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Proof of the Pomeranchuk Theorem and Related Theorems Deduced from Unitarity

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A new general proof of the Pomeranchuk theorem in the form of the average behavior of total cross sections is given. It contains the features of the Martin and Meiman proof and is more precise. By considering the integral of the forward amplitude, this proof can be extended to cover slowly increasing and decreasing total cross sections. The unitarity restriction on the relative behavior of particle-antiparticle total cross sections is studied in general, again without making the smoothness assumption. In particular, using unitarity, it is shown that a modified form of the Pomeranchuk theorem is valid when the total cross sections are asymptotically unbounded.

I. INTRODUCTION

Since the publication of the original paper by Pomeranchuk[1] on the asymptotic equality of particle and antiparticle total cross sections $\sigma, \bar{\sigma}$, there have been many attempts to give a general proof of this theorem with as few assumptions as possible.[2]

To the best of our knowledge, the two most satisfactory proofs are given by Meiman[3] and Martin.[4] Meiman shows that if the forward scattering amplitude f is bounded by E, where E is the laboratory energy of the incident particle, then the set of the limiting values of $\sigma - \bar{\sigma}$ contains zeros, and if $\sigma - \bar{\sigma}$ has a limit, this limit is zero. In the proof given by Martin, the condition $|f|/E \leq C$ assumed by Meiman is replaced by a weaker assumption $|f|/E \ln E \to 0$. Martin shows if the limit $\sigma - \bar{\sigma}$ exists as $E \to \infty$, independently of the oscillations of σ and $\bar{\sigma}$, this limit is zero. Although it is not clear that the Martin method can handle the case where there are severe oscillations (where the Meiman proof is valid), it is quite suitable for the experimental situation where $\Delta\sigma = \bar{\sigma} - \sigma$ does not change sign but contains experimental errors. No assumption on the smoothness of $\Delta\sigma$ within the experimental error is needed. As we shall show below, the proof by Martin can be extended not only to cover the situation when $\Delta\sigma$ has severe oscillations, as in the Meiman proof, but also to give more quantitative results on the distribution of the zeros. The mathematics involved is elementary and transparent.

As was discussed elsewhere, the unsatisfactory aspect of the Pomeranchuk theorem is that we have to assume that $\sigma, \bar{\sigma}$ neither vanish nor become unbounded as $E \to \infty$. From the basic principle of axiomatic field theory we only know that $\sigma, \bar{\sigma}$ are bounded above by the Froissart-Martin[5] bound, $\sigma \leq C \ln^2 E$, while their lower bound under the best condition is $\sigma \geq (E \ln E)^{-2}$ which is far from what is needed to prove the Pomeranchuk theorem. Although present experimental evidences are consistent with the assumption that $\sigma, \bar{\sigma}$ tend to constants (within experimental errors), they cannot exclude the possibility that $\sigma, \bar{\sigma}$ vanish or become unbounded at much higher energy. At this stage of the development in high-energy physics, there is no convincing way to extrapolate the behavior of the total cross section at a lower energy to a much higher energy. The present investigation has its origin in our attempts to give a general proof of the Pomeranchuk theorem for $\Delta\sigma/\sigma$ which covers the cases of slowly decreasing and slowly increasing total cross section without using the smoothness assumption and to study the restriction given by unitarity on the energy dependence of σ and $\bar{\sigma}$.[6]

It will become clear that the proof by Martin cannot be modified for our purpose, without making strong assumptions on the analytic structure of σ and $\bar{\sigma}$ so that the physical assumption or the unitarity condition can be continued to the unphysical region where the oscillation no longer becomes a problem. We are thus led to examine the dispersion integral directly in the physical region. Because we have to deal with the principal-part integration, severe oscillations are produced if no smoothness assumption is made on the behavior of $\sigma, \bar{\sigma}$. To avoid this oscillation problem, we have to consider an averaged behavior of $f(E)$. The basic idea is contained in the bounded-mean-oscillations theorem.[7] Such an averaged behavior was used in the past by Jin and Martin[8] in connection with the property of the Herglotz function to obtain the lower bound for the even-crossing amplitude. It was also used by Khuri and Kinoshita[9] to construct univalent functions for the even-crossing amplitudes from which many interesting results can be obtained. Although the averaged function which we shall use is closely related to those discussed by Khuri and Kinoshita, it is unclear whether their sophisticated method can be adapted to demonstrate the results discussed in the following sections; however, this is unimportant. It will become clear later that we lose no information by performing this averaging; in fact, the strongest statement one can make for $\Delta\sigma$ and $\Delta\sigma/\sigma$ is their average behavior.

In Sec. II, we give an extension of the proof of Martin to give a more precise condition on the averaged behavior of $\Delta\sigma$, in particular the density of the zero when there are severe oscillations. In Sec. III, we show why the proof neither can be extended to other cases, nor a good restriction imposed by unitarity can be obtained. Section IV contains a new proof of the Pomeranchuk theorem (with the usual physical assumption) when $\sigma, \bar{\sigma}$ are bounded above and below by constants. In Sec. V, the Pomeranchuk theorem for $\Delta\sigma/\sigma$ is proved for decreasing total cross sections by using the results of Sec. IV. Section V deals with the general relations between $\sigma, \bar{\sigma}$ as deduced from the unitarity restrictions. In particular, the Pomeranchuk theorem for $\Delta\sigma/\sigma$ is proved when $\sigma, \bar{\sigma}$ are unbounded without any physical assumption.

II. AN EXTENSION OF THE MARTIN PROOF OF THE POMERANCHUK THEOREM

In this section we extend Martin's proof of the Pomeranchuk theorem to give a quantitative condition on the averaged behavior of $\Delta\sigma$. We believe that the result obtained here gives the strongest restriction on the averaged behavior of $\Delta\sigma$, so that the newer proof discussed in Sec. IV must be tested against the results obtained here.

Martin's proof consists in using the crossing symmetry and the analytic properties of the forward scattering amplitude which are deduced from the axiomatic field theory. The crucial point in the paper of Martin is to realize that the usual assumption,

$$\lim_{E\to\infty} \frac{\mathrm{Re} f(E)}{\mathrm{Im} f \ln E} = 0, \tag{1}$$

can be replaced by a slightly different assumption,

$$\lim_{E\to\infty} \frac{|f(E)|}{E \ln E} = 0, \tag{2}$$

for a situation where $\sigma, \bar{\sigma}$ have lower bounds and upper bounds as constants. This enables him to use the condition (2), in the nonphysical region by the Phragmén-Lindelöf theorem. Thus the problem of oscillation can be minimized. Had the form (1) been used, one would have to assume that the ratio $\mathrm{Re} f/\mathrm{Im} f$ be analytic in the upper-half plane before the Martin method could be used. But this would be indeed a very strong assumption.

In the following, for simplicity, we shall assume that dispersion relations exist. A finite region of nonanalyticity can be handled in a straightforward manner as discussed by Martin. Let us denote by $f, \bar{f}$ the forward amplitudes for a particle and its antiparticle on a target. For particles with spin, these must be understood as the spin-averaged amplitudes. f and $\bar{f}$ are polynomially bounded in the complex E plane.[10] On the real axis the Froissart-Martin bound applies; hence, at most, two subtractions are needed for the dispersion relation for $f_A(E) \equiv f(E) - \bar{f}(E)$:

$$f_A(E) = aE + \frac{2E^3}{\pi} \int_{E_0}^{\infty} \frac{\mathrm{Im} f_A(E')dE'}{E'^2(E'^2 - E^2)}, \tag{3}$$

with $\mathrm{Im} f_A(E) = (q/4\pi)\Delta\sigma(E)$, where q is the laboratory momentum of the incident particle. The crossing relations and the assumption (2) for both f and $\bar{f}$, combined with the use of the Phragmén-Lindelöf theorem, enable us to deduce that

$$\lim_{E\to\infty} \frac{|f(E)|}{E \ln E} = 0$$

holds for any complex direction in the upper-half E plane. In particular, setting $E = i|E|$, we must have

$$\lim_{E\to\infty} \frac{f(iE)}{E \ln E} = 0. \tag{4}$$

Using Eq. (3), we have

$$\frac{f_A(iE)}{iE} = -\frac{E^2}{\pi} \int_{E_0}^{\infty} \frac{\mathrm{Im} f_A(E')dE'}{E'^2(E'^2 + E^2)} + a. \tag{5}$$

This is the main result of Martin. However, by a simple manipulation of the right-hand side of Eq. (5), we shall show below that it is possible to get a strong condition on $\Delta\sigma$, when condition (4) is used. For this purpose, let us separate the right-hand side of (5) into two integrals:

$$-\frac{f_A(iE)}{2iE} = \frac{E^2}{\pi} \int_{E_0}^{E_1} \frac{\mathrm{Im} f_A(E')dE'}{E'^2(E'^2 + E^2)} + \frac{E^2}{4\pi^2} \int_{E_1}^{\infty} \frac{\Delta\sigma(E')dE'}{E'(E'^2 + E^2)} + \tfrac{1}{2}a, \tag{6}$$

where E_1 is sufficiently large and the optical theorem is used in the second integral. Taking the limit $E \to \infty$ on the right-hand side of Eq. (6), the first integral is $O(1)$. The second integral can be rewritten as

$$\frac{E^2}{4\pi^2}\int_{E_1}^{E}\frac{\Delta\sigma(E')dE'}{E'(E'^2+E^2)}+\frac{E^2}{4\pi^2}\int_{E}^{\infty}\frac{\Delta\sigma(E')dE'}{E'(E'^2+E^2)} = I_1(E)+I_2(E). \tag{7}$$

We can rewrite $I_2(E)$ in the following form:

$$|I_2(E)| \le \frac{E^2}{4\pi^2}\int_E^{\infty}\frac{|\Delta\sigma(E')|}{E'^3}\left(\frac{E'^2}{E'^2+E^2}\right)dE' \le \frac{E^2}{4\pi^2}\int_E^{\infty}\frac{|\Delta\sigma(E')|}{E'^3}\,dE'.$$

Hence it is $O(1)$ by assumption on σ and $\bar{\sigma}$. The first integral on the left-hand side of (7), $I_1(E)$, can be written as

$$I_1(E)=\frac{1}{4\pi^2}\int_{E_1}^{E}\frac{\Delta\sigma(E')}{E'}\,dE' - \frac{1}{4\pi^2}\int_{E_1}^{E}\frac{\Delta\sigma(E')E'dE'}{E'^2+E^2}. \tag{8}$$

The last term on the right-hand side of Eq. (8) is $O(1)$ since as $E\to\infty$

$$\int_{E_1}^{E}\frac{\Delta\sigma(E')E'dE'}{E'^2+E^2} \le \frac{1}{E^2}\int_{E_1}^{E}|\Delta\sigma(E')|E'dE'.$$

Combining (6), (7), and (8) we have as $E\to\infty$

$$-\frac{f_A(iE)}{2iE}=\frac{1}{4\pi^2}\int_{E_1}^{E}\frac{\Delta\sigma(E')}{E'}\,dE' + \text{terms } O(1). \tag{9}$$

Using (4), (5), and (9) we have the final result:

$$\lim_{E\to\infty}\frac{1}{\ln E}\int_{E_1}^{E}\frac{\Delta\sigma(E')}{E'}\,dE'=0. \tag{10a}$$

By a change of variables $x'=\ln E'$, $x=\ln E$, Eq. (10a) can be rewritten as

$$\lim_{x\to\infty}\frac{1}{x}\int_{x_1}^{x}\Delta\sigma(x')dx' \equiv \langle\Delta\sigma\rangle = 0 \tag{10b}$$

and is simply a condition on the averaged behavior of $\Delta\sigma$.

Equations (10) gives a strong condition on the average behavior of $\Delta\sigma$:

(i) If $\Delta\sigma$ has a limit as $E\to\infty$, this limit is zero.

(ii) If it does not have a limit and if it does not change sign as $E\to\infty$, $\Delta\sigma$ must have an infinite number of zeros; the density of the zeros is such that Eqs. (10) are satisfied.

(iii) If $\Delta\sigma$ changes sign and does not have zero as a limit, it must oscillate indefinitely around the point zero; otherwise we can choose an energy E, sufficiently large, above which $\Delta\sigma$ does not change sign, and the above argument can be used. Even when it oscillates indefinitely around zero, Eqs. (10) require that its "average" value must be zero. This last statement cannot be made in the Meiman proof.

Although the proof given here is not as elegant as Meiman's proof, our result is more powerful since it deals with an averaged behavior. This suggests that it is more advantageous to work with an averaged forward amplitude as we shall do later.

We remark in passing that the Martin proof of the Pomeranchuk theorem for the differential cross section can also be strengthened by the same method as discussed in this section.

III. DIFFICULTY IN THE GENERALIZATION OF THE PROOF

In this section we point out the difficulty in using the Martin proof as extended in the last section to a more general situation. There are two reasons why we want to do this. First, we want to prove the Pomeranchuk theorem in the form $\Delta\sigma/\sigma$ when the cross sections tend to zero slowly like a logarithm, or when they become unbounded as $E\to\infty$. Second, we want to study what are the restrictions imposed by the unitarity on the relative behavior of σ and $\bar{\sigma}$. In particular, when the cross sections become unbounded as $E\to\infty$, we want to give a general proof for the Pomeranchuk theorem in the form $\Delta\sigma/\sigma\to 0$ using only unitarity. For a "Regge" behavior of σ and $\bar{\sigma}$ this last point was shown to be valid by Eden and Kinoshita.[11]

Let us examine the first point. The basic physical assumption is given by Eq. (1) but not Eq. (2). It is only when the total cross sections behave like a constant that they are equivalent. For a situation where $\sigma, \bar{\sigma}$ decrease or increase slowly, we cannot replace (1) by (2), hence we have to deal directly with assumption (1). It is possible to modify assumption (1) by requiring

$$\lim_{E\to\infty}\frac{|f|}{\operatorname{Im}f\,\ln E}=0.$$

To use the Martin method to continue this physical assumption in the complex E plane, we must assume $1/\operatorname{Im}f$ is analytic in the upper-half plane and

$$\lim_{E\to\infty}\left|\frac{\operatorname{Im}f(E+i\epsilon)}{\operatorname{Im}f(-E+i\epsilon)}\right|=1. \tag{11}$$

One can relax (11) by requiring only that it is bounded below and above by constants which are nonzero. Under these conditions one can derive the Pomeranchuk theorem which has a form similar to Eqs. (10), but the assumptions made here are very strong.

The second point is that we want to examine the unitarity condition

$$|f|^2 \underset{E\to\infty}{\leqslant} \frac{1}{16\pi m^2} E^2 \ln^2 E \sigma_{\text{tot}}\,, \tag{12a}$$

$$|\bar{f}|^2 \underset{E\to\infty}{\leqslant} \frac{1}{16\pi m^2} E^2 \ln^2 E \bar{\sigma}_{\text{tot}}\,, \tag{12b}$$

which holds only in the physical region (m is the pion mass). To use Martin's method we must analytically continue this expression in the complex E plane. Again this cannot be done unless $1/\sigma$ and $1/\bar{\sigma}$ are analytic in the upper-half plane and condition (11) is valid.

Another drawback in the Martin method is that we cannot obtain the best restriction due to the unitarity relations (12), even in the special case where $\sigma, \bar{\sigma}$ are bounded above and below by constants. To see this, without loss of generality, let us set $\bar{\sigma} \geqslant \sigma$. Inequalities (12) can be rewritten as

$$\frac{|f(E)|}{E \ln E} \leqslant \frac{(\bar{\sigma}_{\max})^{1/2}}{4\sqrt{\pi}\, m}\,, \tag{13}$$

where $\bar{\sigma}_{\max} = \lim \sup \bar{\sigma}$. Because of the analytic property of the function $f/(E \ln E)$ and the Phragmén-Lindelöf theorem, this inequality should hold in any complex E direction as $|E| \to \infty$. From this we can deduce the upper bound of $\Delta\sigma$ in terms of $\bar{\sigma}_{\max}$. But inequality (13) cannot give a strong condition. It is sufficient to have narrow high peaks in $\bar{\sigma}$, which is unlikely but cannot be excluded experimentally because of the experimental resolution, to weaken the result derived for $\Delta\sigma$. This suggests that an average behavior of $\Delta\sigma$ and σ must be considered.

To sum up this section, let us state the main points. We find that the assumptions on the analytic behavior of $\sigma, \bar{\sigma}$ and (11) are too strong. Even if we accept these assumptions, it is not clear whether the conclusion on $\Delta\sigma$ or $\Delta\sigma/\sigma$ in the form of Eqs. (10) is consistent with the initial assumptions. This brings us to study the averaged behavior of f and $\bar{f}$, which enables us to deduce the Pomeranchuk theorem and related theorems without making the analytic continuation to the unphysical region.

IV. PROOF OF THE POMERANCHUK THEOREM FOR THE AVERAGED TOTAL CROSS SECTIONS

The following proof of the Pomeranchuk theorem is elementary and does not require any sophisticated mathematics. Similarly to Sec. II, we shall restrict ourselves to the situation where $\sigma, \bar{\sigma}$ are bounded from below and above by constants. Let us define the following averaged amplitude:

$$\operatorname{Re} g_A(E) = \frac{1}{\ln E} \int_1^E \frac{\operatorname{Re} f_A(E')}{E'^2} dE'\,, \tag{14a}$$

$$\operatorname{Im} g_A(E) = \frac{1}{\ln E} \int_1^E \frac{\operatorname{Im} f_A(E') dE'}{E'^2}\,, \tag{14b}$$

where we have taken the lower limit of the integration as unity for convenience and $f_A(E)$ is defined by Eq. (3). By interchanging the order of the integration in Eq. (14a), which is always possible, we have the following expression for $g_A(E)$:

$$\operatorname{Re} g_A(E) = \frac{1}{\pi \ln E} \int_{E_0}^{\infty} \frac{\operatorname{Im} f_A(E')}{E'^2} \ln \left| \frac{E'^2}{E'^2 - E^2} \right| dE'\,, \tag{15}$$

where for simplicity we set $\ln(E'^2 - 1) = \ln E'^2$ and we omit a constant term on the right-hand side of Eq. (15). It is clear now that Eq. (15) is much easier to work with than the usual principal part integration, since the singularity at $E' = E$ is only logarithmic. Instead of using the usual physical assumption in the form of Eq. (1), we shall make a weaker assumption,

$$\lim_{E\to\infty} \frac{\operatorname{Re} g(E)}{\operatorname{Im} g(E) \ln E} = 0\,, \tag{16}$$

and a similar expression for $\operatorname{Re} \bar{g}(E)$. For the behavior of $\sigma, \bar{\sigma}$ considered in this section, assumption (16) can be derived from (1) by using the Schwarz inequality.

Let us split the integral on the right-hand side of Eq. (15) into two parts:

$$\operatorname{Re} g_A(E) = \frac{1}{\pi \ln E} \int_{E_0}^{E_1} \frac{\operatorname{Im} f_A(E')}{E'^2} \ln \left| \frac{E'^2}{E'^2 - E^2} \right| dE' + \frac{1}{\pi \ln E} \int_{E_1}^{\infty} \frac{\operatorname{Im} f_A(E')}{E'^2} \ln \left| \frac{E'^2}{E'^2 - E^2} \right| dE'\,, \tag{17}$$

where E_1 is sufficiently large so that the left-hand side of (16) is less than ϵ. We can take the limit $E \to \infty$ inside the first integral. This yields a term of $O(1)$. The second term on the right-hand side of Eq. (17) can be expressed as follows:

$$\int_{E_1}^{\infty} \frac{\Delta\sigma(E')}{E'} \ln \left| \frac{E'^2}{E'^2 - E^2} \right| dE' = 2 \int_{E_1}^{E+a} \left(\ln \frac{E'}{E} \right) \frac{\Delta\sigma(E')}{E'} dE' + \int_{E+a}^{\infty} \frac{\Delta\sigma(E')}{E'} \ln \left| \frac{E'^2}{E'^2 - E^2} \right| dE'$$
$$+ \int_{E_1}^{E-a} \frac{\Delta\sigma(E')}{E'} \ln \left| \frac{E^2}{E'^2 - E^2} \right| dE' + \int_{E-a}^{E+a} \frac{\Delta\sigma(E')}{E'} \ln \left| \frac{E^2}{E'^2 - E^2} \right| dE'\,. \tag{18}$$

In the limit $E \to \infty$, as is shown in the Appendix, we have

$$\operatorname{Re} g_A(E) \underset{E\to\infty}{\sim} \frac{1}{2\pi^2 \ln E} \int_{E_1}^{E} \left(\ln \frac{E'}{E}\right) \frac{\Delta\sigma(E')}{E'} dE' + R(E) \tag{19}$$

with $|R| \leq M$, where M is defined by

$$M(E) \underset{E\to\infty}{\sim} \frac{1}{2\pi^2 E} \int_{E_1}^{E} |\Delta\sigma(E')| dE' + \frac{E^2}{2\pi^2} \int_{E}^{\infty} |\Delta\sigma(E')| \frac{dE'}{E'^3}. \tag{20}$$

For the case discussed in this section $|\Delta\sigma|$ on the average is bounded as $E\to\infty$, hence M defined by Eq. (20) is of $O(1)$. Combining Eq. (19) with the assumption (16), we finally arrive at the condition

$$\lim_{E\to\infty} \frac{1}{\ln^2 E} \int_{E_1}^{E} \left(\ln \frac{E'}{E}\right) \frac{\Delta\sigma(E')}{E'} dE' = 0. \tag{21}$$

This is our main result. We can draw the same conclusions as in Sec. II:

(i) If the limit $\bar\sigma - \sigma = \Delta\sigma$ exists and it is different from zero (let us call it C), then using (21) we have

$$\lim_{E\to\infty} \frac{C}{\ln^2 E}\left(\tfrac{1}{2}\ln^2 \frac{E}{E_1}\right) = 0$$

which implies $C = 0$.

(ii) Even if $\Delta\sigma$ does not have a limit, and if $\Delta\sigma$ does not change sign asymptotically, Eq. (21) requires that $\Delta\sigma$ must have an infinite number of zeros. The density of the zeros must satisfy (21).

(iii) Equation (21) is also valid when $\Delta\sigma$ changes sign as discussed in Sec. II.

The equivalence between Eq. (21) and Eq. (10) can be seen more clearly by rewriting it as

$$\frac{2\pi^2 \operatorname{Re} g_A(E)}{\ln E} \underset{E\to\infty}{\sim} \frac{1}{\ln^2 E} \int_{E_1}^{E} \frac{dE'}{E'} \int_{E_1}^{E'} \frac{\Delta\sigma(E'')}{E''} dE''. \tag{22a}$$

This can be shown straightforwardly by integrating by parts the right-hand side of (22a). By comparing it with Eq. (9), using the definition for $\operatorname{Re} g_A(E)$ as given by Eq. (14a), it is seen that (10) and (21) are equivalent. The final result for $\operatorname{Re} g_A(E)$ in the limit $E\to\infty$ (in the physical region) is the same as if we calculated first $f_A(iE)$ for E sufficiently large, then performed afterward the average integration.

The right-hand side of (22a) is simply the average value of $\Delta\sigma$ over the triangle ABC in the x' and x'' plane (Fig. 1), where we have put $x = \ln E$, $x' = \ln E'$, $x'' = \ln E''$. This is simply the "average of the average" value of $\Delta\sigma$. This result is totally expected since we deal with an integral of the scattering amplitude. Let us denote the right-hand side of (22a) by $\langle\langle\Delta\sigma\rangle\rangle$. We have

$$\langle\langle\Delta\sigma\rangle\rangle \underset{x\to\infty}{\sim} \frac{1}{x^2} \int_{x_1}^{x} dx' \int_{x_1}^{x'} \Delta\sigma(x'')dx''. \tag{22b}$$

Using assumption (16) we have finally

$$\lim_{E\to\infty} \frac{\langle\langle\Delta\sigma\rangle\rangle}{\langle\sigma\rangle} = 0, \tag{22c}$$

where

$$\langle\sigma\rangle = \frac{1}{x} \int_{x_1}^{x} \sigma(x')dx'. \tag{22d}$$

We have not achieved anything better here than those given in Sec. II. The advantage of the method presented here is that it can be extended to prove the Pomeranchuk theorem, when the total cross section becomes unbounded or decreases slowly to zero. It also enables us to study the restriction imposed by the unitarity.

V. POMERANCHUK THEOREM FOR SLOWLY DECREASING TOTAL CROSS SECTIONS

Let us remark first that with assumption (16), there is no difficulty in demonstrating the Pomeranchuk theorem for increasing total cross sections. This case will be proved later, using only the unitarity restrictions, inequalities (12). The proof for the cases of slowly decreasing total cross sections requires however some modification. By slowly decreasing total cross sections σ and $\bar\sigma$ we mean those which make the following integral diverge:

$$\lim_{x\to\infty} \int_{x_1}^{x} \sigma(x')x'^n dx' = \infty, \tag{23}$$

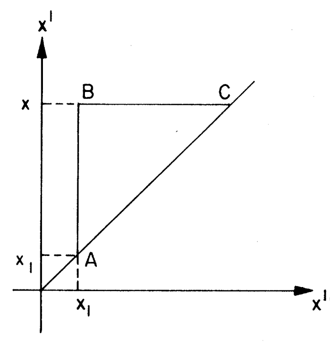

FIG. 1. The region in the $x'x''$ plane yielding $\langle\langle\Delta\sigma\rangle\rangle$, the right-hand side of Eq. (22a).

where $x = \ln E$, $x' = \ln E'$, and n is positive and sufficiently large but finite. We can of course use for the weight function any slowly varying function $\varphi(E)$ defined by Meiman[3] such that it is analytic in the upper-half E plane with the condition $\varphi(E+i0) = \varphi^*(-E+i0)$ for E sufficiently large. For simplicity we have chosen the weight function $(\ln E)^n$. This choice is quite reasonable since the Pomeranchuk theorem can only be proved for a class of slowly decreasing function of the logarithmic type. If $\sigma(E)$ decreases like $E^{-\epsilon}$ where $\epsilon > 0$ but small, the Pomeranchuk theorem in the form of $\Delta\sigma/\sigma$ cannot be demonstrated.

We can follow exactly the same procedure as in Sec. IV provided that $\mathrm{Re}\, g_A(E) \to \infty$ in the limit $E \to \infty$. For this purpose, instead of writing a dispersion relation for f_A as given by Eq. (3), we can write a dispersion relation for $\tilde{f}_A = f_A(E)(\ln E)^m$, where m is positive and sufficiently large such that condition (23) is satisfied. We can now define

$$\mathrm{Re}\, \tilde{g}_A(E) = \frac{1}{\ln E} \int_1^E \frac{\mathrm{Re}\, \tilde{f}_A(E')}{E'^2} dE' .$$

From this we obtain an equation similar to Eq. (19),

$$\mathrm{Re}\, \tilde{g}_A(E) \underset{E\to\infty}{\sim} \frac{1}{2\pi^2 \ln E} \int_{E_1}^{E} \left(\ln \frac{E'}{E} \right) (\ln E')^m \times \frac{\Delta\sigma(E')}{E'} dE' , \qquad (24)$$

provided the right-hand side of (24) dominates over the corresponding integrals defined by Eq. (20). Any $\Delta\sigma(E)$ such that $(\ln E)^m \Delta\sigma(E)$ is bounded below and above by constants will satisfy this requirement. Combining Eq. (24) with the assumption (16), putting $x' = \ln E'$, $x = \ln E$, we have

$$\lim_{x\to\infty} \frac{\int_{x_1}^{x} (x'-x)\, x'^m \Delta\sigma(x') dx'}{\int_{x_1}^{x} x'^m [\sigma(x') + \bar{\sigma}(x')] dx'} = 0 . \qquad (25a)$$

This is our final result on the relation between the relative asymptotic behavior of $\Delta\sigma$ and $\sigma + \bar{\sigma}$. Similar to Eqs. (22a) and (22b), we can rewrite (25a) as

$$\lim_{E\to\infty} \frac{\langle\langle \Delta\sigma_w \rangle\rangle}{\langle \sigma_w + \bar{\sigma}_w \rangle} = 0 , \qquad (25b)$$

where $\Delta\sigma_w$, σ_w, and $\bar{\sigma}_w$ are obtained by multiplying the corresponding values of $\Delta\sigma$, σ, and $\bar{\sigma}$ by the weight function $(\ln E)^m$.

The physical interpretation of this equation will become clear for a class of $\Delta\sigma(x)$ such that

$$\left| \int_{x_1}^{x} (x'-x) x'^m \Delta\sigma(x') dx' \right| \underset{x\to\infty}{\geq} \eta x \left| \int_{x_1}^{x} x'^m \Delta\sigma(x') dx' \right| \qquad (26a)$$

where η is a small, positive but finite constant. Putting this in (25), we have

$$\lim_{x\to\infty} \frac{\int_{x_1}^{x} x'^m \Delta\sigma(x') dx'}{\int_{x_1}^{x} x'^m [\sigma(x') + \bar{\sigma}(x')] dx'} = 0 . \qquad (26b)$$

Hence the average value of $\Delta\sigma/\sigma + \bar{\sigma}$ obeys the Pomeranchuk theorem:

$$\lim_{x\to\infty} \frac{\langle \Delta\sigma_w \rangle}{\langle \sigma_w + \bar{\sigma}_w \rangle} = 0 . \qquad (26c)$$

VI. ASYMPTOTIC THEOREMS DERIVED FROM UNITARITY

Using the Schwarz inequality for $g(E)$ and $\bar{g}(E)$, we have

$$|g(E)| \underset{E\to\infty}{\leq} \frac{\sqrt{2}\ln E}{4\sqrt{3\pi}\, m} \left(\frac{1}{\ln E} \int_{E_1}^{E} \frac{\sigma(E')}{E'} dE' \right)^{1/2} , \qquad (27a)$$

$$|\bar{g}(E)| \underset{E\to\infty}{\leq} \frac{\sqrt{2}\ln E}{4\sqrt{3\pi}\, m} \left(\frac{1}{\ln E} \int_{E_1}^{E} \frac{\bar{\sigma}(E')}{E'} dE' \right)^{1/2} , \qquad (27b)$$

where $g(E)$, $\bar{g}(E)$ are similarly defined as Eqs. (14). It follows that $|g_A(E)|$ is less than the sum of the right-hand sides of (27a) and (27b). This section deals with the consequences imposed by the unitarity condition. We have to examine separately three situations:

(i) Total cross sections become unbounded as $E \to \infty$.

(ii) Total cross sections become bounded but do not tend to zero as $E \to \infty$.

(iii) Total cross sections tend to zero as $E \to \infty$.

Case (i) is most interesting since the Pomeranchuk theorem can be derived. We shall proceed with this case first.

A. Unbounded Total Cross Sections

Equation (19) derived in Sec. IV is valid for this case without any modification. We must however make the following assumptions as $E \to \infty$:

$$\frac{[M(E)]^{-1}}{2\pi^2 \ln E} \int_{E_1}^{E} \left(\ln \frac{E'}{E} \right) \frac{\Delta\sigma(E')}{E'} dE' > 1 , \qquad (28)$$

where M is defined by Eq. (20). As was shown in Sec. IV, when $\Delta\sigma$ is bounded from below and above by constants, (28) can be proved. This is also true for any $\Delta\sigma(E)$ such that $(\ln E)^{-n} \Delta\sigma(E)$ are bounded from below and above by constants. We are unable to prove inequality (28) in general. At this stage we can regard this inequality as an assumption on the asymptotic behavior of $\Delta\sigma$ which is expressed essentially in terms of its moments. Using (28) with Eq. (19) and the unitarity condition (27), we have the following expression:

$$\left| \frac{1}{\ln^2 E} \int_{E_1}^{E} \frac{\Delta\sigma(E')}{E'} \ln \frac{E'}{E} dE' \right| \underset{E\to\infty}{\leq} \frac{\pi^{3/2}}{\sqrt{2}\sqrt{3}\,m} [(\langle\sigma\rangle)^{1/2} + (\langle\bar\sigma\rangle)^{1/2}], \tag{29}$$

where $\langle\sigma\rangle$ and $\langle\bar\sigma\rangle$ are defined by Eq. (22c).

The left-hand side of (29) can again be expressed in the same form as the right-hand side of Eq. (22a). It follows that (29) can be written as

$$|\langle\langle\Delta\sigma\rangle\rangle| \leq \frac{\pi^{3/2}}{\sqrt{2}\sqrt{3}\,m} [(\langle\sigma\rangle)^{1/2} + (\langle\bar\sigma\rangle)^{1/2}]. \tag{30a}$$

Dividing both sides of Eqs. (30) by $\langle\sigma\rangle + \langle\bar\sigma\rangle$, we get

$$\left| \frac{\langle\langle\Delta\sigma\rangle\rangle}{\langle\sigma\rangle + \langle\bar\sigma\rangle} \right| \leq \frac{\pi^{3/2}}{\sqrt{2}\sqrt{3}\,m} \frac{(\langle\sigma\rangle)^{1/2} + (\langle\bar\sigma\rangle)^{1/2}}{\langle\sigma\rangle + \langle\bar\sigma\rangle},$$

and hence

$$\left| \frac{\langle\langle\Delta\sigma\rangle\rangle}{\langle\sigma\rangle + \langle\bar\sigma\rangle} \right| \leq \frac{\pi^{3/2}}{\sqrt{2}\sqrt{3}\,m} \left(\frac{1}{(\langle\sigma\rangle)^{1/2}} + \frac{1}{(\langle\bar\sigma\rangle)^{1/2}} \right) \tag{30b}$$

if both $\langle\sigma\rangle$ and $\langle\bar\sigma\rangle$ are unbounded as $E\to\infty$. [If only one total cross section is bounded, say σ, then on the right-hand side of (30b) only the term $1/\sqrt{\langle\bar\sigma\rangle}$ survives.] We thus finally arrive at the following result:

$$\lim_{E\to\infty} \frac{|\langle\langle\Delta\sigma\rangle\rangle|}{\langle\sigma\rangle + \langle\bar\sigma\rangle} = 0. \tag{31}$$

This result is independent of any physical assumption provided that the high-energy behavior of $\Delta\sigma$ is such that inequality (28) is satisfied.

Similarly to the previous section, we can reexpress the "double average" $\langle\langle\Delta\sigma\rangle\rangle$ in terms of the single average $\langle\Delta\sigma\rangle$ if we restrict ourselves to a less general class of behavior of $\Delta\sigma$ such that

$$\left| \int_{x_1}^{x} (x' - x)\Delta\sigma(x')dx' \right| \geq \eta x \left| \int_{x_1}^{x} \Delta\sigma(x')dx' \right|, \tag{32a}$$

where $x = \ln E$, and $x' = \ln E'$, and η is a small, positive constant. Using this with (29) we have

$$|\langle\Delta\sigma\rangle| \leq \frac{\pi^{3/2}}{\sqrt{2}\sqrt{3}\,\eta m} [(\langle\sigma\rangle)^{1/2} + (\langle\bar\sigma\rangle)^{1/2}] \tag{32b}$$

or

$$\frac{|\langle\Delta\sigma\rangle|}{\langle\sigma\rangle + \langle\bar\sigma\rangle} \underset{E\to\infty}{\leq} \frac{\pi^{3/2}}{\sqrt{2}\sqrt{3}\,\eta m} \left(\frac{1}{\sqrt{\sigma}} + \frac{1}{\sqrt{\bar\sigma}} \right),$$

and hence

$$\lim_{E\to\infty} \frac{\langle\Delta\sigma\rangle}{\langle\sigma\rangle + \langle\bar\sigma\rangle} = 0. \tag{32c}$$

Equations (30b) and (31) exclude also the possibility that the particle total cross section is bounded, while its antiparticle total cross section becomes unbounded.

B. Bounded Total Cross Sections

In this section we deal only with the situation when one total cross section remains finite at infinite energy, while the other can tend to zero or become finite. It is well known that even if $\sigma, \bar\sigma$ have a limit, the Pomeranchuk theorem cannot be demonstrated by using only unitarity. However a restriction can still be made on the magnitude of the $\Delta\sigma$. We generalize this result here without making any assumption on the existence of their limits. *We shall show in particular that it is not possible for σ to tend to zero while $\bar\sigma$ remains finite.*

Although the result of the previous section remains unchanged, i.e., Eqs. (29) and (30) are still valid, the bounds obtained there are not strict enough for our purpose. This comes from the fact that we worked with the bound for $g_A(E)$. It is better to work with those given by Eqs. (27).

For this purpose, let us denote

$$g_S(E) = g(E) + \bar g(E).$$

For the behavior of the total cross sections considered in this section, using the same method as in Sec. IV, it is straightforward to show that $\mathrm{Re}\, g_S(E)$ is $O(1)$. If the Pomeranchuk theorem in the form of Eq. (21) and Eq. (22) is not satisfied, we then have that $g_S(E)$ is negligible compared with $g_A(E)$ in the limit $E\to\infty$,

$$\lim_{E\to\infty} \frac{g_S(E)}{g_A(E)} = 0.$$

Thus, we simply have to replace the left-hand side of Eqs. (27) by $\frac{1}{2}[g_A(E)]$. For definiteness, let us suppose that $\langle\sigma\rangle$ is less than $\langle\bar\sigma\rangle$. Instead of (30a) we now have a better bound:

$$|\langle\langle\Delta\sigma\rangle\rangle| \underset{E\to\infty}{\leq} \frac{\sqrt{2}\,\pi^{3/2}}{\sqrt{3}\,m} (\langle\sigma\rangle)^{1/2}. \tag{33}$$

This inequality asserts that if $\langle\sigma\rangle$ tends to zero, $\langle\langle\Delta\sigma\rangle\rangle$ also tends to zero; hence it is not possible for $\langle\bar\sigma\rangle$ to remain finite. In particular, if σ and $\Delta\sigma$ has a limit, then

$$|\Delta\sigma| \leq \left(\frac{2\sqrt{2}}{\sqrt{3}} \right) \frac{\pi^{3/2}}{m} \sqrt{\sigma}. \tag{34}$$

The right-hand side of (34) is larger than that obtained directly from the ordinary dispersion relation by a factor of $2\sqrt{2}/\sqrt{3}$ because we have used the Schwarz inequality in obtaining inequalities (27).

C. Decreasing Total Cross Sections

We study first the case of slowly decreasing total cross sections. This case was studied in Sec. V with the physical assumption (16) to derive the Pomeranchuk theorem. We study here the restriction due to unitarity. Proceeding as in Secs. VI A and VI B, we have

$$\langle\langle\Delta\sigma_w\rangle\rangle \leqslant \frac{\sqrt{2}\,\pi^{3/2}}{m\sqrt{n+3}}(\ln E)^{n/2}(\langle\sigma_w\rangle)^{1/2}, \qquad (35)$$

where

$$\langle\langle\Delta\sigma_w\rangle\rangle = \frac{1}{\ln^2 E}\int_{E_1}^{E}\frac{\Delta\sigma(E')}{E'}(\ln E')^n \ln\frac{E'}{E}\,dE'$$

and

$$\langle\sigma_w\rangle = \frac{1}{\ln E}\int_{E_1}^{E}\frac{\sigma(E')}{E'}(\ln E')^n\,dE'$$

and $n>0$. In deriving inequality (35) we have assumed that $\langle\sigma_w\rangle$ is smaller than or equal to $\langle\bar{\sigma}_w\rangle$. It is clear that we cannot get a strong condition because of the factor $(\ln E)^{n/2}$ on the right-hand side of (35).

It may be interesting to note however that (35) requires that on the average the asymptotic behavior of σ and $\bar{\sigma}$ cannot be too much different. For example, if $\langle\bar{\sigma}\rangle$ decreases as a logarithm, $\langle\sigma\rangle$ cannot decrease like a power of E. In the special case when $\sigma \to (\ln E)^{-\alpha}$ and $\bar{\sigma}\to(\ln E)^{-\bar{\alpha}}$ with $0<\bar{\alpha}<\alpha$, then inequality (35) acquires that $\alpha \leqslant 2\bar{\alpha}$, a result obtained previously.[6]

When σ and $\bar{\sigma}$ decreases sufficiently fast, say faster than any power of $\ln E$, it is more convenient to write directly a dispersion relation for f and $\bar{f}$ instead of $f_A(E)$ of $f_S(E)$ in order to study the restriction imposed by the unitarity,

$$f(E) = a + \frac{E}{\pi}\int_{E_0}^{\infty}\frac{\mathrm{Im} f(E')dE'}{E'(E'-E)} - \frac{E}{\pi}\int_{E_0}^{\infty}\frac{\mathrm{Im}\bar{f}(E')dE'}{E'(E'+E)}, \qquad (36a)$$

where the number of subtractions is determined by the unitarity condition (12). Let us define

$$\begin{aligned}\mathrm{Re}\, g(E) &= \int_1^E \frac{\mathrm{Re} f(E')-a}{E'}\,dE' \\ &= \int_{E_0}^{\infty}\frac{\mathrm{Im} f(E')}{E'}\ln\left|\frac{E'}{E'-E}\right| dE' \\ &\quad + \int_{E_0}^{\infty}\frac{\mathrm{Im}\bar{f}(E')}{E'}\ln\left|\frac{E'}{E'+E}\right| dE' \\ &= I_1(E)+I_2(E). \end{aligned} \qquad (36b)$$

For E sufficiently large we have

$$I_1(E) \leqslant \frac{1}{4\pi^2}\int_{E_1}^{E}\sigma(E')\ln\frac{E'}{E}\,dE' + M', \qquad (37a)$$

$$M' = \frac{\ln E}{4\pi^2 E}\int_{E_1}^{E}\sigma(E')E'dE' + \frac{E\ln E}{4\pi^2}\int_E^{\infty}\frac{\sigma(E')}{E'}\,dE' + O\left(\frac{\sigma(E)}{E}\ln E\right). \qquad (37b)$$

On the other hand, because of the positivity of the total cross section $\bar{\sigma}$, we have

$$|I_2(E)| > \ln 2\int_{E_1}^{E}\bar{\sigma}(E')dE'.$$

If we now require that

$$\lim_{E\to\infty}\frac{\int_{E_1}^{E}\sigma(E')\ln(E'/E)dE' + M'}{\int_{E_1}^{E}\bar{\sigma}(E')dE'} = 0, \qquad (38)$$

then the asymptotic value of $g(E)$ is given by $I_2(E)$. This condition is valid when $\sigma(E)$ goes to zero sufficiently faster than $\bar{\sigma}(E)$.

Using the Schwarz inequality for $g(E)$ and $\bar{g}(E)$ together with Eq. (12), we have

$$|g(E)| \leqslant \frac{E^{1/2}}{4\sqrt{\pi}\,m}\ln E\left(\int_{E_1}^{E}\sigma(E')dE'\right)^{1/2}.$$

Hence if (38) is valid, we have

$$\int_{E_1}^{E}\bar{\sigma}(E')dE' \leqslant \frac{E^{1/2}\ln E}{4\ln 2\sqrt{\pi}\,m}\left(\int_{E_1}^{E}\sigma(E')dE'\right)^{1/2}. \qquad (39)$$

This is our final result. It shows that the particle-antiparticle total cross sections cannot have a totally arbitrary energy dependence. In particular, if $\sigma\sim E^{-\alpha}$, $\bar{\sigma}\sim E^{-\bar{\alpha}}$ with $\bar{\alpha}<\alpha$, then inequality (39) requires that $\alpha < 2\bar{\alpha}$. Although the result given in this section is weak, it is interesting that unitarity can still give some constraints.

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APPENDIX

We give here the details of the calculation which leads to Eqs. (19) and (20). Let us denote respectively the integrals on the right-hand side of Eq. (18) H_1, H_2, H_3, and H_4. Let us first consider $H_2(E)$:

$$H_2(E) = \int_{E+a}^{\infty}\frac{\Delta\sigma(E')}{E'^3}E'^2\ln\left|\frac{E'^2}{E'^2-E^2}\right| dE'. \qquad (A1)$$

Let us set $E'^2 = y'$ and $E^2 = y$ and consider the following function:

$$h(y') = y' \ln \left| \frac{y'}{y'-y} \right| .$$

Its derivative with respect to y' is

$$h'(y') = 1 + \ln \left| \frac{y'}{y'-y} \right| - \frac{y'}{y'-y} .$$

Hence it is a monotonically decreasing function for $y'/(y'-1) > 1$. Hence for E sufficiently large,

$$E'^2 \ln \left| \frac{E'^2}{E'^2 - E^2} \right| < E^2 \ln E^2 ,$$

and we have finally

$$H_2(E) < 2E^2 \ln E \int_E^\infty \frac{|\Delta\sigma(E')|}{E'^3} dE' . \qquad \text{(A2)}$$

$H_3(E)$ can be rewritten as

$$H_3(E) = \int_{E_1}^{E-a} \frac{\Delta\sigma(E')}{E'} \ln \left| \frac{E^2}{E'^2 - E^2} \right| dE' . \qquad \text{(A3a)}$$

Similarly in the region of integration of (A3), we have

$$\frac{1}{E'^2} \ln \frac{E^2}{E'^2 - E^2} \leqslant \frac{1}{E^2} \ln E^2$$

for E sufficiently large. Using this inequality, we have

$$|H_3(E)| \leqslant \frac{2 \ln E}{E^2} \int_{E_1}^{E} |\Delta\sigma(E')| E' dE' . \qquad \text{(A3b)}$$

Let us now consider H_4:

$$\begin{aligned} |H_4(E)| &\leqslant \int_{E-a}^{E+a} \frac{|\Delta\sigma(E')|}{E'} \ln \left| \frac{E^2}{E'^2 - E^2} \right| dE' \\ &\leqslant \frac{2 \ln E}{E-a} \int_{E-a}^{E+a} |\Delta\sigma(E')| dE' \\ &\quad + \frac{1}{E-a} \int_{E-a}^{E+a} |\Delta\sigma(E')| \ln |E'^2 - E^2| dE' \\ &\leqslant 4\left(\frac{a}{E}\right) \ln E [\limsup |\Delta\sigma(E)|] \\ &\quad + 2\left(\frac{a}{E}\right) \ln 2E [\limsup |\Delta\sigma(E)|] . \qquad \text{(A4)} \end{aligned}$$

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