation in those variables has no analog of the form (4.12).

As is most clearly displayed in Fig. 1, our representation is particularly suitable for deriving various sum rules connecting the three channels. If needed, the asymptotic part of the amplitude can be damped at will by introducing enough factors $(z-z_k)$ raised to appropriate powers. We shall postpone a detailed investigation of such practical sum rules to a later paper. In the next two sections we shall limit ourselves to deriving a set of sum rules involving only the absorptive parts and their derivatives.

Before proceeding with this task we have to settle the problem of what the admissible values of the parameter a are. In this section we have so far taken a to be in some neighborhood of $a=\frac{4}{3}$. In Sec. III we noted an admissible set of values for a, i.e., $0 \leq a < 4$. Equation (4.12) is valid for these values. But we can also use Eq. (4.12) to perform an analytic continuation in a by taking into account the analyticity properties of $\bar{A}(z,a)$. This will give us a larger domain of admissible values which we shall calculate below.

We have to investigate the analyticity in a of the right-hand side of Eq. (4.12) for fixed z such that $z \notin V(a)$.

The first point to note is that the subtraction terms do not produce any singularities, since $f_1(a)$ and $f_2(a)$ are polynomials of the first and second order, respectively. This is easily seen by substituting Eq. (3.6) in the Taylor expansion of $F(s,t)$ at $s=t=\frac{4}{3}$ and identifying with the expansion (4.11).

It is clear that the three pieces of the dispersion integral in Eq. (4.12) corresponding to the three channels have the same analytic properties in a. Thus we can consider only the piece corresponding, say, to $V_1(a)$. Splitting this range further as indicated by Eq. (4.6), and making the substitution $z' \to 1/z'$ in the part $V_1^-(a)$, we find that the singularities of $\bar{F}(z,a)$ in a are the same as those of

$$I(z,a) \equiv \int_{V_1^+(a)} dz' \frac{z'^3-1}{z'^3}\left[\frac{\bar{A}(z',a)}{z'-z} - z'\frac{\bar{A}(1/z',a)}{1/z'-z}\right]. \tag{4.17}$$

At this point, in order to remove the dependence on a in the integration range, we return to the integration variable s'. For definiteness we first assume a to be real and $a < \frac{4}{3}$. Then, using Eqs. (3.6) and (4.7), one finds that Eq. (4.17) takes the form

$$I(z,a) = \frac{1}{4-3a}\int_4^\infty ds' \frac{(z'^3-1)^3}{z'^3(z'-1)^3(z'+1)}\left[\frac{A_1(s',t_+(s',a))}{z'-z} + z'\frac{A_1(s',t_-(s',a))}{1/z'-z}\right], \tag{4.18}$$

where

$$z' = z'(s',a) = -\frac{1}{2(s'-\frac{4}{3})}\{(s'+\tfrac{8}{3}-3a) - [3(s'-a)]^{1/2} \times \begin{cases} i(s'+3a-\frac{16}{3})^{1/2}\} & \text{for } s' \geq \frac{16}{3}-3a \text{ (case I)}, \\ (\frac{16}{3}-3a-s')^{1/2}\} & \text{for } 4 \leq s' \leq \frac{16}{3}-3a \text{ (case III)}, \end{cases} \tag{4.16$'$}$$

and

$$t_\pm(s',a) = -\tfrac{1}{2}\left[(s'-4) \mp \frac{s'-\frac{4}{3}}{(s'-a)^{1/2}} \times \begin{cases} (s'+3a-\frac{16}{3})^{1/2}] & \text{(case I)}, \\ i(\frac{16}{3}-3a-s')^{1/2}] & \text{(case III)}. \end{cases}\right. \tag{4.14$'$}$$

Now in order to isolate the potential singularity at $s' = \frac{16}{3}-3a$, which seems to appear according to Eqs. (4.16′) and (4.14′), it is convenient to rewrite Eq. (4.18) as

$$I(z,a) = \frac{1}{4-3a}\int_4^\infty \frac{ds'}{z^2 - z(z'+1/z')+1}\left[g_+(z',z)A_+(s',a) + g_-(z',z)(s'+3a-\tfrac{16}{3})^{1/2}\frac{A_-(s',a)}{(s'+3a-\frac{16}{3})^{1/2}}\right], \tag{4.19}$$

with

$$A_\pm(s',a) = A_1(s',t_+(s',a)) \pm A_1(s',t_-(s',a)),$$

$$g_\pm(z',z) = \frac{1}{2}\frac{(z'^3-1)[(1\pm z'^3) - z'(1\pm z')z]}{z'^4(z'-1)^3(z'+1)}.$$

It is shown in the Appendix that for all $s' \geq 4$, $A_+(s',a)$ and $A_-(s',a)/(s'+3a-\frac{16}{3})^{1/2}$ are holomorphic functions of a in a domain containing the open interval $]-28.19,4[$. That this is also true for the whole of the integrand in Eq. (4.19) can be checked as follows: $(z'+1/z')$ and $g_+(z',z)$ are invariant under the substitution $z' \to 1/z'$. From Eq. (4.16′) this means that they are regular at $s' = \frac{16}{3}-3a$. On the other hand, since $g_-(z',z)$ changes sign when $z' \to 1/z'$, $g_-(z',z)(s'+3a-\frac{16}{3})^{1/2}$ is regular at this point too. Furthermore, a repeat of the argument used in fixed-transfer dispersion relations shows that the integral in Eq. (4.19) is absolutely convergent in the analyticity domain of the integrand. Thus we conclude that, at fixed a, $I(z,a)$ is holomorphic in a neighborhood of $]-28.19,\frac{4}{3}[$. The analytic continuation to $[\frac{4}{3},4[$ presents no difficulty. Then it follows from Hartog's theorem that $\bar{F}(z,a)$ is holomorphic in $\{z,a | a \in \mathfrak{N}, z \notin V(a)\}$, where $\mathfrak{N}$ is some complex neighborhood of $]-28.19,4[$. In turn, $F(s,t)$ is holomorphic in a domain G of the form

$$G = \{s,t \,|\, (s-a)(t-a)(u-a) = (\tfrac{4}{3}-a)^3;\; s,t,u = 4-s-t \neq \rho;\; a \in \mathfrak{N},\; \rho \geq 4\}. \tag{4.20}$$

It is interesting to note the existence of points (s,t) in the domain (4.20) which are not contained in the extremal domains of the form (2.5) obtained by the Mandelstam-Martin method. Indeed the parameters a_i in Eq. (2.5) are strictly restricted to be larger than -28, but our lower limit is a

$=-28.19$. This improvement is not very significant, because it can also be obtained (and even better) by applying our method to hyperbolas of the form $(s-a)(t-a)=A$ instead of cubics. For this reason, the full extent of the domain (4.20) for complex a will not be worked out here. For the purpose of analytic continuation, it is not clear that one gains very much by using simple analytic hypersurfaces which have the full symmetry of the (unknown) domain of holomorphy. On the other hand, the method of analytic completion based on the theorem of "removal of cuts" and the tube theorem, so powerful in the derivation of domains of the form (2.5),[3] does not apply directly to domains of the form (2.4), at least not in an obvious way. For this paper we shall be content with the simple result of more practical interest, namely the validity of Eq. (4.12) for $-28.19<a<4$.

V. SUM RULES FOR $\pi^0\pi^0 \to \pi^0\pi^0$ SCATTERING

We shall first specialize the representation (4.12) to the case of neutral pion scattering and use the resulting representation to derive a set of physical necessary and sufficient conditions on the absorptive part that guarantee full crossing symmetry. These conditions also lead to potentially useful sum rules.

The amplitude for $\pi^0\pi^0$ scattering, $F_0(s,t,u)$, is completely symmetric in s, t, and u. In terms of our variable z, it is clear from Eq. (3.6) that $\overline{F}_0(z,a)$ is a function of z^3, and instead of Eq. (4.11) we have a Taylor expansion of the form

$$\overline{F}_0(z,a)=\sum_{n=0}^{\infty}\alpha_n(a)z^{3n}. \tag{5.1}$$

Thus the representation (4.12) becomes

$$\overline{F}_0(z,a)=\alpha_0+\frac{z^3}{(1-z^3)\pi}\int_{V(a)}dz'\,\frac{z'^3-1}{z'^3(z'-z)}\,\overline{A}(z',a). \tag{5.2}$$

As we already noted in the previous section, the coefficients $\alpha_n(a)$ are polynomials in a. By expanding the right-hand side of Eq. (5.2) we get sum rules for $\alpha_n(a)$ for $n\geq 1$,

$$\alpha_n(a)=\frac{1}{\pi}\int_{V(a)}dz'\overline{A}(z',a)\frac{1-z'^{-3n}}{z'}\,,\quad n\geq 1. \tag{5.3}$$

Using the full symmetry of F_0 and Eqs. (4.7), we can reduce the integration path to $V_1^+(a)$ only. Furthermore, at this stage we shall for simplicity limit ourselves to case I, $\frac{4}{9}<a<\frac{4}{3}$. The final results of this section can later be suitably extended to all admissible values of a. With $z'=\exp(i\phi)$, we get from Eq. (5.3)

$$\alpha_n(a)=-\frac{6}{\pi}\int_{2\pi/3}^{\phi_0(a)}d\phi\,\sin(3n\phi)\overline{A}(e^{i\phi},a)\,,\quad n\geq 1. \tag{5.4}$$

In the present case it turns out to be convenient to use the variables $\overline{s}_k$ and $\overline{a}$ defined by

$$\begin{aligned}\overline{s}_k&=s_k-\tfrac{4}{3}\,,\\ \overline{a}&=a-\tfrac{4}{3}\,.\end{aligned} \tag{5.5}$$

We also introduce homogeneous variables similar to those given by Wanders,[8]

$$\begin{aligned}x&=\tfrac{1}{27}(\overline{s}\,\overline{t}+\overline{u}\,\overline{t}+\overline{s}\,\overline{u})\,,\\ y&=\tfrac{1}{27}(\overline{s}\,\overline{t}\,\overline{u})\,.\end{aligned} \tag{5.6}$$

Simple algebra relates x and y to our variables $\overline{a}$ and z,

$$\begin{aligned}x&=\overline{a}^2\frac{z^3}{(z^3-1)^2}\,,\\ y&=\overline{a}^3\frac{z^3}{(z^3-1)^2}\,.\end{aligned} \tag{5.7}$$

A necessary and sufficient condition for $F_0(s,t,u)$ to have full crossing symmetry is for F_0 to be a function of x and y only. Thus in some neighborhood of the symmetry point, $x=y=0$, one can write

$$F_0(s,t,u)=\sum_{p,q=0}^{\infty}C_{pq}x^p y^q. \tag{5.8}$$

The coefficients C_{pq} are independent, and Eq. (5.8) contains all the crossing information for any set $\{C_{pq}\}$.

In order to obtain the crossing restrictions on the absorptive part, we shall first use Eq. (5.8) to relate the C_{pq}'s to the coefficients $\alpha_n(a)$ of Eq. (5.1). Then using the representation (5.4) for $\alpha_n(a)$ we get sum rules involving $A(s,t)$.

Using Eq. (5.7) we have

$$\overline{F}_0(z,a)=\sum_{p,q=0}^{\infty}C_{pq}\,\overline{a}^{(2p+3q)}\,\nu^{p+q}\,, \tag{5.9}$$

where

$$\nu=\frac{z^3}{(z^3-1)^2}\,. \tag{5.10}$$

For a fixed a, $\frac{4}{9}<a<\frac{4}{3}$, $\overline{F}_0(z,a)$ has a simple expansion in powers of ν,

$$\overline{F}_0(z,a)=\sum_{n=0}^{\infty}(\overline{a})^{2n}\mathcal{P}_n(\overline{a})\nu^n\,, \tag{5.11}$$

where $\mathcal{P}_n(\overline{a})$ are polynomials of degree n determined by $C_{p,q}$ according to

$$\mathcal{P}_n(\overline{a})\equiv\sum_{q=0}^{n}C_{n-q,q}\,\overline{a}^{\,q}\,. \tag{5.12}$$

To obtain relations between the α_n's and the

$C_{p,q}$'s we have to compare the coefficients of each power of z in Eqs. (5.1) and (5.11). It turns out to be more convenient to expand z^3 in terms of ν in Eq. (5.1) and compare coefficients of powers of ν. Inverting Eq. (5.10) we obtain

$$z^3 = \nu\left[\frac{1-(1+4\nu)^{1/2}}{2\nu}\right]^2 . \tag{5.13}$$

We need the expansion of the $2n$th power of the term in the bracket above. This turns out to be a standard expansion related to the so-called Catalan numbers,[11]

$$\left[\frac{1-(1+4\nu)^{1/2}}{2\nu}\right]^{2n} = \sum_{m=0}^{\infty}\frac{n}{(m+n)}\binom{2m+2n}{m}(-1)^m\nu^m . \tag{5.14}$$

Using Eqs. (5.14) and (5.13) in (5.1) and exchanging orders of summation, we obtain

$$\overline{F}_0(z,a) = \alpha_0 + \sum_{n=1}^{\infty}\left[\sum_{m=1}^{n}\alpha_m(a)\frac{m}{n}\binom{2n}{n-m}(-1)^m\right](-1)^n\nu^n . \tag{5.15}$$

Comparing this with Eq. (5.11) we get our relations between the α_n's and the C_{pq}'s,

$$(\overline{a})^{2n}\mathcal{P}_n(\overline{a}) = (-1)^n\sum_{m=1}^{n}\alpha_m(a)\frac{m}{n}\binom{2n}{n-m}(-1)^m . \tag{5.16}$$

We now substitute Eq. (5.4) for $\alpha_n(a)$ and use the fact that $m\alpha_m(a)$ is even in m, as is $\binom{2n}{n-m}$. After some algebra this leads to

$$(\overline{a})^{2n}\mathcal{P}_n(\overline{a}) = \frac{6}{\pi}(-2)^{n-1}\times\int_{2\pi/3}^{\phi_0(a)} d\phi\,\overline{A}(e^{i\phi},a)(1-\cos 3\phi)^{n-1}\sin 3\phi ,\quad n\geq 1 . \tag{5.17}$$

Finally, we rewrite this set of sum rules in terms of the more familiar s and t variables. From Eq. (3.6), for $z=e^{i\phi}$ we have

$$\cos\phi = -\frac{\overline{s}-3\overline{a}}{2\overline{s}} . \tag{5.18}$$

By substituting Eq. (5.18) into Eq. (5.17) we have

$$\frac{(-27)^n}{\pi}\int_{8/3}^{\infty} d\overline{s}\,\frac{A(\overline{s},\overline{t}(\overline{s},\overline{a}))}{\overline{s}^{2n+1}}\left(1-\frac{\overline{a}}{\overline{s}}\right)^{n-1}\left(2-\frac{3\overline{a}}{\overline{s}}\right) = -\mathcal{P}_n(\overline{a}),\quad n\geq 1 , \tag{5.19}$$

where

$$\overline{t}(\overline{s},\overline{a}) = \tfrac{1}{2}\overline{s}\left[-1+\left(\frac{\overline{s}+3\overline{a}}{\overline{s}-\overline{a}}\right)^{1/2}\right] . \tag{5.20}$$

According to our discussion at the end of Sec. IV, these equations are valid for $-29.52<\overline{a}<\frac{8}{3}$.

To get sum rules from Eq. (5.19) one can evaluate the left-hand side for different values of $\overline{a}$ and eliminate the polynomials on the right. These sum rules will only involve $A(\overline{s},\overline{t}(\overline{s},\overline{a}))$ for different values of a. Another way to proceed would be to differentiate Eq. (5.19) with respect to $\overline{a}$ $(n+1)$ times and obtain relations involving $A(\overline{s},\overline{t}(\overline{s},\overline{a}))$ and its derivatives with respect to $\overline{t}$.

We give an example of the first kind of sum rule obtained by setting $\overline{a}=b$, $-b$, and 0 in Eq. (5.19) with $n=1$:

$$\int_{8/3}^{\infty}\frac{d\overline{s}}{\overline{s}^3}\left[A(\overline{s},\overline{t}(\overline{s},b))\left(2-\frac{3b}{\overline{s}}\right) + A(\overline{s},\overline{t}(\overline{s},-b))\left(2+\frac{3b}{\overline{s}}\right)\right] = 4\int_{8/3}^{\infty}\frac{d\overline{s}}{\overline{s}^3}A(\overline{s},\overline{t}=0) . \tag{5.21}$$

This equation, which follows from two-variable analyticity and crossing, is the type of sum rule that was sought in Ref. 1 for $W_2(\nu,q^2)$, the absorptive part of the Compton amplitude. Here in Eq. (5.21), for admissible values of b, the absorptive part $A(\overline{s},\overline{t})$ is either physical or determined from physical partial waves through the Legendre expansion.

But perhaps the main value of Eq. (5.19) lies not in sum rules like (5.21) but elsewhere. Namely, we shall see below that in a sense Eq. (5.19) provides us with a set of necessary and sufficient conditions that guarantee full crossing symmetry.

The problem is the following: Supposing we are given an s-channel absorptive part $A_1(s,t)$ with the necessary analyticity properties in t, how can we make sure it is the absorptive part of a symmetric amplitude?. Of course we can easily choose the u-channel absorptive part $A_3(u,t)$ such that $A_1\equiv A_3$. This takes care of $s\to u$ crossing. However, it also effectively determines $F_0(s,t)$ modulo subtractions, via fixed-t dispersion relations. The difficult part to guarantee now is that the F_0 determined by $A_1=A_3=A$ should also have a t-channel absorptive part $A_2(t,s)$ which is identical with A_1 and A_3. The converse of the steps used in this section gives us a way to achieve this aim. Namely, suppose one starts with an $A(\overline{s},\overline{t})$ that satisfies Eq. (5.19) and defines a set of polynomials $\mathcal{P}_n(\overline{a})$. We can use the coefficients of these polynomials to uniquely define a function $G_0(s,t)$ through a series of the form

$$G_0(s,t) = \sum_{n=0}^{\infty}\sum_{q=0}^{n}\beta_q^{(n)}x^{n-q}y^q , \tag{5.22}$$

where $\mathcal{P}_n(\overline{a}) = \sum_{q=0}^{n}\beta_q^{(n)}\overline{a}^q$. The function $G_0(s,t)$ thus defined will obviously be crossing-symmetric. By reversing the argument of this section, substituting Eq. (5.17) for $\mathcal{P}_n(\overline{a})$, and carrying out the summation over n we get back a representation identical to (5.2) for $\overline{G}_0(z, a)$. This will define a crossing-symmetric function whose three-channel absorptive parts are identical and given by the $A(\overline{s}, \overline{t})$ we started with.

We can convert Eq. (5.19) into a set of sum rules for the absorptive parts of the partial-wave amplitudes. As we shall see below this could be of some use for the problem of deriving bounds. Along the s-channel cut $A(\overline{s}, \overline{t}(\overline{s}, \overline{a}))$ has a convergent partial-wave expansion

$$A(\overline{s}, \overline{t}(\overline{s}, \overline{a})) = \left(\frac{4(\overline{s}+\frac{4}{3})}{(\overline{s}-\frac{8}{3})}\right)^{1/2} \times \sum_{\substack{l=0\\ l \text{ even}}}^{\infty} (2l+1) a_l(\overline{s}) P_l(\xi^{1/2}(\overline{s}, \overline{a})). \tag{5.23}$$

The variable ξ is given by

$$\xi = \cos^2\theta_s = \left(1 + \frac{2t}{s-4}\right)^2.$$

From Eq. (5.20) we have

$$\xi(\overline{s}, \overline{a}) = \left(\frac{\overline{s}}{\overline{s}-\frac{8}{3}}\right)^2 \left(\frac{\overline{s}+3\overline{a}}{\overline{s}-\overline{a}}\right). \tag{5.24}$$

We also define $\xi_0 = \xi(\overline{s}, \overline{a}=0) = [\overline{s}/(\overline{s}-\frac{8}{3})]^2$. We want to substitute Eq. (5.23) into Eq. (5.19) and expand in powers of $\overline{a}$. Setting the coefficients of $\overline{a}^m$ equal to zero for $m \geq n+1$ gives us the desired sum rules. We first expand $P_l(\xi^{1/2})$ about the point ξ_0 and get

$$P_l(\xi^{1/2}) = \sum_{j=0}^{l/2} \frac{p_l^{(j)}(\xi_0)}{j!}(\xi - \xi_0)^j, \quad l \text{ even}, \tag{5.25}$$

where

$$p_l^{(j)}(\xi_0) = \frac{d^j}{d\xi^j} P_l(\xi^{1/2})\bigg|_{\xi=\xi_0}. \tag{5.26}$$

From Eq. (5.24) we have

$$\xi(\overline{s}, \overline{a}) - \xi_0 = 4\xi_0\left(\frac{\overline{a}}{\overline{s}-\overline{a}}\right). \tag{5.27}$$

Using Eqs. (5.27), (5.25), and (5.23) in (5.19) we obtain

$$\frac{(-27)^n}{\pi}\int_{8/3}^{\infty} \frac{d\overline{s}}{\overline{s}^{2n+1}}\left[\frac{4(\overline{s}+\frac{4}{3})}{(\overline{s}-\frac{8}{3})}\right]^{1/2} \sum_{\substack{l=0\\ l \text{ even}}}^{\infty} (2l+1)a_l(\overline{s}) \sum_{j=0}^{l/2} \frac{p_l^{(j)}(\xi_0)}{j!}(4\xi_0)^j \left(1-\frac{\overline{a}}{\overline{s}}\right)^{n-j-1}\left(\frac{\overline{a}}{\overline{s}}\right)^j\left(2-\frac{3\overline{a}}{\overline{s}}\right) = -\mathcal{P}_n(\overline{a}). \tag{5.28}$$

We now expand $(1-\overline{a}/\overline{s})^{n-j-1}$ and rearrange summations. The coefficients of $\overline{a}^m$ for $m \geq n+1$ on the left should vanish, giving us, after some algebra,

$$\int_{8/3}^{\infty} \frac{d\overline{s}}{\overline{s}^{2n+m+1}}\left[\frac{4(\overline{s}+\frac{4}{3})}{(\overline{s}-\frac{8}{3})}\right]^{1/2} \sum_{l=2n}^{\infty}\sum_{j=n}^{m} (2l+1)a_l(\overline{s}) p_l^{(j)}(\xi_0)(4\xi_0)^j \frac{(3j-m-2n)}{(j!)(m-j)!(j-n)!} = 0, \quad \text{for } m \geq n+1;\ n \geq 1 \tag{5.29}$$

with $\xi_0(\overline{s}) = [\overline{s}/(\overline{s}-\frac{8}{3})]^2$. This gives us a sum rule for each pair of positive integers (m, n) with $m \geq n+1$. The absorptive parts $a_l(\overline{s})$ are physical and satisfy $0 \leq a_l(\overline{s}) \leq 1$. Roskies[5] derived a similar (and equivalent) set of sum rules for the $a_l(\overline{s})$ using different techniques. Although his results and ours agree for a few low values of (m, n), his conditions, as noted in Ref. 5, are not all independent. On the contrary, the relations given by Eq. (5.29) are independent, as is clear from the fact that the coefficients of $\mathcal{P}_n(\overline{a})$ uniquely determine the C_{pq}'s in Eq. (5.8).

The set of conditions (5.29) is also simpler than those of Ref. 5. This raises for us the following important question: Can Eq. (5.29) be used as an additional input in seeking to improve bounds like the Froissart bound? The input used in many of the proofs is (i) positivity, $0 \leq a_l(\overline{s}) \leq 1$; (ii) convergence in the Martin-Lehmann ellipses; (iii) polynomial boundedness. Clearly, crossing symmetry and analyticity demand that $\{a_l(\overline{s})\}$ also satisfy Eq. (5.29). Kinoshita, Loeffel, and Martin[12] constructed a set of $a_l(s)$ satisfying (i), (ii), and (iii), which saturated the Froissart bound. It would be quite valuable to construct a set $\{a_l(\overline{s})\}$ which satisfied not only (i), (ii), and (iii) but also (5.29) such that Froissart bound would again be saturated. If such a set can be found then the problem of "improving" the Froissart bound would become considerably more difficult, since we would have included the main remaining general condition, i.e., crossing, that is available to us, without improving the bound. Of course, a completely satisfactory example should also include the strict unitarity in the elastic region. But it is very doubtful that this last requirement could change the status of the Froissart bound. The problem of improvement, if indeed any is possible, will have to depend on detailed dynamical input which will take it outside the field of derivation of bounds from a few general principles.

VI. SUM RULES FOR CHARGED-PION SCATTERING

We do not intend to discuss here the scattering of charged pions in full details. The method of Sec. V can be applied to this case in a straightforward way, and we shall content ourselves with a few examples. In the neutral case, sum rules were derived by identifying the coefficients of z^{3n}, $n=1,2,\ldots,$ in Eqs. (5.1) and (5.2). In the general case, we have to identify the coefficients of z^p, $p=3,4,\ldots,$ in Eqs. (4.11) and (4.12). It turns out that no further condition is obtained when $p=3n$. These sum rules are identical to those given in the neutral case, whatever isospin combination is considered. On the other hand, two new (and independent) sets of conditions are obtained when $p=3n+1$ and $p=3n+2$, which again do not depend on the choice of a particular isospin amplitude. For instance, the simplest one, corresponding to $p=4$, is found to be

$$f_4^{s1}(a)-f_1^{s1}(a)=\frac{3i\sqrt{3}\,\bar{a}^2}{2\pi}\int_{8/3}^{\infty}\frac{d\bar{s}}{\bar{s}^3}\left[\left(1-9\,\frac{\bar{a}}{\bar{s}}+9\,\frac{\bar{a}^2}{\bar{s}^2}\right)(2A^0-5A^2)+9\left(\frac{\bar{s}-\bar{a}}{\bar{s}+3\bar{a}}\right)^{1/2}\left(1+3\,\frac{\bar{a}}{\bar{s}}-9\,\frac{\bar{a}^2}{\bar{s}^2}\right)A^1\right], \tag{6.1}$$

where

$$\bar{F}^{s1}(z,a)=\sum_{n=0}^{\infty}f_n^{s1}(a)z^n, \tag{6.2}$$

is the amplitude with isospin $I=1$ in the s channel, and

$$A^I\equiv A_s^{sI}(\bar{s},\bar{t}(\bar{s},\bar{a})),\quad I=0,1,2. \tag{6.3}$$

is the s-channel absorptive part of the amplitude with isospin I in the same channel. Moreover, we have

$$f_4^{s1}(a)-f_1^{s1}(a)=\bar{a}^2\mathcal{P}_2(\bar{a}),$$

where $\mathcal{P}_2(\bar{a})$ is an arbitrary quadratic polynomial in $\bar{a}$. Further differentiations (or differences) with respect to $\bar{a}$ have to be carried out in Eq. (6.1) in order to get useful sum rules, or partial-wave constraints. The latter, of course, will be equivalent to (although different from) the set given by Roskies.[5]

Extra conditions supplement the rigorous ones derived until now if one adds phenomenological inputs restricting the high-energy behavior of the amplitudes. For instance, it is commonly assumed that the fixed-s dispersion relations for the amplitude with isospin 1 in the s channel require only one subtraction. In our z plane, this means that we can write contour integrals for $(z^2+z+1)\bar{F}^{s1}(z,a)$ instead of $(z^3-1)\times\bar{F}^{s1}(z,a)$. Then, taking into account the antisymmetry of F^{s1} in the interchange $t\leftrightarrow u$ [which implies $\bar{F}^{s1}(z,a)=-\bar{F}^{s1*}(z^*,a)$], Eq. (4.12) is replaced by

$$\bar{F}^{s1}(z,a)=-\frac{z}{2(1+z+z^2)}\,\frac{1}{\pi}\int_{V(a)}dz'\bar{A}^{s1}(z',a)\frac{z'^2+z'+1}{z'^2}\,\frac{z'+z}{z'-z}, \tag{6.4}$$

from which it follows that

$$f_1^{s1}(a)=\frac{i\sqrt{3}\,\bar{a}}{2\pi}\int_{8/3}^{\infty}\frac{d\bar{s}}{\bar{s}^2}\left[(2A^0-5A^2)-3\,\frac{\bar{s}-3\bar{a}}{[(\bar{s}-\bar{a})(\bar{s}+3\bar{a})]^{1/2}}A^1\right], \tag{6.5}$$

with the definitions (6.2) and (6.3).

As $f_s^{s1}(a)$ is proportional to $\bar{a}$, the simplest sum rule resulting from Eq. (6.5) is obtained by differentiating once with respect to $\bar{a}$ and using Eq. (4.14′) (case I):

$$12\int_{8/3}^{\infty}\frac{d\bar{s}}{[(\bar{s}-\bar{a})(\bar{s}+3\bar{a})]^{3/2}}A^1=-\int_{8/3}^{\infty}\frac{d\bar{s}}{(\bar{s}+3\bar{a})^{1/2}(\bar{s}-\bar{a})^{3/2}}\left[\left(2\,\frac{\partial A^0}{\partial\bar{t}}-5\,\frac{\partial A^2}{\partial\bar{t}}\right)-3\frac{\bar{s}-3\bar{a}}{[(\bar{s}-\bar{a})(\bar{s}+3\bar{a})]^{1/2}}\,\frac{\partial A^1}{\partial\bar{t}}\right]. \tag{6.6}$$

The "fixed transfer" version of sum rules like (6.1) and (6.6) has been discussed extensively elsewhere.[5,9] As in our case the argument goes through in essentially the same way; it will not be pursued here.

Note added in proof. One should note that Eq. (4.13) is tremendously simplified in the fully symmetric case, $\pi^0\pi^0 \to \pi^0\pi^0$. In that case one obtains

$$F_0(\bar{s},\bar{t},\bar{u}) = \alpha_0 + \frac{1}{\pi}\int_{8/3}^{\infty} \frac{d\bar{s}'}{\bar{s}'} A(\bar{s}'; \bar{t}_+(\bar{s}'; \bar{s},\bar{t},\bar{u}))\, H(\bar{s}'; \bar{s},\bar{t},\bar{u}),$$

where $H(\bar{s}'; \bar{s},\bar{t},\bar{u}) = [\bar{s}(\bar{s}'-\bar{s})^{-1} + \bar{t}(\bar{s}'-\bar{t})^{-1} + \bar{u}(\bar{s}'-\bar{u})^{-1}]$, and $\bar{t}_+(\bar{s}'; \bar{s},\bar{t},\bar{u}) = t_+(\bar{s}'; \bar{a})$ with $\bar{a} = \bar{s}\,\bar{t}\,\bar{u}(\bar{s}\,\bar{t} + \bar{t}\bar{u} + \bar{s}\,\bar{u})^{-1}$ and $\bar{t}_+(\bar{s}', \bar{a})$ given in Eq. (5.20). This representation holds for *any* point (s,t) for which $\tau \equiv \bar{t}_+(\bar{s}'; \bar{s},\bar{t},\bar{u}) + \frac{4}{3}$ lies in the Martin-Lehmann ellipses $E(s')$ for $A(s',\tau)$ given in Eq. (A2). The similarity of this representation to the Cini-Fubini approximation[4] is striking. This representation follows most directly from Eq. (5.2) by transforming from the (z, a) variables to the s, t, u variables.,

ACKNOWLEDGMENTS

We are indebted to F. J. Dyson and G. Wanders for useful comments.

APPENDIX

We want to find the range I of real values of a such that the two functions $A_+(s',a)$ and $A_-(s',a)/(s'+3a-\frac{16}{3})^{1/2}$ are analytic for $a\in I$, $\forall s'\geq 4$. Here

$$A_\pm(s',a) = A_1(s', t_+(s',a)) \pm A_1(s', t_-(s',a)), \quad \text{(A1)}$$

where $A_1(s',t)$ is the absorptive part of the amplitude in the s channel and $t_\pm(s',a)$ are defined by Eq. (4.14). Equation (A1) shows that the singularities at $a = \frac{16}{9} - \frac{1}{3}s'$ due to the square-root branch point in $t_\pm(s',a)$ cancel each other in the expressions for A_+ and $A_-/(s'+3a-\frac{16}{3})^{1/2}$. This means that I is simply given by

$$\mathrm{I} = \{a \,|\, t_\pm(s',a) \in E(s'), \quad s'\geq 4\},$$

where $E(s')$ is the analyticity domain in t of $A_1(s',t)$, which contain the Martin-Lehmann ellipse.[3] More precisely, one can take

$$E(s') = \begin{cases} E\left(0, 4-s' \,\middle|\, 16 + \dfrac{64}{s'-4}\right), & 4\leq s' < 16 \\ E\left(0, 4-s' \,\middle|\, \dfrac{256}{s'}\right), & 16 \leq s' \leq 32, \\ E\left(0, 4-s' \,\middle|\, 4 + \dfrac{64}{s'-16}\right), & s'\geq 32 \end{cases} \quad \text{(A2)}$$

where $E(f_1,f_2|d)$ stands for the (open) elliptic dis with foci $t=f_1$, $t=f_2$ and right extremity $t=d$. If $a\geq \frac{4}{9}$, $t_\pm(s',a)$ are real for all $s'\geq 4$. Thus the condition for $a\in \mathrm{I}$ is in this case

$$t_+(s',a) < \begin{cases} 16 + \dfrac{64}{s'-4}, & 4\leq s' < 16 \\ \dfrac{256}{s'}, & 16\leq s' \leq 32\,. \\ 4 + \dfrac{64}{s'-16}, & s'\geq 32 \end{cases} \quad \text{(A3)}$$

It is easily checked that Eq. (A3) is satisfied when $\frac{4}{9} < a < 4$, the two curves $t = t_+(s',4)$ and $t = 4+64/(s'-16)$ touching only at $s'=\infty$. If $a<\frac{4}{9}$, $t_\pm(s',a)$ are still real for $s'\geq \frac{16}{3}-3a$, with $t_\pm(s',a)\in E(s')$ in this range. For $s' < \frac{16}{3}-3a$, however, $t_\pm(s',a)$ become complex, and the condition $t_\pm(s',a)\in E(s')$ now means that $\mathrm{Im}\,t_+(s',a)$ has to be less than the half-length of the small axis of the ellipse $E(s')$. From Eqs. (4.14) and (A2), this is equivalent to

$$\frac{1}{2}\frac{s'-\frac{4}{3}}{(s'-a)^{1/2}}\left(\tfrac{16}{3}-3a-s'\right)^{1/2} < \begin{cases} \dfrac{4\sqrt{s'}\,(s'+4)}{s'-4}, & 4\leq s'<16 \quad \text{(A4}'\text{)} \\ \dfrac{16}{s'}(s'^2-4s'+256)^{1/2}, & 16\leq s'\leq 32 \quad \text{(A4}''\text{)} \\ \dfrac{2\sqrt{s'}\,(s'-8)}{s'-16}, & s'\geq 32\,. \quad \text{(A4}'''\text{)} \end{cases}$$

Numerical evaluation shows that (A4′) is satisfied for all $a<\frac{4}{9}$. On the other hand, (A4″) is true for $a > -37.09$, and (A4‴) for $a>-28.19$ (in the last case, the inequality is saturated for $a=-28.19$, $s'=46.48$). Collecting the results, we find that $\mathrm{I} =]-28.19, 4[$.

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-3505.

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PHYSICAL REVIEW D VOLUME 6, NUMBER 10 15 NOVEMBER 1972

Reformulation of the Crossing-Unitarity Equation in Terms of Partial Waves*

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The Mandelstam construction of crossing-symmetric, unitarity amplitudes for pion-pion scattering is reformulated in terms of partial waves. The amplitude is obtained by solving a set of nonlinear equations for the physical partial-wave amplitudes. A conformal mapping of the $\cos\theta$ plane, introduced by Ciulli, Cutkosky, and Deo, is used to implement crossing symmetry. It is shown that the equations have a solution, which may be constructed by means of a convergent iteration.

I. INTRODUCTION

During the past few years, substantial experimental information on low-energy π-π scattering has accumulated.[1] At the same time, new theoretical techniques are leading toward a phenomenology in which the constraints of crossing symmetry, unitarity, and analyticity are exploited more completely than heretofore. For instance, Martin[2] has obtained some interesting inequalities for partial-wave amplitudes in the region $0<s<4$ as consequences of analyticity, crossing symmetry, and positivity of absorptive parts. Balachandran and Nuyts,[3] Roskies,[4] Basdevant *et al.*,[5] and others have derived necessary and sufficient conditions for crossing symmetry in the form of linear moment relations between partial waves in the same region. Roy,[6] Steiner,[7] Basdevant *et al.*,[8] and Yen and Roskies[9] have introduced crossing constraints which refer to the physical region alone. Ciulli,[10] Cutkosky and Deo,[11] Prešnajder and Pišút,[12] Ross,[13] and others are following a somewhat different program in which analytic approximation theory and statistical analysis are used to design "optimal" methods of data fitting which respect the presumed analyticity properties of the amplitude. The crossing constraints have narrowed the choice of acceptable models of low-energy π-π scattering, while optimal data fitting appears to be helpful in reducing the ambiguities of phase-shift analysis, coupling-constant determinations, and the like.

Several workers[14] have constructed generalized effective-range models of low-energy π-π scattering, in which the parameters were fixed, or at least restricted, by crossing constraints. In