PHYSICAL REVIEW D VOLUME 6, NUMBER 10 15 NOVEMBER 1972

Low's Problem and Its Investigation by Means of Power-Series Expansions

I. P. Nedelkov*
International Centre for Theoretical Physics, Miramare-Trieste, Italy
(Received 24 August, 1971; revised manuscript received 16 February 1972)

In this paper Low's problem is understood as a generalization of problems which are solved by dispersion integral equations of Chew and Low, Shirkov, Chew and Mandelstam, and the like. The abstract definition of Low's problem is based on the properties of analyticity, crossing symmetry, and unitarity of the unknown functions $h^{\alpha}(z)$ which are physically interpreted as partial-wave amplitudes. In this paper we show that a relevant formulation of the abstract problem of Low can be given by a nonlinear system of algebraic equations. This system can be solved numerically by the classical methods of nonlinear functional analysis, Newton's method, and the method based on the Banach-Cacciopoli theorem. In both cases the calculations are applied directly to $h^{\alpha}(z)$. This has an advantage over the conventional N/D method in that it is not necessary to control the appearance of zeros in the denominator. Another advantage is the possibility of resolving problems when there are cuts in the physical plane.

I. INTRODUCTION

By Low's problem we mean the well-known problem of S-matrix theory – the determination of N functions $h^{\alpha}(z)$, $\alpha = 1, 2, \ldots, N$ satisfying certain conditions of analyticity, crossing symmetry, and unitarity. The precise definition will be given in Sec. II.

Low's problem in particular cases is usually examined by means of a system of nonlinear integral equations of the Cauchy type. In the present paper another way of treating this problem is suggested: its reduction to a nonlinear, infinite algebraic system, which is convenient both for theoretical investigations and numerical calculations.

If it is possible to represent the functions $h^{\alpha}(z)$ by the Cauchy integral, then Low's problem is reduced to Low's integral equation (1). However, it the condition of analyticity is weaker, so that functions $h^{\alpha}(z)$ can no longer be represented by the Cauchy integral, the possibility of expanding them in power series often remains. In this case the relevant formulation of Low's problem will be algebraic.

Low's equation is normally written as the system (1) of nonlinear singular integral equations.[1] Upon appropriate choice of the crossing matrix $C^{\alpha\beta}$ and of the constants λ_{α}, $\alpha = 1, 2, \ldots, N$, Eq. (1) assumes a specific form, and to a greater or lesser degree it can describe various real processes in which the $h^{\alpha}(z)$ represent the partial scattering amplitudes. Typical examples of such equations are the equation of Chew and Low[1] describing pion-nucleon scattering at low energies, the equation of Chew and Mandelstam,[2] and the equation of Shirkov[3,4] by means of which pion-pion scattering is studied.

The algebraic formulation of Low's problem is closely related to the possibility of expressing $h^{\alpha}(z)$ by power series. Representing $h^{\alpha}(z)$ by power-series expansion, we satisfy the first condition defining Low's problem, the condition of analyticity. The imposition of the remaining two conditions of unitarity and crossing symmetry lead to the system of nonlinear algebraic equations (14) which have to be satisfied by the coefficients of the power series.

The nonlinear system (14) is the algebraic analog of Low's integral equation. Algebraic systems of the type (14) have been subject to few investigations. Thus, for example, for $N = 1$ in a particular Low problem closely related to the equation of Castillejo, Dalitz, and Dyson (CDD), the algebraic system was used to determine its numerical solution by means of Newton's method.[5] The possibility of applying Newton's method for the numerical solution of the dispersion integral equations was also suggested in Ref. 6.

Algebraic formulation of Low's problem is also given in Ref. 7, where a more general algebraic system is derived which is equivalent to Low's Eq. (1) in particular cases. The generalization consists in the following: Eq. (1) is satisfied by solutions $h^{\alpha}(z)$ which are analytic in the cut plane with the eventual exception of the point $z = 0$. Functions $h^{\alpha}(z)$ which correspond to the solution of the algebraic system are analytic in the cut plane z from which a whole region s, not only the point $z = 0$, can be removed. Therefore, while the solutions of (1) make it possible to investigate scattering amplitudes with only one simple pole, the algebraic system in Ref. 7 is also suitable for the cases when $h^{\alpha}(z)$ has more complicated singularities on the cut plane.

It is proved in the present paper that the formulation of Low's problem by means of the conditions for analyticity, unitarity and crossing symmetry is equivalent to the algebraic formulation contained in the system (14). In this manner an analytical apparatus for its numerical solution has been indicated.

Numerical treatment of similar problems by Low's integral equation is effected by means of the N/D method or the method of the inverse Low amplitude,[8] which are typical for dispersion relations. One of the main objects of the present work is to popularize the idea that numerical results can also be obtained in the given case through classical general methods of nonlinear functional analysis, namely Newton's method and a method of calculation based on the Banach-Cacciopoli fixed-point theorem, which will be called the Low-amplitude method in this paper. This method has the advantage of eliminating automatically the necessity of watching for the appearance of zeros in the physical plane of $h^{\alpha}(z)$. Moreover, as we shall point out, this method is suitable both for solving the algebraic system and for solving Low's integral equation for which the N/D method was preferred so far.

The Low-amplitude method is advantageous when dealing with nonresonant (adiabatic) solutions.

At the same time the N/D method is definitely to be preferred for numerical investigation of resonant solutions (solutions containing CDD poles). As was shown in Ref. 9 the Low-amplitude method is also applicable for numerical treatment of resonant solutions. In its present-day form, however, it is less efficient than the N/D method for this class of problems. An illustration of the possibilities of the Low amplitude method is the determination of the upper bound of the coupling constant $f^2_{\max}$ for the adiabatic solution of the Chew-Low equation. This was performed in Ref. 9 by applying repeatedly the Banach-Cacciopoli method directly to the integral equation (1). The value obtained for $f^2_{\max}$ in this way was $f^2_{\max}=0.07$, whereas the experimental value referring, however, to the resonance solution is $f^2=0.087$. An $f^2_{\max}$ defined in another way has been determined in Ref. 8, which investigates the conditions of applicability of some fixed-point theorems.

In the present paper the algebraic system has been simplified to the extent of its being suitable to prove the uniqueness and existence of the solutions of Low's problem. These proofs will be presented in another publication.[10] In some sense they are more general than the corresponding proofs of Warnock[8,11] and Atkinson[12] for the integral equations. The generalization is reduced to the fact that the algebraic system also permits solutions $h^{\alpha}(z)$ with poles, cuts, and other singularities in the physical plane, while there can be one pole for solutions of integral equations at most.

The present work has been written along the following lines:

In Sec. II the algebraic system (14) is derived (from an appropriate development of $h^{\alpha}(z)$ into a power series).

The conditions under which the basic and algebraic formulation of Low's problem are equivalent are studied in Sec. III.

Section IV discusses the question of the numerical solution of Low's problem both by the Low amplitude method and by Newton's method.

The essential results of the work are summarized in Sec. V.

II. DERIVATION OF THE SYSTEM OF ALGEBRAIC EQUATIONS

The questions discussed in the present paper are closely connected with Low's Eq. (1) which in its integral form is[1]

$$h^{\alpha}(z)=\frac{\lambda_{\alpha}}{z}+\frac{1}{\pi}\int_1^{\infty}dz'f(z')\times\left(\frac{|h^{\alpha}(z')|^2}{z'-z}+\frac{\sum_{\beta=1}^{N}C^{\alpha\beta}|h^{\beta}(z')|^2}{z'+z}\right). \qquad (1)$$

In (1) the unknown functions $h^{\alpha}(z)$ $(\alpha=1,2,\ldots,N)$ are the partial scattering amplitudes (for instance, of the p waves of pion-nucleon scattering); $C^{\alpha\beta}$ $(\alpha,\beta=1,2,\ldots,N)$ is the crossing matrix which should be equal to the square root of the unit N-row matrix, though otherwise arbitrary, and λ_{α} $(\alpha=1,2,\ldots,N)$ are values proportional to the coupling constant f^2. The λ_{α} satisfy the condition

$$\lambda_{\alpha}=-\sum_{\beta=1}^{N}C^{\alpha\beta}\lambda_{\beta}.$$

The remaining notation is as follows: z' is a real number $1\leqslant z'\leqslant\infty$; $f(z')$ is a given function of z'; $z=x+iy$ is a point in the complex cut plane p with cuts $-\infty\leqslant x\leqslant -1$ and $1\leqslant x\leqslant\infty$; $h^{\alpha}(z)$ are related to the phase shifts δ_{α} by the formula

$$h^{\alpha}(x)=f^{-1}(x)\exp[i\delta_{\alpha}(x)]\sin\delta_{\alpha}(x).$$

Equation (1) is equivalent to the following problem.[1] Determine the functions $h^{\alpha}(z)$ if it is known that they satisfy the following conditions.

(a′) Analyticity: The functions $h^{\alpha}(z)$, $z\in(p-g)$, where g is the point $z=0$, are analytic.

(b) $h^{\alpha *}(z) = h^{\alpha}(z^*)$.

(c) Unitarity: $\mathrm{Im}\,h^{\alpha}(x) = f(x)|h^{\alpha}(x)|^2, \quad 1 \leq x \leq \infty$.

(d) Crossing symmetry:

$h^{\alpha}(-x - i0) = \sum_{\beta=1}^{N} C^{\alpha\beta} h^{\beta}(x + i0)$.

(e) Behavior at infinity: The integrals in (1) converge. The contribution of the contour integrals $\int [h^{\alpha}(z)/z] dz$ taken on a semicircle with an infinite radius in the upper half-plane is zero.

Let us denote by s some arbitrary subregion of the physical cut plane p. s can contain a cut or, in general, be a part of the physical plane in which $h^{\alpha}(z)$ are not analytic. On the assumption of this condition the region of analyticity of $h^{\alpha}(z)$ is $p - s$. On the whole it is smaller than specified in condition (a′). The latter should be replaced by the condition

(a) Analyticity: The functions $h^{\alpha}(z)$, $z \in (p - s)$ are analytic.

We shall use the name Low's problem for the problem (a), (b), (c), (d): it is a generalization of the problem (a′), (b), (c), (d), (e) which is equivalent to Eq. (1).

It is more convenient to discuss Low's problem in the complex plane of the auxiliary variable $Z = X + iY = \mathrm{Re}^{i\varphi}$ related to the original variable z by means of the conformal transformation

$$z = \frac{2Z}{Z^2 + 1}.$$

This transformation changes the cut plane p in the interior P of the unit circle of the Z plane. The cuts $-\infty \leq x \leq -1$ and $1 \leq x \leq \infty$ of the original plane are mapped into the unit circle C_0 of the Z plane.

After the conformal transformation the function $f(x)$, $1 \leq x \leq \infty$ is transformed into the odd function $F(\varphi)$, $-\frac{1}{2}\pi \leq \varphi \leq \frac{1}{2}\pi$, and the region s into the region S. After the transformation the problem (a), (b), (c), (d) is reformulated in the following way.

Find the functions $H^{\alpha}(Z)$, $\alpha = 1, 2, \ldots, N$, which satisfy the conditions:

(A) Analyticity: $H^{\alpha}(Z)$ are analytic in the annular region

$$1 \geq r_i \leq |Z| \leq r_e \geq 1. \tag{2}$$

$$\text{(B)}\quad H^{\alpha *}(Z) = H^{\alpha}(Z^*). \tag{3}$$

(C) Unitarity:

$$\mathrm{Im}\,H^{\alpha}(\varphi) = F(\varphi)|H^{\alpha}(\varphi)|^2, \quad -\tfrac{1}{2}\pi \leq \varphi \leq \tfrac{1}{2}\pi \tag{4}$$

where $F(\varphi)$ is an odd function.

(D) Crossing symmetry:

$$H^{\alpha}(\varphi + \pi) = \sum_{\beta=1}^{N} C^{\alpha\beta} H^{\beta}(\varphi), \quad -\pi \leq \varphi \leq \pi \tag{5}$$

where the matrix elements $C^{\alpha\beta}$ are real and satisfy the condition

$$\sum_{\gamma=1}^{N} C^{\alpha\gamma} C^{\gamma\beta} = \delta_{\alpha\beta}.$$

Here and below $H^{\alpha}(\varphi)$ means $H^{\alpha}(e^{i\varphi})$.

Condition (A) is a modification of the condition according to which the functions $H^{\alpha}(Z)$, $Z \in (P - S)$ are analytic. Conditions (A), (B), (C), and (D), being very convenient for the investigation of Low's problem, will be treated as its basic formulation.

Further on, the index α will be accepted to take values $1, 2, \ldots, N$.

Hölder-continuous functions will be used in our further exposition.

A periodic function $G(\varphi)$ of period 2π will be called Hölder-continuous periodic function of period 2π and of order ϵ or briefly a Hölder-continuous function if it satisfies the following condition: $|G(\varphi') - G(\varphi)| \leq K|\varphi' - \varphi|^{\epsilon}$, where φ' and φ appurtain to a segment $[\psi_1, \psi_2]$, $\psi_1 < -\pi$, $\psi_2 > \pi$ and K and ϵ are positive constants ($\epsilon \leq 1$). The functions $H^{\alpha}(\varphi)$ are supposed to be Hölder-continuous.

The function $F(\varphi)$ is supposed to satisfy the following conditions:

(1) $|F(\varphi') - F(\varphi)| \leq K_1|\varphi_1 - \varphi|^{\epsilon_1}$,

where K_1 and ϵ_1, $\epsilon_1 \leq 1$ are positive constants and φ' and φ belong to the segment $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

(2) $F(\pm\frac{1}{2}\pi) = 0$.

Under these conditions the auxiliary function $\tilde{F}(\varphi)$ which is defined by the equalities

$$\tilde{F}(\varphi) = F(\varphi), \quad -\tfrac{1}{2}\pi \leq \varphi \leq \tfrac{1}{2}\pi,$$

$$\tilde{F}(\varphi) = 0, \quad \tfrac{1}{2}\pi \leq \varphi \leq \tfrac{3}{2}\pi$$

is also Hölder-continuous.

In order to give the problem (A), (B), (C), (D) an algebraic form we proceed in the following way:

In view of condition (A), we develop $H^{\alpha}(Z)$ in the Laurent series

$$H^{\alpha}(Z) = \sum_{n=-\infty}^{\infty} H_n^{\alpha} Z^n \tag{6}$$

from which, having in mind that, in view of Eq. (3), the coefficients of the power-series expansion H_n^{α} are real, we obtain for $|Z| = 1$

$$\mathrm{Re}\,H^{\alpha}(\varphi) = \sum_{n=-\infty}^{+\infty} H_n^{\alpha} \cos n\varphi, \tag{7}$$

$$\mathrm{Im}\,H^{\alpha}(\varphi) = \sum_{n=-\infty}^{\infty} H_n^{\alpha} \sin n\varphi. \tag{8}$$

It is necessary to differentiate between series (7) and (8) and the Fourier series for $\mathrm{Re}\,H^{\alpha}(\varphi)$ and $\mathrm{Im}\,H^{\alpha}(\varphi)$:

$$\mathrm{Re}\,H^{\alpha}(\varphi) = C_0^{\alpha} + \sum_{\nu=1}^{\infty} C_{\nu}^{\alpha} \cos \nu\varphi, \tag{9}$$

$$\mathrm{Im}H^\alpha(\varphi)=\sum_{\nu=1}^{\infty} S_\nu^\alpha \sin\nu\varphi\,. \tag{10}$$

The coefficients of the series (7), (8), (9), and (10) are related by the equations

$$\begin{aligned}&C_0^\alpha=H_0^\alpha,\\ &\left.\begin{aligned}C_\nu^\alpha&=H_\nu^\alpha+H_{-\nu}^\alpha,\\ S_\nu^\alpha&=H_\nu^\alpha-H_{-\nu}^\alpha,\end{aligned}\right\}\ \nu=1,2,\ldots,\infty\,.\end{aligned}\tag{11}$$

In order to satisfy conditions (C) and (D), the following formula is applied:

$$S_\nu^\alpha=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\varphi\,\sin\nu\varphi\,\mathrm{Im}H^\alpha(\varphi)+\frac{1}{\pi}\int_{\pi/2}^{3\pi/2}d\varphi\,\sin\nu\varphi\,\mathrm{Im}H^\alpha(\varphi)\,. \tag{12}$$

In view of (4), (5) and (11), Eq. (12) becomes

$$H_\nu^\alpha=H_{-\nu}^\alpha+\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\varphi\,\sin\nu\varphi\,F(\varphi)\,|H^\alpha(\varphi)|^2+(-1)^\nu\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}d\varphi\,\sin\nu\varphi\,F(\varphi)\sum_{\beta=1}^{N}C^{\alpha\beta}|H^\beta(\varphi)|^2$$

or

$$H_\nu^\alpha=H_{-\nu}^\alpha+\frac{1}{\pi}\int_{-\pi}^{\pi}d\varphi\,\sin\nu\varphi\,\tilde F(\varphi)|H^\alpha(\varphi)|^2+\frac{(-1)^\nu}{\pi}\int_{-\pi}^{\pi}d\varphi\,\sin\nu\varphi\,\tilde F(\varphi)\sum_{\beta=1}^{N}C^{\alpha\beta}|H^\beta(\varphi)|^2\,,$$

$$\alpha=1,2,\ldots N;\ \ \nu=1,2,\ldots,\infty\,. \tag{13}$$

The further transformation of Eq. (13) proceeds as follows: After introducing the series (7) and (8) into the expression

$$L^\alpha(\varphi)=|H^\alpha(\varphi)|^2=|\mathrm{Re}H^\alpha(\varphi)|^2+|\mathrm{Im}H^\alpha(\varphi)|^2$$

we have the series

$$\sum_{k=-\infty}^{\infty}L_k^\alpha\cos k\varphi\,,$$

where

$$L_k^\alpha=\sum_{m=-\infty}^{\infty}H_{m+k}^\alpha H_m^\alpha\,.$$

Then the last series, multiplied by $\tilde F(\varphi)$, is introduced into (13) to get the basic formula

$$H_\nu^\alpha=H_{-\nu}^\alpha+\sum_{k=-\infty}^{\infty}F(\nu;k)\sum_{m=-\infty}^{\infty}E_\nu^\alpha(H_m;H_{m+k})\,,$$

$$\alpha=1,2,\ldots,N;\ \ \nu=1,2,\ldots,\infty \tag{14}$$

where

$$F(\nu;k)=\frac{1}{\pi}\int_{-\pi}^{\pi}d\varphi\,\sin\nu\varphi\,\cos k\varphi\,\tilde F(\varphi) \tag{15}$$

and

$$E_\nu^\alpha(H_m;H_{m+k})=H_m^\alpha H_{m+k}^\alpha+(-1)^\nu\sum_{\beta=1}^{N}C^{\alpha\beta}H_m^\beta H_{m+k}^\beta\,. \tag{16}$$

To the system (14) is to be added the equation

$$H_0^\alpha=\sum_{\beta=1}^{N}C^{\alpha\beta}H_0^\beta,\quad \alpha=1,2,\ldots,N$$

which is derived by integrating (5) with respect to φ from $-\pi$ to π.

The formal operations leading to (14) can be justified with the help of standard theorems of the theory of Fourier series.[13,14] (In Refs. 13 and 14 they speak of the Lipschitz condition instead of the Hölder condition.) The main points are: (1) $\mathrm{Re}H^\alpha(\varphi)$ and $\mathrm{Im}H^\alpha(\varphi)$ being Hölder-continuous, the same holds for $L^\alpha(\varphi)$. Hence, the last expression is equal to its Fourier series

$$\sum_{k=-\infty}^{\infty}L_k^\alpha\cos k\varphi\,.$$

(2) The multiplication of Fourier series is carried out in the usual way because the functions are Hölder-continuous and therefore quadratic integrable. (3) Termwise integration is justified because the series, being series of Hölder-continuous functions, are uniformly convergent.

In particular cases the system (14) is the algebraic equivalent of the integral equation (1). Equation (14) was obtained by applying power-series expansions. It expresses nonlinear algebraic relations between the coefficients of these series. Hence, according to the basic formulation of Low's problem, it can also be considered as the algebraic formulation of this problem.

III. EQUIVALENCE OF THE TWO FORMULATIONS OF LOW'S PROBLEM

Low's problem (A), (B), (C), (D) cannot be reduced to the integral equation (1) in the general case, since the condition of analyticity is weakened. However, as can be seen from the following assertion, it can be solved by means of the algebraic system (14), provided certain conditions are observed.

Theorem. Let $H^\alpha(\varphi)$, $\alpha=1,2,\ldots,N$, be periodic functions of period 2π satisfying the Hölder condition of order ϵ, $0<\epsilon\leq 1$, in the interval $[-\pi-\eta,\pi+\eta]$, where η is some positive number. Let $F(\varphi)$, $-\frac{1}{2}\pi\leq\varphi\leq\frac{1}{2}\pi$, satisfy the Hölder condition of order ϵ, $0<\epsilon\leq 1$ and let $F(\pm\frac{1}{2}\pi)=0$. Let the functions $H^\alpha(Z)$ satisfy the conditions of the problem (A), (B), (C), (D).

Then the coefficients H_n^α ($\alpha=1,2,\ldots,N$; $n=-\infty,\ldots,-2,-1,0,1,2,\ldots,\infty$) of the series (6) will satisfy the algebraic system (14).

With certain modification of the conditions of the theorem the opposite assertion is also true: Let the system (14) have real roots H_n^α ($\alpha = 1, 2, \ldots, N; n = 0, \pm 1, \pm 2, \ldots, \pm\infty$) satisfying the following conditions:

(1) The series

$$\sum_{n=1}^{\infty} H_n^\alpha \sin n\varphi \text{ and } \sum_{n=1}^{\infty} H_{-n}^\alpha \sin n\varphi$$

are convergent on the whole interval $-\pi \leq \varphi \leq \pi$ to certain functions $V_+^\alpha(\varphi)$ and $V_-^\alpha(\varphi)$, respectively, which are known to satisfy the Hölder condition of order ϵ, $\epsilon < 0 < 1$, on the interval $[-\pi - \eta, \pi + \eta]$ where η is some positive number.

(2) $H_n^\alpha = (-1)^n \sum_{\beta=1}^{N} C^{\alpha\beta} H_n^\beta, \quad n = 0, -1, -2, \ldots, -\infty.$

Then the series (6) converge to the functions $H^\alpha(Z)$ which satisfy the last three conditions of the problem (A), (B), (C), (D).

If besides that the roots of (14) satisfy the conditions

$$H_n^\alpha \leq He^{-\theta_e n}, \quad n = 0, 1, 2, \ldots, \infty$$

$$H_n^\alpha \leq He^{-\theta_i n}, \quad n = -1, -2, \ldots, -\infty$$

where H is a positive constant, and the positive constants θ_e and θ_i obey the conditions $\theta_e > |\ln r_e|$ and $\theta_i > |\ln r_i|$, then the functions $H^\alpha(Z)$ satisfy condition (A) too.

The proof of the first part of the theorem is contained in the derivation of (14).

In order to prove the second part of the theorem we substitute (15) into (14), then multiply by $\sin\nu\varphi$ and sum from $\nu = 1$ to $\nu = \infty$:

$$\sum_{\nu=1}^{\infty} H_\nu^\alpha \sin\nu\varphi - \sum_{\nu=1}^{\infty} H_{-\nu}^\alpha \sin\nu\varphi = P^\alpha(\varphi) + \sum_{\beta=1}^{N} C^{\alpha\beta} Q^\beta(\varphi), \tag{17}$$

where

$$P^\alpha(\varphi) = \frac{1}{\pi} \sum_{\nu=1}^{\infty} \sin\nu\varphi \sum_{k=-\infty}^{+\infty} L_k^\alpha \int_{-\pi}^{\pi} d\psi \sin\nu\psi \cos k\psi \, \tilde{F}(\psi) \tag{18}$$

and

$$Q^\alpha(\varphi) = \frac{(-1)^\nu}{\pi} \times \sum_{\nu=1}^{\infty} \sin\nu\varphi \sum_{k=-\infty}^{+\infty} L_k^\alpha \int_{-\pi}^{\pi} d\psi \sin\nu\psi \cos k\psi \, \tilde{F}(\psi). \tag{19}$$

According to Privalov's theorem (see Ref. 13, Chap. VIII, Sec. 13) if

$$V_+^\alpha(\varphi) = \sum_{\nu=1}^{\infty} H_\nu^\alpha \sin\nu\varphi, \quad -\pi \leq \varphi \leq \pi$$

is Hölder-continuous of order ϵ, $0 < \epsilon < 1$, then its conjugate function

$$U_+^\alpha(\varphi) = \sum_{\nu=1}^{\infty} H_\nu^\alpha \cos\nu\varphi, \quad -\pi \leq \varphi \leq \pi$$

is also Hölder-continuous of the same order ϵ. Similar statements are also valid for $V_-^\alpha(\varphi)$ and $U_-^\alpha(\varphi)$. From here it follows that the expression

$$[U_+^\alpha(\varphi) + U_-^\alpha(\varphi) + H_0^\alpha]^2 + [V_+^\alpha(\varphi) + V_-^\alpha(\varphi)]^2$$

is Hölder-continuous and therefore it coincides with its Fourier series

$$\sum_{k=-\infty}^{+\infty} \left[\sum_{m=-\infty}^{+\infty} H_{m+k}^\alpha H_m^\alpha\right] \cos k\varphi.$$

On the other hand, as

$$L_k^\alpha = \sum_{m=-\infty}^{+\infty} H_{m+k}^\alpha H_m^\alpha$$

by definition, the series

$$\sum_{k=-\infty}^{\infty} L_k^\alpha \cos k\varphi = L^\alpha(\varphi)$$

is uniformly convergent, and its sum $L^\alpha(\varphi)$ is a Hölder-continuous function of order ϵ, $0 < \epsilon < 1$.

The fact that the series $\sum_{k=-\infty}^{\infty} L_k^\alpha \cos k\varphi$ is uniformly convergent allows us to change the order of summation and integration in the expression for $P^\alpha(\varphi)$:

$$P^\alpha(\varphi) = \frac{1}{\pi} \sum_{\nu=1}^{\infty} \sin\nu\varphi \int_{-\pi}^{\pi} d\psi \sin\nu\psi \left(\sum_{k=-\infty}^{\infty} L_k^\alpha \cos k\psi\right) \tilde{F}(\psi),$$

$$P^\alpha(\varphi) = \frac{1}{\pi} \sum_{\nu=1}^{\infty} \sin\nu\varphi \int_{-\pi}^{\pi} d\psi \sin\nu\psi \, L^\alpha(\psi) \tilde{F}(\psi). \tag{20}$$

As $L^\alpha(\psi)$ and $\tilde{F}(\psi)$ are Hölder-continuous, their product $L^\alpha(\psi)\tilde{F}(\psi)$ is also Hölder-continuous. Then, according to a theorem of the theory of Fourier series, the Fourier series (18) converges to the function $L^\alpha(\varphi)\tilde{F}(\varphi)$:

$$P^\alpha(\varphi) = L^\alpha(\varphi)\tilde{F}(\varphi). \tag{21}$$

If in Eq. (19) we substitute $\varphi + \pi$ for φ, and take into consideration that

$$(-1)^\nu \sin\nu(\varphi + \pi) = \sin\nu\varphi,$$

then the right-hand side of (19) becomes identical with the right-hand side of (18). Then, with regard to (21) we have

$$Q^\alpha(\varphi + \pi) = L^\alpha(\varphi)\tilde{F}(\varphi) \quad . \tag{22}$$

If in (22) we substitute $\varphi + \pi$ for φ we obtain

$$Q^\alpha(\varphi) = L^\alpha(\varphi + \pi)\tilde{F}(\varphi + \pi). \tag{23}$$

Introducing (21) and (23) into (17) we obtain

$$\mathrm{Im}H^{\alpha}(\varphi)=L^{\alpha}(\varphi)\tilde{F}(\varphi)+\sum_{\beta=1}^{N}C^{\alpha\beta}L^{\alpha}(\varphi+\pi)\tilde{F}(\varphi+\pi), \tag{24}$$

where

$$\mathrm{Im}H^{\alpha}(\varphi)=\sum_{n=-\infty}^{\infty}H_n^{\alpha}\sin n\varphi=V_+^{\alpha}(\varphi)-V_-^{\alpha}(\varphi)$$

is Hölder-continuous of order ϵ, $0<\epsilon<1$.

Let φ belong to the $[-\frac{1}{2}\pi,\frac{1}{2}\pi]$. Then introducing φ into Eq. (24) we obtain

$$\mathrm{Im}H^{\alpha}(\varphi)=L^{\alpha}(\varphi)\tilde{F}(\varphi),\quad -\tfrac{1}{2}\pi\leqslant\varphi\leqslant\tfrac{1}{2}\pi. \tag{25}$$

If in (24) we substitute $\varphi+\pi$ for φ, with φ belonging to the interval $[-\frac{1}{2}\pi,\frac{1}{2}\pi]$ we obtain

$$\mathrm{Im}H^{\alpha}(\varphi+\pi)=\sum_{\beta=1}^{N}C^{\alpha\beta}L^{\beta}(\varphi)\tilde{F}(\varphi),\quad -\tfrac{1}{2}\pi\leqslant\varphi\leqslant\tfrac{1}{2}\pi. \tag{26}$$

Comparing Eqs. (25) and (26) we see that

$$\mathrm{Im}H^{\alpha}(\varphi+\pi)=\sum_{\beta=1}^{N}C^{\alpha\beta}\,\mathrm{Im}H^{\beta}(\varphi),\quad -\pi\leqslant\varphi\leqslant\pi. \tag{27}$$

In the latter equality the interval $[-\frac{1}{2}\pi,\frac{1}{2}\pi]$ has been expanded to the interval $[-\pi,\pi]$. This is possible because we have

$$\sum_{\gamma=1}^{N}C^{\alpha\gamma}C^{\gamma\beta}=\delta_{\alpha\beta}.$$

Taking into account that $\mathrm{Im}H^{\alpha}(\varphi)$ is Hölder-continuous we could derive from (27) the following relations:

$$(-1)^n(H_n^{\alpha}-H_{-n}^{\alpha})=\sum_{\beta=1}^{N}C^{\alpha\beta}(H_n^{\beta}-H_{-n}^{\beta}),\quad n=1,2,\ldots,\infty.$$

As according to the conditions of the theorem

$$(-1)^nH_{-n}^{\alpha}=\sum_{\beta=1}^{N}C^{\alpha\beta}H_{-n}^{\beta},\quad n=0,1,2,\ldots,\infty$$

from the latter equality it follows that

$$(-1)^nH_n^{\alpha}=\sum_{\beta=1}^{N}C^{\alpha\beta}H_n^{\beta},\quad n=-\infty,\ldots,-2,-1,0,1,2,\ldots,\infty.$$

Multiplying by $\cos n\varphi$ and summing from $n=-\infty,\ldots,-2,-1,0,1,2,\ldots,\infty$, we obtain

$$\sum_{n=-\infty}^{\infty}H_n^{\alpha}\cos n(\varphi+\pi)=\sum_{\beta=1}^{N}C^{\alpha\beta}\left(\sum_{n=-\infty}^{\infty}H_n^{\beta}\cos n\varphi\right).$$

Since it was shown above that the series

$$\sum_{n=1}^{\infty}H_n^{\alpha}\cos n\varphi \text{ and } \sum_{n=1}^{\infty}H_{-n}^{\alpha}\cos n\varphi$$

are convergent in the interval $[-\pi,\pi]$, then on both sides of the latter equality there are series which converge to functions conjugate to $\mathrm{Im}H^{\alpha}(\varphi+\pi)$ and $\mathrm{Im}H^{\alpha}(\varphi)$. From the latter equality and (27) we have

$$H^{\alpha}(\varphi+\pi)=\sum_{\beta=1}^{N}C^{\alpha\beta}H^{\beta}(\varphi),\quad -\pi\leqslant\varphi\leqslant\pi. \tag{28}$$

From what was said above it also follows that $H^{\alpha}(\varphi)$, $-\pi\leqslant\varphi\leqslant\pi$, is Hölder-continuous of order ϵ, $0<\epsilon<1$.

Now we are in a position to verify the conditions of the converse of the theorem.

If the last condition of the theorem is also fulfilled (this condition should also be added to the theorem in Ref. 15), then the functions $H^{\alpha}(Z)$ are analytic in the region $r_i\leqslant|Z|\leqslant r_e$, i.e., the condition (A) of the problem (A), (B), (C), (D) is satisfied. This is so because in that region the series $\sum_{n=-\infty}^{+\infty}H_n^{\alpha}Z^n$ is majorized by the expression

$$\sum_{m=0}^{\infty}H\exp\left(m\ln\frac{r_i}{Z}\right)+\sum_{m=1}^{\infty}H\exp\left(m\ln\frac{Z}{r_e}\right)$$

in which both series are convergent.

Condition (B) is also satisfied because H_n^{α} are real numbers.

Conditions (C) and (D) are also satisfied; this could be seen from (25) and (28). With this observation, the proof of the theorem is completed.

Condition (1) of the second part of the theorem actually means that the functions $\mathrm{Im}H^{\alpha}(\varphi)$ are Hölder-continuous. According to Privalov's theorem it follows from the above condition that $\mathrm{Re}H^{\alpha}(\varphi)$ and hence $H^{\alpha}(\varphi)$ are Hölder-continuous – a condition which is included in the first part of the theorem. The first part of the theorem has been formulated under the condition of Hölder continuity for $H^{\alpha}(\varphi)$ with the purpose of simplifying the exposition. This condition, however, contains some unnecessary information, and could eventually be replaced by the following condition: $\mathrm{Im}H^{\alpha}(\varphi)$ is Hölder-continuous. Superfluous information exists also in the condition

$$\sum_{n=-\infty}^{\infty}H_n^{\alpha}\exp\left(i\frac{\pi}{2}n\right)=0$$

from the analogous theorem in Ref. 15, and this condition could be omitted.

Condition (2) of the second part of the theorem is a generalization of the equalities $\lambda^{\alpha}=-\sum_{\beta=1}^{N}C^{\alpha\beta}\lambda^{\beta}$ which the residues in Eq. (1) obey. It should also be added in the theorem from Ref. 15.

Let us suppose that we know the conditions under which the solutions H_n^{α} of the algebraic system (14) exist and are unique. This would be sufficient to prove the existence and uniqueness of solutions of the problem (A), (B), (C), (D) provided condition (1) of the second part of the theorem is satisfied. It turns out that such a possibility exists. In Ref. 13 (Chap. II, Sec. 3) it is proved that if $H_n^\alpha = O(1/|n|^{1+\epsilon})$ then the series

$$\sum_{n=1}^{\infty} H_n^\alpha \sin n\varphi \text{ and } \sum_{n=1}^{\infty} H_{-n}^\alpha \sin n\varphi$$

converge to Hölder-continuous functions of order ϵ, $0<\epsilon<1$.

The equivalence theorem of the present work is based mainly on the assumption that $H^\alpha(\varphi)$ are Hölder-continuous functions.

In the corresponding theorem in Ref. 15 the condition $\sum_{n=-\infty}^{\infty} |H_n^\alpha| < \infty$ is imposed instead of the above assumption. This condition is satisfied if the functions $\sum_{n=-\infty}^{\infty} H_n^\alpha Z^n$ and hence, the functions $H^\alpha(\varphi)$ are continuous (see Chap. VIII, Sec. 12 of Ref. 13) and have a bounded variation. (See the Hardy-Littlewood theorem in Chap. VIII, Sec. 12, in Ref. 13).

If the formulation of the second part of the theorem in Ref. 15 is completed as suggested above, then the method of proof will be similar to the way the corresponding proofs were made here.

As stated above, the condition of $H^\alpha(\varphi)$ being Hölder-continuous can be interchanged with the condition of $H^\alpha(\varphi)$ having a bounded variation. This can also be extended to the function $F(\varphi)$. In this case we would have an analogous theorem in which the functions $H^\alpha(\varphi)$ and $F(\varphi)$ would have a bounded variation – the first in the interval $[-\pi, \pi]$, and the second in the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$.

IV. NUMERICAL SOLUTION OF LOW'S PROBLEM

After the equivalence between Low's problem and the system (14) has been established, the problem of the latter's numerical solution can be discussed.

In order to obtain numerical solutions of (14) the infinite limits are replaced by finite ones. The correctness of this procedure could be established theoretically; however, we have limited ourselves to numerical experiments confirming it.

The system (14) was successfully solved numerically in connection with Low's problem, which corresponds to the equation of Chew and Low.[7] In this case $C^{\alpha\beta}$ and λ_α are equal to

$$C^{\alpha\beta} \to \frac{1}{9}\begin{pmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix}; \quad \lambda_\alpha \to \frac{2}{3} f^2 \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix}. \tag{29}$$

Here f^2 is the coupling constant and the index α is determined by the isotopic spin I and the full moment of momentum J of the pion-nucleon system in the following manner:

$$\alpha = 1 \to I=\tfrac{1}{2}, \quad J=\tfrac{1}{2}$$
$$\alpha = 2 \to I=\tfrac{1}{2}, \quad J=\tfrac{3}{2}$$
$$\alpha = 3 \to I=\tfrac{3}{2}, \quad J=\tfrac{3}{2}.$$

The function $f(x)$ is determined by the equation $f(x) = p^3 v^2(p)$ where $p=(x^2-1)^{1/2}$ ($m_\pi = c = \hbar = 1$; m_π is the mass of the pion) is the pion's impulse in the c.m. system, and $v^2(p)$ is the cutoff function. The cutoff function can be selected in different ways.[16,17] A cutoff function

$$v^2(p) = e^{-kp}(1+a^2p^2)^{-1}, \quad a=0.27 \tag{30}$$

with $k \ll 1$ can be used for the numerical solution. It coincides with the function given in Ref. 16 for $k=0$.

In order to facilitate the numerical solution and theoretical investigation, (14) will be rewritten in such a way that it will be clear that the H_n^α values, $n=0,1,2,\ldots,\infty$, are related to the interior of the circle C_0 and the H_n^α values, $n=-1,-2,\ldots,-\infty$ to the region outside the circle (the interior of C_0 corresponds to the physical sheet of the original variable z and the region outside C_0 to the second Riemann sheet). With the notation

$$T_\xi^\alpha = H_n^\alpha, \quad n=\xi=0,1,2,\ldots,\infty, \qquad R_{-\nu}^\alpha = H_n^\alpha, \quad -n=\nu=1,2,\ldots,\infty$$

Eqs. (14) become:

$$T_0^\alpha = \sum_{\beta=1}^{N} C^{\alpha\beta} T_0^\beta, \quad \alpha=1,2,\ldots,N,$$

$$T_\nu^\alpha = \sum_{\xi,\eta} F(\nu;\eta-\xi)E_\nu^\alpha(T_\xi;T_\eta) + 2\sum_{\lambda,\xi} F(\nu;\xi+\lambda)E_\nu^\alpha(R_{-\lambda};T_\xi) + \sum_{\lambda,\mu} F(\nu;\lambda-\mu)E_\nu^\alpha(R_{-\lambda};R_{-\mu}) + R_{-\nu}^\alpha,$$

$$\xi,\eta=0,1,2,\ldots,\infty; \quad \lambda,\mu,\nu=1,2,\ldots,\infty; \quad \alpha=1,2,\ldots,N. \tag{31}$$

In order to apply the fixed-point theorem in (31), we take $T_0^\alpha = t_0^\alpha + \tau_0^\alpha$, $T_\nu^\alpha = t_\nu^\alpha + \tau_\nu^\alpha$ where the numbers τ_0^α, τ_ν^α correspond to an approximate solution and are known, and t_0^α, t_ν^α are unknown small corrections. After the substitution, (31) becomes

$$t_0^\alpha + \tau_0^\alpha = \sum_{\beta=1}^{N} C^{\alpha\beta}(t_0^\beta + \tau_0^\beta), \quad \alpha = 1, 2, \ldots, N;$$

$$t_\nu^\alpha = \sum_{\xi=0}^{\xi\max} \sum_{\eta=0}^{\eta\max} F(\nu; \eta - \xi) E_\nu^\alpha(t_\xi; t_\eta) + 2 \sum_{\xi=0}^{\xi\max} \sum_{\eta=0}^{\eta\max} F(\nu; \eta - \xi) E_\nu^\alpha(\tau_\eta; t_\xi)$$

$$+ 2 \sum_{\lambda=1}^{\lambda\max} \sum_{\xi=0}^{\xi\max} F(\nu; \xi + \lambda) E_\nu^\alpha(R_{-\lambda}; t_\xi) + 2 \sum_{\lambda=1}^{\lambda\max} \sum_{\xi=0}^{\xi\max} F(\nu; \xi + \lambda) E_\nu^\alpha(R_{-\lambda}; \tau_\xi)$$

$$+ \sum_{\xi=0}^{\xi\max} \sum_{\eta=0}^{\eta\max} F(\nu; \eta - \xi) E_\nu^\alpha(\tau_\xi; \tau_\eta) + \sum_{\lambda=1}^{\lambda\max} \sum_{\mu=1}^{\mu\max} F(\nu; \lambda - \mu) E_\nu^\alpha(R_{-\lambda}; R_{-\mu}) + R_{-\nu}^\alpha - \tau_\nu^\alpha,$$

$$\alpha = 1, 2, \ldots, N; \quad \nu = 1, 2, \ldots, \nu_{\max}. \qquad (32)$$

As has been pointed out earlier, infinite sums are replaced by finite sums in the system (32) which is intended for numerical calculations.

The numerical solution of (32) for a problem corresponding to the equation of Chew and Low has been found by means of a fixed-point method in the Banach-Cacciopoli variant which can be considered as an algebraic version of the Low-amplitude method used in Ref. 9. The solutions obtained have been compared with the numerical solutions of the integral equation (1). The latter were obtained[9] applying the Low-amplitude method with $k = 0$ in the cutoff function (30). The maximum difference between the two solutions was 0.5%.

Reference 9 seems to be the only work in which the numerical solution $h^\alpha(z)$ of the integral equation (1) is determined by the help of the Low-amplitude method. The inverse Low-amplitude method[8] and the N/D method are preferred instead. In these methods the unknown functions are the auxiliary function $S^\alpha(z) = 1/h^\alpha(z)$ in the first case and $N^\alpha(z)$ and $D^\alpha(z) = N^\alpha(z)/h^\alpha(z)$ in the second. If the denominator $h^\alpha(z) = 0$, $S^\alpha(z)$ and $D^\alpha(z)$ may not correspond to the correct solution of Eq. (1). That is why the Low-amplitude method in which $h^\alpha(z)$ never appears as a denominator has the advantage of eliminating the necessity of searching for zeros in $h^\alpha(z)$. Therefore it is convenient for calculations with large values of the coupling constant where surprises are likely to crop up. Such an investigation has been carried out in Ref. 9 (it was also proposed in Ref. 8), by successive application of the Low-amplitude method in the following manner: f^2 is given values $f^2{}_0 < f^2{}_1 < f^2{}_2 < \cdots < f^2{}_{\max}$, $f^2{}_0$ being a sufficiently small positive number. The numerical solution of (1), $h_0^\alpha(z)$, is determined in the first place with the maximum possible precision for $f^2{}_0$. Then starting from $h_0^\alpha(z)$ as a zero-order approximation, we find the solution of (1), $h_1^\alpha(z)$, for $f^2 = f^2{}_1$, etc.

The results obtained in this manner are shown in Fig. 1. The value $f^2 = 0.07$ is the highest possible value of f^2 for the adiabatic solution of the equation of Chew and Low which can be obtained in this way. This value is close to the experimental 0.087 for the resonance solution.

It is also to be noted that for $f^2 = f^2{}_{\max}$ the graph in Fig. 1 indicates the presence of a corner in the δ_1 curve which probably may be related to a cut in the $h^1(z)$ plane. To prove this let us accept that for $f^2 = f^2{}_{\max}$ the curve behaves as a corner in the neighborhood of the maximum. To some extent Fig. 1 gives grounds for such a hypothesis. This neighborhood will be considered in the plane of the auxiliary variable $\zeta = \xi + i\eta$ which coincides partially with the first sheet of the physical plane for $\eta > 0$, and for $\eta < 0$ with the second sheet (the half line $\xi \geq 1$, $\eta = 0$ in the plane corresponds to the upper side of the physical cut). In these conditions let us consider the harmonic functions $U(\xi, \eta) = \mathrm{Re}\, h^1(z)$ and $V(\xi, \eta) = \mathrm{Im}\, h^1(z)$, supposing that in the vicinity of the corner $\zeta \equiv z$. Obviously the curves $U(\xi, 0)$ and $V(\xi, 0)$ would also have corners. The functions $U(\xi, \eta)$ and $V(\xi, \eta)$ can be considered approximately as harmonic for $\eta > 0$ because $\eta > 0$ corresponds to points in the first sheet. If we suppose that $U(\xi, 0)$ and $V(\xi, 0)$ are known, then we can use the well-known Poisson's integral to find the solution of the Dirichlet problem for the upper half-plane, i.e., the functions $U(\xi, \eta)$ and $V(\xi, \eta)$. The fact that $U(\xi, 0)$ and $V(\xi, 0)$ have corners leads to logarithmic terms in the expressions for $U(\xi, \eta)$

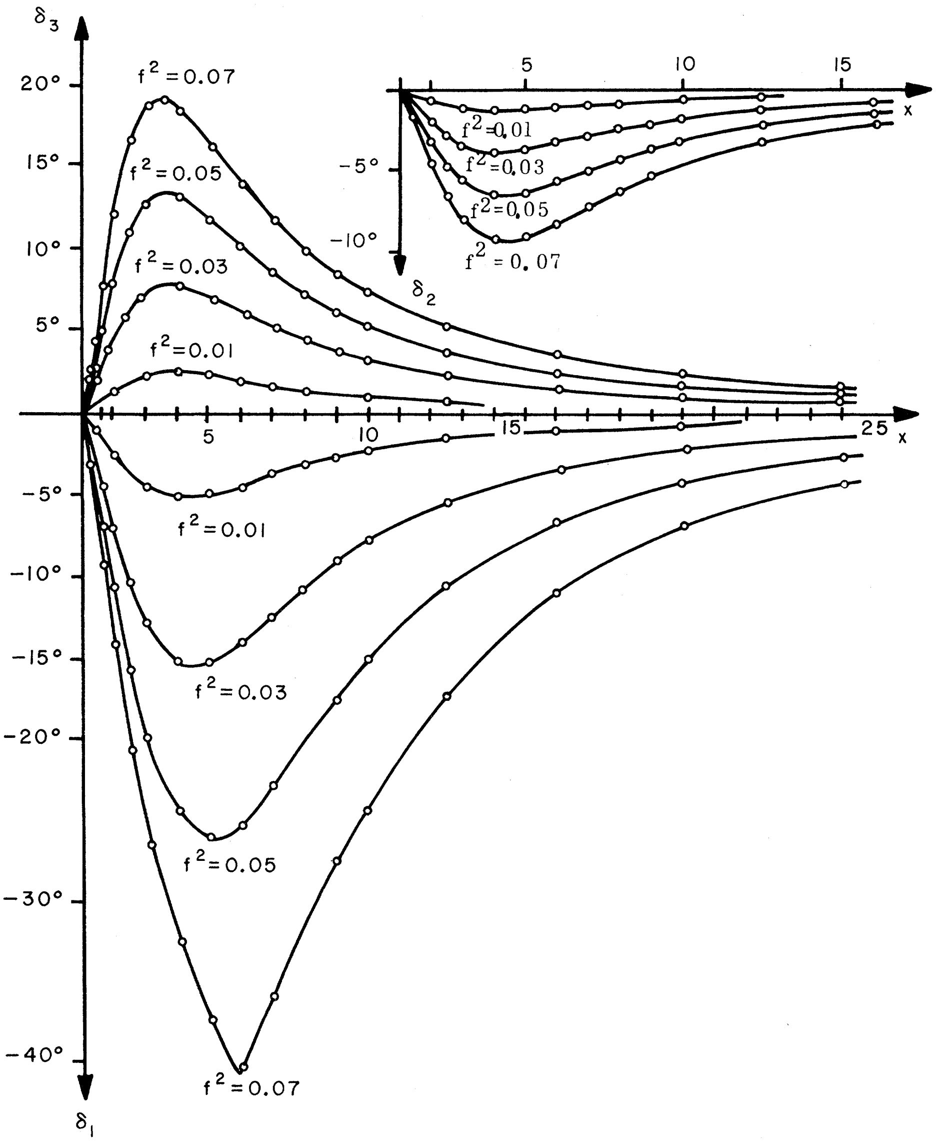

FIG. 1. Numerical solution of Low's equation.

and $V(\xi, \eta)$, $\eta > 0$. The next step is to make an analytic continuation of $U(\xi, \eta)$ and $V(\xi, \eta)$ into the lower half-plane $\eta < 0$. This being done, it is easy to see that these logarithmic terms can be interpreted as having been generated by a cut lying in the second sheet of the variable z. This cut touches the O-x axis in the vicinity of the maximum of $h^1(z)$. For $f^2 > f^2{}_{\max}$ this cut would probably pass into the upper plane, thus violating the condition of analyticity which is indispensable for the integral equation.

The system (14), after its transformation into (32), was also solved numerically by Newton's method. A solution was found for a problem corresponding to the equation of Chew and Low with the data adduced earlier. It differs from solutions obtained by other methods by several tenths of 1%. The first experiments were unsuccessful because precision decreased rapidly upon increasing the number of iterations. As proved later, the reason this occurred was the truncation of (32). If the $\xi_{\max}$, $\eta_{\max}$, $\mu_{\max}$, $\nu_{\max}$, and $\lambda_{\max}$ were infinite in (32), t_ν^α would have been real (H_ν^α and τ_ν^α are real by definition). However, as the sums in (32) are truncated the solutions of the truncated equation become complex. By substituting t_ν^α with $t'^\alpha_\nu + i t''^\alpha_\nu$ in (32) and separating the real and the imaginary part, we obtain the system of equations

$$
\begin{aligned}
t_0'^{\alpha} + \tau_0^{\alpha} &= \sum_{\beta=1}^{N} C^{\alpha\beta}(t_0'^{\beta} + \tau_0^{\beta}),\\
t_\nu'^{\alpha} &= \sum_{\xi=0}^{\xi\max}\sum_{\eta=0}^{\eta\max} F(\nu;\eta-\xi)[E_\nu^{\alpha}(t_\xi';t_\eta') - E_\nu^{\alpha}(t_\xi'';t_\eta'')]\\
&\quad + 2\sum_{\xi=0}^{\xi\max}\sum_{\eta=0}^{\eta\max} F(\nu;\eta-\xi)E_\nu^{\alpha}(\tau_\eta;t_\xi') + 2\sum_{\lambda=1}^{\lambda\max}\sum_{\xi=0}^{\xi\max} F(\nu;\lambda+\xi)E_\nu^{\alpha}(R_{-\lambda};t_\xi')\\
&\quad + 2\sum_{\lambda=1}^{\lambda\max}\sum_{\xi=0}^{\xi\max} F(\nu;\lambda+\xi)E_\nu^{\alpha}(R_{-\lambda};\tau_\xi) + \sum_{\xi=0}^{\xi\max}\sum_{\eta=0}^{\eta\max} F(\nu;\eta-\xi)E_\nu^{\alpha}(\tau_\xi;\tau_\eta)\\
&\quad + \sum_{\lambda=1}^{\lambda\max}\sum_{\mu=1}^{\mu\max} F(\nu;\lambda-\mu)E_\nu^{\alpha}(R_{-\lambda};R_{-\mu}) + R_{-\nu}^{\alpha} - \tau_\nu^{\alpha},\\
t_0''^{\alpha} &= \sum_{\beta=1}^{N} C^{\alpha\beta} t_0''^{\beta},\\
t_\nu''^{\alpha} &= \sum_{\xi=0}^{\xi\max}\sum_{\eta=0}^{\eta\max} F(\nu;\eta-\xi)[E_\nu^{\alpha}(t_\xi'';t_\eta') + E_\nu^{\alpha}(t_\xi';t_\eta'')] + 2\sum_{\xi=0}^{\xi\max}\sum_{\eta=0}^{\eta\max} F(\nu;\eta-\xi)E_\nu^{\alpha}(\tau_\eta;t_\xi'')\\
&\quad + 2\sum_{\lambda=1}^{\lambda\max}\sum_{\xi=0}^{\xi\max} F(\nu;\lambda+\xi)E_\nu^{\alpha}(R_{-\lambda};t_\xi''),\quad \alpha=1,2,\ldots,N;\ \nu=1,2,\ldots,\nu_{\max}.
\end{aligned}
\tag{33}
$$

Iterations using Newton's method can be effected with arbitrary precision in the system (33). We assume $H_\nu^{\alpha} \approx H_\nu'^{\alpha}$ for an approximate solution, as the values of $H_\nu''^{\alpha}$ are very small. It is absolutely necessary to take the latter into account in order to guarantee the convergence of the process of successive approximations, though not in order to improve considerably the precision of H_ν^{α}.

Methods for the solution of dispersion integral equations other than the N/D method and the inverse Low amplitude method are described in Refs. 18–21.

V. CONCLUSION

The S-matrix problem, called Low's problem in this paper, is defined in Sec. II in an abstract way by the conditions (A), (B), (C), (D).

The main point of the investigation is the proof of the theorem in Sec. III, which guarantees a one-to-one correspondence between the abstract definition of Low's problem and the algebraic one given by the system (14). The latter offers a possibility of solving Low's problem numerically. In Sec. IV two methods are proposed for this purpose. The first is the well known Newton's method of successive linearizations. The second is based on the Banach-Cacciopoli principle of contracting mapping and leads to an iterative numerical algorithm. It is to be stressed that iteration is the main way to get numerical solutions of the related integral equations of dispersive type. However, in this case iterations are indirect because $h^{\alpha}(z)$ are represented either as N/D or as $1/(F(z))$. In this paper we have shown that the Low-amplitude method, where the iterations are applied directly to the partial-wave scattering amplitude $h^{\alpha}(z)$, is suitable for numerical treatment of the algebraic system as well as for the corresponding integral equation. It is noteworthy that this method is suitable not only for investigation of the adiabatic solutions. As has been proved in Ref. 9 it gives numerically the resonance solutions as well, i.e., solutions which depend on CDD poles. However, in this case the N/D method is more advantageous.

A reinvestigation of the numerical procedures described here is in progress, and will soon be submitted for publication.

Note added in proof. Professor A. Martin kindly drew my attention to the fact that the existence of cuts on the second sheet of $h^{\alpha}(z)$ contradicts its analyticity of the first sheet. (See also A. Martin, *Problems of Theoretical Physics,* essays dedicated to Nicolai N. Bogoliubov on the occasion of his sixtieth birthday (Nauka Publishing House, Moscow, 1969).

In Sec. IV the appearance of a corner in the δ_1 plot was tentatively interpreted as a manifestation of the existence of a cut in the second sheet of $h^1(z)$. If the corner which was conjectured on the basis of a numerical analysis really exists, the assertion about the existence of a cut must be modified. As Professor A. Martin suggests then instead of a cut there must exist suitably arranged poles which approximately simulate a cut.

ACKNOWLEDGMENTS

The author is very grateful to Professor Abdus Salam and Professor P. Budini as well as to the

International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. Thanks are also due to Professor L. Bertocchi and Professor G. Furlan, and also to W. Bantchev for interesting discussions.

*Institute of Physics of the Bulgarian Academy of Sciences, Sofia, Bulgaria.

[1]S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper, Row, New York, 1961).
[2]G. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).
[3]V. V. Serebryakov and D. V. Shirkov, Fortschr. Physik. 13, 227 (1965).
[4]D. V. Shirkov, V. V. Serebryakov, and V. A. Meshcheryakov, *Dispersion Theories of Strong Interactions at Low Energies* (North-Holland, Amsterdam, 1969).
[5]P. Kh. Byrnev, V. A. Meshcheryakov, and I. P. Nedialkov, Zh. Eksp. Teor. Fiz. 46, 663 (1964) [Sov. Phys. JETP 19, 449 (1964)].
[6]R. L. Warnock in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.*, (Gordon and Breach, New York, 1969), Vol. II D.
[7]I. P. Nedialkov, JINR, Dubna Report No. E-1294, 1963 (unpublished).
[8]H. McDaniel and R. L. Warnock, Nuovo Cimento 64A, 905 (1963).
[9]I. P. Nedialkov and G. J. Pentchev, JINR, Dubna Report No. P-1445, 1963 (unpublished).
[10]I. P. Nedelkov, ICTP, Trieste, Internal Report No. IC/71/91 (unpublished).
[11]R. L. Warnock, Phys. Rev. 170, 1323 (1968).
[12]D. Atkinson, J. Math. Phys. 8, 2281 (1967).
[13]N. K. Bari, *Trigonometrical Series* (State Publishing House for Physical and Mathematical Literature, Moscow, 1961).
[14]G. M. Fikhtengolts, *The Fundamentals of Mathematical Analysis* (Pergamon, London, 1965), Vol. II.
[15]I. P. Nedelkov, ICTP, Trieste, Report No. IC/71/86 1971 (unpublished).
[16]W. M. Layson, Nuovo Cimento 20, 1207 (1961).
[17]G. Salzman and F. Salzman, Phys. Rev. 108, 1619 (1957).
[18]P. K. Chang, Nucl. Phys. (to be published).
[19]P. Byrnev, R. Denchev, and Kim Tse, JINR, Dubna Report No. P-1284, 1963 (unpublished).
[20]G. Steinbrecher, Phys. Rev. 174, 1794 (1968).
[21]Steinbrecher informed me that he had now a numerical version of the method described in Ref. 20.

PHYSICAL REVIEW D VOLUME 6, NUMBER 10 15 NOVEMBER 1972

Algebra of Fields on the Light Cone*

Frank L. Feinberg†
Laboratory for Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 13 April 1972)

A consistent light-cone formulation of the algebra of fields is derived by the use of Schwinger's quantum action principle. The current commutators are extracted from the theory for a massive Yang-Mills field interacting with fermions by an invariant coupling. Deep-inelastic scaling sum rules are found which are formally valid and which relate $F_2(w)$ $[=F_L(w)]$ to the Fourier transform of the diagonal matrix elements of a bilocal operator. In addition, fixed-mass sum rules (valid if Class-II graphs do not contribute) are derived. These relate integrals of various structure functions to the bilocal operator mentioned above. In the Born approximation these fixed-mass sum rules are not valid, except at the physically interesting point $q^2=0$.

I. INTRODUCTION

In 1949 Dirac[1] pointed out that it is feasible to quantize a theory on a hyperplane tangent to the light cone (a method he calls the "front form"). During the past few years techniques which use such a quantization have been developed[2-8] in order to understand better high-energy processes. Several authors[2,5] have shown that for quantum electrodynamics the light-cone quantization formally gives the same S-matrix expansion as in a conventionally quantized system. Here we take the light-cone theory of vector mesons as the proper theory and therefore do not need to question the consistency of the two methods of quantization. There are many advantages to a light-cone formalism not found by conventional quantization methods. For example, Cornwall and