frames, our polar angle ϕ_c equals π minus Mueller's.

[13]The MRE in Eq. (4) does not exist when ω_2 and ω_4 are equal to β_{bc}, which is imaginary for large (s/M^2) and finite t. Hence, we have to first continue to the region where β_{bc} is real, expand $T(p_a, p_b, p_c)$ into an MRE (or a multiparticle partial-wave expansion if θ_2 is real), collect the Toller angle β_{bc} (or Treiman-Yang angle) and the residue function (or partial-wave amplitude) into a function of t and β_{bc}, and then continue the amplitude T back to the region concerned in the context.

[14]A. H. Mueller, in *Multiperipheral Dynamics*, 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G. J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971), Vol. 5, p.48.

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Pion Radius and Isovector Nucleon Radii in the Limit of Small Pion Mass*

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We study the pion radius and the isovector nucleon radii in the limit in which the pion mass approaches zero. The leading terms in this limit are calculated.

I. INTRODUCTION

Despite some interesting suggestions to the contrary, it is generally agreed that the only satisfactory way to understand the successes of current algebra and partial conservation of axial-vector current (PCAC) is in terms of the proximity of the real world to a fictitious world which is invariant – in the Nambu-Goldstone sense – under the chiral SU(2)⊗SU(2) group. The pion is presumed to be the massless Goldstone boson in the symmetry limit; its mass, μ, must therefore stem entirely from the symmetry-breaking terms in the hadronic Hamiltonian.[1]

One can probe the transition from the real world to the symmetry limit by allowing $\mu \to 0$. In general one would expect infrared singularities to arise in this limit, preventing the transition from being a terribly "smooth" transition.[2] The pseudoscalar nature of the pion guarantees, however, that in a Yukawa-type theory of pions and nucleons these singularities are rather benign when compared, for example, to the infrared singularities encountered in perturbative treatments of quantum electrodynamics. (Perhaps this is the reason why pionic infrared problems have received very little attention in the literature.) Nevertheless, singularities can, and do, arise in parameters *which are experimentally relevant in the real world.*

In this paper we investigate the singularities in pion and nucleon radii in the limit $\mu \to 0$. We show that if the electromagnetic form factors satisfy dispersion relations with no more than one subtraction (so that the radii are in principle calculable in terms of other parameters) the two-pion contribution to the dispersion integrals yields singular terms proportional to $\ln\mu$ in the radii; for the Pauli radius of the nucleon there is an additional μ^{-1} singularity.[3] Furthermore, subject to certain innocuous assumptions – which are stated below – the coefficients of the μ^{-1} and $\ln\mu$ terms admit of precise computation. We conjecture that no singularities emerge from higher intermediate states and that our computation of the singular terms in the radii is therefore exact.

Our results, derived in Secs. II and III, are as follows:

$$\langle r_\pi^{\,2}\rangle = \frac{1}{8\pi^2}\,\frac{1}{f_\pi^{\,2}}\,\ln\frac{q_0}{\mu} + \text{terms finite in the limit } \mu \to 0. \tag{1.1}$$

$$\langle r_1^{\,2}\rangle^V = \left(\frac{g^2}{4\pi}\,\frac{3}{\pi M^2} + \frac{1}{8\pi^2}\,\frac{1-g_A^{\,2}}{f_\pi^{\,2}}\right)\ln\frac{q_0}{\mu} + \text{f.t.'s} \tag{1.2}$$

$$(\kappa_p - \kappa_n)\langle r_2^{\,2}\rangle^V = \left(\frac{g^2}{4\pi}\,\frac{1}{2M^2}\right)\frac{M}{\mu} + \left\{\frac{g^2}{4\pi}\,\frac{6}{\pi M^2} + \frac{1}{4\pi^2}\left[\frac{\partial}{\partial\nu}A^{(-)}(\nu,0)\right]_{\nu=0}\right\}\ln\frac{q_0}{\mu} + \text{f.t.'s}. \tag{1.3}$$

Here f_π is the pion decay constant, q_0 is an unknown parameter with the dimension of mass, g is the pion-nucleon coupling constant, g_A is the axial-vector to vector ratio in nucleon β decay ($\cong 1.22$), M is the nucleon mass, κ_p and κ_n are the anomalous magnetic moments of the proton and the neutron, respectively. $A^{(-)}$ is one of the standard covariants in pion-nucleon scattering.[4] In Eqs. (1.2) and (1.3) and in the rest of the paper "f.t.'s" stands for "terms finite in the limit $\mu \to 0$."

In Sec. IV we discuss the possible relevance of these results to the real world.

II. THE PION RADIUS

A. Notation and Definitions

Let $j_\mu(x)$ be the electromagnetic current density, e the unit of (positive) electric charge and $|q, a\rangle$ an invariantly normalized one-pion state of 4-momentum q and isotopic index a. The pion form factor, $F_\pi(t)$, is defined via

$$\langle q_2, b|j_\mu(0)|q_1, a\rangle = ie(q_2+q_1)_\mu \epsilon^{3ab} F_\pi(t), \qquad t=(q_2-q_1)^2 \leq 0 \tag{2.1}$$

or

$$\langle q_2, b; \bar{q}_1, \bar{a}|j_\mu(0)|0\rangle = ie(q_2-\bar{q}_1)_\mu \epsilon^{3\bar{a}b} F_\pi(t), \qquad t=(q_2+\bar{q}_1)^2 \geq 0, \tag{2.2}$$

with $F_\pi(0)=1$.

If $F_\pi(t)$ satisfies a dispersion relation with no more than one subtraction, we can write for the mean square radius of the pion

$$\tfrac{1}{6}\langle r_\pi^{\,2}\rangle \equiv F_\pi{}'(0) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt\,\frac{\mathrm{Im}F_\pi(t)}{t^2}\;, \tag{2.3}$$

where the spectral function $\mathrm{Im}F_\pi(t)$ is given by the unitarity relation:

$$\begin{aligned} ie(q_2-\bar{q}_1)_\mu \mathrm{Im}F_\pi(t)\epsilon^{3\bar{a}b} &= \tfrac{1}{2}\sum_n (2\pi)^4\delta^4(q_2+\bar{q}_1-q_n) \\ &\quad\times \langle q_2, b; \bar{q}_1, \bar{a}|T|n\rangle\langle n|j_\mu(0)|0\rangle, \end{aligned} \tag{2.4}$$

T being the usual collision matrix.

The $2\pi \to 2\pi$ matrix elements of T may be expressed in terms of three invariant amplitudes A, B, and C via

$$\begin{aligned} &\langle q_2', d; q_1', c|T|q_1, a; q_2, b\rangle \\ &\quad = A(s,t,u)\delta_{ab}\delta_{cd} + B(s,t,u)\delta_{ac}\delta_{bd} + C(s,t,u)\delta_{ad}\delta_{bc}\,, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} t &= (q_1+q_2)^2, \\ s &= (q_1'-q_1)^2, \\ u &= (q_2'-q_1)^2, \\ s+t+u &= 4\mu^2. \end{aligned}$$

These amplitudes satisfy the crossing relations

$$\begin{aligned} A(t,s,u) &= B(s,t,u) \\ &= C(u,s,t). \end{aligned} \tag{2.6}$$

Substituting Eqs. (2.5) and (2.2) in Eq. (2.4) one obtains the 2π contribution to the spectral function:

$$\mathrm{Im}F_\pi(t) = F_\pi(t) f_{11}(t)^*\theta(t-4\mu^2), \tag{2.7}$$

where $f_{11}(t)$, the $\pi\pi$ amplitude in the $I=J=1$ state, is defined via

$$f_{11}(t) = \frac{q}{32\pi\sqrt{t}}\int_{-4q^2}^{0}\frac{ds}{2q^2}\left(1+\frac{s}{2q^2}\right)(B-C), \qquad q^2 \equiv \tfrac{1}{4}(t-4\mu^2). \tag{2.8}$$

If we write

$$f_{11}(t) = e^{i\delta_{11}}\sin\delta_{11}, \tag{2.9}$$

the unitarity condition ensures that δ_{11} is real in the interval $4\mu^2 \leq t \leq 16\mu^2$; it is in fact the usual phase shift.

Finally, we note that the threshold behavior of $f_{11}(t)$ is specified by[5]

$$\frac{\sqrt{t}}{q^3} f_{11}(t) = a_{11} + O(q^2), \tag{2.10}$$

a_{11} being the $\pi\pi$ scattering length in the $I=J=1$ state. Current algebra and PCAC predict that[6]

$$a_{11} = \frac{1}{12\pi f_\pi^{\,2}}\;. \tag{2.11}$$

B. Singular Part of the 2π Intermediate State Contribution

The expression for the radius may be written in the form

$$\tfrac{1}{6}\langle r_\pi^{\,2}\rangle = I_1 + I_2, \tag{2.12}$$

$$I_1 = \frac{1}{\pi}\int_{4\mu^2}^{\lambda^2} dt\,\frac{\mathrm{Im}F_\pi(t)}{t^2}\;, \tag{2.13}$$

$$I_2 = \frac{1}{\pi}\int_{\lambda^2}^{\infty} dt\,\frac{\mathrm{Im}F_\pi(t)}{t^2}\;, \tag{2.14}$$

where λ is some real parameter – with the dimension of mass – independent of μ.

Under the assumption that $\mathrm{Im}F_\pi(t)$ is bounded on the cut, for all t and (real) μ, $|I_2|$ is also bounded; singularities, if any, must therefore lurk in I_1. In order to examine I_1 it is convenient to rewrite it in the form

$$I_1 = \frac{1}{4\pi}\int_0^{q_0}\frac{q^4\,dq}{(q^2+\mu^2)^{5/2}}\,\Phi(q^2), \tag{2.15}$$

$$\Phi(q^2) \equiv F_\pi(t)\frac{f_{11}(t)^*\sqrt{t}}{q^3}, \qquad q_0 \equiv q(\lambda^2). \tag{2.16}$$

Now as $\mu \to 0$, the integrand in Eq. (2.15) becomes singular at the lower limit unless Φ happens to vanish. However,

$$\Phi(0)=F_\pi(4\mu^2)a_{11}$$

$$\to\frac{1}{12\pi f_\pi^{\,2}}\text{ as }\mu\to 0. \tag{2.17}$$

We expect, therefore, a logarithmic divergence in the limit $\mu\to 0$. The divergent part can be isolated immediately, provided one can assume that $\Phi(q^2)$ is "slowly varying" in the vicinity of $q=0$. One obtains:

$$I_1=\frac{1}{4\pi}\Phi(0)\ln\frac{q_0}{\mu}+\text{f.t.'s}. \tag{2.18}$$

A set of conditions which are sufficient, *but not necessary*, to ensure that $\Phi(q^2)$ may be regarded as "slowly varying" in the interval $0\le q\le m$ are $|\Phi(m^2)|<\infty$, $|q\Phi'(q^2)|<\infty$. Equation (2.18) then follows from (2.15) by partial integration. Hence

$$\tfrac{1}{6}\langle r_\pi^{\,2}\rangle=\frac{1}{48\pi^2}\frac{1}{f_\pi^{\,2}}\ln\frac{q_0}{\mu}+\text{f.t.'s}. \tag{2.19}$$

Note that there is no way to assign a value to the scale parameter q_0.

To see how things work out in a simple model with elastic unitarity in the physical region, we consider the model of Brown and Goble.[7] In this model

$$\frac{\sqrt{t}}{q^3}f_{11}(t)=\left\{\frac{1}{a_{11}}+r_{11}q^2+\frac{q^3}{\sqrt{t}}\left[\frac{2}{\pi}\ln\left(\frac{\sqrt{t}+2q}{2\mu}\right)-i\right]\right\}^{-1}, \tag{2.20}$$

with

$$r_{11}=-\frac{1}{\pi}\ln\frac{m_\rho}{\mu}-\frac{4}{a_{11}m_\rho^{\,2}}+O\left(\frac{\mu}{m_\rho}\right), \tag{2.21}$$

$$\Gamma_\rho=\frac{m_\rho^{\,3}}{96\pi f_\pi^{\,2}}+O\left(\frac{\mu}{m_\rho}\right). \tag{2.22}$$

Equations (2.21) and (2.22) guarantee that the amplitude in Eq. (2.20) has a resonance at mass m_ρ with width Γ_ρ.

The form factor can be determined explicitly by noticing that, *in this model*, $F_\pi(t)$ and $f_{11}(t)\sqrt{t}/q^3$ have the same singularities; both are real analytic and polynomially bounded in the t plane cut from $4\mu^2$ to ∞ and their phases are identical over the entire cut. Furthermore, $f_{11}(t)\sqrt{t}/q^3$ has no zeros for any finite t. Hence $[F_\pi(t)/f_{11}(t)\sqrt{t}/q^3]$ is a polynomial; if we assume for the sake of simplicity that $F_\pi(t)$ also has no finite zeros, we obtain:

$$F_\pi(t)=\left[\frac{1}{a_{11}}-\mu^2\left(r_{11}+\frac{1}{\pi}\right)\right]f_{11}(t)\frac{\sqrt{t}}{q^3}. \tag{2.23}$$

The constant in Eq. (2.23) has been chosen to ensure that $F_\pi(0)=1$.

One can check explicitly that all the assumptions needed to derive Eq. (2.19) are satisfied. The radius obtained by direct differentiation of Eq. (2.23) indeed agrees with (2.19).

It is important to emphasize that Eq. (2.19) is *not* true in arbitrary nonchiral models of the $\pi\pi$ interaction. The simplest example is furnished by the $\lambda\phi^4$ theory. In this theory the only parameter with the dimension of mass is μ; one expects, therefore, that the radius would blow up as $1/\mu^2$ rather than $\ln\mu$.[8]

III. ISOVECTOR NUCLEON RADII

A. Notation and Definitions

We write the electromagnetic current in terms of its isoscalar and isovector pieces.

$$j_\mu(x)=j_\mu^S(x)+j_\mu^V(x), \tag{3.1}$$

normalizing the pieces via $(2/e)\int j_0^S(x)d^3x=Y$, the total hypercharge, and $(1/e)\int j_0^V(x)d^3x=I_3$, the 3-component of isospin. The four form factors associated with the nucleon are defined by

$$\langle p_2|j_\mu^a(0)|p_1\rangle=e\,\bar u(p_2)M^a\times\left[\gamma_\mu F_1^a(t)+\frac{i\sigma_{\mu\nu}}{2M}(p_2-p_1)^\nu F_2^a(t)\right]u(p_1)$$

$$a=S\text{ or }V. \tag{3.2}$$

Here the $|P_i\rangle$ are invariantly normalized nucleon states, the spinors are normalized by $\bar u(p)u(p)=1$, M is the nucleon mass and the internal matrix, M^a, is defined to be $\frac{1}{2}$ for $a=S$ and $\frac{1}{2}\tau_3$ for $a=V$. Our normalization of the current is such that $F_1^S(0)=F_1^V(0)=1$. Also, $F_2^S(0)=(\kappa_p+\kappa_n)\cong 0.12$ and $F_2^V(0)=(\kappa_p-\kappa_n)\cong 3.7$.

If the form factors satisfy dispersion relations with no more than one subtraction, the mean square radii of the nucleon

$$\langle r^2\rangle_i^a\equiv 6F_i^{a\prime}(0)/F_i^a(0)\quad(a=S\text{ or }V;\ i=1,2) \tag{3.3}$$

are determined by

$$F_i^{a\prime}(0)=\frac{1}{\pi}\int_{4\mu^2}^\infty\frac{\operatorname{Im}F_i^a(t)}{t^2}\,dt. \tag{3.4}$$

The spectral functions in Eq. (3.4) are determined by equations analogous to (2.4),

$$\operatorname{Im}F_i^a(t)=-\tfrac{1}{2}\sum_n P_i^\mu(2\pi)^4\delta^4(\bar p_1+p_2-p_n)\times\langle\bar p_1,p_2|T|n\rangle\langle n|j_\mu^a(0)|0\rangle, \tag{3.5}$$

P_i^μ being an appropriate projection operator.

In the present paper we restrict ourselves to the isovector radii. The only relevant intermediate states are therefore those with G parity $+1$ (2π, 4π, 6π, etc.).

To handle the 2π intermediate state contribution we shall use the usual decomposition of the $\pi\pi \to N\bar{N}$ amplitude in terms of invariant amplitudes [4]:

$$\langle \bar{p}_1, p_2|T|q_1, a; q_2, b\rangle = -\bar{u}(p_2)[A + \tfrac{1}{2}\gamma\cdot(q_1 - q_2)B]v(\bar{p}_1), \tag{3.6}$$

with

$$A = A^{(+)}\delta_{ab}\tfrac{1}{2}[\tau_a, \tau_b]A^{(-)}, \tag{3.7}$$

$$B = B^{(+)}\delta_{ab}\tfrac{1}{2}[\tau_a, \tau_b]B^{(-)}. \tag{3.8}$$

Equations (2.2) and (3.6) substituted into (3.5) yield

$$\mathrm{Im}F_1^V(t) = \frac{1}{4\pi}\frac{q^3}{\sqrt{t}}\left\{\frac{M}{pq}\, a_1^{(-)}(t) - \frac{M^2}{p^2}\, b_2^{(-)}(t) - \tfrac{1}{3}[b_2^{(-)}(t) - b_0^{(-)}(t)]\right\}F_\pi^*(t), \tag{3.9}$$

$$\mathrm{Im}F_2^V(t) = \frac{1}{4\pi}\frac{q^3}{\sqrt{t}}\left[-\frac{M}{pq}\, a_1^{(-)}(t) + \frac{M^2}{p^2}\, b_2^{(-)}(t)\right]F_\pi^*(t), \tag{3.10}$$

where

$$\begin{aligned} q = [\tfrac{1}{4}(t - 4\mu^2)]^{1/2}, \quad p &= [\tfrac{1}{4}(t - 4M^2)]^{1/2} \ \text{ for } t \geq 4M^2 \\ &= +i[\tfrac{1}{4}(4M^2 - t)]^{1/2} \ \text{ for } t \leq 4M^2, \end{aligned} \tag{3.11}$$

and

$$\begin{pmatrix} a_l^{(-)}(t) \\ b_l^{(-)}(t)\end{pmatrix} = \int_{-(p+q)^2}^{-(p-q)^2} \frac{ds}{2pq}\, P_l\!\left(\frac{s + \frac{1}{2}t - M^2 - \mu^2}{2pq}\right)\begin{pmatrix} A^{(-)}(s, t) \\ B^{(-)}(s, t)\end{pmatrix}. \tag{3.12}$$

We shall find it convenient to study the small-q behavior of these partial-wave amplitudes, for the process $\pi\pi \to N\bar{N}$, by relating them to the amplitudes for the crossed reaction $\pi + N \to \pi + N$, through the so-called Froissart-Gribov representation. Since both reactions are described by the same invariant functions A and B evaluated over different domains of s and t, the requisite relationships can be established via fixed-t dispersion relations (assumed to be valid without subtractions):

$$A^{(-)}(\nu, t) = \frac{2\nu}{\pi}\int_{\nu_0}^{\infty} d\nu'\, \frac{\mathrm{Im}\,A^{(-)}(\nu', t)}{\nu'^2 - \nu^2}, \tag{3.13}$$

$$B^{(-)}(\nu, t) = \frac{g^2}{M}\frac{\nu_B}{\nu_B{}^2 - \nu^2} + \frac{2}{\pi}\int_{\nu_0}^{\infty} \nu' d\nu'\, \frac{\mathrm{Im}B^{(-)}(\nu', t)}{\nu'^2 - \nu^2}, \tag{3.14}$$

where

$$\nu = \frac{1}{2M}(s - M^2 - \mu^2 + \tfrac{1}{2}t), \qquad \nu_0 = \mu + \frac{t}{4M}, \qquad \nu_B = \frac{t - 2\mu^2}{4M}, \tag{3.15}$$

and g is the πN coupling constant. [Note that $s = (p_1 + q_1)^2$, $t = (q_1 - q_2)^2$ for $\pi N \to \pi N$ and, $s = (-\bar{p}_1 + q_1)^2$, $t = (\bar{q}_1 + q_2)^2$ for $\pi\pi \to N\bar{N}$.]

Equations (3.12), (3.13), and (3.14) lead immediately to the following expressions for the partial-wave amplitudes:

$$a_1^{(-)}(t) = \frac{4iM}{Pq\pi}\int_{\nu_0}^{\infty} d\nu'\, \mathrm{Im}\,A^{(-)}(\nu', t)\left[1 - \frac{\nu' M}{Pq}\, \tan^{-1}\!\left(\frac{Pq}{\nu' M}\right)\right], \tag{3.16}$$

$$b_0^{(-)}(t) = g^2\frac{2}{Pq}\, \tan^{-1}\!\left(\frac{4Pq}{t - 2\mu^2}\right) + \frac{4M}{\pi Pq}\int_{\nu_0}^{\infty} d\nu'\, \mathrm{Im}B^{(-)}(\nu', t)\tan^{-1}\!\left(\frac{Pq}{\nu' M}\right), \tag{3.17}$$

$$\begin{aligned} b_2^{(-)}(t) = {} & g^2\frac{t - 2\mu^2}{4P^2q^2}\left\{3 - \left[\frac{3(t - 2\mu^2)^2}{16P^2q^2} + 1\right]\frac{4Pq}{t - 2\mu^2}\, \tan^{-1}\!\left(\frac{4Pq}{t - 2\mu^2}\right)\right\} \\ & + \frac{2M^2}{\pi P^2q^2}\int_{\nu_0}^{\infty} d\nu'\, \mathrm{Im}B^{(-)}(\nu', t)\left\{3 - \left[\frac{3\nu'^2M^2}{P^2q^2} + 1\right]\frac{Pq}{\nu' M}\, \tan^{-1}\!\left(\frac{Pq}{\nu' M}\right)\right\}. \end{aligned} \tag{3.18}$$

These expressions are appropriate for t in the interval $4\mu^2 \leq t \leq 4M^2$ so that $P = -ip$ is real.

B. Singular Parts of the 2π Contribution

Convenient expressions for the 2π contribution to the radii are obtained by substituting the partial-wave amplitudes in Eqs. (3.16), (3.17), and (3.18) into the expressions (3.9) and (3.10) for the spectral functions which, in turn, are substituted into Eq. (3.4). As in the pion case, the singular parts of the radii stem from integration in the neighborhood of the threshold, i.e.,

$$F_i^{a\prime}(0)=\frac{1}{\pi}\int_{4\mu^2}^{\lambda^2}\frac{\operatorname{Im}F_i^a(t)}{t^2}\,dt+\text{f.t.'s}\,. \tag{3.19}$$

In order to isolate the most singular terms we shall deem it legitimate to set $F_\pi(t)=1$ in the integrand; the terms thereby ignored are expected to be of order $F_\pi'(0)\times$ (terms retained) i.e., $\mu\ln\mu$ or $\mu^2\ln^2\mu$. The contribution of the nucleon pole term in Eqs. (3.17) and (3.18) is then immediately obtainable, by explicit integration [9]

$$F_1^{V\prime}(0)_{\text{pole}}=\frac{g^2}{8\pi^2M^2}\ln\frac{M}{\mu}+\text{f.t.'s}, \tag{3.20}$$

$$F_2^{V\prime}(0)_{\text{pole}}=\frac{g^2}{48\pi M^2}\,\frac{M}{\mu}-\frac{g^2}{4\pi^2M^2}\ln\frac{M}{\mu}+\text{f.t.'s}\,. \tag{3.21}$$

The nonpole (np) or continuum contribution can be written in the form

$$F_1^{V\prime}(0)_{(\text{np})}=\frac{1}{4\pi^3}\int_0^{q_0}dq\,\frac{q^2}{(q^2+\mu^2)^{5/2}}\int_\mu^\infty d\omega'[\operatorname{Im}A^{(-)}(\omega',q)+\nu'\operatorname{Im}B^{(-)}(\omega',q)]\left[1-\frac{\nu'}{q}\tan^{-1}\!\left(\frac{q}{\nu'}\right)\right]+\text{f.t.'s}\,, \tag{3.22}$$

$$\begin{aligned}F_2^{V\prime}(0)_{(\text{np})}=-\frac{1}{4\pi^3}\int_0^{q_0}dq\,\frac{q^2}{(q^2+\mu^2)^{5/2}}\int_\mu^\infty d\omega'\Big\{&\operatorname{Im}A^{(-)}(\omega',q)\left[1-\frac{\nu'}{q}\tan^{-1}\!\left(\frac{q}{\nu'}\right)\right]\\&+\tfrac{1}{2}\nu'\operatorname{Im}B^{(-)}(\omega',q)\left[3-\left(\frac{3\nu'^2}{q^2}+1\right)\frac{q}{\nu'}\tan^{-1}\!\left(\frac{q}{\nu'}\right)\right]\Big\}+\text{f.t.'s}.\end{aligned} \tag{3.23}$$

Here we have introduced the variable $\omega'=\nu'-t/4M$ and displayed $A^{(-)}$ and $B^{(-)}$ as functions of ω' and q rather than ν' and t. Also, we have set $(P/M)=1$; this entails neglecting terms of order μ^2/M^2.

To proceed further we assume that $\operatorname{Im}A^{(-)}(\omega',q)$ and $\operatorname{Im}B^{(-)}(\omega',q)$ are "slowly varying" – in the sense mentioned earlier, after Eq. (2.17) – functions of q around $q=0$ and interchange the order of integration in Eqs. (3.22) and (3.23). Then, as shown in detail in the Appendix, the singular parts of $F_i^{V\prime}(0)$ can be readily extracted. We find

$$F_1^{V\prime}(0)_{(\text{np})}=\frac{1}{12\pi^3}\left(\ln\frac{M}{\mu}\right)\int_\mu^\infty\frac{d\omega}{\omega^2}\,[\operatorname{Im}A^{(-)}(\omega,0)+\omega\operatorname{Im}B^{(-)}(\omega,0)]+\text{f.t.'s}, \tag{3.24}$$

$$F_2^{V\prime}(0)_{(\text{np})}=-\frac{1}{12\pi^3}\left(\ln\frac{M}{\mu}\right)\int_\mu^\infty\frac{d\omega}{\omega^2}\,\operatorname{Im}A^{(-)}(\omega,0)+\text{f.t.'s}\,. \tag{3.25}$$

The integrals on the right-hand side of Eqs. (3.24) and (3.25) can be recognized as first derivatives at the origin of forward pion-nucleon amplitudes.

Combining the pole and continuum contributions, Eqs. (3.20), (3.21), (3.24), and (3.25), we obtain:

$$\langle r_1^2\rangle^V=\left\{\frac{g^2}{4\pi}\,\frac{3}{\pi M^2}+\frac{1}{4\pi^2}\,\frac{\partial}{\partial\nu}[A^{(-)}(\nu,0)+\nu\tilde{B}^{(-)}(\nu,0)]\Big|_{\nu=0}\right\}\ln\frac{M}{\mu}+\text{f.t.'s}, \tag{3.26}$$

$$(\kappa_p-\kappa_n)\langle r_2^2\rangle^V=\left(\frac{g^2}{4\pi}\,\frac{1}{2M^2}\right)\frac{M}{\mu}-\left[\frac{g^2}{4\pi}\,\frac{6}{\pi M^2}+\frac{1}{4\pi^2}\,\frac{\partial}{\partial\nu}A^{(-)}(\nu,0)\Big|_{\nu=0}\right]\ln\frac{M}{\mu}+\text{f.t.'s}\,. \tag{3.27}$$

Here $\tilde{B}^{(-)}$ is defined to be $B^{(-)}$ less the pole term. Note that the combination of derivatives in Eq. (3.26) is determined by the Adler-Weisberger formula [10]

$$\frac{\partial}{\partial\nu}[A^{(-)}(\nu,0)+\nu\tilde{B}^{(-)}(\nu,0)]\Big|_{\nu=0}=\frac{1}{2f_\pi^{\,2}}(1-g_A^{\,2})\,. \tag{3.28}$$

Unfortunately the derivative in Eq. (3.27) is not determined by current algebra or any other model-independent argument.

IV. CONCLUDING REMARKS

(1) The foregoing results pertain only to the 2π intermediate state in the dispersion integrals. How about the 4π and higher states? We conjecture that these contributions are finite in the limit $\mu \to 0$; our conjecture is based on a simple counting of powers of q in the phase-space integrals in Eqs. (2.4) and (3.5). It does not seem possible to substantiate this conjecture without making many assumptions about higher point functions; even in simplified models the labor involved appears to be prohibitive. Incidentally, this phase-space suppression of singularities has been encountered in another context by Pagels and Zepeda[11]; they found that the 3π-state contribution to the $\pi N\bar{N}$ form factor is free of singularities in the limit $\mu \to 0$.

(2) Having isolated the singular parts of the pion radius and the isovector nucleon radii we are naturally led to ask: Are these results just mathematical curiosities or can one use them to obtain meaningful estimations of the radii in the real world where μ is small but nonvanishing? There is a difficulty of principle here stemming from the fact that the mass scale in the logarithmic singularities cannot be fixed by our analysis, and a change of scale changes the numerical values of both the singular and the regular parts of the radius.

The numerical situation can be gleaned from the following relationships [obtained by substituting numerical values[12] for the various parameters in Eqs. (1.1)–(1.3)]

$$\langle r_\pi^2\rangle \cong \left(0.06 \ln\frac{q_0}{\mu} + \text{f.t.'s}\right) \text{ F}^2 \cong (0.4-1.0?) \text{ F}^2, \tag{4.1}$$

$$\langle r_1^2\rangle^V = \left(0.6 \ln\frac{q_0}{\mu} + \text{f.t.'s}\right) \text{ F}^2 \cong 0.52 \text{ F}^2, \tag{4.2}$$

$$\langle r_2^2\rangle^V = \left[0.6\left(1 - \tfrac{1}{2}\ln\frac{q_0}{\mu}\right) + \text{f.t.'s}\right] \text{ F}^2 \cong 0.72 \text{ F}^2. \tag{4.3}$$

Here F stands for fermi (10^{-13} cm), the numbers after the second equality sign are experimental values[13] and the question mark in Eq. (4.1) is indicative of the fact that the data on the pion radius are extremely crude. (Note that the μ^{-1} term in the Pauli radius has been replaced by its numerical value).

It is obvious that the singular terms, with the logarithms expressed in terms of a single mass scale, cannot – for any choice of scale – account for the experimental values of all three radii. This is not surprising if one remembers that the singular terms stem solely from threshold contributions to the dispersion integrals. One cannot ignore such important medium energy effects as the ρ-resonance contribution (~0.4 F^2 in all the radii). Also, high-energy effects may not be entirely negligible for the Dirac radius. Proper inclusion of the various contributions is, of necessity, a highly model-dependent undertaking and lies outside the scope of the present paper.

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APPENDIX

In this appendix we sketch the derivation of Eqs. (3.26) and (3.27). With the assumptions that (a) $\text{Im}A^{(-)}(\omega', q)$ and $\text{Im}B^{(-)}(\omega, q)$ can be regarded as slowly varying functions of q near $q=0$, and (b) the order of integration in Eqs. (3.22) and (3.23) can be interchanged, and with neglect of terms which are potentially of order μ/M, we obtain

$$F_1^{V\prime}(0)_{(\text{np})} = \frac{1}{12\pi^3}\int_\mu^\infty \frac{d\omega}{\omega^2}[\text{Im}A^{(-)}(\omega,0) + \omega\,\text{Im}B^{(-)}(\omega,0)]\int_0^{q_0} \frac{q^4 dq}{(q^2+\mu^2)^{5/2}} f_1(\omega,q) + \text{f.t.'s}, \tag{A1}$$

$$\begin{aligned} F_2^{V\prime}(0)_{(\text{np})} = &-\frac{1}{12\pi^3}\int_\mu^\infty \frac{d\omega}{\omega^2}\,\text{Im}A^{(-)}(\omega,0)\int_0^{q_0} \frac{q^4 dq}{(q^2+\mu^2)^{5/2}} f_1(\omega,q) \\ &-\frac{1}{12\pi^3}\int_\mu^\infty \frac{d\omega}{\omega}\,\text{Im}B^{(-)}(\omega,0)\int_0^{q_0} \frac{q^4 dq}{(q^2+\mu^2)^{5/2}} f_2(\omega,q) + \text{f.t.'s}, \end{aligned} \tag{A2}$$

where

$$f_1(\omega, q) = \frac{3\omega^2}{q^2}\left[1 - \frac{\omega}{q}\tan^{-1}\left(\frac{q}{\omega}\right)\right], \tag{A3}$$

$$f_2(\omega, q) = \frac{3\omega^2}{2q^2}\left[3 - \left(\frac{3\omega^2}{q^2}+1\right)\frac{q}{\omega}\tan^{-1}\left(\frac{q}{\omega}\right)\right]. \tag{A4}$$

The integrals over q can be evaluated explicitly. Let

$$I_i \equiv \int_0^{q_0} \frac{q^4 dq}{(q^2+\mu^2)^{5/2}} f_i(\omega, q) \quad (i = 1 \text{ or } 2). \tag{A5}$$

Then

$$I_1 = -\frac{\omega^2}{\omega^2-\mu^2}\frac{\lambda(\lambda^2+\omega^2)}{(\lambda^2+\mu^2)^{3/2}} + \frac{\omega^3}{(\lambda^2+\mu^2)^{1/2}}\tan^{-1}\left(\frac{\lambda}{\omega}\right) + \left(\frac{\omega^2}{\omega^2-\mu^2}\right)^{3/2}\frac{1}{2}\ln\frac{\omega(\lambda^2+\mu^2)^{1/2}+\lambda(\omega^2-\mu^2)^{1/2}}{\omega(\lambda^2+\mu^2)^{1/2}-\lambda(\omega^2-\mu^2)^{1/2}}, \tag{A6}$$

$$I_2 = -\frac{3}{2}\frac{\omega^2}{\lambda^2+\mu^2}\frac{\lambda}{(\lambda^2+\mu^2)^{1/2}} + \frac{3}{2}\frac{\omega}{(\lambda^2+\mu^2)^{1/2}}\left(\frac{\omega^2-\mu^2/3}{\lambda^2+\mu^2}+1\right)\tan^{-1}\left(\frac{\lambda}{\omega}\right). \tag{A7}$$

As $\mu \to 0$ we have

$$I_1 \to +\ln\left(\frac{\lambda}{\mu}\right) - \frac{\lambda^2+\omega^2}{\lambda^2} + \left(\frac{\omega}{\lambda}\right)^3\tan^{-1}\left(\frac{\lambda}{\omega}\right) + \frac{1}{2}\ln\frac{\omega^2}{\lambda^2+\omega^2}, \tag{A8}$$

$$I_2 \to -\frac{3}{2}\frac{\omega^2}{\lambda^2} + \frac{3}{2}\frac{\omega}{\lambda}\left(1+\frac{\omega^2}{\lambda^2}\right)\tan^{-1}\left(\frac{\lambda}{\omega}\right). \tag{A9}$$

So long as the I_i become large no faster than $\ln\omega$ at $\omega = 0$, no infrared singularity can emerge from the ω integration. This follows from the fact that near $\omega = 0$, $\mathrm{Im}A^{(-)}(\omega, 0) \sim O(\omega^2)$ and $\mathrm{Im}B^{(-)}(\omega, 0) \sim O(\omega)$. The only singularity in the $\mu \to 0$ limit is therefore the one isolated in the first term on the right-hand side of (A8). Consequently,

$$F_1^{V'}(0)_{(\mathrm{np})} = \frac{1}{12\pi^3}\left(\ln\frac{M}{\mu}\right)\int_\mu^\infty \frac{d\omega}{\omega^2}[\mathrm{Im}A^{(-)}(\omega, 0) + \omega\,\mathrm{Im}B^{(-)}(\omega, 0)] + \text{f.t.'s}, \tag{A10}$$

$$F_2^{V'}(0)_{(\mathrm{np})} = -\frac{1}{12\pi^3}\left(\ln\frac{M}{\mu}\right)\int_\mu^\infty \frac{d\omega}{\omega^2}\mathrm{Im}A^{(-)}(\omega, 0) + \text{f.t.'s}. \tag{A11}$$

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