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Operator Droplet Model at Finite Momenta*

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The operator droplet model at finite momenta is discussed, using an infinite set of generalized s-channel ladder graphs. The model is shown to follow from the assumption that the centers of mass of the scattering systems move with constant velocities. The conditions for the validity of this assumption are investigated and shown to be closely related to the conditions for the validity of Glauber theory for the nonrelativistic scattering of composite systems.

I. INTRODUCTION

The operator droplet model[1] has been used extensively to discuss the high-energy scattering of elementary particles.[2-4] When applied to the scattering of two nuclei it is equivalent to Glauber theory[5,6] provided the meson degrees of freedom are ignored: One simply has the option of working with many-body wave functions (the usual form of Glauber theory) or with the equivalent second-quantized theory (the operator droplet model). The operator droplet model, however, is more general than Glauber theory in that it applies to the scattering of any two high-energy systems, whether or not they are composed of a definite number of constituents. It may, for example, be possible to use it to extend Glauber theory so as to take into account in a systematic way the meson degrees of freedom of nuclei.

Until recently the validity of the operator droplet model was simply postulated as a natural extension of Glauber theory. A recent paper by Chang,[7,8] however, demonstrated that the model could be obtained as the infinite-momentum limit of the set of all generalized s-channel ladder graphs. It is not clear, however, how relevant this derivation might be for finite momenta, and in particular for generalizing Glauber theory, which is usually applied at momenta down to 1 GeV/c and below.

In this paper the possibility that the operator droplet model might be valid at finite momenta is discussed, using the same set of graphs. Besides furnishing a certain amount of foundation for a generalization of Glauber theory, this "derivation" makes more explicit the approximations and assumptions involved. Furthermore, the condition for the irrelevance of the time ordering of the operators is directly related to the vanishing of equal-time commutators, rather than the light-cone commutators as required in Chang's discussion.

In Sec. II the set of graphs under discussion is defined, and it is shown that the operator droplet model is obtained simply by assuming that the centers of mass of the high-energy systems move with constant velocity. Section III contains a detailed analysis of this assumption and the corresponding conditions on the nature of the interaction and the mass spectra. The paper concludes with a summary of its results and a discussion of some possible applications of the operator droplet model at finite momenta.

II. THE MODEL

Although it is likely that the operator droplet idea can be applied to more general processes, only the scattering of two high-energy systems, denoted by s and s', moving in opposite directions, will be considered here. The state of the system s is specified by giving its internal state, labeled by an index α, and its momentum $\vec{p}$. Its energy is then

$$E(\alpha, \vec{p}) = [M(\alpha)^2 + \vec{p}^2]^{1/2},$$

where $M(\alpha)$ is the mass of s in the internal state α.

The states of s' are described in an analogous way.

The two systems can scatter via an interaction of the form

$$\int d^4x[\, g\rho(x)+g'\rho'(x)]\,\phi(x),$$

where $\phi(x)$ is a scalar field and ρ and ρ' are commuting scalar operators acting on the states of the system s and s', respectively.[9] Only the contributions to the scattering amplitude from graphs of the type shown in Fig. 1 are considered: For the process

$$\alpha_i(\vec{p}_i)+\alpha_i'(\vec{p}_i')\to\alpha_f(\vec{p}_f)+\alpha_f'(\vec{p}_f')$$

the scattering amplitude T is given by

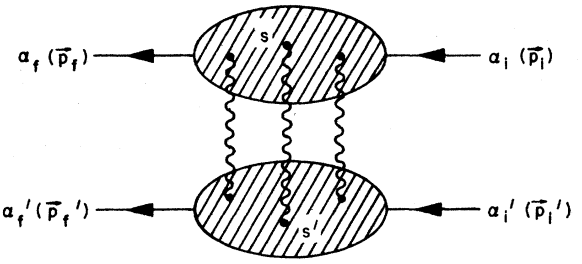

FIG. 1. The third-order ladder diagram. The upper blob represents the system s, initially in its internal state α_i with total momentum $\vec{p}_i$ and finally in its internal state α_f with total momentum $\vec{p}_f$. The lower blob similarly represents the system s'. The wavy lines represent the interaction between the two systems due to the exchange of a particle with propagator $D(x-x')$.

$$i(2\pi)^4\delta^4(p_f+p_f'-p_i-p_i')T(\alpha_f,\alpha_f';\alpha_i,\alpha_i';\vec{p}_f,\vec{p}_f';\vec{p}_i,\vec{p}_i')$$

$$=\left\langle \vec{p}_f,\alpha_f;\vec{p}_f',\alpha_f'\right|\left\{\mathcal{T}\exp\left[-igg'\int d^4x\,d^4x'\,\rho(x)D(x-x')\rho(x')\right]-1\right\}\left|\vec{p}_i,\alpha_i;\vec{p}_i',\alpha_i'\right\rangle. \tag{1}$$

Here $\mathcal{T}$ indicates the time-ordered product, which, of course, can be taken independently for the ρ and ρ' operators, while $D(x-x')$ is the propagator of the field ϕ.

The operator droplet model is obtained by the simple replacement

$$\rho(x)=\rho(t,\vec{r})\approx\rho(0,\vec{r}-\vec{v}t), \tag{2}$$

where $\vec{v}$ is a velocity typical of the system s, with the analogous replacement

$$\rho'(x')\approx\rho'(0,\vec{r}'-\vec{v}'t') \tag{3}$$

for the system s'. These replacements amount to assuming that the systems travel with constant velocities, unaffected by their interactions with each other except in their internal degrees of freedom: As far as the motions of their centers of mass are concerned the systems act as external sources. The conditions under which these replacements are justified are discussed in more detail in Sec. III.

With these replacements for ρ and ρ' the integral in the exponent in Eq. (1) is simply

$$\int dt\,dt'\,d^3r\,d^3r'\rho(0,\vec{r}-\vec{v}t)D(t-t',\vec{r}-\vec{r}')\rho'(0,\vec{r}'-\vec{v}'t')=\int dt\,dt'\,d^3r\,d^3r'\rho(0,\vec{r})D(t-t',\vec{r}-\vec{r}'+\vec{v}t-\vec{v}'t')\rho'(0,\vec{r}'). \tag{4}$$

Now, choosing a reference frame such that $\vec{v}$ and $\vec{v}'$ are in the $+z$ and $-z$ directions, respectively,

$$\int dt\,dt'\,D(t-t',\vec{r}-\vec{r}'+\vec{v}t-\vec{v}'t')=(v+v')^{-1}\int d\tau\,d\zeta\,D(\tau;\vec{r}_\perp-\vec{r}_\perp',\zeta), \tag{5}$$

independent of z and z'. Here $\tau=t-t'$, $\zeta=z-z'+vt+v't'$, and $\vec{r}_\perp$ and $\vec{r}_\perp'$ are the projections of $\vec{r}$ and $\vec{r}'$ on the x-y plane. Finally, defining

$$\chi(\vec{r}_\perp-\vec{r}_\perp')=-gg'(v+v')^{-1}\int d\tau\,d\zeta\,D(\tau;\vec{r}_\perp-\vec{r}_\perp',\zeta), \tag{6}$$

$$\sigma(\vec{r}_\perp)=\int dz\,\rho(0,\vec{r}), \tag{7}$$

and

$$\sigma'(\vec{r}_\perp')=\int dz'\,\rho'(0,\vec{r}'), \tag{8}$$

and assuming that ρ and ρ' commute with themselves at equal times so that the time ordering can be ig-

nored,[10] we have

$$i(2\pi)^4\delta^4(p_f+p_f'-p_i-p_i')T(\alpha_f,\alpha_f';\alpha_i,\alpha_i';\vec{p}_f,\vec{p}_f';\vec{p}_i,\vec{p}_i')$$
$$=\left\langle \vec{p}_f,\alpha_f;\vec{p}_f',\alpha_f'\left|\left\{\exp\left[i\int d^2r_\perp d^2r_\perp'\sigma(\vec{r}_\perp)\chi(\vec{r}_\perp-\vec{r}_\perp')\sigma'(\vec{r}_\perp')\right]-1\right\}\right|\vec{p}_i,\alpha_i;\vec{p}_i',\alpha_i'\right\rangle. \tag{9}$$

Note that because of the approximations which have been made the energy and longitudinal momentum of each system are conserved.

A useful alternate expression for T can be obtained by Lorentz-transforming each system to the frame where its longitudinal momentum is approximately zero. Because ρ and ρ' are assumed to transform as scalars and because, as will be shown in Sec. III, the energy differences in the average rest frames are ignored, under these transformations

$$\sigma(\vec{r}_\perp)\to\gamma^{-1}\sigma(\vec{r}_\perp) \tag{10}$$

and

$$\sigma'(\vec{r}_\perp')\to\gamma'^{-1}\sigma'(\vec{r}_\perp), \tag{11}$$

where

$$\gamma=(1-v^2)^{-1/2} \tag{12}$$

and

$$\gamma'=(1-v'^2)^{-1/2}. \tag{13}$$

We thus obtain (assuming an invariant-state normalization)

$$i(2\pi)^4\delta^4(p_f+p_f'-p_i-p_i')T(\alpha_f,\alpha_f';\alpha_i,\alpha_i';\vec{p}_f,\vec{p}_f';\vec{p}_i,\vec{p}_i')$$
$$=\left\langle \vec{p}_f^{\,(0)},\alpha_f^{(0)};\vec{p}_f^{\,(0)'},\alpha_f^{(0)'}\left|\left\{\exp\left[i(\gamma\gamma')^{-1}\int d^2r_\perp d^2r_\perp'\sigma(\vec{r}_\perp)\chi(\vec{r}_\perp-\vec{r}_\perp')\sigma'(\vec{r}_\perp')\right]-1\right\}\right|\vec{p}_i^{\,(0)},\alpha_i^{(0)};\vec{p}_i^{\,(0)'},\alpha_i^{(0)'}\right\rangle, \tag{14}$$

where the superscripts (0) indicate quantities in the new frames where the longitudinal momenta of the two systems approximately vanish.

The evaluation of expressions such as Eq. (14) has been discussed in detail by Lee.[4] For the scattering of two nuclei, when the meson degrees of freedom are ignored and many-body wave functions are used, the exponent becomes simply[1]

$$i(\gamma\gamma')^{-1}\sum_{n,n'}\chi(\vec{r}_{n\perp}-\vec{r}_{n'\perp}),$$

where n and n' label the constituent nucleons in the two nuclei, and $\vec{r}_n$ and $\vec{r}_{n'}$ are the positions of these nucleons relative to a common origin.

III. THE APPROXIMATION

It was shown above that the operator droplet model is obtained from the sum of all generalized s-channel ladder diagrams by the simple replacements

$$\rho(x)\approx\rho(0,\vec{r}-\vec{v}t)$$

and

$$\rho'(x')\approx\rho'(0,\vec{r}'-\vec{v}'t'),$$

where $\vec{v}$ and $\vec{v}'$ are the characteristic velocities of the two systems s and s'. In this section the nature of the approximations involved in these replacements will be discussed.

Since the time-ordered exponential in Eq. (1) can be expanded in powers of ρ and ρ', and complete sets of intermediate states inserted between the operators, the replacements above are equivalent to the assumption that (concentrating on the system s for brevity)

$$\langle \vec{p}_k \alpha_k | \rho(x) | \vec{p}_j \alpha_j \rangle \approx \langle \vec{p}_k \alpha_k | \rho(0, \vec{r} - \vec{v}t) | \vec{p}_j \alpha_j \rangle \quad (15)$$

for all such matrix elements which contribute significantly to the integrals or sums over the $\vec{p}$'s and α's.

Since by translational invariance

$$\langle \vec{p}_k \alpha_k | \rho(x) | \vec{p}_j \alpha_j \rangle = \exp[i(p_k - p_j)\cdot x] \times \langle \vec{p}_k \alpha_k | \rho(0) | \vec{p}_j \alpha_j \rangle , \quad (16)$$

the replacement will be valid if a $\vec{v}$ can be found such that

$$(E_k - E_j)t \approx \vec{v}\cdot(\vec{p}_k - \vec{p}_j)t \quad (17)$$

or

$$(E_k - \vec{v}\cdot\vec{p}_k)t \approx (E_j - \vec{v}\cdot\vec{p}_j)t \quad (18)$$

for each non-negligible matrix element. If the t is canceled the last relation, when multiplied by

$$\gamma = (1 - v^2)^{-1/2},$$

becomes

$$E_k^{(0)} = E_j^{(0)} , \quad (19)$$

where

$$E_j^{(0)} = \gamma(E_j - \vec{v}\cdot\vec{p}_j) \quad (20)$$

is the energy of the jth state in the frame moving with velocity $\vec{v}$ relative to the original frame. Since $\vec{v}$ is a typical velocity of the system in the original frame, the new frame is just the "average rest frame" for the system s.

It is convenient at this stage to write Eq. (19) as

$$E_j^{(0)} = M \quad (21)$$

for all j, and to define

$$E = \gamma M \quad (22)$$

and

$$\vec{p} = \vec{v}\gamma M . \quad (23)$$

Then M, E, and $\vec{p}$ can be thought of as, respectively, the typical mass, energy, and momentum of the system s. With this notation Eq. (20) becomes

$$E_j = E + \vec{v}\cdot(\vec{p}_j - \vec{p}) \quad (24)$$

for all j. It is easy to show that if these relations hold in any Lorentz frame they will hold in all frames moving parallel to $\vec{v}$ relative to the first frame, provided the appropriate transformed values are used for v, E, and p.

The approximate expression (24) can be derived from the exact expression

$$E_j = [M(\alpha_j)^2 + \vec{p}_j^{\,2}]^{1/2}$$

provided a few simple conditions are satisfied: If $E_j^2 - E^2 \ll E^2$ then

$$E_j = [E^2 + (E_j^2 - E^2)]^{1/2} \approx E + (E_j^2 - E^2)/2E - (E_j^2 - E^2)^2/8E^3 + \cdots . \quad (25)$$

Now

$$E_j^2 - E^2 = \vec{p}_j^{\,2} - \vec{p}^{\,2} + M^2(\alpha_j) - M^2 = 2\vec{p}\cdot(\vec{p}_j - \vec{p}) + (\vec{p}_j - \vec{p})^2 + M^2(\alpha_j) - M^2 , \quad (26)$$

so that

$$E_j = E + \vec{v}\cdot(\vec{p}_j - \vec{p}) + (\vec{p}_j - \vec{p})^2/2E + [M^2(\alpha_j) - M^2]/2E - (E_j^2 - E^2)^2/8E^3 + \cdots . \quad (27)$$

A necessary condition for the validity of the substitution leading to the operator droplet model is therefore that a $\vec{v}$ and an M can be found such that

$$\{(\vec{p}_j - \vec{p})^2/2E + [M^2(\alpha_j) - M^2]/2E - (E_j^2 - E^2)^2/8E^3 + \cdots\}\Delta t \ll 1 \quad (28)$$

for all $\vec{p}_j$ and $M(\alpha_j)$ contributing significantly to the integrals, where Δt is the range of t over which there are significant contributions to the integrals over time.

To study this problem in more detail it is simplest to suppose that the substitution *is* valid for one of the systems, say s', and then investigate the conditions under which it will also be valid for the other system, s. The system s can be considered to move in the operator field

$$\Phi(x) = g' \int d^4x' D(x - x')\rho'(x') ; \quad (29)$$

when the replacement is valid for the system s' this simplifies to

$$\Phi(x) \approx g' \int dt'\, d^3r' D(t - t'; \vec{r} - \vec{r}')\rho'(0, \vec{r}' - \vec{v}'t') = g' \int dt'\, d^3r' D(t - t'; \vec{r} - \vec{r}' - \vec{v}'t')\rho'(0, \vec{r}') = \gamma \int d^3r' V(|\vec{r} - \vec{r}'|_{v'})\rho'(0, \vec{r}') , \quad (30)$$

where

$$V(r) = g' \int dt D(t, \vec{r}) \tag{31}$$

is the static spherically symmetric external field ϕ produced by a stationary point source of strength g' at the origin. The argument of V in the integral is simply the Lorentz transform of $|\vec{r} - \vec{r}'|$ from the average rest frame of the system s':

$$|\vec{r} - \vec{r}'|_{\vec{v}'} = [(\vec{r}_\perp - \vec{r}'_\perp)^2 + \gamma'^2(z - z' - v' t')^2]^{1/2} . \tag{32}$$

Since ρ' is assumed to transform as a scalar, and because all states of the system s' are assumed to have approximately the same energy M' in the average rest system, when the s' states are transformed to the average rest system we can make the replacement

$$\rho(0, \vec{r}') \to \rho(0, \vec{r}'_L),$$

where $z'_L = \gamma z'$ and $\vec{r}'_L = \vec{r}'$. It is thus clear that $\Phi(x)$ is just the potential operator produced by a system with internal degrees of freedom, the center of mass of which is moving with the constant velocity $\vec{v}'$. Furthermore, this approximate potential operator is stationary in the frame where the composite system is at rest because we have ignored the energy differences among the different internal states. [If s' is a nucleus, and its meson degrees of freedom are ignored, then in its average rest frame $\Phi(x)$ is simply $\sum_{n'} V(|\vec{r} - \vec{r}_{n'}|)$ – the motions of its constituent nucleons during the interaction are ignored.]

In the average rest frame of the system s', then, the system s will be moving with velocity $v_L = (v + v')/(1 + vv')$ through a static potential of range R which will be, roughly, the sum of the range of the potential V and the radius of the internal states of s'. During the interaction the momentum of the system s will change by amounts of the order of $1/R$. The duration of the interaction in this frame is about R/v_L, and thus condition (28) will be satisfied if

$$(1/R)^2 (2E_L)^{-1} (R/v_L) = (2p_L R)^{-1} \ll 1$$

and

$$[M^2(\alpha_j) - M^2](2E_L)^{-1}(R/v_L) = \{[M^2(\alpha_j) - M^2]/2M\}(R/\gamma_L v_L) \ll 1.$$

[The third term in (28) will generally automatically be small if the first term is small.]

The first condition is familiar from the nonrelativistic eikonal approximation, while the second condition requires that the excitation energies of the various states be small enough that the internal motions can be ignored during the interaction.[11] It is difficult to proceed further with this condition on the masses without a detailed model for the internal states of the two systems which would give their masses and form factors. Experimental information can be utilized, however, to estimate the probability of exciting different internal states in different types of collisions. For example, if the incident particle is a pion, both M^2 and ΔM^2 are perhaps best taken of the order of 1 GeV^2, and the operator droplet model would therefore probably not be valid unless the pion's laboratory energy were at least several GeV.

IV. DISCUSSION

It has been shown above that the conditions for the validity of the operator droplet model at finite momenta are closely related to those for the validity of Glauber theory in nonrelativistic scattering: First, the momentum of each system in the frame where the other is approximately at rest should be much larger than the inverse range of the force field produced by the other system, which will depend on both its distribution of matter and the range of the elementary interaction. The second condition reflects the fact that the effective spread in the mass squared of a system determines the rate at which its constituents move relative to the center of mass. This rate is required to be small enough that during the collision the constituents of each system can be taken as static in the average rest frames.

Without a detailed model for the internal states of the colliding systems these conditions are necessarily somewhat qualitative. It should be possible, however, to use a mixture of experimental information and experience gained from crude models[12] to make reasonable estimates of the conditions under which the operator droplet model might be valid. It should of course be remembered that all of the analysis above is based on a restricted set of graphs and an overly simple underlying field theory; it is not yet clear whether the model can be extended to more realistic situations.

One of the original motivations for this work was to see whether Glauber theory could be generalized so as to include in a consistent way the meson degrees of freedom of nuclei. It appears that this might well be possible using a version of the operator droplet model; the real difficulty will be the long-standing problem of developing a useful theory of the individual nuclear systems which includes their meson degrees of freedom.

Another possible application of the operator

droplet model is to the scattering of systems which may contain substantial amounts of several different types of internal states. For example, the ρ meson for some purposes can be thought of as a quark-antiquark pair, whereas it nearly always decays into two pions (which might be thought of as two quark pairs); it may also have some admixture of nucleon-antinucleon states. All of these different pieces of the ρ's internal states could presumably be accommodated in the operator droplet model. It might then be possible to analyze experiments where ρ's are produced on nuclei to decide on the relative importance of the various pieces.[13]

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[9]It would probably be more reasonable to consider vector exchanges, especially for the infinite-momentum limit, but the scalar case is notationally simpler and differs only by trivial energy-dependent factors in the approximation used.

[10]If ρ and ρ' do not commute with themselves at equal times, the time ordering must be retained: The resulting expression may still be useful, however, if the commutation relations are simple enough.

[11]A clear discussion of this point for nonrelativistic scattering can be found in Ref. 5, pp. 364–368.

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