Note on the Self-Consistent Calculation of π-π Resonance Parameters

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A self-consistent calculation of π-π resonance parameters is attempted in a model suggested by Balázs. Two variations of the model are considered. The input in this calculation is the experimental masses and widths of the ρ, f_0, and g resonances and the inelasticities, known as well as arbitrary, in the corresponding partial waves. Bootstrap possibilities in different isospin and partial-wave states are discussed.

I. INTRODUCTION

Over a considerable period of time the equivalent-potential method has been in use for the self-consistent π-π scattering problems. Calculations have been performed using Schrödinger, Lippmann-Schwinger, and Bethe-Salpeter equations[1,2]; recently Balázs[1] has suggested that the method can be used in connection with an on-shell integral form of the Schrödinger equation in momentum space. The suggested nonlinear equation has the following form:

$$f(q^2, t) = V(t, s) + \frac{1}{\pi}\int_0^\infty \frac{dq'^2}{q'^2 - q^2}\,\frac{q'}{4\pi} \times \int d\Omega\, f^*(q'^2, t_1) f(q'^2, t_2). \tag{1}$$

It may also be given a relativistic form by rewriting the kernel in a slightly different manner.[2] Here $V(t, s)$ is the Fourier transform of the potential; q is the momentum in the c.m. system; and $-t$, $-t_1$, and $-t_2$ are the squares of the momentum transfer between the initial and the final, the initial and the intermediate, and the intermediate and the final states, respectively. The solution of this equation gives exactly the same physical amplitude as the Schrödinger equation.

The equation should however be written in a slightly different manner to bring out the fact that it has to be used in conjunction with the equivalent potential. Peculiarities of that potential will be mentioned shortly afterwards; the relevant point here is that the equation should involve an extra parameter, the c.m. energy squared s, which will have to be treated as fixed in the process of solution. In other words Eq. (1) should be rewritten in the following form:

$$f(q^2, t, s) = V_l(t, s) + \frac{1}{\pi}\int_0^\infty \frac{dq'^2}{q'^2 - q^2}\,\frac{q'}{4\pi} \times \int d\Omega\, f^*(q'^2, t_1, s) f(q'^2, t_2, s). \tag{1'}$$

In solving this we can treat s as fixed, and the solutions are then obtained for various values of q^2. Each value of s will therefore generate a series of solutions, and many different such series are obtained for various values of s. The correct physical amplitude is given when $q^2 = \frac{1}{4}s - 1$. From the construction of the potential in (8) it will be clear that this prescription will make the integral appearing in Eq. (1) convergent.

Another obvious approach could be to allow the on-shell relationship between s and q^2 in the potential to remain effective at all stages during the process of solution. With the potential conceived in this way the integral term in (1), however, will need a cutoff.

We shall apply Eq. (1) in a self-consistent π-π scattering calculation; Balázs has suggested several techniques, among which is the powerful Padé-approximant method, for solving this nonlinear equation. We, however, follow the more conventional N/D approach; the solutions are then somewhat approximate but much easier to handle. The method can be easily generalized to take into account inelastic effects in the partial waves. Briefly, we proceed in the following manner: We project out the partial-wave amplitude from Eq. (1); then, taking the imaginary part, we find

$$\mathrm{Im} f_l(q^2) = q\,|f_l(q^2)|^2. \tag{2}$$

We now generalize this result by writing, instead of (2), the relation

$$\mathrm{Im} f_l(q^2) = q\,|f_l(q^2)|^2 + \frac{1 - \eta_l^2(q^2)}{4q}, \tag{3}$$

where $\eta_l(q^2)$ is the inelasticity function, defined in the well-known manner through an inelastic unitarity relation. This makes (3) identical with the standard inelastic unitarity relation for the partial waves.

Working with relation (3), and assuming that the only left-cut contribution to the amplitude comes from the potential term, while unitarity introduces the right cut, we can write down a dispersion relation for the amplitude on these cuts:

$$f_l(q^2,s)=B_l^U(q^2,s)+\frac{1}{\pi}\int_P dq'^2\,\frac{q'|f_l(q'^2)|^2}{q'^2-q^2}+\frac{1}{4\pi}\int_I dq'^2\,\frac{1-\eta_l^2(q'^2)}{q'(q'^2-q^2)}$$

$$=B_l^U(q^2,s)+B_l^I(q^2)+\frac{1}{\pi}\int_{\text{"}P\text{"}} dq'^2\,\frac{q'|f_l(q'^2,s)|^2}{q'^2-q^2}, \qquad (4)$$

where B_l^U denotes the contribution from the left cut and B_l^I is the contribution of the inelastic right cut starting from $q^2=3m_\pi^2$. "P" denotes the elastic right cut. There is an approximation, however, in Eq. (4), as we have assumed that the entire left-cut contribution comes from the inhomogeneous term in (1); this is not really the case, as may be seen by iterating Eq. (1); the contribution coming from the integral term has in fact been neglected. We now adopt the N/D method in the form of Frye and Warnock[3] equations for solving (4). If the unitary amplitude that leads to (3) is divided by q^{2l} to remove the threshold zeros and then written in the N/D form, then the following two equations are easily obtained:

$$N_l(q^2)=\frac{1}{\pi}\int dq'^2\,\frac{\mathrm{Im}f_l(q'^2,s)D(q'^2,s)}{q'^2-q^2}+\frac{1}{\pi}\int dq'^2\,\frac{1-\eta_l(q'^2)}{2q'^{2l+1}}\,\frac{\mathrm{Re}D(q'^2)}{q'^2-q^2} \qquad (5)$$

and

$$D(q^2)=1+\frac{q^2-q_0^2}{\pi}\times\int dq'^2\left[\frac{-2q'^{2l+1}}{1+\eta(q'^2)}\right]\frac{\mathrm{Re}N(q'^2)}{(q'^2-q^2)(q'^2-q_0^2)}. \qquad (6)$$

To simplify the calculations, we make the determinantal approximation by putting $D(q^2)=1$ in (5) and then substitute the value of $N(q^2)$ so obtained back in (6). Then using the Poincaré-Bertrand formula to reduce the double principal-value integral into a single one, we can finally write

$$D(q^2,s)=D_{\mathrm{I}}+D_{\mathrm{II}}, \qquad (7)$$

where

$$D_{\mathrm{I}}=1-\frac{q^2-q_0^2}{\pi}\int_P dq'^2\,\frac{2q'^{2l+1}}{1+\eta(q'^2)}\,\frac{B_l^U(q'^2,s)}{(q'^2-q^2)(q'^2-q_0^2)},$$

$$D_{\mathrm{II}}=\frac{1-\eta(q^2)}{1+\eta(q^2)}+\frac{q^2-q_0^2}{\pi^2}\int_P\frac{1-\eta(q''^2)}{q''^{2l+1}}\left[\frac{f_l(q''^2)-f_l(q^2)}{q''^2-q^2}\right]dq''^2,$$

and

$$f_l(X)=\int_I\frac{q'^{2l+1}}{1+\eta(q'^2)}\,\frac{dq'^2}{(q'^2-q_0^2)(q'^2-X)}.$$

It is to be noted that the divergence difficulties that plague the Frye and Warnock equations as η tends to zero disappear in this approximation. Equation (7), together with an expression for $B_l(q^2,s)$ to be developed in the next section and a set of values of η as a function of q^2 in various channels, are all that are needed for our calculations.

II. THE POTENTIAL

The potential to be used in conjunction with our equations is an analog of the equivalent potential defined by Charap and Fubini.[4] It may be introduced in the following manner: Let the t-channel amplitude be written as

$$A(t,s)=\sum_l(2l+1)a_l(t)P_l(\cos\theta_t);$$

in this expression we shall retain partial waves only up to $l=3$. For $a_l(t)$ a pole form is taken and the residues may be directly linked with the widths of the corresponding resonances. The exchanges are ρ, f_0, and g. The g meson is a π-π resonance at 1663 ± 20 MeV having a width of 111 ± 30 MeV (Ref. 5); its exchange was incorporated in a bootstrap calculation earlier.[6] Following the usual prescription for equivalent potentials we replace the t variable by the square of the mass of the corresponding exchanges whenever it appears in the numerator of the expression. The potential is now constructed by projecting out the s-channel partial amplitude and taking into account the appropriate isospin recoupling coefficients for π-π scattering; the result is

$$B_l^U(q^2)=\frac{1}{(q^2)^{l+1}\sqrt{s}}\left[3C_1 q_{t_1}{}^2 G_\rho{}^2 P_1(\cos\theta_{t_1})Q_l(a_1)+5C_2 q_{t_2}{}^4 G_{f_0}{}^2 P_2(\cos\theta_{t_2})Q_l(a_2)\right.$$
$$\left.+7C_3 q_{t_3}{}^6 G_g{}^2 P_3(\cos\theta_{t_3})Q_l(a_3)\right]. \tag{8}$$

Here C_1, C_2, and C_3 are the isospin recoupling coefficients that are obtained from the following isospin crossing matrix for the simultaneous contribution from t- and u-channel exchanges:

$$\begin{pmatrix} \frac{2}{3} & 2 & \frac{10}{3} \\ \frac{2}{3} & 1 & -\frac{5}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{pmatrix}.$$

In (8),

$$a_i=1+\frac{m_i^2}{2q^2},\quad \cos\theta_{t_i}=1+\frac{2s}{m_i^2-4},$$

and

$$q_{t_i}{}^2=\tfrac{1}{4}m_i^2-1,$$

where m_i, $i=1, 2, 3$, stand for the masses of the ρ, f_0, and g, respectively. The relationship between Γ and G^2 is given by the expression

$$\Gamma=\frac{q_R{}^{2l+1}G^2}{E_R{}^2},$$

with R denoting the resonance position. The entire expression so obtained has been divided by $\frac{1}{2}\sqrt{s}$ to construct the nonrelativistic potential.

First we discuss the results for which the on-shell relationship between s and q^2 is allowed to remain effective in the process of solution. Considering the fact that the potential is constructed from exchanges in the two crossed channels, this method is seen to be a simplified nonrelativistic version of the similar N/D calculation of the π-π problem made earlier by Zachariasen and Zemach.[7] We designate this method as method I.

The other possibility is to use expression (8) in the form of an equivalent potential. As has been stated earlier, s in (8) is now to be treated as a constant inside the integrals. This makes the D integral in (7) convergent. This approach has also been applied; we call this method II.

Finally, the output width is to be calculated from the formula

$$\Gamma=\frac{-2q_R{}^{2l+1}B_l^U(q_R{}^2)}{E_R[1+\eta(q_R{}^2)]\,dD/ds|_{s=s_R}}.$$

III. RESULTS

First we discuss and display the results obtained in method I. The zeros of ReD are located and the output width is worked out using formula (7). The inputs in this calculation are the widths of the resonances and also the inelasticity parameters in the corresponding partial waves. Some inelasticity data are now available for π-π scattering in the $l=1$, $I=1$, and $l=2$, $I=0$ channels from experiment[8] up to an energy of 1.4 BeV. For the rest of the energies the inelasticities must be suitably extrapolated, as in Fig. 2. The plots of these experimental data with several arbitrary extrapolations are shown in Figs. 2 and 3 for the ρ and f_0 channels, respectively. As input we take the masses and widths of the resonances (in units of $m_\pi=1$) lying within their experimental limit, viz.,

$$m_\rho{}^2=29,\quad m_{f_0}{}^2=81,\quad m_g{}^2=140,$$
$$\Gamma_\rho=1.0,\quad \Gamma_{f_0}=1.42,\quad \Gamma_g=0.71.$$

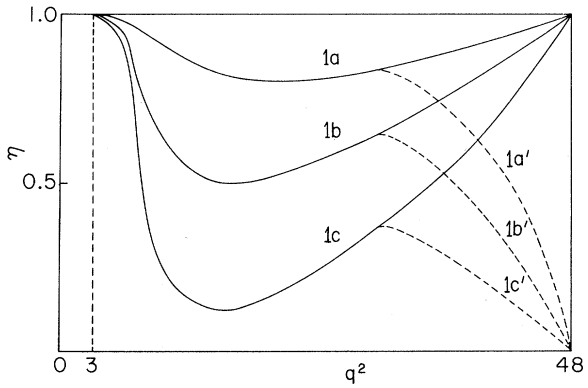

FIG. 1. Arbitrary inelasticity curves for π-π scattering. For f_0 the cutoff is 70 but the same forms of the inelasticity curves are used.

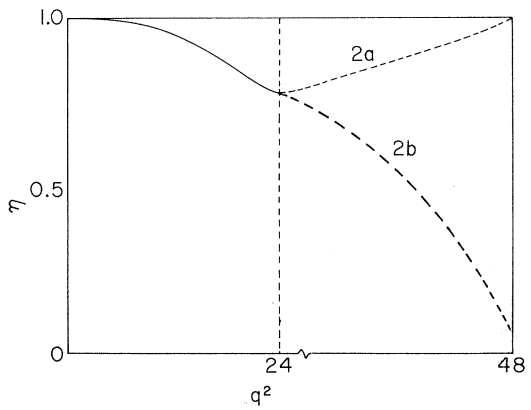

FIG. 2. The inelasticity curves for the ρ channel with extrapolations.

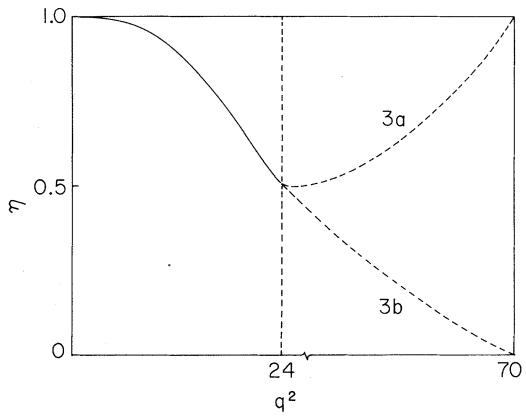

FIG. 3. The inelasticity curves for the f_0 channel with extrapolations.

To get an idea of the effectiveness of the role played by inelasticity, several arbitrary curves for inelasticities were first drawn, each with two different extrapolations at higher energies. The results so obtained are collected in Table I, where for all the entries the input values are the ones mentioned above.

The subtraction point was the same for all the three channels and was fixed at $q_0^2 = -5.0$. The fixed values for the cutoff for the ρ, f_0, and g channels were, respectively, 48, 70, and 48; these values were chosen such that the output quantities may remain near the input quantities. The results imply the existence of self-consistent output values for masses and coupling but do not indicate the possibility of a simultaneous bootstrap[9] with the same value of cutoff in all the three channels. The variation of the output parameters with different cutoffs is considerable; it is far more pronounced than that observed by Zachariasen and Zemach.[7] (In their result, the $\rho\pi\pi$ width is off by a factor of 4, as has been noted by Capps.[10]) Our output parameters may be moved up and down the energy scale relatively easily by a variation of the cutoff; for example, in a calculation with Fig. 2(a) and $q_0^2 = -3$, for a change of cutoff from 48 to 70, the ρ-resonance position changes from 22.5 to 13; widths are similarly affected. This behavior of the solution has been instrumental in finding the self-consistent values. The variation with the subtraction point is less pronounced; for a change in its value from -2 to -10, the change in the output values was only a few percent. The choice of inelasticities also affects the results considerably, as can be seen from Table I. It is possible that a simultaneous variation of the functional form of η and the cutoff might lead to a simultaneous bootstrap with the cutoff the same in all the channels. But it still may not be very realistic because the actual inelasticities in the ρ and f_0 channels are quite small. As is seen from Table I, two different higher-energy extrapolations in the curves of Fig. 1 do not lead to markedly different results.

TABLE I. Squares of the output masses and the output widths for fixed values of the input, all calculated in energy units $m_\pi = 1$. In the mass-squared values, the decimal figures have been neglected. Numbers of inelasticity curves correspond to Fig. 1.

Inelasticity curve	Resonance position $l=1$	$l=2$	$l=3$	Width $l=1$	$l=2$	$l=3$
1*a*	29	68	191	1.57	2.14	0.54
1*a'*	25	57	200	1.29	1.36	0.71
1*b*	27	81	180	1.43	2.64	0.36
1*b'*	23	71	186	1.00	2.50	0.50
1*c*	19	110	168	0.64	3.92	0.31
1*c'*	18	104	172	0.54	3.50	0.54

The curves of Figs. 2 and 3, where the experimental points are plotted only up to $q^2 = 24$, are to be used for ρ and f_0, respectively. With the same cutoff and q_0^2 and the same input as above, the two different extrapolations in each case cause less than 15% variation in the output values, showing thereby that for a calculation on low-energy resonances the inelasticity contribution at lower energies assumes the most important role. The average output masses in these two cases are 25 and 54, respectively, and the average widths are 1.32 and 1.78.

In the next part of our calculation (i.e., in method II) for the sake of simplicity we put $\eta = 1$ throughout; D_{II} is now vanishing and D_I is a convergent integral. Here again simultaneous self-consistency could not be achieved, but the self-consistent output parameters may be found in the individual channels. For example, fixing the other input parameters of the potential at their experimental values, we have ρ-self-consistent output quantities at $m_\rho^2 = 13$ and $\Gamma_\rho = 1.11$. However, if the previously mentioned input values are used for each of ρ, f_0, and g, then the output masses and widths in their respective channels are 21, 66, 206 and 2.28, 2.14, 2.07, respectively. The only adjustable parameter is now q_0^2, but variation with respect to it is very small; its value was fixed at $q_0^2 = -20$.

In conclusion, we mention that our model is a simple one, and we have made rather weak assumptions to solve the equations; no doubt a better solution is called for. The N/D technique used here is easy to apply and more accurate solutions could also be found easily, but fundamentally it will still remain an approximate method. The Padé-approximant method, on the other hand, al-

though complicated numerically, appears to be a more attractive alternative.

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Generalized Renormalizable Gauge Formulation of Spontaneously Broken Gauge Theories

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The spontaneously broken gauge theory is formulated in the generalized renormalizable gauge (R_ξ gauge). A parameter ξ can be adjusted to include existing gauges, U gauge, R gauge, and 't Hooft-Feynman gauge as special cases. Three applications of the R_ξ-gauge formulation are given. First we compute the weak correction to the muon magnetic moment unambiguously in the existing models for leptons. Secondly, we discuss the large-momentum-transfer limit of the Pauli magnetic form factor of the muon. Finally, we discuss the static charge of the neutrino, and show that an appropriate regularization makes it vanish.

I. INTRODUCTION

The possibility of constructing a unified theory of weak and electromagnetic interactions in terms of a spontaneously broken gauge symmetry has attracted a great deal of attention lately, following the works of Weinberg[1] and 't Hooft.[2] In this paper we shall present a formulation of spontaneously broken gauge theories (SBGT) which is particularly suited for practical calculations. In this formulation the gauge condition one adopts is a generalization of the one used by 't Hooft and depends on a parameter ξ which can vary continuously from 0 to ∞. In this gauge, which we shall call generically the R_ξ gauge, the massive-vector-boson propagator is precisely the one invented by Lee and Yang in their discussion[3] of the ξ-limiting process:

$$\Delta_{\mu\nu}(p,\xi) = -i\left[g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\frac{p_\mu p_\nu}{p^2 - M^2/\xi}\right]\frac{1}{p^2-M^2}$$
$$= -i\left(g_{\mu\nu} - \frac{1}{M^2}p_\mu p_\nu\right)\frac{1}{p^2-M^2}$$
$$-i\frac{1}{M^2}p_\mu p_\nu\frac{1}{p^2-M^2/\xi}. \qquad (1.1)$$

The difference between the R_ξ-gauge formulation of SBGT and the ξ-limiting process applied to the electrodynamics of massive vector bosons is this: