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Singularities in Complex Angular Momentum and Helicity*

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Simple representations of multiparticle amplitudes are proposed which we hope will be useful in working out the implications of Regge asymptotic behavior in practical applications. These generalized Sommerfeld-Watson transforms are integrals over complex angular momentum and helicity. We discuss the various types of singularities in complex angular momentum and helicity and the asymptotic limits of multiparticle amplitudes which they will determine. We discuss Regge-pole and Regge-cut singularities in the angular momentum plane and generalize the concept of wrong-signature–nonsense fixed poles to multiparticle amplitudes. We find that the absence of simultaneous discontinuities in overlapping channel invariants in the physical region requires that the relevant singularities in complex helicity be completely determined by the singularities in complex angular momentum. Therefore the only truly dynamical singularities are those in complex angular momentum.

I. INTRODUCTION

The implications of Regge behavior for four-particle amplitudes has been well understood for several years and we now have a practical lore of "Regge theory."[1] With the increasing experimental data on multiparticle amplitudes it has become increasingly important to have a comparable practical lore for multiparticle amplitudes.

While the generalization of the assumption of Regge-pole or Regge-cut singularities in the complex angular momentum plane to multiparticle amplitudes and its consequences is quite natural and well understood,[2] there is an entirely new feature in multiparticle amplitudes which is not so well understood. This is the Toller angle (ω) or, alternatively, helicity dependence of the amplitude. It is therefore this feature we will need to concentrate upon. We shall see that in order to be able to specify the consequences of Regge behavior[3] fully, in addition to singularities in the complex angular momentum plane, we will need to discuss singularities in the complex helicity plane. These singularities determine the dependence on the Toller angle and have a number of interesting consequences: They determine the behavior of multi-Regge vertices for $\cos\omega \to \infty$,[4] they determine the asymptotic behavior of amplitudes in regions distinct from the multi-Regge asymptotic region (helicity-pole limits),[5-7] and they are intimately related to the generalization of the concepts of "nonsense zeros" and "wrong-signature–nonsense fixed poles" to multi-Regge amplitudes.

In this paper we propose representations for multiparticle amplitudes which we hope will be useful in working out the implications of singularities in complex angular momentum and helicity in practical applications; for example, the phenomenology of exclusive and inclusive cross sections, finite-energy sum rules, etc. These representations are essentially multiple Sommerfeld-Watson transforms in angular momentum and helicity. A given representation is appropriate for the exhibition of the asymptotic behavior of the amplitude in certain limits. The asymptotic limits for which our representations are appropriate are those which determine a unique partial-wave expansion in which pushing the Sommerfeld-Watson contours to the left yields the desired asymptotic behavior; these are the Regge- and helicity-pole asymptotic limits. As a trivial example, the Regge limit, $s \to \infty$ with fixed t, of the four-particle amplitude determines the t-channel partial-wave analysis as the one whose Sommerfeld-Watson transform yields the asymptotic behavior.

Here we will not attempt to prove the existence of our representations from basic S-matrix principles and we will be content to assume the behavior at infinity in complex angular momentum and helicity necessary for the contour distortions made in obtaining asymptotic behaviors. For motivation for the representations we shall rely heavily on White's thorough study of the five-particle amplitude[4,8] in which the importance of obtaining Sommerfeld-Watson transforms in helicity has been particularly stressed. For motivation for the types of singularities to be expected in angular momentum and helicity, we shall of necessity rely on models to a certain extent.

In order to illustrate our approach we summarize some of our results. In Sec. II we review the situation for amplitudes for four spinless particles, emphasizing certain simple points which, however, will be vital in the generalization to multiparticle amplitudes. The amplitude can be conveniently represented by the Sommerfeld-Watson-type formula

$$A^{\tau}(s;\ t)=\frac{1}{2\pi i}\int dj\ \Gamma(-j)(-s)^{j}a^{\tau}(j;\ t)\,. \tag{1.1}$$

A^{τ} is an amplitude with only right-hand cuts in s, e.g., the signatured amplitude. Equation (1.1) differs from the usual Sommerfeld-Watson formula in that certain messy kinematic and group-theoretical factors have been absorbed into the partial-wave amplitude a^{τ}. In general our representations will be given in terms of the channel invariants rather than the group-theoretical variables of the partial-wave analyses. Since the amplitudes have simple analytic behavior in terms of the invariants, this avoids kinematic singularities and exhibits the essential features of the representations. The representations will thus be rather like generalized Khuri representations[9] and are, of course, identical to the partial-wave representations to leading asymptotic order.

The contour in j in (1.1) is taken to lie between the poles in $\Gamma(-j)$ and the ("dynamic") singularities in a^{τ} which are assumed to be of the following types:

(i) Regge poles,

$$a^{\tau}(j;\ t)\approx\frac{\beta(t)}{j-\alpha(t)}\ ; \tag{1.2a}$$

(ii) Regge cuts,

$$a^{\tau}(j,\ t)\approx\int_{-\infty}^{\alpha(t)}d\alpha\,\frac{\beta(\alpha;\ t)}{j-\alpha}\ ; \tag{1.2b}$$

and

(iii) fixed poles (wrong-signature nonsense),

$$a^{\tau}(j;\ t)\approx\frac{\beta(t)}{j-J}\,, \tag{1.2c}$$

with

$$J=-1,\ -2,\ -3,\ \ldots\quad\text{and}\quad\tau=(-1)^{J+1}\,.$$

The asymptotic behavior in s is calculated by sweeping the j contour in (1.1) to the left to pick up the contributions of the singularities (1.2). For example, a Regge pole (1.2a) gives the contribution,

$$A^{\tau}(s;\ t)\sim(-s)^{\alpha(t)}\Gamma[-\alpha(t)]\,\beta(t)\,. \tag{1.3}$$

In the absence of singularities in β, A^{τ} will have poles only for $\alpha(t)=0,\ 1,\ 2,\ \ldots$. The absence of poles for negative integral $\alpha(t)$ is the manifestation of the existence of nonsense zeros when multiplicative fixed poles are not present. Although the fixed poles do not contribute to the asymptotic behavior of the amplitude because they are wrong signature, their existence can be tested with sum rules.

In Sec. III we discuss the five-particle case in some detail, using heavily White's application[4,8] of the Sommerfeld-Watson transform to the five-particle amplitude. We motivate the following expression:

$$A^{\tau_1\tau_2}(s_1,\ s_2,\ s;\ t_1,\ t_2)=\left(\frac{1}{2\pi i}\right)^3\int dm\int dj_1\int dj_2\Gamma(-m)\Gamma(-j_1+m)\Gamma(-j_2+m)$$
$$\times(-s_1)^{j_1-m}(-s_2)^{j_2-m}(-s)^{m}a^{\tau_1\tau_2}(j_1,j_2,m;\ t_1,\ t_2)\,, \tag{1.4}$$

where $A^{\tau_1\tau_2}$ has only right-hand cuts in the total energy s and the subenergies s_1 and s_2. The contour of integration is roughly such that the explicit singularities in the Γ functions lie to the right and the ("dynamical") singularities in $a^{\tau_1\tau_2}$ lie to the left. There are two types of singularities in the helicity in arising from the gamma functions, however. Those from $\Gamma(-m)$ lie to the right of the contour whereas those from $\Gamma(-j_1-m)\times\Gamma(-j_2+m)$ lie to the left (see Fig. 5).

We shall suggest on the basis of the absence of simultaneous discontinuities in overlapping channel invariants in the physical region that, in the asymptotic limits for which (1.4) is appropriate, there will be no contribution from singularities in m to the left of the contour in $a^{\tau_1\tau_2}$. This has the important consequence that the singularities in helicity which determine the asymptotic limit $s/s_1s_2\to\infty$ are completely determined by the singularities in angular momentum through their pinching the contour of integration with the singularities in $\Gamma(-j_1+m)\Gamma(-j_2+m)$. In particular, helicity poles are determined completely by Regge poles at $j_i=\alpha(t_i)\equiv\alpha_i$ to lie at $m=\alpha_i,\ \alpha_i-1,\ \alpha_i-2,\ \ldots$. Therefore, the only fundamental dynamical singularities are those in the angular momentum plane.

The partial-wave amplitude $a^{\tau_1\tau_2}$ is thus assumed to have singularities in the angular momenta of the types:

(i) Regge poles,

$$a^{\tau_1\tau_2} \approx \frac{R(j_2, m; t_1, t_2)}{j_1 - \alpha_1}; \tag{1.5a}$$

(ii) Regge cuts,

$$a^{\tau_1\tau_2} \approx \int_{-\infty}^{\alpha_1} d\alpha \, \frac{R(\alpha, j_2, m; t_1, t_2)}{j_1 - \alpha}; \tag{1.5b}$$

(iii) nonsense–wrong-signature fixed poles of the two types

$$a^{\tau_1\tau_2} \approx \frac{R(j_2, m; t_1 t_2)}{j_1 - J_1}, \tag{1.5c}$$

with

$$J_1 = -1, -2, -3, \ldots \quad \text{and} \quad \tau_1 = (-1)^{J_1+1},$$

and

$$a^{\tau_1\tau_2} \approx \frac{R(j_2, m; t_1, t_2)}{j_1 - m - J_1}, \tag{1.5d}$$

with

$$J_1 = -1, -2, -3, \ldots \text{ and } \tau_1\tau_2 = (-1)^{J_1+1}.$$

These two types of fixed poles correspond to nonsense with respect to the helicities at the vertices to the left and right of j_1, respectively.

Let us mention a few consequences of this proposal. A double-Regge pole

$$a^{\tau_1\tau_2} \approx \frac{\beta(m; t_1, t_2)}{(j_1 - \alpha_1)(j_2 - \alpha_2)}, \tag{1.6}$$

where, for the reasons discussed above, $\beta(m; t_1, t_2)$ can have no singularities in m, gives an asymptotic contribution

$$A^{\tau_1\tau_2} \sim (-s_1)^{\alpha_1}(-s_2)^{\alpha_2}\left(\frac{1}{2\pi i}\right) \times \int dm \Gamma(-m)\Gamma(-\alpha_1 + m) \times \Gamma(-\alpha_2 + m)\left(-\frac{s}{s_1 s_2}\right)^m \beta(m; t_1, t_2). \tag{1.7}$$

The behavior for large $\eta \equiv s/s_1 s_2$ is obtained by pushing the m contour to the left and picking up the poles in helicity in the second two Γ functions,

$$A^{\tau_1\tau_2} \sim (-s_1)^{\alpha_1}(-s_2)^{\alpha_2} \times [(-\eta)^{\alpha_1}\Gamma(-\alpha_1)\Gamma(\alpha_1 - \alpha_2)\beta(\alpha_1; t_1, t_2) + (-\eta)^{\alpha_2}\Gamma(-\alpha_2)\Gamma(\alpha_2 - \alpha_1)\beta(\alpha_2; t_1 t_2)]. \tag{1.8}$$

We see, as mentioned above, that the helicity-pole locations are completely determined by the Regge-pole locations.[10]

There is also a rather new type of asymptotic limit for which the representation (1.4) is appropriate – the helicity-pole limit,[5,6,7]

$$s_1 \to \infty, \quad s/s_1 \to \infty; \quad t_1, t_2, s_2 \text{ fixed}.$$

This limit also uniquely determines the partial-wave analysis of (1.4) and, assuming dominant Regge poles at $j_i = \alpha_i$, we have

$$A^{\tau_1\tau_2} \sim (-s_1)^{\alpha_1}\left[\left(\frac{-s}{-s_1}\right)^{\alpha_2}\Gamma(-\alpha_2)\Gamma(-\alpha_1 + \alpha_2)\beta(\alpha_2; t_1, t_2) + \left(\frac{-s}{-s_1}\right)^{\alpha_1}\Gamma(-\alpha_1)\left(\frac{1}{2\pi i}\right)\int dj_2 \Gamma(-j_2 + \alpha_1)(-s_2)^{j_2 - \alpha_1} R(j_2, \alpha_1; t_1, t_2)\right], \tag{1.9}$$

where R and β are defined by (1.5a) and (1.6). We shall discuss some features of the interesting structure of (1.9) in Sec. III.

The nonsense–wrong-signature fixed poles, (1.5c) and (1.5d), have the effect of eliminating nonsense zeros if they occur multiplicatively with Regge poles. This will be discussed further in Sec. III. We note that it would be particularly interesting to test for the existence of such fixed poles by using sum rules for their residues.

In Sec. III we also discuss the signature associated with the complex helicity which we have suppressed above and its implications.

In Sec. IV we discuss some aspects of amplitudes with more than five particles. A new feature arises in such amplitudes. There are nonlinear relations among the channel invariants due to the four-dimensionality of space-time which must be taken into account. We believe the correct treatment of these relations is intimately connected with the satisfaction of the requirement of no simultaneous discontinuities in overlapping invariants. We illustrate what we believe is the correct way to handle this problem by a discussion of the multiperipheral and triple-Regge limits of the six-particle amplitude.

II. FOUR-PARTICLE AMPLITUDE

In order to illustrate our approach with a familiar example we give a brief discussion of the amplitude for four spinless equal-mass particles.

Of the three channel invariants for the four-particle amplitude (s, t, and u – see Fig. 1), two may be chosen as independent variables (say s and t). Since we wish to discuss the limit $s \to \infty$ with t fixed, it is natural to group the channel invariants into two classes:

$$\begin{aligned} &\text{“}t\text{” invariants: } t, \\ &\text{“}s\text{” invariants: } s, u. \end{aligned} \tag{2.1}$$

The usual normal-threshold singularities in the "s" invariants lead to right-hand and left-hand cuts in s.

It is well known that in order to obtain a partial-wave amplitude with suitable Carlsonian continuation to complex angular momentum, one must perform the partial-wave analysis on an amplitude with only right-hand or only left-hand cuts. Such amplitudes can be obtained by writing a dispersion relation for the full amplitude,

$$\begin{aligned} A(s;t) &= \frac{1}{\pi}\int_R \frac{\mathrm{Im}A_R(s';t)ds'}{s'-s} + \frac{1}{\pi}\int_L \frac{\mathrm{Im}A_L(u';t)du'}{u'-u} \\ &\equiv A_R(s;t) + A_L(u;t). \end{aligned} \tag{2.2}$$

One can then do the partial-wave analysis on the amplitudes A_R and A_L since they have cuts in only one direction. It is more conventional to introduce signatured amplitudes

$$A^\tau(s;t) \equiv \tfrac{1}{2}[A_R(s;t) + \tau A_L(s;t)], \tag{2.3}$$

where $\tau \equiv \pm 1$, and perform the partial-wave analysis on them. The full amplitude can be reconstructed using

$$A(s;t) = \sum_{\tau=\pm 1} [A^\tau(s;t) + \tau A^\tau(u;t)]. \tag{2.4}$$

We see that the net effect of introducing signature is to introduce two different amplitudes giving two degrees of freedom corresponding to the two distinct cuts in s. As far as we shall be concerned, doing partial-wave analyses on A_R and A_L or A^+ and A^- are equally acceptable[11]; for example, the assumption of dominance of either A^+ or A^- is simply equivalent to a statement that $A_R(s,t)$ is

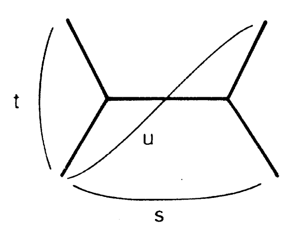

FIG. 1. Channel invariants for four-particle amplitude.

equal or opposite in sign to $A_L(s,t)$. For mostly historical reasons we shall use the signature notation. We shall assume the amplitudes with only right-hand cuts which we partial-wave analyze have in other respects the same singularity structure as the full amplitude. This important assumption greatly restricts the possible singularities of the partial-wave amplitudes.

The asymptotic limit of interest determines an appropriate set of group variables in which a Sommerfeld-Watson transform gives the desired asymptotic behavior. In this case, we perform an O(3) partial-wave analysis in the t-channel center-of-mass system. We then have

$$s = -\tfrac{1}{2}(t-4m^2) + \tfrac{1}{2}(t-4m^2)\cos\theta \tag{2.5}$$

and the asymptotic behavior in $\cos\theta$ gives the asymptotic behavior in s.

We therefore begin with the partial-wave expansion for $A^\tau(s;t)$,

$$A^\tau(s;t) = \sum_{j=0}^{\infty}(2j+1)a^\tau(j;t)P_j(\cos\theta), \tag{2.6}$$

where, using (2.3),

$$\begin{aligned} a^\tau(j;t) = \frac{1}{\pi}\int &[\mathrm{Im}A_R(s';t) + \tau\,\mathrm{Im}A_L(s';t)] \\ &\times Q_j(\cos\theta')d\cos\theta'. \end{aligned} \tag{2.7}$$

Obtaining the Sommerfeld-Watson transform of (2.6) gives

$$A^\tau(s;t) = -\frac{1}{2i}\int \frac{dj\,(2j+1)a^\tau(j;t)P_j(-\cos\theta)}{\sin\pi j}. \tag{2.8}$$

In order to be able to push the contour to the left of $\mathrm{Re}\,j = -\tfrac{1}{2}$ we need to perform Mandelstam's trick which effectively replaces

$$\frac{P_j(-\cos\theta)}{\sin\pi j}$$

by

$$-\frac{Q_{-j-1}(-\cos\theta)}{\pi\cos\pi j}.$$

Since here we will be interested only in leading asymptotic behaviors, we replace Q_{-j-1} by its asymptotic form

$$Q_{-j-1}(-\cos\theta) \sim \sqrt{\pi}\,\frac{\Gamma(-j)}{\Gamma(-j+\frac{1}{2})}(-2\cos\theta)^j \tag{2.9}$$

and write

$$A^\tau(s;t) = \frac{1}{2\pi i}\int dj\,\Gamma(-j)(-s)^j a^\tau(j;t). \tag{2.10}$$

In obtaining (2.10) we have redefined a^τ by absorbing various nonsingular factors.[12] Kinematic singularities have been taken into account in a natural way by extracting the threshold behavior from a^τ to convert $\cos\theta$ to s.[13] The contour of integration is taken to separate the singularities in $\Gamma(-j)$ from those in a^τ. Equation (2.10) is particularly convenient since it emphasizes the fundamental features of the Sommerfeld-Watson transform [for example, the factor $\Gamma(-j)$ yields the sum over nonnegative integral j in (2.6) when the contour is closed to the right]. It is also sufficient for phenomenology with leading asymptotic behavior.

The asymptotic behavior of A^τ is obtained by pushing the j contour of integration to the left,[14] thereby picking up the singularities in a^τ. At present our intuition about the types and locations of singularities in complex angular momentum rests mostly on models and rigorous results from two-body unitarity. This leads us to expect the following types of singularities (see Fig. 2):

(i) Regge poles,

$$a^\tau(j;t)\approx\frac{\beta(t)}{j-\alpha(t)}\,; \tag{2.11a}$$

(ii) Regge cuts,

$$a^\tau(j;t)\approx\int_{-\infty}^{\alpha(t)}d\alpha\,\frac{\beta(\alpha;t)}{j-\alpha}\,; \tag{2.11b}$$

(iii) nonsense–wrong-signature fixed poles,

$$a^\tau(j;t)\approx\frac{\beta(t)}{j-J}\,, \tag{2.11c}$$

with

$$J=-1,\,-2,\,-3,\,\ldots\quad\text{and}\quad\tau=(-1)^{J+1}\,.$$

For simple singularities like the above, the residues α and β must be regular in t for $t<4m^2$. Singularities in t would lead to singularities in A^τ which are inconsistent with our assumption that it have the same analytic structure as the full amplitude. The singularities (2.11) can, however, occur in a "multiplicative" manner, i.e., in such a way that the individual residues have singularities in t which cancel in the full partial-wave amplitude. A particularly well-known example is a multiplicative fixed pole and Regge pole,

$$a^\tau(j;t)\approx\frac{\tilde\beta(t)}{[j-\alpha(t)][j-J]}=\frac{\tilde\beta(t)}{[\alpha(t)-J]}\,\frac{1}{j-\alpha(t)}-\frac{\tilde\beta(t)}{[\alpha(t)-J]}\,\frac{1}{j-J}\,, \tag{2.12}$$

where $\tilde\beta$ is regular for $t<4m^2$. Whereas a Regge-pole contribution normally behaves like

$$\begin{aligned}&A^\tau(s;t)\sim\Gamma[-\alpha(t)]\,\beta(t)(-s)^{\alpha(t)}\,,\\&A(s;t)\sim\Gamma[-\alpha(t)]\,\beta(t)(e^{-i\pi\alpha(t)}+\tau)s^{\alpha(t)}\,,\end{aligned} \tag{2.13}$$

and thus A vanishes for $\alpha(t)$ a negative integer (nonsense) of wrong signature, with (2.12) $A(s,t)$ will be finite at such a point. The multiplicative fixed pole is thus correlated with the absence of the nonsense zero. Multiplicative singularities need not be of the simple form (2.12) – any form for a^τ in which the individual poles or cuts have singularities not present in a^τ itself is permissible.

Let us discuss the fixed poles (2.11c) in more detail since their generalization to many-particle amplitudes will present the most difficulty. From (2.7) and (2.9) we see that $a^\tau(j,t)$ has a potential singularity for j negative integral. Such a "group-theoretical" singularity only makes such in singularity in a^τ appear more or less natural, it could always have a vanishing residue (at least outside the region where two-body unitarity holds). Thus while such factors provide heuristic reasons for fixed poles they do not guarantee their existence. From unitarity we know that fixed poles actually cannot be present unless they are masked, for example, by moving Regge cuts. Since such mechanisms have been found only for wrong signature we assume only wrong-signature fixed poles. These, of course, have the property that they do not give any asymptotic fixed power behavior the full amplitude. Their existence can be checked, however, by sum rules, e.g.,

$$\frac{1}{\pi}\int ds'\,(s')^{-J-1}[\mathrm{Im}A_R(s';t)+(-1)^{J+1}\mathrm{Im}A_L(s';t)]=\Gamma(-J)\beta(t)(-1)^J\,. \tag{2.14}$$

The existence of multiplicative fixed poles can also be implicitly checked by the absence of nonsense zeros as discussed above.

The above discussion can be generalized to the case of arbitrary spin particles[1] or representa-

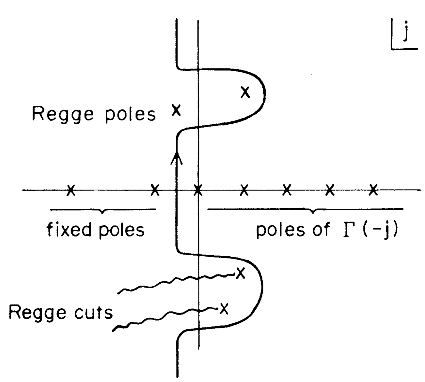

FIG. 2. Singularities in complex angular momentum.

tions corresponding to (2.10) can be obtained directly by taking the residues of the multiparticle amplitudes below at nonzero spin poles. We briefly discuss this in Sec. IV.

III. FIVE-PARTICLE AMPLITUDE

A. Double-Regge Asymptotic Limit

There are ten different channel invariants in the five-particle amplitude (see Fig. 3) of which five are independent. These ten invariants naturally fall into four groups:

$$\text{“}t\text{” invariants: } t_1, t_2;$$
$$\text{“}s\text{” invariants:} \quad s_1, u_1; \quad s_2, u_2; \quad s, s', u, u'. \tag{3.1}$$

All "s" invariants in the same group are asymptotically proportional in the double-Regge region and we choose as independent variables the first member of each group along with t_1 and t_2.

In general one expects, along with other singularities, normal threshold singularities in the "s" invariants. In the double-Regge region these will yield right-hand and left-hand singularities in s_1, s_2, and s. Thus a decomposition of the amplitude into terms with cuts on only one side analogous to (2.2) would be expected to consist of eight terms (corresponding to a choice of either right-hand or left-hand cuts for each of s_1, s_2, and s). We shall not discuss possible generalizations of (2.2) here

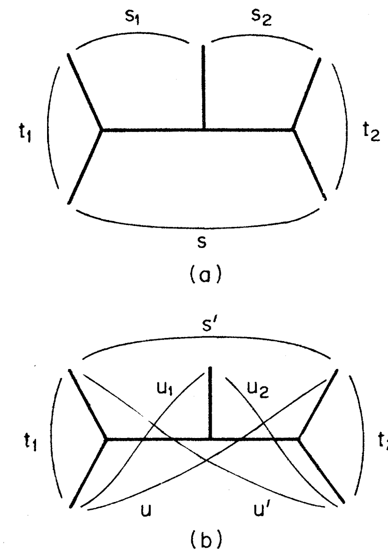

FIG. 3. Channel invariants for five-particle amplitude.

but instead rely on the development of signatured amplitudes given by White.[8] We refer the reader to his paper for discussion of some of the many difficulties involved in a rigorous definition. We only give here the generalization of (2.4). Instead of the variable s we use $\eta \equiv s/s_1 s_2$ as the fifth independent variable – the reason for this will become clear when we discuss the partial-wave analysis below. We have[15]

$$A(s_1, s_2, \eta; t_1, t_2) = \sum_{\tau_1, \tau_2, \tau_{12} = \pm 1} \{[A^{\tau_1\tau_2\tau_{12}}(s_1, s_2, \eta; t_1, t_2) + \tau_1 A^{\tau_1\tau_2\tau_{12}}(-s_1, s_2, \eta; t_1, t_2) + \tau_2 A^{\tau_1\tau_2\tau_{12}}(s_1, -s_2, \eta; t_1, t_2)$$
$$+ \tau_1\tau_2 A^{\tau_1\tau_2\tau_{12}}(-s_1, -s_2, \eta; t_1, t_2)] + \tau_{12}(\eta \to -\eta)\}. \tag{3.2}$$

We see that there are now three signatures,[4] and thus a total of eight amplitudes, giving 8 degrees of freedom corresponding to the eight combinations of right-hand and left-hand cuts in s_1, s_2, and s. The signatures τ_1 and τ_2 are the ordinary signatures for the angular momentum in t_1 and t_2 channels, whereas τ_{12} is a new signature for the helicity at the central vertex.[4] From now on we shall discuss the partial-wave analysis of the signatured amplitudes. If one prefers, one can think of the analysis as applying to an amplitude with only right-hand cuts in s_1, s_2, and η which would be used directly in a generalization of (2.2) which might hold at least asymptotically.

The appropriate set of group variables in which Sommerfeld-Watson transforms lead to the desired double-Regge asymptotic behavior is given by a double O(3) analysis in the t_1- and t_2-channel centers-of-mass systems. We have

$$s_1 = 2m^2 + \tfrac{1}{2}(t_2 - m^2 - t_1) + \frac{1}{2}\left[\frac{(t_1 - 4m^2)\lambda(t_1, t_2, m^2)}{t_1}\right]^{1/2} \cos\theta_1,$$
$$s_2 = 2m^2 + \tfrac{1}{2}(t_1 - m^2 - t_2) + \frac{1}{2}\left[\frac{(t_2 - 4m^2)\lambda(t_1, t_2, m^2)}{t_2}\right]^{1/2} \cos\theta_2,$$
$$s = 2m^2 + \tfrac{1}{4}(m^2 - t_1 - t_2) + \frac{1}{4}\left[\frac{(t_1 - 4m^2)\lambda(t_1, t_2, m^2)}{t_1}\right]^{1/2} \cos\theta_1 + \frac{1}{4}\left[\frac{(t_2 - 4m^2)\lambda(t_1, t_2, m^2)}{t_2}\right]^{1/2} \cos\theta_2 \tag{3.3}$$
$$+ \frac{1}{4}\left[\frac{(t_1 - 4m^2)(t_2 - 4m^2)}{t_1 t_2}\right]^{1/2} (m^2 - t_1 - t_2)\cos\theta_1\cos\theta_2 - \tfrac{1}{2}[(t_1 - 4m^2)(t_2 - 4m^2)]^{1/2} \sin\theta_1 \sin\theta_2 \cos\omega,$$

where $\lambda(a,b,c)=a^2+b^2+c^2-2ab-2bc-2ca$. From (3.3) we see that s_1 and s_2 are directly related to the angles θ_1 and θ_2 whereas s involves also the Toller angle ω. In the double-Regge region we have

$$\eta \equiv \frac{s}{s_1 s_2} \sim \frac{2(t_1t_2)^{1/2}\cos\omega + m^2 - t_1 - t_2}{\lambda(t_1, t_2, m^2)}, \tag{3.4}$$

and thus η is simply related to the Toller angle.

The double partial-wave expansion for $A^{\tau_1\tau_2\tau_{12}}(s_1, s_2; t_1, t_2)$ is

$$A^{\tau_1\tau_2\tau_{12}} = \sum_{m=-\infty}^{\infty} \sum_{j_1=|m|}^{\infty} \sum_{j_2=|m|}^{\infty} (2j_1+1)(2j_2+1) \times a^{\tau_1\tau_2\tau_{12}}(j_1, j_2, m; t_1, t_2) \times d^{j_1}_{om}(\cos\theta_1) d^{j_2}_{mo}(\cos\theta_2) e^{im\omega}. \tag{3.5}$$

Following Goddard and White, we obtain the Sommerfeld-Watson transform of (3.5) in two steps. First a Sommerfeld-Watson transform is performed on the helicity m. We have, suppressing irrelevant labels,

$$\begin{aligned} A(z) &= \sum_{m=0}^{\infty} a^>(m) z^m - \sum_{m=-\infty}^{-1} a^<(m) z^m \\ &= -\frac{1}{2i}\int_{C^>} dm\, \frac{a^>(m)(-z)^m}{\sin\pi m} \\ &\quad -\frac{1}{2i}\int_{C^<} dm\, \frac{a^<(m)(-z)^m}{\sin\pi m}, \end{aligned} \tag{3.6}$$

where $z \equiv e^{i\omega}$ and the contours are shown in Fig. 4. We have assumed the analog of (2.2),

$$A(z) = \int_0^{1-\epsilon} \frac{\mathrm{Im}A(z')dz'}{z'-z} + \int_{1+\epsilon}^{\infty} \frac{\mathrm{Im}A(z')dz'}{z'-z},$$

which gives

$$a^>(m) = \int_{1+\epsilon}^{\infty} (z')^{-m-1}\,\mathrm{Im}A(z')dz',$$

$$a^<(m) = \int_0^{1-\epsilon} (z')^{-m-1}\,\mathrm{Im}A(z')dz'.$$

The amplitudes $a^>(m)$ and $a^<(m)$ have good asymptotic behaviors in the right-half and left-half m planes respectively. We may thus write (see Fig. 4)

$$A(z) = -\frac{1}{2i}\int_C dm\, \frac{a^>(m)+a^<(m)}{\sin\pi m}(-z)^m. \tag{3.7}$$

We now would like to translate from $\omega(z)$ to η. From (3.3) we see that η is even in ω (in $z \to 1/z$) and thus only even functions of ω can be written as functions of η. The possibility that the amplitude is odd in ω corresponds to the possibility of a dependence on the pseudoscalar object $\epsilon^{\mu\nu\lambda\sigma}P_{1\mu}P_{2\nu} \times P_{3\lambda}P_{4\sigma}$. Here for simplicity we shall assume there is no such dependence.[16] One easily sees that in this case $a^<(m) = -a^>(-m)$ and that the amplitude can be written equivalently to (3.7) as

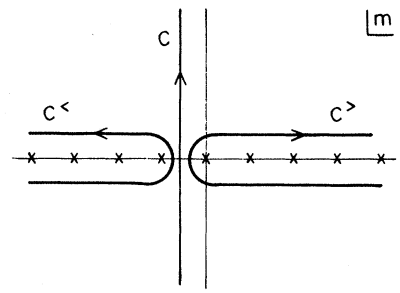

FIG. 4. Integration contours in complex helicity plane.

$$A(\eta) = -\frac{1}{2i}\int_C dm\, \frac{a(m)}{\sin\pi m}(-\eta)^m. \tag{3.8}$$

We note that the Fourier expansion (3.6) is obtained by closing the contour C to the right. To obtain the asymptotic behavior for large η, on the other hand, we sweep the contour to the left. It appears that there are fixed inverse powers of η arising from the poles in $(\sin\pi m)^{-1}$. However, there is no reason to expect these generally and, indeed, we shall argue below that they cannot be present. Consequently we finally rewrite (3.9) as

$$A(\eta) = \frac{1}{2\pi i}\int_C dm\, \Gamma(-m)(-\eta)^m a(m). \tag{3.9}$$

The second step in the Sommerfeld-Watson transform of (3.5) is to perform the usual partial-wave analysis of

$$a(m) = a^{\tau_1\tau_2\tau_{12}}(s_1, s_2, m; t_1, t_2).$$

This analysis must be done for complex helicity m so that, for m positive integral, one reobtains (3.5). The correct complete set of functions to use are the $d^{j_i}_{om}(\cos\theta_i)$ [equivalently the Jacobi polynomials $P^{(m,m)}_{j_i-m}(\cos\theta_i)$]. Then one obtains, analogously to (2.6),

$$a(m) = \sum_{j_1=m}^{\infty}\sum_{j_2=m}^{\infty} (2j_1+1)(2j_2+1)a(j_1, j_2, m; t_1, t_2) \times d^{j_1}_{om}(\cos\theta_1) d^{j_2}_{mo}(\cos\theta_2). \tag{3.10}$$

We note that if the helicity m is nonintegral then the sums are over nonintegral j_i as well. The Sommerfeld-Watson transform on (3.10) then gives analogously to (2.8),

$$a(m) = \left(\frac{1}{2i}\right)^2 \int dj_1 \int dj_2 (2j_1+1)(2j_2+1)\frac{a(j_1, j_2, m; t_1, t_2) d^{j_1}_{om}(-\cos\theta_1) d^{j_2}_{mo}(-\cos\theta_2)}{\sin\pi(j_1-m)\sin\pi(j_2-m)}. \tag{3.11}$$

Finally, using the generalization of the Mandelstam trick[1] and taking the leading asymptotic form, we have

$$A^{\tau_1\tau_2\tau_{12}}(s_1, s_2, \eta; t_1, t_2) = \left(\frac{1}{2\pi i}\right)^3 \int dm \int dj_1 \int dj_2 \Gamma(-m)\Gamma(-j_1+m)\Gamma(-j_2+m)(-s_1)^{j_1}(-s_2)^{j_2}(-\eta)^m \times a^{\tau_1\tau_2\tau_{12}}(j_1, j_2, m; t_1, t_2)\,. \tag{3.12}$$

In obtaining (3.12) we have extracted certain "kinematic" singularities like $[\Gamma(-j_1+m)]^{1/2}$ from the partial-wave amplitude, as well as usual threshold behaviors.[13] The advantage of this is, of course, that (3.12) expresses the amplitude in terms of the invariants in terms of which it has the simplest analyticity properties. We shall assume suitable asymptotic behavior in j_1, j_2, and m to allow the neglect of all contours at infinity. In fact so far simultaneous Carlsonian behavior has only been established for either j_1 or j_2 and m.[8]

We now need to discuss the location of the contour of integration in the j_1, j_2, and m planes. Roughly speaking the contour is such that the singularities needed to reproduce the partial-wave series (3.5) lie to the right [e.g., the singularities in $\Gamma(-m)$, $\Gamma(-j_1+m)$, and $\Gamma(-j_2+m)$] and the "dynamical" singularities in $a^{\tau_1\tau_2\tau_{12}}$ lie to the left.[17] It is, of course, difficult to produce diagrams of the contour since it is a three-dimensional surface in a six-dimensional space. We show in Fig. 5 sections in the complex j_1 and m planes when the j_1 and m integrals, respectively, are carried out first and also a section in the real part of j_1 and m plane for arbitrary order of integration.

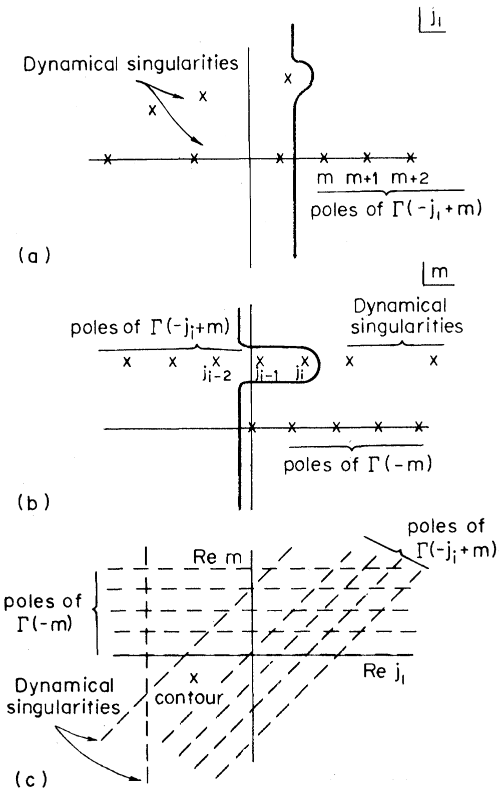

FIG. 5. Integration contour for Eq. (3.12). (a) Complex j_1 plane when j_1 integration performed first. (b) Complex m plane when m integration performed first. (c) Real part of m and real part of j_1 plane.

Suppose the j_1 integrations in (3.12) are performed first. Clearly $a^{\tau_1\tau_2\tau_{12}}$ cannot have any singularities in j_i to the right of the contour, since when the contour is closed to the right we must reproduce the partial-wave series (3.10). However, $a^{\tau_1\tau_2\tau_{12}}$ can have singularities in j_i to the left of the contour which will contribute to the asymptotic behavior. Now consider the integration of the resultant function over m. Again there cannot be singularities to the right of the contour, since closing the contour to the right must reproduce (3.6). Such singularities will clearly be absent if $a^{\tau_1\tau_2\tau_{12}}$ itself has no singularities whose position depends upon m, but this is not a necessary condition. It is necessary that there be no j_i-independent singularities in m to the right of the contour. However, j_i-dependent singularities could be present, since they would be washed out in the j_i integrations if they cannot pinch against the singularities in the $\Gamma(-j_i+m)$. The fixed-pole singularities discussed below are such singularities. The only singularities to the left of the contour in the final m integration to be expected are the singularities arising from the pinching of the dynamical singularities in the j_i against the singularities in $\Gamma(-j_i+m)$. This is because the integrand of (3.12) has the form

$$(-s_1)^{j_1-m}(-s_2)^{j_2-m}(-s)^m\,. \tag{3.13}$$

The natural interpretation of the phase of this expression for positive s_1, s_2, and s is that it represents the usual physical region unitarity (e.g., normal threshold, etc.) singularities in these channels. Thus in an asymptotic limit for which a term of the form (3.13) dominates, e.g., $s_1, s_2, s \to \infty$, this expression would represent an amplitude with simultaneous discontinuities in s_1 and s_2. Since such physical-region simultaneous discontinuities in overlapping-channel invariants are prohibited by the Steinmann relations,[18] it must be that either $j_1 - m$ of $j_2 - m$ is a nonnegative integer.[19] Only singularities in m arising from pinches of the j_i integrations against the $\Gamma(-j_i+m)$ satisfy this. We therefore see that *the locations of the singularities in helicity which determine asymptotic behavior are completely determined by the singularities in angular momentum*. These singularities are at

$$m = \alpha - N , \tag{3.14}$$

where α is the location of a dynamical singularity in j_1 or j_2 and N is a non-negative integer. The required singularity structure of the amplitude thus has a very simple expression in complex helicity language.

It is also interesting to imagine performing the m integration first. If the contour is pushed to the left, we pick up only the poles in the $\Gamma(-j_i+m)$ and simultaneous discontinuities in overlapping variables in (3.13) are clearly absent as before. On the other hand, if the contour is closed to the right, it is important to notice, as emphasized by White,[8] that the required partial-wave sums over j_i are produced by pinches of the m contour for integral j_i by the poles in $\Gamma(-j_i+m)$ and $\Gamma(-m)$. The fixed-pole singularities discussed below, or other j_i-dependent singularities in m to the right of the contour, will give no contribution to the partial-wave series in j_i since they do not cause pinches against $\Gamma(-m)$.

As is the case for the four-particle amplitude, a more precise specification of what types of dynamical singularities to expect must depend at present largely on conjectures based on model calculations. We expect the following types of singularities in the angular momentum plane j_1 (or j_2):

(i) Regge poles,

$$a^{\tau_1\tau_2\tau_{12}}(j_1, j_2, m; t_1, t_2) \approx \frac{R(j_2, m; t_1, t_2)}{j_1 - \alpha_1} ; \tag{3.15a}$$

(ii) Regge cuts,

$$a^{\tau_1\tau_2\tau_{12}}(j_1, j_2, m; t_1, t_2) \approx \int_{-\infty}^{\alpha_1} d\alpha \frac{R(\alpha, j_2, m; t_1, t_2)}{j_1 - \alpha} ; \tag{3.15b}$$

(iii) nonsense–wrong-signature fixed poles of two types,

(a)

$$a^{\tau_1\tau_2\tau_{12}}(j_1, j_2, m; t_1, t_2) \approx \frac{R(j_2, m; t_1, t_2)}{j_1 - J_1} , \tag{3.15c}$$

with

$$J_1 = -1, -2, -3, \ldots \quad \text{and} \quad \tau_1 = (-1)^{J_1+1} ,$$

(b)

$$a^{\tau_1\tau_2\tau_{12}}(j_1, j_2, m; t_1, t_2) \approx \frac{R(j_2, m; t_1, t_2)}{j_1 - m - J_1} , \tag{3.15d}$$

with

$$J_1 = -1, -2, -3, \ldots \quad \text{and} \quad \tau_1\tau_2 = (-1)^{J_1+1} .$$

We expect the residues above to be entire functions of m. We expect no singularities in m other than those already present in (3.15d) above.

The Regge-pole and -cut singularities are quite natural generalizations of (2.11a) and (2.11b); the fixed-pole singularities deserve further comment, however. The two types of fixed poles correspond to nonsense with respect to the two vertices which the angular momentum j_i couples. Thus the vertex to the left of j_1 has two spinless particles and the first nonsense point is $j_1 = -1$ – this gives (3.15c). The vertex to the right of j_1 has (complex) helicity m and the first nonsense point is $j_1 = m - 1$ – this gives (3.15d). The rule $\tau_1\tau_2 = (-1)^{J_1+1}$ defining wrong signature for this case is a rather natural generalization of the rule for vertices with spinless particles. For j_2 integral and $\tau_2 = (-1)^{j_2}$ we expect particle poles in t_2. The maximum helicity of these states is j_2 and thus the nonsense points are $j_1 = j_2 + J_1$ and the usual rule gives $\tau_1 = (-1)^{j_2+J_1+1}$ or $\tau_1\tau_2 = (-1)^{J_1+1}$.

As is the case for the four-particle amplitude, the existence of fixed poles (3.15c) and (3.15d) is suggested by "group theoretical" singularities in the functions defining the partial-wave amplitudes.[20,21] The expression analogous to (2.9) is

$$e_{mn}^{-j-1}(-\cos\theta) \sim \sqrt{\pi}\, \frac{[\Gamma(m-j)\Gamma(-m-j)\Gamma(n-j)(-n-j)]^{1/2}}{\Gamma(-j)\Gamma(-j+\frac{1}{2})} \times (-2\cos\theta)^j , \tag{3.16}$$

which for $n=0$ clearly exhibits a singularity at $j = m+J$.[22] Such arguments only make the existence of fixed poles appear natural; whether or not they are present depends, of course, on dynamics. We shall see below that the presence of multiplicative fixed poles has the very direct and simple consequence of removing certain "kinematic" nonsense zeros (which in general need not be present) from the asymptotic behavior of the amplitude.

Let us now discuss the behavior of the scattering amplitude which the various singularities (3.15) lead to. First consider the contribution of a double-Regge pole

$$a^{\tau_1\tau_2\tau_{12}} \approx \frac{\beta(m; t_1, t_2)}{(j_1-\alpha_1)(j_2-\alpha_2)}:$$

$$A^{\tau_1\tau_2\tau_{12}} \sim (-s_1)^{\alpha_1}(-s_2)^{\alpha_2}\left[(-\eta)^{\alpha_1}\sum_{i=0}^{\infty}\Gamma(-\alpha_1+i)\Gamma(\alpha_1-\alpha_2-i)\beta(\alpha_1-i, t_1, t_2)\frac{\eta^{-i}}{i!} + (-\eta)^{\alpha_2}\sum_{i=0}^{\infty}\Gamma(-\alpha_2+i)\Gamma(\alpha_2-\alpha_1-i)\beta(\alpha_2-i; t_1, t_2)\frac{\eta^{-i}}{i!}\right]. \tag{3.17}$$

We have closed the m contour to the left in order to exhibit the singularities of the double-Regge vertex for large η. The singularities $(-\eta)^{\alpha_1}$ and $(-\eta)^{\alpha_2}$ are well known to result in most all model calculations.[23] Goddard and White[4] have previously noted that the assumption that $a^{\tau_1\tau_2\tau_{12}}$ has no singularities in m yields (3.17). This property, as we have discussed above, is actually required by the Steinmann relations so that the amplitude has the behavior

$$A^{\tau_1\tau_2\tau_{12}} \sim (-s)^{\alpha_1}(-s_2)^{\alpha_2-\alpha_1}\Gamma(-\alpha_1)\Gamma(\alpha_1-\alpha_2)\beta(\alpha_1; t_1, t_2) + (-s)^{\alpha_2}(-s_1)^{\alpha_1-\alpha_2}\Gamma(-\alpha_2)\Gamma(\alpha_2-\alpha_1)\beta(\alpha_2; t_1, t_2) \tag{3.18}$$

and thus no double discontinuities in s_1 and s_2. That the Steinmann relations lead to an expression of the form (3.18) has been pointed out previously by Halliday[24] and DeTar and Weis.[7,25] The two singularity structures in (3.18) correspond to the two types of tree diagrams exhibiting singularities in the *asymptotic* invariants (see Fig. 6). We note that each term in (3.17) has "spurious" singularities for $\alpha_1-\alpha_2$ integral. It is clear from (3.12) that the amplitude does not have such unphysical singularities since the singularities at $m=\alpha_1$ and $m=\alpha_2$ do not pinch the contour. It is easy to check that they cancel between the two terms in (3.17).

It is important to reemphasize the interpretation of (3.17). The asymptotic behavior in s_1 and s_2 is controlled by the singularities in angular momentum whereas the asymptotic behavior in η is controlled by the singularities in helicity. Equation (3.17) is a concrete example of the general discussion above, since it shows that in the double-Regge limit the singularities in helicity are determined completely by the Regge-pole locations. Therefore, in a sense the helicity singularities are kinematical whereas the angular momentum singularities are dynamical. It is reassuring that one does not need further dynamics beyond that needed to determine the angular momentum singularities in order to determine the helicity-pole locations.[26]

We now consider the contribution of a fixed pole of the form (3.15d) multiplying a double-Regge pole:

$$a^{\tau_1\tau_2\tau_{12}} \approx \frac{\beta(m; t_1, t_2)}{(j_1-\alpha_1)(j_1-m-J_1)(j_2-\alpha_2)},$$

which yields

$$A^{\tau_1\tau_2\tau_{12}} \sim \frac{\Gamma(-\alpha_2)}{\alpha_1-\alpha_2-J_1}\left[\Gamma(\alpha_2-\alpha_1)(-s)^{\alpha_2}(-s_1)^{\alpha_1-\alpha_2} - \Gamma(-J_1)(-s)^{\alpha_2}(-s_1)^{J_1}\right]\beta(\alpha_2; t_1, t_2) - \frac{\Gamma(-\alpha_1)\Gamma(\alpha_1-\alpha_2)}{J_1}(-s)^{\alpha_1}(-s_2)^{\alpha_2-\alpha_1}\beta(\alpha_1; t_1, t_2) + \text{terms of lower order in } \eta. \tag{3.19}$$

The fixed pole (3.15d) thus leads to a fixed-power behavior $(-s_1)^{J_1}$. For a multiplicative fixed pole, the Regge pole and fixed poles have residues with compensating spurious singularities at $\alpha_1-\alpha_2-J_1=0$. We thus see that "nonsense" spurious singularities have a rather different compensation mechanism than do "sense" spurious singularities [i.e., in (3.19) both terms have $m=\alpha_2$, whereas in (3.18) one term has $m=\alpha_1$ and one $m=\alpha_2$].[21,27] Additive fixed poles give a contribution like that above except with no spurious singularity.

Amplitudes with fixed poles of the form (3.15d) are found in a number of models with third-double-spectral functions. We have studied the dual-resonance model for the five-line amplitude in detail. There are 12 contributions corresponding to the 12 possible orderings of the external lines. Of these, 4 exhibit the double-Regge behavior (3.17) [with $\beta(m; t_1, t_2)\equiv 1$] and 4 have *additive* fixed poles in j_1 and j_2 or the form (3.15d).[28] These additive fixed poles are the generalizations of the well-known additive fixed poles in the "third-double-spectral" term, $B[-\alpha(s), -\alpha(u)]$ in the four-particle amplitude.[29] Multiplicative fixed poles of the form (3.15d) have been found in ϕ^3 perturbation theory models of Regge poles with nonplanar "third-double-spectral function" couplings.[30,31] These authors have actually treated the six-particle amplitude but we can specialize their results by taking the residue of one Reggeon at $\alpha=0$. [They also discussed the helicity-pole limit (see Sec. III B) but the results for the Regge limit are essentially the same.] Gordon[30] has explicitly calculated the fixed-pole and Regge-pole contributions and they indeed have the form (3.19). His calculations are completely consistent with our general expressions.

It is amusing to observe that possible multiplicative fixed poles of the form

$$\frac{1}{j_1-m-J}\,\frac{1}{j_2-m-J},$$

whether or not they multiply Regge poles or cuts, do not give a behavior of the amplitude different from that of a single fixed pole. This is because the m contour receives contributions only from the singularities in $\Gamma(-j_1+m)$ and $\Gamma(-j_2+m)$ which convert one of the two factors above into a regular factor. Thus in addition to the behaviors like (3.19) the only other type of behavior is that arising from the fixed poles of the form (3.15c). Although these fixed poles have not been found in the models we have studied explicitly, in general they may be necessary for consistency of the rules for many-particle amplitudes (see Sec. IV). The poles give a distinctive contribution; for example

$$a^{\tau_1\tau_2\tau_{12}}\approx\frac{\beta(m;t_1,t_2)}{(j_1-J_1)(j_1-\alpha_1)(j_2-\alpha_2)}$$

yields

$$\begin{aligned}A^{\tau_1\tau_2\tau_{12}}\sim\frac{1}{\alpha_1-J_1}\{&[\Gamma(-\alpha_1)\Gamma(\alpha_1-\alpha_2)(-s)^{\alpha_1}(-s_2)^{\alpha_2-\alpha_1}\beta(\alpha_1;t_1,t_2)+\Gamma(-\alpha_2)\Gamma(\alpha_2-\alpha_1)(-s)^{\alpha_2}(-s_1)^{\alpha_1-\alpha_2}\beta(\alpha_2;t_1,t_2)]\\&-[\Gamma(-J_1)\Gamma(J_1-\alpha_2)(-s)^{J_1}(-s_2)^{\alpha_2-J_1}\beta(J_1;t_1,t_2)+\Gamma(-\alpha_2)\Gamma(\alpha_2-J_1)(-s)^{\alpha_2}(-s_1)^{J_1-\alpha_2}\beta(\alpha_2;t_1,t_2)]\}\\&+\text{terms of lower order in }\eta.\end{aligned}\qquad(3.20)$$

We see that in (3.18), $\mathrm{Disc}_s\,\mathrm{Disc}_{s_2}A^{\tau_1\tau_2\tau_{12}}$ has zeros for α_1 negative integral. Such nonsense zeros would not seem to be an absolute necessity, and these fixed poles allow them to be absent (for wrong signature) – compare with (2.13) and (3.19) [which allows the $\mathrm{Disc}_{s_1}A^{\tau_1\tau_2\tau_{12}}$ to be finite for $\alpha_1-\alpha_2$ negative integral].

Although one can easily see that because the fixed poles occur at wrong signature they do not contribute to the asymptotic behavior of the amplitude, their existence can be checked experimentally using sum rules analogous to (2.14) or finite-energy sum rules. Such experimental tests would be extremely interesting. We remark that in writing finite-energy sum rules for multiparticle processes it is clearly essential to take into proper account the singularity structures (3.17), (3.19), and (3.20) in order to correctly exploit the analyticity of the amplitude.

One can easily compute the contribution of the various types of singularities to the full amplitude using (3.2). For double-Regge poles the case where there is no left-hand cut in η was worked out some time ago by Drummond *et al.*[23] The general case has been studied recently by Roth[32] and it was found that the only effect was to take the even (odd) part of the first sum in (3.17) for even (odd) values of $\tau_1\tau_{12}$ similarly the even (odd) part of the second sum for even (odd) values of $\tau_2\tau_{12}$.[33] In the past the possibility of left-hand cuts in η has usually been ignored. Our general discussion above should make this appear to be a peculiar assumption, since generally there is no reason not to have left-hand cuts in s in the same term with right-hand cuts in s_1 and s_2. The reason for the neglect of such contributions in the past is undoubtedly due to the fact that one considered only planar-type models. Indeed Roth[32] has shown that certain nonplanar models (e.g., the nonplanar dual model of Mandelstam[34]) have left-hand cuts in η [but, of course, still have the behavior (3.17)]. The question then arises whether or not we expect Regge couplings to have a definite signature τ_{12} rather than being some arbitrary mixture of $\tau_{12}=\pm1$. At the present time this is an interesting open question. We feel, however, that the answer is probably no, since certain models like the ordinary dual model for mesons or Mandelstam's dual quark model[34] do not have definite τ_{12} signature. We shall therefore neglect vertex signatures in the remainder of the paper.

B. Helicity Pole and More General Asymptotic Limits

It is very interesting to ask whether the asymptotic behavior in other limits than the double-Regge limit is determined by the singularities in j_1, j_2, and m, discussed above. Of course, the single-Regge limits $s_1\to\infty$ (s_1/s fixed) and $s_2\to\infty$ (s_2/s fixed) are determined by the singularities in

j_1 and j_2, respectively. More interesting, however, is the limit $\eta\to\infty$ (e.g., $s\to\infty$ with s_1 and s_2 as well as t_1 and t_2 fixed). This limit also specifies the unique partial-wave analysis of Fig. 3(a) and Eq. (3.12). Only this partial-wave analysis has the property that there are two fixed momentum transfers in the limit. We therefore expect the singularities in m determine the asymptotic behavior as $\eta\to\infty$. Thus Regge poles in j_1 and j_2 [see (3.15a) and above Eq. (3.17)] lead to the behavior

$$A^{\tau_1\tau_2}\sim(-s)^{\alpha_1}\Gamma(-\alpha_1)\int dj_2\Gamma(-j_2+\alpha_1)(-s_2)^{j_2-\alpha_1}R(j_2,\alpha_1;t_1,t_2)$$
$$+(-s)^{\alpha_2}\Gamma(-\alpha_2)\int dj_1\Gamma(-j_1+\alpha_2)(-s_1)^{j_1-\alpha_2}\,R(j_1,\alpha_2;t_1,t_2)\,.$$

This formula shows clearly that the Regge poles determine the asymptotic behavior. The ordinary dual-resonance model has this behavior: 4 of the 12 contributions have both of the above terms, 4 contributions have only one of the terms, and 4 contributions are expected to decrease exponentially.

Intermediate between the $\eta\to\infty$ limit and the double-Regge limit is the so-called helicity-pole limit.[5,6,7] The helicity-pole limit is

$$s_1\to\infty,\qquad s/s_1\to\infty,\qquad t_1,t_2,s_2\text{ fixed}\,. \tag{3.21}$$

The limit $s_1\to\infty$ requires the j_1 partial-wave analysis and the limit $s/s_1\to\infty$ requires the j_2 partial-wave analysis ($s/s_1\to0$ requires partial-wave analysis in the s_2-channel instead). In this case it is useful to rewrite (3.12):

$$A^{\tau_1\tau_2}(s_1,s_2,s;t_1,t_2)=\left(\frac{1}{2\pi i}\right)^3\int dm\int dj_1\int dj_2\,\Gamma(-m)\Gamma(-j_1+m)\Gamma(-j_2+m)(-s_1)^{j_1}\left(\frac{-s}{-s_1}\right)^m(-s_2)^{j_2-m}$$
$$\times a^{\tau_1\tau_2}(j_1,j_2,m;t_1,t_2)\,. \tag{3.22}$$

Let us compute the contribution of a Regge pole (3.15a) in the limit $s_1\to\infty$,

$$A^{\tau_1\tau_2}\sim(-s_1)^{\alpha_1}\left(\frac{1}{2\pi i}\right)^2\int dm\int dj_2\,\Gamma(-m)\Gamma(-\alpha_1+m)\Gamma(-j_2+m)\left(\frac{-s}{-s_1}\right)^m(-s_2)^{j_2-m}R(j_2,m;t_1,t_2)\,. \tag{3.23}$$

The limit $s/s_1\to\infty$ is then determined by the leading singularities in m. There is a singularity at $m=\alpha_1$ and, in addition, singularities at $m=\alpha_2^{(i)}$ due to Regge poles (or cuts) in j_2 pinching the j_2 contour of integration against the singularities in $\Gamma(-j_2+m)$. Retaining only the dominant poles at $m=\alpha_1$ and $m=\alpha_2$, we obtain

$$A^{\tau_1\tau_2}\sim(-s_1)^{\alpha_1}\left[\left(\frac{-s}{-s_1}\right)^{\alpha_2}\Gamma(-\alpha_2)\Gamma(\alpha_2-\alpha_1)\beta(\alpha_2;t_1,t_2)\right.$$
$$\left.+\left(\frac{-s}{-s_1}\right)^{\alpha_1}\Gamma(-\alpha_1)\left(\frac{1}{2\pi i}\right)\int dj_2\Gamma(-j_2+\alpha_1)(-s_2)^{j_2-\alpha_1}R(j_2,\alpha_1;t_1,t_2)\right]. \tag{3.24}$$

The two terms in (3.24) correspond to the two tree diagrams in Fig. 6. The first term is determined completely by the leading Regge behavior and has the structure of a three-point vertex.[35] On the other hand, the second term is a contour integral over j_2 and can have pole and cut singularities in s_2 as well as t_2. This term therefore has the structure of a four-point function. Since only the first term has a discontinuity in s_1, this discontinuity is determined completely by the usual behavior even though the limit is not a Regge limit.

The behavior (3.24) is a special case of the helicity-pole limit for single-particle inclusive cross sections where one Regge trajectory is at its spin-zero pole.[36] It has already been argued that the behavior of the discontinuity is determined by the leading Regge poles.[5] Also the existence and structure of the two terms in (3.24) has been discussed in some detail using the dual-resonance model as a guide.[7,37,38] We see that, as suggested there, this structure is very general.

Some authors have regarded the second term in (3.24) as anomalous.[39] However, the above analysis shows clearly that it is due just to the usual poles in complex helicity arising in the Sommerfeld-Watson transform. Perhaps some of the confusion is due to the fact that the limit is often spoken of as a double-Regge limit (triple-Regge in the case of inclusive cross sections) and thus one naively expects only the single term $(-s_1)^{\alpha_1}$

FIG. 6. Tree diagrams representing possible simultaneous discontinuities in asymptotic variables in five-particle amplitude.

$\times(-s/-s_1)^{\alpha_2}$. As we have emphasized before,[7] the limit is an helicity-pole limit, not a multi-Regge limit. However, for the discontinuity discussed above, it may be more or less permissible to speak of the limit as "double-Regge" (or "triple-Regge") limit since its behavior is completely determined by the Regge poles.

Clearly, fixed-pole and Regge-cut contributions can easily be calculated. The zero in (3.24) in the discontinuity in s_1 for $\alpha_1-\alpha_2$ negative integral can be removed by a multiplicative fixed pole of the form $1/(j_1-m-J_t)$ for wrong signature.

IV. MANY-PARTICLE AMPLITUDES

At the level of the six-particle amplitude an entirely new feature arises. Due to the four-dimensionality of space-time not all invariants formed from more than four independent four-vectors are independent. The Gram-determinant constraints between the invariants are nonlinear and thus extremely complicated.[40] Unfortunately, we shall see that these constraints play an important role and must be taken into account in proposed representations of many-particle amplitudes. Indeed they require important modifications of the expressions one is lead to by simple Sommerfeld-Watson transform analysis. These modifications are generally of two types, modifications of "propagators" and of "vertices," which we shall illustrate by studying two limits of the six-particle amplitude.

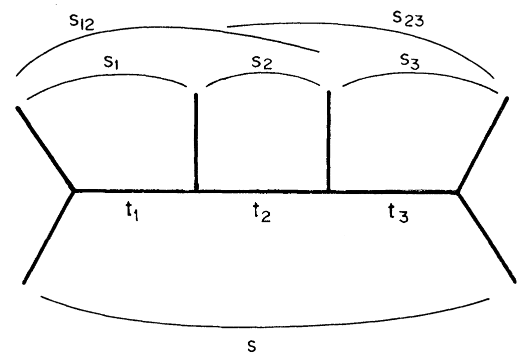

FIG. 7. Channel invariants for six-particle amplitude in linear triple-Regge limit.

A. The Multiperipheral Asymptotic Limit

We first study the linear triple-Regge limit of the six-particle amplitude. Linear relations among the invariants reduce the 25 channel invariants to 9 linearly independent variables which we choose as in Fig. 7. However, there is one nonlinear constraint which reduces the number of independent variables to the appropriate $3N-10=8$. The conventional choice of variables is the three "t" invariants (t_1, t_2, t_3) and five of the six "s" invariants $(s_1, s_2, s_3, s_{12}, s_{23})$. As usual we expect both right-hand and left-hand cuts in the "s" invariants which the signature decomposition will represent.

The appropriate set of group variables in which Sommerfeld-Watson transforms lead to the desired linear triple-Regge asymptotic behavior is clearly given by triple O(3) analysis in the t_1, t_2, and t_3 channels' center-of-mass systems. We shall not give the expression for the invariants in terms of the group variables since these are easy generalizations of (3.3). The triple partial-wave expansion for the amplitude with only right-hand cuts is[41]

$$A^{\tau_1\tau_2\tau_3}(s_1, s_2, s_3, \eta_{12}, \eta_{23}; t_1, t_2, t_3)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{j_1=|m|}^{\infty}\sum_{j_2=\max\{|m|,|n|\}}^{\infty}\sum_{j_3=|n|}^{\infty}(2j_1+1)(2j_2+1)(2j_3+1)$$
$$\times a^{\tau_1\tau_2\tau_3}(j_1, j_2, j_3, m, n; t_1, t_2, t_3)d^{j_1}_{0m}(\cos\theta_1)$$
$$\times d^{j_2}_{mn}(\cos\theta_2)d^{j_3}_{n0}(\cos\theta_3)e^{im\omega_{12}}e^{im\omega_{23}}, \tag{4.1}$$

where $\eta_{ij}=s_{ij}/s_is_j$. The Sommerfeld-Watson transform is obtained as in Sec. III: First the helicities m and n are transformed, then the angular momenta j_1, j_2, and j_3. We then arrive at the generalization of (3.12):

$$A^{\tau_1\tau_2\tau_3}=\left(\frac{1}{2\pi i}\right)^5\int dm\int dn\int dj_1\int dj_2\int dj_3\Gamma(-j_1+m)\Gamma(-m)\left[\frac{\Gamma(-j_2+m)\Gamma(-j_2+n)}{\Gamma(-j_2)}\right]\Gamma(-n)\Gamma(-j_3+n)$$
$$\times(-s_1)^{j_1}(-s_2)^{j_2}(-s_3)^{j_3}(-\eta_{12})^m(-\eta_{23})^n a^{\tau_1\tau_2\tau_3}, \tag{4.2}$$

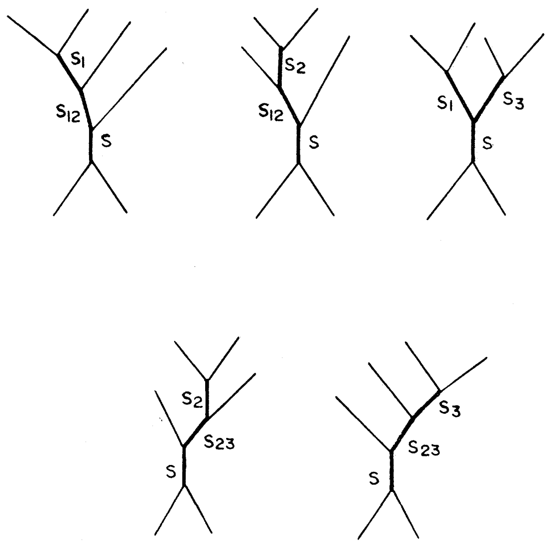

FIG. 8. Tree diagrams representing possible simultaneous discontinuities in asymptotic variables in the linear triple-Regge limit.

where the contours are drawn analogously to Fig. 5. We have assumed that A is even in ω_{12} and ω_{23} for simplicity and, as usual, extracted various kinematic singularities from the partial-wave amplitude. The only really new aspect of (4.2) is the dependence on j_2 which now couples to two complex helicities. The factor

$$\left[\frac{\Gamma(-j_2+m)\Gamma(-j_2+m)}{\Gamma(-j_2)}\right]$$

arises from the function $e_{mn}^{-j-1}(-\cos\theta)$ in the Mandelstam-Sommerfeld-Watson transform and the "kinematic" group-theoretical singularities in $a^{\tau_1\tau_2\tau_3}$ [see Eq. (3.16)]. Unfortunately, (4.2) is not a proper representation of A, since it does not satisfy the Steinmann relations. This problem has already been discussed in some detail.[42] The possible sets of simultaneous discontinuities in the "s" invariants are shown in the tree diagrams of Fig. 8. Clearly, we must distinguish which side of cuts due to singularities in s we are on. Thus in order to obtain an amplitude which satisfies the Steinmann relations, we must distinguish the various cuts in the independent variables according to their origin as, say, normal thresholds in the dependent-channel invariants. It must be the case that taking the asymptotic limit on different sides of these cuts will lead to different phases.

Instead of the dependent variables it is convenient to use

$$\phi \equiv \frac{s s_2}{s_{12} s_{23}} \tag{4.3}$$

in the linear triple-Regge limit $\phi = 1$. However, dependence on its phase $e^{2n\pi i}$ must be included to specify which side of cuts due to singularities in s we are on. Indeed the full amplitude receives contributions from signatured amplitudes with $\phi = e^{\pm 2\pi i}$ as well as $\phi = 1$.[24,43] We suggest that the appropriate way to incorporate the dependence on ϕ is to insert a factor

$$\left[\frac{\sin\pi(m-j_2)\sin\pi n}{\sin\pi j_2 \sin\pi(m-n)}\phi^m + \frac{\sin\pi(n-j_2)\sin\pi m}{\sin\pi j_2 \sin\pi(n-m)}\phi^n\right] \tag{4.4}$$

into the integrand of (4.3). For $\phi = 1$, this factor is unity but for $\phi = e^{\pm 2\pi i}$ it is not. Our motivation for the factor (4.4) comes from the study of the dual-resonance model[43] and the ladder model.[24] We found[42] that both models could be written in the form (4.2) with (4.4) and that this gives satisfaction of the Steinmann relations and factorization of the full amplitude (as well as certain discontinuities). We therefore believe the appropriate representation of the six-line amplitude in the linear triple-Regge limit is

$$\begin{aligned}
A^{\tau_1\tau_2\tau_3}(s_1, s_2, s_3, \eta_{12}, \eta_{23}; t_1, t_2, t_3) &= \left(\frac{1}{2\pi i}\right)^5 \int dm \int dn \int dj_1 \int dj_2 \int dj_3 \Gamma(-j_1+m)\Gamma(-m)\left[\frac{\Gamma(-j_2+m)\Gamma(-j_2+n)}{\Gamma(-j_2)}\right]\Gamma(-n)\Gamma(-j_3+n) \\
&\quad \times \left[\frac{\sin\pi(m-j_2)\sin\pi n}{\sin\pi j_2 \sin\pi(m-n)}\phi^m + \frac{\sin\pi(n-j_2)\sin\pi m}{\sin\pi j_2 \sin\pi(n-m)}\phi^n\right] \\
&\quad \times (-s_1)^{j_1}(-s_2)^{j_2}(-s_3)^{j_3}(-\eta_{12})^m(-\eta_{23})^n a^{\tau_1\tau_2\tau_3}(j_1, j_2, j_3, m, n; t_1, t_2),
\end{aligned} \tag{4.5}$$

where the contours are drawn as in Fig. 5 and $a^{\tau_1\tau_2\tau_3}$ has no singularities in m and n other than the fixed-pole singularities discussed below.

We expect the usual Regge-pole and -cut singularities in j_1, j_2, and j_3. The nonsense-wrong-signature fixed poles in j_1 and j_3 are expected to be the same as (3.15c) and (3.15d). The fixed poles in j_2 will correspond to nonsense with respect to m or n, however, and thus will be of the two types

$$a^{\tau_1\tau_2\tau_3} \approx \frac{R}{j_2 - m - J_2}, \tag{4.6a}$$

with

$$J_2 = -1, -2, -3, \ldots \quad \text{and} \quad \tau_1\tau_2 = (-1)^{J_2+1},$$

and

$$a^{\tau_1\tau_2\tau_3} \approx \frac{R}{j_2 - n - J_2}, \tag{4.6b}$$

with

$$J_2 = -1, -2, -3, \ldots \quad \text{and} \quad \tau_2\tau_3 = (-1)^{J_2+1}.$$

We note that the singularity (4.6b) will lead a fixed pole of the form (3.15c) if the residue of a pole at $j_3 = 0$ is taken. Also a multiplicative pole of the form

$$a^{\tau_1\tau_2\tau_3} \approx \frac{R}{(j_2 - m - J)(j_2 - n - J)} \tag{4.7}$$

could lead to a fixed double pole in the four-line amplitude obtained by taking the residue at poles at $j_1 = 0$ and $j_3 = 0$. We thus should extend the zeros of $[\Gamma(-j_2)]^{-1}$ in (4.5) to the left as well and take instead

$$a^{\tau_1\tau_2\tau_3} \approx \frac{R(j_2 - J)(j_2 - J - 1)\cdots(j_2 + 1)}{(j_2 - m - J)(j_2 - n - J)}. \tag{4.8}$$

B. The Triple-Regge Limit and the Triple-Regge Vertex

The most general vertex occurring in a multiple partial-wave analysis has three general angular momenta and, correspondingly, the most general element occurring in Regge-asymptotic limits is the triple-Regge vertex.[44] Thus far we have studied only the double-Regge–single-particle vertex, so we now turn to a discussion of the novel aspects of the triple-Regge vertex.

The appropriate channels for the triple partial-wave analysis are shown in Fig. 9. There are six natural angle variables corresponding to the six "s" invariants. These are the polar and azimuthal angles of each of the pairs of external lines measured in a frame where the corresponding momentum transfer, t_i, is at rest. The constraint that reduces these nine variables to the eight independent variables is simply the invariance of the configuration of momenta under a simultaneous translation of all three azimuthal angles. If we call the azimuthal angles φ_i, the amplitude will only depend on the differences $\varphi_i - \varphi_j$. Indeed the invariants s_{ij} in Fig. 6 are given by expressions like (3.3) with the Toller angles

$$\omega_{ij} = \varphi_i - \varphi_j. \tag{4.9}$$

It clearly follows that

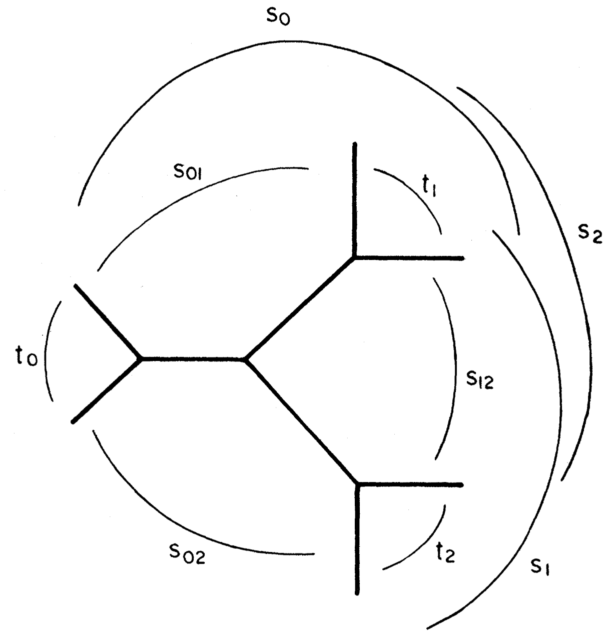

FIG. 9. Choice of variables for triple-Regge limit.

$$\omega_{01} + \omega_{12} + \omega_{20} = 0, \tag{4.10}$$

which shows that only two of the three s_{ij} are independent. While the constraint is a simple linear constraint on the Toller angles, it is a complicated nonlinear constraint on the s_{ij}. We remark that (4.10) is equivalent to the requirement of helicity conservation at the central vertex in the frame where the three momentum transfers are collinear.

The conventional approach to the partial-wave analysis would be to express the amplitude in terms of two of the Toller angles.[45] If, for example, ω_{12} is eliminated the normal threshold singularities in s_{12} will lead to extremely complicated singularity structure in ω_{01} and ω_{20} (or s_{01} and s_{20}). To avoid this so as to be able to clearly distinguish singularities according to their origin as ordinary unitarity singularities (e.g., normal thresholds) in various channels as in the preceding subsection, we will give a representation of the amplitude in terms of an overcomplete set of variables, e.g., all three ω_{ij}. Thus the Fourier transform of the amplitude will be written

$$A = \sum_{m_{ij}=-\infty}^{\infty} e^{im_{01}\omega_{01}} e^{im_{12}\omega_{12}} e^{im_{20}\omega_{20}} A_{m_{01}, m_{12}, m_{20}}. \tag{4.11}$$

Since the helicities m_i associated with the momentum transfers t_i are the Fourier transforms of the φ_i, we have from (4.9),

$$m_0 = m_{01} - m_{20},$$
$$m_1 = m_{12} - m_{01}, \tag{4.12}$$
$$m_2 = m_{20} - m_{12}.$$

As usual we assume for simplicity that the amplitude is a function of the invariants only. Since s_{ij} is even in ω_{ij}, $A_{m_{01},m_{12},m_{20}}$ must be even in the m_{ij}.[46] We may therefore write instead of (4.11),

$$A = \sum_{m_{ij}=0}^{\infty} (e^{im_{01}\omega_{01}} + e^{-im_{01}\omega_{01}})(e^{im_{12}\omega_{12}} + e^{-im_{12}\omega_{12}}) \times (e^{im_{20}\omega_{20}} + e^{-im_{20}\omega_{20}}) A_{m_{01},m_{12},m_{20}}. \tag{4.13}$$

With this form we see that $A_{m_{01},m_{12},m_{20}}$ contributes to helicities $m_0 = \pm m_{01} \pm m_{20}$, etc. Since sense values of the helicities are $j_0 \geq m_0$, we see that for each value of the m_{ij} in (4.13) we must have

$$j_0 \geq m_{01} + m_{20},$$
$$j_1 \geq m_{12} + m_{01}, \tag{4.14}$$
$$j_2 \geq m_{20} + m_{12}.$$

When the partial-wave analysis in the j_i is performed this leads us to suggest the generalization of (3.12),[41]

$$A^{\tau_0\tau_1\tau_2}(s_0, s_1, s_2, \eta_{01}, \eta_{12}, \eta_{20}; t_0, t_1, t_2) = \left(\frac{1}{2\pi i}\right)^6 \int dm_{01} \int dm_{12} \int dm_{20} \int dj_0 \int dj_1 \int dj_2$$
$$\times \Gamma(-m_{01})\Gamma(-m_{12})\Gamma(-m_{20})\Gamma(-j_0 + m_{01} + m_{20})$$
$$\times \Gamma(-j_1 + m_{12} + m_{01})\Gamma(-j_2 + m_{20} + m_{12})$$
$$\times (-s_0)^{j_0}(-s_1)^{j_1}(-s_2)^{j_2}(-\eta_{01})^{m_{01}}(-\eta_{12})^{m_{12}}(-\eta_{20})^{m_{20}}$$
$$\times a^{\tau_0\tau_1\tau_2}(j_0, j_1, j_2, m_{01}, m_{12}, m_{20}; t_0, t_1, t_2). \tag{4.15}$$

As usual $a^{\tau_0\tau_1\tau_2}$ is assumed not to have singularities in m_{ij} other than fixed-pole singularities.

We do not regard the above motivation for (4.15) as completely convincing. Indeed a large part of the motivation for (4.15) comes from the study of the dual-resonance model.[7] It was found that a triple-Regge-pole contribution of the form arising from (4.15) gives an amplitude with no simultaneous discontinuities in overlapping channel invariants. The distortion of the m_{ij} contours to exhibit the singularities in the η_{ij} is quite complicated in this case and we shall only restate the answer[7] here:

$$A^{\tau_0\tau_1\tau_2} \sim (-s_0)^{\alpha_0-\alpha_1-\alpha_2}(-s_{01})^{\alpha_1}(-s_{20})^{\alpha_2}\Gamma(-\alpha_1)\Gamma(-\alpha_2)\Gamma(\alpha_1+\alpha_2-\alpha_0)\beta(\alpha_1, 0, \alpha_2; t_0, t_1, t_2)$$
$$+ (-s_1)^{\alpha_1-\alpha_2-\alpha_0}(-s_{12})^{\alpha_2}(-s_{01})^{\alpha_0}\Gamma(-\alpha_2)\Gamma(-\alpha_0)\Gamma(\alpha_2+\alpha_0-\alpha_1)\beta(\alpha_0, \alpha_2, 0; t_0, t_1, t_2)$$
$$+ (-s_2)^{\alpha_2-\alpha_0-\alpha_1}(-s_{20})^{\alpha_0}(-s_{12})^{\alpha_1}\Gamma(-\alpha_0)\Gamma(-\alpha_1)\Gamma(\alpha_0+\alpha_1-\alpha_2)\beta(0, \alpha_1, \alpha_0; t_0, t_1, t_2)$$
$$+ \tfrac{1}{2}(-s_{01})^{(\alpha_0+\alpha_1-\alpha_2)/2}(-s_{12})^{(\alpha_1+\alpha_2-\alpha_0)/2}(-s_{20})^{(\alpha_2+\alpha_0-\alpha_1)/2}\Gamma(\tfrac{1}{2}(\alpha_2-\alpha_0-\alpha_1))\Gamma(\tfrac{1}{2}(\alpha_0-\alpha_1-\alpha_2))\Gamma(\tfrac{1}{2}(\alpha_1-\alpha_2-\alpha_0))$$
$$\times \beta(\tfrac{1}{2}(\alpha_0+\alpha_1-\alpha_2), \tfrac{1}{2}(\alpha_1+\alpha_2-\alpha_0), \tfrac{1}{2}(\alpha_2+\alpha_0-\alpha_1); t_0, t_1, t_2). \tag{4.16}$$

The four terms in (4.16) correspond to the four possible combinations of cuts in nonoverlapping variables (see Fig. 10). Therefore the consistency with this requirement is perhaps the most important feature of (4.15). We refer the reader to Ref. 7 for a further discussion of the structure of (4.16).

The generalization of the remainder of the discussion of Sec. III to this case is straightforward. The fixed-pole singularities of the type (3.15d) become[47]

$$a^{\tau_0\tau_1\tau_2} \approx \frac{\beta}{j_0 - m_{01} - m_{20} - J_0}, \tag{4.17}$$

with

$$J_0 = -1, -2, -3, \ldots \quad \text{and} \quad \tau_0\tau_1\tau_2 = (-1)^{J_0+1}.$$

One can also define an helicity-pole limit[5-7] as

$$s_0, s_{01}/s_0, s_{20}/s_0 \to \infty,$$
$$s_1, s_2, s_{12}, t_0, t_1, t_2 \text{ fixed}, \tag{4.18}$$

and obtain expressions analogous to (3.24). This limit has been of particular interest in single-particle inclusive cross sections where s_0 is the missing mass (M^2) and s_{01} and s_{20} the total energy (s). We should note that Patrascioiu has pointed out that this limit does not uniquely define a tree configuration.[48] However, we do not expect other contributions not suggested by (4.15) [i.e., not analogous to (3.24)] which may therefore be present to contribute to the missing mass (s_0) discontinuity. In the absence of a multiplicative fixed

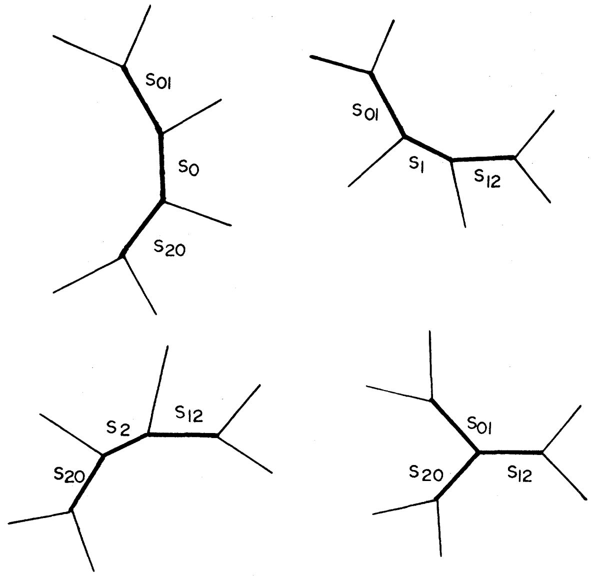

FIG. 10. Tree diagrams representing possible simultaneous discontinuities in asymptotic variables in the triple-Regge limit.

pole of the type (4.17), the inclusive cross section will vanish at $\alpha_0 - \alpha_1 - \alpha_2 = J_0$. The possibility of such fixed poles has been discussed before and models with such fixed poles have been exhibited.[30,31] The results of these model calculations can be seen to be consistent with our general rules.

We close this section with a remark on the four-particle amplitudes for particles with spin implicit in our representations. For example, taking the residue of the pole at $j_1 = J_1$ and $j_3 = J_3$ in (4.5), we have for the maximum helicity amplitude

$$A_{J_1 J_2} \sim \frac{\Gamma(-\alpha_2 + J_1)\Gamma(-\alpha_2 + J_2)}{\Gamma(-\alpha_2)} \tilde{\beta}_{J_1}(t_2)\tilde{\beta}_{J_3}(t_2)(-s_2)^{\alpha_2} . \tag{4.19}$$

Comparing with the standard Regge asymptotic form

$$A_{J_1 J_2} \sim \beta_{J_1}(t_2)\beta_{J_3}(t_2) e_{J_1 J_3}{}^{-\alpha_2 - 1}(-\cos\theta_2), \tag{4.20}$$

we see that, if the $\tilde{\beta}$ are finite at $\alpha_2 = J_2$,

$$\beta_{\text{sense}} \approx 1 ,$$

$$\beta_{\text{nonsense}} \approx (-\alpha_2 + J_2)^{1/2} ,$$

near $\alpha_2 = J_2$. This is the usual behavior for a "sense-choosing" Regge trajectory.[1] Other behaviors are obtained by vanishing or singular behavior of $\tilde{\beta}$ at $\alpha_2 = J_2$. Thus a "nonsense-choosing" trajectory has the behavior

$$\beta_{\text{sense}} \approx (-\alpha_2 + J_2)^{1/2} ,$$

$$\beta_{\text{nonsense}} \approx 1,$$

or

$$\tilde{\beta}_{\text{sense}} \approx (-\alpha_2 + J_2)^{1/2} ,$$

$$\tilde{\beta}_{\text{nonsense}} \approx (-\alpha_2 + J_2)^{-1/2} .$$

For wrong signature the nonsense-choosing trajectory can have residues singular by an extra $(-\alpha_2 + J_2)^{-1/2}$ if there are nonsense-wrong-signature fixed poles.

V. CONCLUSION

We have studied the five-particle amplitude in detail and proposed a simple representation which allows one to readily find the implications of complex angular momentum and complex helicity singularities. We have also proposed representations for the six-particle amplitude. However, in this case we have had to deal with the new feature of nonlinear relations between the channel invariants. We believe this aspect needs further study before one can be sure whether or not our method for handling them is unique. Furthermore, the study of further models would probably be useful in order to gain further insight about the presence or absence of the various types of singularities suggested here.[49]

From a practical point of view, one could obtain plausible representations for amplitudes with greater than six particles by using the multi-Regge limit of the dual-resonance model essentially as the kernel of the representation [compare for example (3.12), (4.5), and (4.15) with Eqs. (A9) of Ref. 7, (A4) of Ref. 42, and (3.7) of Ref. 7 respectively].[50] The kernel provides the proper angular momentum and helicity structure and the proper structure in overlapping channel invariants and the partial-wave amplitude allows the insertion of general dynamical singularities (Regge poles, Regge cuts, and fixed poles). In this way one could suggest representations for arbitrary tree diagrams.[51] We expect, however, that the essential features of such representations have already been exhibited in the five- and six-particle amplitudes discussed here.

Note added. Since the submission of this paper, a report by Abarbanel and Schwimmer[52] which treats many of the same issues has appeared. From this report it is clear that the original version of the first paragraph in Sec. III B was in error and I have therefore corrected it. This report also gives the important criterion for when helicity-pole limit is in the physical region: It is in the physical region if there is a tree graph with a vertex of momenta t_1, t_2, and t_3 with $\lambda(t_1, t_2, t_3) = t_1^2 + t_2^2 + t_3^2 - 2t_1t_2 - 2t_2t_3 - 2t_3t_1 < 0$.

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[1]Among the concepts in this lore one encounters nonsense zeros, wrong-signature–nonsense fixed poles, kinematic singularities, conspiracies, etc. For recent reviews see P. D. B. Collins, Phys. Reports 1C, 103 (1971) and W. Dreschler, Fortschr. Phys. 18, 305 (1970).

[2]See, for example, N. F. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. 163, 1572 (1967); M. Toller, Nuovo Cimento 62A, 341 (1969).

[3]By Regge behavior we mean Regge-pole, Regge-cut, or fixed-pole behavior.

[4]P. Goddard and A. R. White, Nuovo Cimento 1A, 645 (1971).

[5]This limit is experimentally accessible in inclusive cross sections; see C. E. DeTar, C. E. Jones, F. E. Low, C.-I Tan, J. H. Weis, and J. E. Young, Phys. Rev. Letters 26, 675 (1971).

[6]C. E. Jones, F. E. Low, and J. E. Young, Phys. Rev. D 4, 2358 (1971).

[7]C. E. DeTar and J. H. Weis, Phys. Rev. D 4, 3141 (1971).

[8]A. R. White, Nucl. Phys. B39, 432 (1972); B39, 461 (1972).

[9]N. N. Khuri, Phys. Rev. 132, 914 (1963).

[10]Goddard and White, Ref. 4, were the first to obtain the expression (1.8) through the complex-helicity analysis. However, they tacitly assumed the absence of singularities in m in $\beta(m; t, t_2)$ which we have argued is a plausible consequence of the absence of simultaneous discontinuities in overlapping channel invariants.

[11]Historically, one of the important reasons for introducing signature was that it diagonalizes the two-body unitarity equations and thus makes natural the assumption that Regge poles occur in amplitudes of definite signature. We do not know whether this generalizes to many-particle unitarity (see Ref. 8 for recent progress in this program). Experimentally, many Regge trajectories seem to be degenerate in signature ("exchange degenerate") which suggests that dynamics may not respect signature.

[12]The singularities of $\cos\pi j$ for j half-integral can be shown to cancel out using (2.9).

[13]The kinematic singularities at $t=0$ we have extracted are appropriate to nondegenerate (Toller quantum number $M=0$) trajectories. The expression can be generalized to $M\neq 0$ trajectories.

[14]Throughout we assume suitable behavior for $j\to\infty$ to allow neglect of contributions from contours at infinity.

[15]We give the expression for the double-Regge region where $u_1 \approx -s_1$, etc., but an expression of the same form holds generally.

[16]The case where there is such a dependence can easily be treated by applying the analysis here to $\tilde{A}$ where $A = \epsilon^{\mu\nu\lambda\sigma}P_{1\mu}P_{2\nu}P_{3\lambda}P_{4\sigma}\tilde{A}$. An amplitude with both even and odd parts is forbidden by parity conservation. In the double-Regge limit evenness in ω is equivalent to the requirement that the product of the natural parities of the three objects at the central vertex is positive.

[17]The fixed poles discussed below are an exception since they lie to the right of the m contour.

[18]O. Steinmann, Helv. Phys. Acta 33, 257 (1960); 33, 347 (1960); H. Araki, J. Math. Phys. 2, 163 (1960); H. P. Stapp, Phys. Rev. D 3, 3177 (1971).

[19]We believe negative integral powers of the s_i should be interpreted as representing cuts, since sum rules for the coefficients of such powers require nonvanishing discontinuities [see, for example, Eq. (2.14)]. However, this point needs further study since it is conceivable that discontinuities outside the physical region could lead to such powers without violating the Steinmann relations. We thank Richard Brower for discussions on this point.

[20]This has been noted by V. N. Gribov, I. Ya Pomeranchuk, and K. A. Ter-Martirosyan [Phys. Rev. 139B, 184 (1965)] and by H. D. I. Abarbanel and M. Green [Phys. Letters 38B, 90 (1972)] and seems to have occurred to numerous others.

[21]Fixed poles analogous to (3.14d) have also been suggested for applications to inclusive reactions by F. E. Low (private communication) and DeTar and Weis (Ref. 7).

[22]The square-root singularity is converted into a pole by the extraction of "kinematic" factors from $a^{\tau_1\tau_2\tau_{12}}$ in obtaining (3.12).

[23]I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Nucl. Phys. B11, 383 (1969) and Phys. Letters 28B, 676 (1969).

[24]I. G. Halliday, Nucl. Phys. B33, 285 (1971).

[25]These authors also give an expansion equivalent to (3.17). However, they include terms with positive powers of η in the sums which lead to terms like $(-s)^{\alpha_1+k}\,(-s_1)^{-k} \times(-s_2)^{\alpha_2-\alpha_1-k}$ and hence are probably absent; see footnote 19 above.

[26]A specific example of determining helicity-pole locations from Regge-pole locations has been given previously in Ref. 5. See also Sec. III B.

[27]We are, of course, talking about wrong-signature nonsense. For right-signature nonsense there are no fixed poles and the usual compensation mechanism operates.

[28]The remaining four terms appear to give no contribution to signatured amplitudes in the double-Regge limit.

[29]D. I. Fivel and P. K. Mitter, Phys. Rev. 183, 1240 (1969); V. Alessandrini and D. Amati, Phys. Letters 29B, 193 (1969); D. Sivers and J. Yellin, Rev. Mod. Phys. 43, 125 (1971).

[30]D. Gordon, Phys. Rev. D 5, 2102 (1972).

[31]A. H. Mueller and T. L. Trueman, Phys. Rev. D 5,

2115 (1972).

[32]R. Roth, Phys. Rev. D 6, 2274 (1972).

[33]If $\tau_1 = -\tau_2$ this would have the effect of canceling the leading behavior of one of the two terms. In this case the usual "signature" factor coming from the sum of the first four terms in (3.2) cancels the spurious pole at $\alpha_1 = \alpha_2$.

[34]S. Mandelstam, Phys. Rev. D 1, 1720 (1970).

[35]Note that this term has no poles for α_1 integer and thus does not appear at physical particle poles in j_1. The reasoning of J. Arafune, H. Sugawara, and Y. Hara [Phys. Letters 37B, 92 (1972)] is thus incorrect, since implicit in their argument for the absence of this term was the treatment of the Pomeranchukon as a particle, whereas it is actually a Regge trajectory at a wrong-signature integer.

[36]The invariant s_1 corresponds to the missing mass and s to the total energy.

[37]Some features are also discussed by S.-J. Chang, D. Gordon, F. E. Low, and S. B. Treiman, Phys. Rev. D 4, 3055 (1971). The existence of these two terms has also been motivated recently on group-theoretical grounds by C. E. Jones, F. E. Low, and J. E. Young, Phys. Rev. D 6, 640 (1972). We do not understand the connection, if any, between their work and ours.

[38]For other Sommerfeld-Watson transform approaches see Y. Iwasaki and S. Yazaki, Phys. Letters 39B, 361 (1972), and R. Jengo, Lett. Nuovo Cimento 3, 335 (1972).

[39]See, for example, H. Fujisaki, Rikkyo report, 1972 (unpublished) and Jengo, Ref. 38.

[40]For a nice discussion of these constraints see R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge, 1966).

[41]As usual we suppress the Toller-angle signatures.

[42]J. H. Weis, Phys. Rev. D 5, 1043 (1972).

[43]J. H. Weis, Phys. Rev. D 4, 1777 (1971).

[44]P. V. Landshoff and W. J. Zakrzewski, Nucl. Phys. B12, 216 (1969); M. N. Misheloff, Phys. Rev. 184, 1732 (1969); P. Goddard and A. R. White, Nucl. Phys. B17, 45 (1970); B17, 88 (1970).

[45]See, for example, White, Ref. 8.

[46]If we had eliminated one ω_{ij} we would not have been able to assume that A is even in the remaining ω_{ij} due to the nonlinear nature of the constraints on the s_{ij}. One can easily see that if A were even in that case,

$$A_{m_0,m_1} = A_{-m_0,m_1} = A_{m_0,-m_1} = A_{-m_0,-m_1},$$

whereas parity invariance only requires

$$A_{m_0,m_1} = A_{-m_0,-m_1}.$$

[47]We should also note that the wrong-signature rule $\tau_0\tau_1\tau_2 = (-1)^{J_0+1}$ has been independently obtained by M. B. Einhorn, J. Ellis, and J. Finkelstein, Phys. Rev. D 5, 2063 (1972).

[48]A. Patrascioiu, Phys. Rev. D (to be published). Other possible tree diagrams correspond to interchanging the two momenta forming s_{12} in Fig. 9.

[49]For example, recently there has been a great deal of interest in the possible collision of a Pomeranchukon pole, a two-Pomeranchukon Regge cut, and a nonsense fixed pole at $t=0$. See, e.g., H. D. I. Abarbanel and M. Green, Phys. Letters 38B, 90 (1972); P. Goddard and A. R. White, *ibid.* 38B, 93 (1972); H. D. I. Abarbanel, this issue, Phys. Rev. D 6, 2788 (1972); J. B. Bronzan, Phys. Rev. D 6, 1130 (1972); I. J. Muzinich, F. E. Paige, T. L. Trueman, and L.-L. Wang, *ibid.* 6, 1048 (1972).

[50]The ladder model could also be used but the dual model gives directly a kernel with no extra smearing integrations.

[51]Gordon, Ref. 30, has also suggested using the dual-resonance model in a similar fashion.

[52]H. D. I. Abarbanel and A. Schwimmer, this issue, Phys. Rev. D 6, 3018 (1972).