$$\frac{A^{-\sigma}}{B} G^{m+1\,0}_{0\,2m+1}(A/B|\,-2, -2+\sigma, \ldots, -2+\sigma, \sigma, -1+\sigma, \ldots, -1+\sigma, -1).$$

The limit of this expression exists and equals

$$\frac{1}{B} G^{m+1\,0}_{0\,2m+1}(A/B|\,-2, \ldots, -2, 0, -1, \ldots, -1). \tag{B10}$$

[1]H. Lehmann and K. Pohlmeyer, Commun. Math. Phys. 20, 101 (1971).

[2]Abdus Salam and J. Strathdee, Phys. Rev. D 2, 2869 (1970).

[3]J. G. Taylor, University of Southampton report, 1971 (unpublished).

[4]M. K. Volkov, Commun. Math. Phys. 7, 289 (1968).

[5]Recently some work on these questions has been done by Z. Horvâth and G. Pôcsik, Eötvös University, Budapest report (unpublished); Lett. Nuovo Cimento 2, 1146 (1971). Their approach to nonpolynomial theories appears to be different from the usual ones and we have not yet managed to understand it.

[6]In Ref. 3 Professor Taylor has claimed these statements for the exponential interactions to all orders.

[7]A. Jaffe, Phys. Rev. 158, 1454 (1967).

[8]H. Epstein, V. Glaser, and A. Martin, Commun. Math. Phys. 13, 257 (1969).

[9]Abdus Salam, in *Nonpolynomial Lagrangians, Renormalisation and Gravity,* 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971), Vol. I.

[10]R. J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, England, 1966).

[11]C. S. Meijer, Proc. Koninkl. Ned. Akad. Wetenschap. 49, 227 (1946).

[12]Details are well known and are contained in Ref. 10.

[13]See Appendix B.

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# Discontinuities Across Branch Cuts in the Angular Momentum Plane

Henry D. I. Abarbanel*
*National Accelerator Laboratory, Batavia, Illinois 60510*†
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Integral equations for coupled-particle and Reggeon partial-wave amplitudes are presented. A construction of these equations proceeds from the unitarity relation using the notion of two-Reggeon irreducibility. From these equations, which are appropriate matrix elements of a Lippmann-Schwinger equation in two-dimensional nonrelativistic quantum mechanics, we demonstrate that the discontinuity across the two-Reggeon cut in particle scattering is equal to an integral over Reggeon-particle absorptive parts actually measurable in single-particle inclusive reactions. This provides one with a handle on the magnitude of Regge cuts. Finally we make a little model of coupled Reggeon and particle "states" and solve for the allowed partial-wave amplitudes when a pole and a two-Reggeon cut are close by. This has clear relevance for the physics of diffraction scattering near $l=1$ and $t=0$.

## I. INTRODUCTION

It has been known for some years that multiparticle production leads to the presence of branch cuts in the $l$ plane whenever moving poles in $l$ are present.[1] Very little, however, is known about constraints on such cuts and, more interesting to phenomenological analyses, the magnitude of the discontinuity across them. About the only property we are certain of is that the discontinuity at the tip of the cut must vanish and be nonanalytic there.[2] This result, shown by Bronzan and Jones some time ago, is a nice consequence of the origin of the branch cuts in multiparticle states, for the phase space in such states vanishes so rapidly at thresholds that the discontinuity across inelastic cuts vanishes there. Continuing those inelastic cuts in $l$ leads to the stated result.

In this paper I would like to present a decomposition of the particle-wave amplitude for particle scattering, call it $F(l,t)$, which exhibits in a convenient manner the branch cut in $l$ arising from the presence of two moving poles in the $t$ channel. The procedure is to take the definition of $F(l,t)$ in terms of $s$- and $u$-channel absorptive parts and decompose them, in the spirit of the Bethe-Salpeter equation, into their two-Reggeon reducible and irreducible pieces.[3] When this is done and $F(l, t)$ is evaluated one finds for it a two-dimensional equation which relates it to Reggeon-particle absorptive parts. There is an attractive analogy with the Lippmann-Schwinger equation in two-dimensional nonrelativistic quantum theory which we will draw.[4] In essence the angular momentum $l$ plays the role of the energy, while the free Green's function is governed by the Regge trajectories.[5] The "potential" gives rise to the irreducible parts.

Using this construction we show that the discontinuity of $F(l, t)$ across the two-Reggeon branch cut is given in terms of an integral over Reggeon-particle absorptive parts which, at $t=0$, can be directly measured in inclusive reactions. This gives one a handle on the *magnitude of Regge cuts* of which heretofore we have been ignorant.

Pursuing the nonrelativistic analogy, we write the $T$-matrix equations for Reggeon-particle and Reggeon-Reggeon partial-wave equations. All of them are appropriate matrix elements in a particle and Reggeon space of a familiar looking equation,

$$T = V + VD_0T ,$$

where the operator $T$ yields full partial-wave amplitudes, $V$ yields two-Reggeon irreducible parts, and $D_0$ propagates two Reggeons. Clearly, although we do not do it in this paper, one can also separate out the three or four, etc. Reggeon states in the $t$ channel and write a hierarchy of more and more complicated equations identical to what one has become accustomed to in many-body nonrelativistic physics.[6]

Finally we take the coupled Reggeon and particle equations and make a number of approximations on them so as to be able to solve them in what is essentially an effective-range approximation. The case of particular interest is that when a pole and a two-Reggeon cut are nearby and the joint effect on the dynamics is significant.

Just a bibliographical note before we launch into our results: Many of the basic ideas here are present in various manifestations of the multiperipheral model[7] or the Reggeon calculus.[8] In a sense this work serves to clarify many of the results of those models and then goes on to use the dynamical framework of two-dimensional quantum mechanics to suggest methods to attack coupled $l$-plane poles and cuts.

The paper is organized so the basic construction of nonrelativistic $T$-matrix equations for partial-wave amplitudes is presented first. Next the connection of the Reggeon-particle and Reggeon-Reggeon absorptive parts with inclusive reactions is discussed, and at the end a little model of a pole and a neighboring two-Reggeon cut is analyzed. The latter may be relevant to the problem of understanding the $l$ plane near $l=1$, that is, the Pomeranchukon or diffraction scattering.

## II. DISCONTINUITY ACROSS TWO-REGGEON CUTS

### A. Spinless-Particle Scattering

It is a familiar property of partial-wave amplitudes that when two or more Reggeons can occur in the channel in which the partial wave is taken, a branch cut in the $l$ plane transpires. In this section we wish to consider the partial-wave amplitudes, $F(l, t)$, in the $t$ channel for the scattering of spinless particles carrying zero isospin or other internal quantum numbers. Our goal is to give a decomposition of the contributions to $F(l, t)$, which is especially convenient for exhibiting the branch cut coming from the presence of two Reggeons in the $t$ channel.

We begin by recalling the definition of the signatured partial-wave amplitudes for the process $a+b \to a'+b'$ at energy $s=(p_a+p_b)^2$ and momentum transfer $t=(p_a-p_b')^2$:

$$F^{(\pm)}(l, t) = \int_{\bar{s}}^{\infty} ds\, Q_l(y)[A_s(s, t) \pm A_u(s, t)], \tag{1}$$

in which $\bar{s}$ is some appropriate threshold, $y$ is the continued cosine of the crossed-channel scattering angle, and $A_s$ and $A_u$ are the $s$ and $u$ absorptive parts, at fixed $t$, of the scattering amplitude. We will normalize things so that when the process $a+b \to a'+b'$ is elastic and the masses of $a$ and $b$ are $m_a$ and $m_b$, respectively, then the $ab$ total cross section is given at energy $s$ by

$$\sigma_T(s) = \frac{A_s(s, 0)}{\Delta^{1/2}(s, m_a^2, m_b^2)} , \tag{2}$$

where

$$\Delta(x, y, z) = (x+y-z)^2 - 4xy . \tag{3}$$

Concentrate now on the $s$-channel absorptive part and consider the contribution to it coming from the physical intermediate state of $N$ particles with momentum $p_1$ to $p_N$. This is given via the unitarity relation as

$$A_s(s, t) = \sum_{N=2}^{\infty} A_s^{(N)}(s, t), \tag{4}$$

with

$$A_s^{(N)}(s,t)=\frac{1}{2}\int \frac{d^3p_1}{(2\pi)^3 2p_{10}}\cdots\frac{d^3p_N}{(2\pi)^3 2p_{N0}}(2\pi)^4\delta^4\left(p_a+p_b-\sum_{j=1}^{\infty}p_j\right)T_{2\to N}(p_a+p_b\to p_1+\cdots p_N)$$
$$\times T^*_{2\to N}(p'_a+p'_b\to p_1+\cdots+p_N), \qquad (5)$$

and is shown in Fig. 1. Our basic assumption is that when any of the subenergies, $s_{ij}=(p_i+p_j)^2$, in the production matrix element $T_{2\to N}$ becomes large, its behavior in $s_{ij}$ is governed by the exchange of a factorized $l$-plane pole at $l=\alpha(t_{ij})$.[9] We can then write meaningfully all $T_{2\to N}$ as a piece containing $(s_{ij})^{\alpha(t_{ij})}$ for the dependence on the subenergy $s_{ij}$ plus a piece which has other, for example resonant, behavior in $s_{ij}$. Now we may split all contributions to $A^{(N)}$ into those which can be cut into two disjoint pieces by snipping two and only two Regge exchanges and, of course, those that cannot. (In carrying out this construction it is important to extract out the $s^{\alpha}$ behavior in every subenergy.) We will call the first class of contributions to $A^{(N)}$ two-Reggeon *reducible*, and the second class two-Reggeon *irreducible*.

In the reducible contributions, it is further convenient to collect together all those contributions to the left of the leftmost pair of Reggeon lines that can be cut to split the unitarity graph into two disjoint parts. This collection of contributions is, by definition, also two-Reggeon irreducible. It corresponds to particle $a$ plus a Reggeon going to a set of particles, say $N_1$, then going back to particle $a'$ plus another Reggeon. It is distinguished by the fact that none of the subenergies among the $N_1$ particles has the Regge-pole asymptotic behavior $s_{ij}{}^{\alpha(t_{ij})}$ of the trajectory $\alpha(t)$ which we are considering. (See Fig. 2.)

The collection of contributions to the right of the special two-Reggeon state we have distinguished consists of a particle $b$ plus a Reggeon to go to particle $b'$ plus another Reggeon proceeding through an intermediate state of $N-N_1$ physical particles. It contains any number of pairs of Reggeons which can be cut to separate it into two disjoint parts. Clearly it is two-Reggeon reducible.

We will label the irreducible contribution to the particle-particle absorptive part as $I^{(N)}$, the irreducible contribution to the Reggeon-particle absorptive part as $J^{(N_1)}$, and the reducible contribution to the Reggeon-particle absorptive part

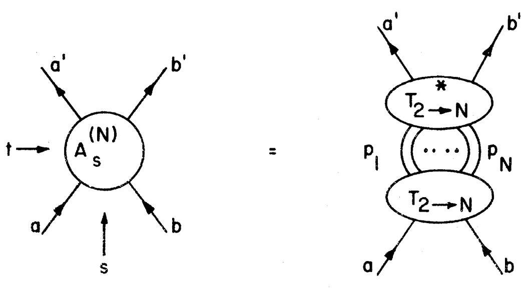


FIG. 1. The $N$-particle contribution to the $s$-channel absorptive part coming from unitarity.

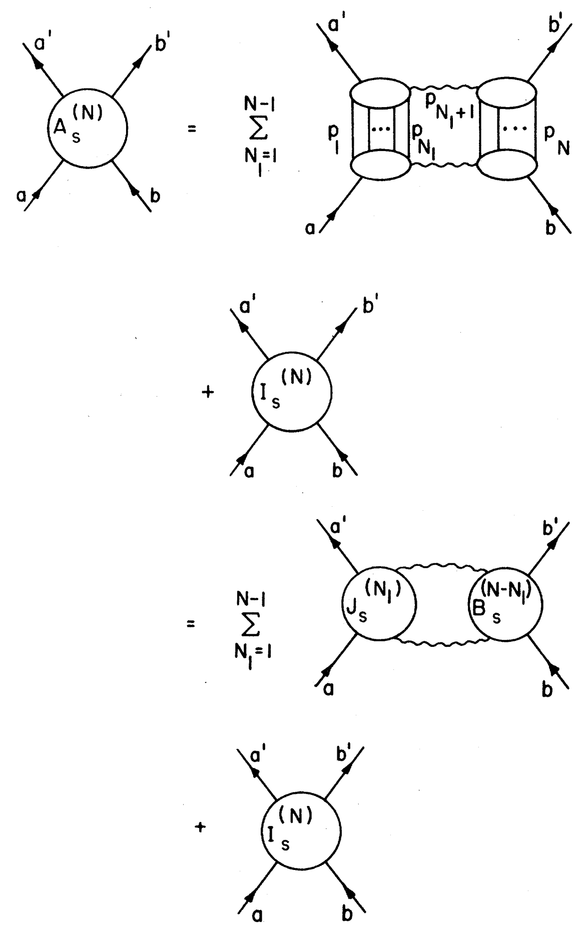


FIG. 2. The decomposition of $A_s^{(N)}$ into its two-Reggeon irreducible part. The reducible contribution is further split by separating off the two-Reggeon irreducible part of the Reggeon-particle absorptive part called $J^{(N_1)}$. This object is distinguished by having all Reggeon contributions removed from each subenergy which can be formed from the momenta $p_1,\ldots,p_{N_1}$. The full Reggeon-particle absorptive part, $B$, appears also.

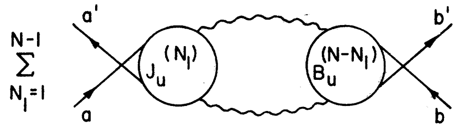


FIG. 3. Contribution to the $s$-channel absorptive part coming from the product of two $u$-channel discontinuities.

$B^{(N-N_1)}$. The $N$-particle contribution to $A_s(s,t)$ now takes the symbolic form

$$A_s^{(N)} = I_s^{(N)} + \sum_{N_1=1}^{N-1} [J_s^{(N_1)} D B_s^{(N-N_1)} + J_u^{(N_1)} D B_u^{(N-N_1)}], \tag{6}$$

where we have noted that two $u$-channel contributions to $J$ and $B$ can result in a piece of $A_s$. (See Fig. 3.) In (6) $D$ is a two-Reggeon propagator to be made explicit shortly.

If we now define

$$J_{s,u} = \sum_{N_1=1}^{\infty} J_{s,u}^{(N_1)},$$
$$I_{s,u} = \sum_{N=1}^{\infty} I_{s,u}^{(N)}, \tag{7}$$

and

$$B_{s,u} = \sum_{N-1}^{\infty} B_{s,u}^{(N)}, \tag{8}$$

then from (4) we have

$$A_s = I_s + J_s D B_s + J_u D B_u. \tag{9}$$

We can carry out the same decomposition of the $u$-channel absorptive part to write

$$A_u = I_u + J_u D B_s + J_s D B_u. \tag{10}$$

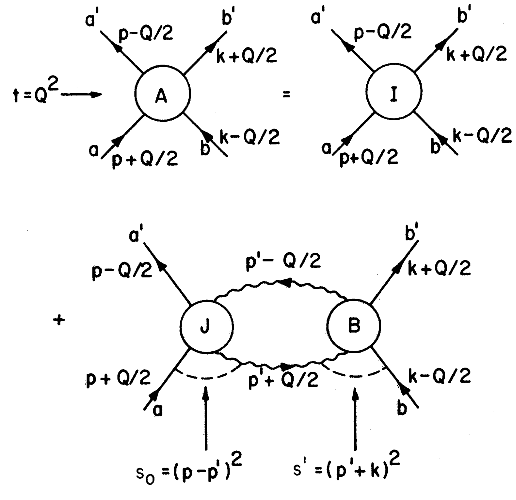


FIG. 4. Kinematics used in carrying out the integration of Eq. (1). An equation of this form holds for each signature.

The combinations $A_s \pm A_u = A^{(\pm)}$ which we need to evaluate $F^{(\pm)}(l,t)$ then satisfy

$$A^{(\pm)} = I^{(\pm)} + J^{(\pm)} D B^{(\pm)}, \tag{11}$$

where

$$I^{(\pm)} = I_s \pm I_u, \text{ etc.}$$

To give a more quantitative definition to Eq. (11) we label momenta as in Fig. 4 and write (dropping signature labels)

$$A(p,Q,k) = I(p,Q,k) + \int d^4p' J(p,Q,p') \left(\frac{s}{s_0 s'}\right)^{\alpha((p'+Q/2)^2)+\alpha((p'-Q/2)^2)} B(p',Q,k), \tag{12}$$

where

$$s = (p+k)^2, \tag{13}$$
$$s_0 = (p-p')^2, \tag{14}$$

and

$$s' = (p'+k)^2. \tag{15}$$

One recognizes $s/s_0 s'$ as essentially the "crossed cosine" for the production "process" $a+b \to \text{mass } \sqrt{s_0} + \text{mass } \sqrt{s'}$. The other factors in a Reggeon propagator such as $\sin \pi\alpha$ and complications due to signature which are functions of $(p' \pm \frac{1}{2}Q)^2$ only have been absorbed symmetrically into the functions $J$ and $B$. With the definitions given in (12), $J$ and $B$ are real functions.

Now we may carry out the integration indicated in Eq. (1) by using the detailed kinematic lore developed in Ref. 10. The crossed cosine, $y$, is

$$y = \cosh\Theta = \frac{[p-(p\cdot Q)Q/Q^2]\cdot[k-(k\cdot Q)Q/Q^2]}{\{[p-(p\cdot Q)Q/Q^2]^2[k-(k\cdot Q)Q/Q^2]^2\}^{1/2}} \geq 1, \tag{16}$$

and by repeating the steps in Ref. 10 we arrive at the equation valid for $l$ far enough to the right,

$$F(l,t,u,z,v,\zeta)=I(l,t,u,z,v,\zeta)+\tfrac{1}{4}\int_{-\infty}^{0}du'\int_{-1}^{+1}\frac{dz'}{(1-z'^2)^{1/2}}\int_{\bar{s}}^{\infty}ds_0\int_{\bar{s}}^{\infty}ds'\int_{-\infty}^{+\infty}d\psi\, J(\Theta_0,t,u,z,u',z')B(\Theta',t,u',z',v,\zeta)\times\left(e^{[\alpha(t_1)+\alpha(t_2)]\psi}+e^{-[\alpha(t_1)+\alpha(t_2)]\psi}\right)Q_l(\cosh\Theta)\,, \tag{17}$$

where we have defined the variables

$$u=p^2\,,\qquad u'=p'^2\,,\qquad v=k^2\,, \tag{18}$$

$$z=p\cdot Q/(ut)^{1/2}\,,\qquad z'=p'\cdot Q/(u't)^{1/2}\,,\qquad \zeta=k\cdot Q/(vt)^{1/2}\,, \tag{19}$$

and $\cosh\Theta$ and $\cosh\Theta'$ are given by expressions like (16), with $p$ and $p'$ replacing $p$ and $k$ for $\Theta_0$, and $p'$ and $k$ replacing $p$ and $k$ for $\Theta'$. Further we have found it convenient to call

$$t_1=(p'+\tfrac{1}{2}Q)^2 \quad \text{and} \quad t_2=(p'-\tfrac{1}{2}Q)^2\,. \tag{20}$$

The variable $\psi$ is a $y$-boost angle which is related to the subenergy variables $\Theta$, $\Theta_0$, and $\Theta'$ by

$$\cosh\Theta=\cosh\Theta_0\cosh\Theta'+\sinh\Theta_0\sinh\Theta'\cosh\psi\,. \tag{21}$$

This connection has allowed us to replace $s/s_0s'$ in the propagator numerator by a combination of $e^{\psi}$ and $e^{-\psi}$, which reflects the fact that the range of variation of $\psi$ is $-\infty$ to $+\infty$. Such a replacement can be regarded as altering the definition of what we mean by Regge asymptotic behavior; that is, when $s$ becomes large, $s/s_0s'\propto e^{\psi}$ to leading order in $s$, so we might just as well have taken the propagator numerator to be $e^{[\alpha(t_1)+\alpha(t_2)]\psi}$ from the very start. There is a small conceptual advantage in this replacement, for the spin analysis of Ref. 10 tells us immediately that $e^{[\alpha(t_1)+\alpha(t_2)]\psi}$ indicates we are dealing with Reggeons of helicity (analytically continued to be eigenvalues of $y$ boosts) $\alpha(t_1)$ for the lower line with mass $t_1$ and $-\alpha(t_2)$ for the upper Reggeon.[11] The quantity $B$, therefore, is the Reggeon-particle $b$ absorptive part of a definite helicity amplitude in the $b\bar{b}'$ channel.

Now using the addition theorem, Eq. (41) of Ref. 10, we may carry out the $\psi$ integration involved in (17) with the result [setting $h=\alpha(t_1)+\alpha(t_2)$]

$$\tfrac{1}{4}\int_{-\infty}^{+\infty}d\psi[e^{h\psi}+e^{-h\psi}]Q_l(\cosh\Theta)=e^l_{0h}(\cosh\Theta_0)e^l_{h0}(\cosh\Theta')\,, \tag{22}$$

where $e^l_{mn}(\cosh\Theta)$ is the second-kind function on the SO(1, 2) invariance group of $Q$.[10,12] The function we want has the explicit form

$$e^l_{0h}(\cosh\Theta)=\frac{[\Gamma(l-h+1)\Gamma(l+h+1)]^{1/2}}{2\Gamma(2l+2)}[\sinh\tfrac{1}{2}\Theta\cosh\tfrac{1}{2}\Theta]^{-h}[\sinh\tfrac{1}{2}\Theta]^{-2(l-h+1)}\times{}_2F_1\left(l+1,\,l-h+1;\,2l+2;\,\frac{-1}{\sinh^2(\frac{1}{2}\Theta)}\right)\,, \tag{23}$$

and what will be important for us is that $e^l_{0h}/[\Gamma(l-h+1)]^{1/2}$ is well behaved at $l=h-1, h-2, \ldots$.

Now we define partial-wave amplitudes from the Reggeon-particle absorptive parts by

$$J(l,t,u,z,u',z')=\int_{\bar{s}}^{\infty}ds_0\frac{e^l_{0h}(\cosh\Theta_0)}{[\Gamma(l-h+1)]^{1/2}}J(\Theta_0,t,u,z,u',z') \tag{24}$$

and

$$G(l,t,u',z',v,\zeta)=\int_{\bar{s}}^{\infty}ds'\frac{e^l_{h0}(\cosh\Theta')}{[\Gamma(l-h+1)]^{1/2}}B(\Theta',t,u',z',v,\zeta)\,. \tag{25}$$

Then the equation for $F(l,t)$ reads

$$F(l,t,u,z,v,\zeta)=I(l,t,u,z,v,\zeta)+\int_{-\infty}^{0}du'\int_{-1}^{+1}\frac{dz'}{(1-z'^2)^{1/2}}J(l,t,u,z,u',z')\Gamma(l-\alpha(t_1)-\alpha(t_2)+1)G(l,t,u',z',v,\zeta). \tag{26}$$

The $l$-plane cut arises from the integration over the $\Gamma(l-\alpha(t_1)-\alpha(t_2)+1)$. The leading cut in the $l$ plane comes from the neighborhood of $l=\alpha(t_1)+\alpha(t_2)-1$ with other two-Reggeon branch lines displaced a whole unit in $l$. For the contributions in the $l$ plane lying farthest to the right we may make the approximation

$$\Gamma(l-\alpha(t_1)-\alpha(t_2)+1)\approx\frac{1}{l-\alpha(t_1)-\alpha(t_2)+1}, \tag{27}$$

so that in the neighborhood of this leading two-Reggeon cut the equation for $F(l,t)$ reads

$$F(l,t,u,z,v,\zeta)=I(l,t,u,z,v,\zeta)+\int_{-\infty}^{0}du'\int_{-1}^{+1}\frac{dz'}{(1-z'^2)^{1/2}}\frac{J(l,t,u,z,u',z')G(l,t,u',z',v,\zeta)}{l-\alpha(t_1)-\alpha(t_2)+1}. \tag{28}$$

A more transparent cast is given to the whole equation if we define the two-dimensional spacelike vectors

$$\vec{q}=(0,\sqrt{-t}\,),\qquad |\vec{q}|^2=-t, \tag{29}$$

$$\vec{p}=\sqrt{-u}\,(\sin\phi,\ \cos\phi),\qquad z=\cos\phi,\ |\vec{p}|^2=-u, \tag{30}$$

$$\vec{p}\,'=\sqrt{-u'}\,(\sin\phi',\ \cos\phi'),\qquad z'=\cos\phi',\ |\vec{p}\,'|^2=-u', \tag{31}$$

and

$$\vec{k}=\sqrt{-v}\,(\sin\eta,\ \cos\eta),\qquad \zeta=\cos\eta,\qquad |\vec{k}|^2=-v. \tag{32}$$

Then we find the equation takes the form

$$F(l,\vec{p},\vec{q},\vec{k})=I(l,\vec{p},\vec{q},\vec{k})+\int d^2p'\,\frac{J(l,\vec{p},\vec{q},\vec{p}\,')G(l,\vec{p}\,',\vec{q},\vec{k})}{l-\alpha(t_1)-\alpha(t_2)+1}, \tag{33}$$

where

$$-t_1=(\vec{p}\,'+\tfrac{1}{2}\vec{q})^2 \tag{34}$$

and

$$-t_2=(\vec{p}\,'-\tfrac{1}{2}\vec{q})^2. \tag{35}$$

Equation (33) now exhibits the leading two-Reggeon branch cut possessed by $F(l,t)$. The branch point is at $\alpha_c(t)=2\alpha(\frac{1}{4}t)-1$, as usual, and with our conventions the branch line runs to the left in $l$. We define the physical value of $F(l,t)$ to the left of $l=\alpha(t)$ by replacing $l$ by $l+i\epsilon$, $\epsilon>0$. This is the right prescription for acquiring the physical partial-wave amplitude when we look at our branch line in the $t$ plane.

The virtue of the form (33) of our equation for $F(l,t)$ is in its interpretation as a nonrelativistic Lippmann-Schwinger-like equation in a two-dimensional space. We have three types of "particle" coupled together in this equation: spinless objects of $(\text{mass})^2=(p\pm\frac{1}{2}Q)^2$ and $(k\pm\frac{1}{2}Q)^2$, and objects of spin $\alpha(t)$, helicity $\pm\alpha(t)$, and $(\text{mass})^2=t$. Let us consider $F(l,\vec{p},\vec{q},\vec{k})$ as the matrix element of a transition operator $T$ taking us from a state of particles with momentum $\vec{p}$ to a particle state of momentum $\vec{k}$, all in the presence of an "external field" specified by the passive vector $\vec{q}$. Also we can think of $I(l,\vec{p},\vec{q},\vec{k})$ as the same matrix element of a potential $V$. Similarly, $G(l,\vec{p},\vec{q},\vec{k})$ is the matrix element of the transition operator $T$ between the Reggeon state with momentum $\vec{p}\,'$ and the particles of momentum $\vec{k}$, and also $J$ is a particle-Reggeon matrix element of $V$. The analogy of the free Green's function is of course $[l-\alpha(t_1)-\alpha(t_2)+1]^{-1}$, and, indeed, if we call $E=l-1$ then we are led to define

$$D_0(E)^{-1}=E-H_0, \tag{36}$$

where the free Hamiltonian $H_0$ is

$$H_0=[\alpha(-(\vec{p}+\tfrac{1}{2}\vec{q})^2)-1]+[\alpha(-(\vec{p}-\tfrac{1}{2}\vec{q})^2)-1]. \tag{37}$$

With this notation the basic equation (33) takes the familiar form

$$T(E)=V(E)+V(E)D_0(E)T(E), \tag{38}$$

where one takes the matrix element between particles of momentum $\vec{p}$ and particles of momentum $\vec{k}$ to recover (33). To evaluate the discontinuity across the two-Reggeon branch cut we recall that the solution of (38) is

$$T(E)=V(E)+V(E)D(E)V(E), \quad (39)$$

with

$$D(E)^{-1}=D_0(E)^{-1}-V(E) \quad (40)$$

$$=E-H_0-V(E). \quad (41)$$

We want to evaluate

$$T(E+i\epsilon)-T(E-i\epsilon)=V(E)[D(E+i\epsilon)-D(E-i\epsilon)]V(E), \quad (42)$$

remembering that $V(E)$ does not contain the two-Reggeon cut. Now we note[6]

$$D(E+i\epsilon)-D(E-i\epsilon)=-2i\epsilon D(E+i\epsilon)D(E-i\epsilon) \quad (43)$$

$$=-2i\epsilon[1+D(E+i\epsilon)V(E)]D_0(E+i\epsilon)D_0(E-i\epsilon)[1+V(E)D(E-i\epsilon)] \quad (44)$$

$$=[1+D(E+i\epsilon)V(E)][D_0(E+i\epsilon)-D_0(E-i\epsilon)][1+V(E)D(E-i\epsilon)], \quad (45)$$

and then we are led to

$$T(E+i\epsilon)-T(E-i\epsilon)=-2\pi i T(E+i\epsilon)\delta(E-H_0)T(E-i\epsilon), \quad (46)$$

which is a familiar result. Upon taking the matrix element of this unitarity relation between $\vec{p}$ and $\vec{k}$ we arrive at

$$\mathrm{Im}F(E,\vec{p},\vec{q},\vec{k})=-\pi\int d^2p'G(E,\vec{p},\vec{q},\vec{p}\,')G^*(E,\vec{p}\,',\vec{q},\vec{k}) \times\delta\Big(E-[\alpha(-(\vec{p}\,'+\tfrac{1}{2}\vec{q})^2)-1]-[\alpha(-(\vec{p}-\tfrac{1}{2}\vec{q})^2)-1]\Big), \quad (47)$$

showing that the discontinuity across the leading two-Reggeon cut in particle scattering is given by a well-defined integral over Reggeon-particle "scattering." Since we will show below that the latter objects are in fact measurable in inclusive reactions, the relation (47) has significant physical content.

Because of the δ function in (47) we may write out the form of $G$ a bit more explicitly by evaluating (25) at $l=\alpha(t_1)+\alpha(t_2)-1$:

$$G(E,\vec{p}\,',\vec{q},\vec{k})|_{E=H_0}=\frac{2^{E+1}}{[\Gamma(2E+4)]^{1/2}}\int_{\bar{s}}^{\infty}ds'\frac{B(\Theta',\vec{p}\,',\vec{q},\vec{k})}{[\sinh\Theta']^{\alpha(t_1)+\alpha(t_2)}}, \quad (48)$$

which we recognize as proportional to the residue of the fixed pole at $l=\alpha(t_1)+\alpha(t_2)-1$ in the $t$-channel partial-wave amplitude for Reggeon of helicity $\alpha(t_1)$ plus Reggeon of helicity $-\alpha(t_2)\to$ particle plus antiparticle. So we may say that the residues of a certain fixed poles in Reggeon-particle scattering set the scale for discontinuities across two-Reggeon cuts. In various models[13] this result has been known for some time, although the form quoted is slightly incorrect since one is told that the cut contribution to $F(l,t)$ is given as a two-dimensional integral over the product of fixed-pole residues. This, we now see, is only true for the discontinuity and would involve immense double counting if taken to be correct for $F(l,t)$ itself.

### B. Reggeon-Particle "Scattering"

If we apply our arguments about two-Reggeon irreducibility to the Reggeon-particle absorptive part $B(s,\vec{p},\vec{q},\vec{k})$ then we may decompose $B$ into its irreducible part $J$ plus an integral over the Reggeon-Reggeon two-Reggeon irreducible part, call it $K$, and $B$ itself. In our symbolic notation of Eqs. (6)–(11) we find

$$B^{(\pm)}=J^{(\pm)}+K^{(\pm)}DB^{(\pm)}. \quad (49)$$

(See Fig. 5.) Again we may carry out the integration indicated in Eq. (25) to learn

$$G(l,\vec{p},\vec{q},\vec{k})=J(l,\vec{p},\vec{q},\vec{k})+\int d^2p'K(l,\vec{p},\vec{q},\vec{p}\,')\frac{G(l,\vec{p}\,',\vec{q},\vec{k})}{l-\alpha(t_1)-\alpha(t_2)+1}, \quad (50)$$

employing the approximation $\Gamma(x)\approx 1/x$ once more. In (50) the partial-wave amplitude $K(l,\vec{p},\vec{q},\vec{p}\,')$ of the Reggeon-Reggeon two-Reggeon irreducible part is given by

$$K(l,\vec{p},\vec{q},\vec{p}')=\int_{\bar{s}}^{\infty} ds_0 \frac{e^{l}_{\alpha(t_a)+\alpha(t_a'),\,\alpha(t_1)+\alpha(t_2)}(\cosh\Theta_0)}{[\Gamma(l-\alpha(t_a)-\alpha(t_a')+1)\Gamma(l-\alpha(t_1)-\alpha(t_2)+1)]^{1/2}} K(\Theta_0,\vec{p},\vec{q},\vec{p}') , \tag{51}$$

where we have set $-t_a=(\vec{p}+\frac{1}{2}\vec{q})^2$ and $-t_a'=(\vec{p}-\frac{1}{2}\vec{q})^2$.

If we call the full Reggeon-Reggeon absorptive part $C(s,\vec{p},\vec{q},\vec{k})$ for the $t$-channel helicity transition

$$\text{Reggeon}[\alpha(t_a)]+\text{Reggeon}[-\alpha(t_a')]\to\text{Reggeon}[-\alpha(t_b)]+\text{Reggeon}[\alpha(t_b')],$$

where $-t_b=(\vec{k}+\frac{1}{2}\vec{q})^2$ and $-t_b'=(\vec{k}-\frac{1}{2}\vec{q})^2$, then we are led to define the full partial-wave amplitude as

$$H(l,\vec{p},\vec{q},\vec{k})=\int_{\bar{s}}^{\infty} ds \frac{e^{l}_{\alpha(t_a)+\alpha(t_a'),\,\alpha(t_b)+\alpha(t_b')}(\cosh\Theta)C(s,\vec{p},\vec{q},\vec{k})}{[\Gamma(l-\alpha(t_a)-\alpha(t_a')+1)\Gamma(l-\alpha(t_b)-\alpha(t_b')+1)]^{1/2}} . \tag{52}$$

Repeating the arguments we used to find the discontinuity across the two-Reggeon cut in $F(l,t)$, we now can evaluate the discontinuity across the same cut in $G(l,t)$:

$$\text{Im}G(E,\vec{p},\vec{q},\vec{k})=-\pi\int d^2p'H(E,\vec{p},\vec{q},\vec{p}')G^*(E,\vec{p}',\vec{q},\vec{k}) \times\delta\left(E-[\alpha(-(\vec{p}'+\tfrac{1}{2}\vec{q})^2)-1]-[\alpha(-(\vec{p}'-\tfrac{1}{2}\vec{q})^2)-1]\right) . \tag{53}$$

Using the δ function once more we find that the value of $H(E)$ "on shell" is

$$H(E,\vec{p},\vec{q},\vec{k})|_{l=\alpha(t_b)+\alpha(t_b')-1}=\tfrac{1}{2}[\Gamma(m+n)\Gamma(2n)]\int_{\bar{s}}^{\infty} ds \frac{C(s,\vec{p},\vec{q},\vec{k})}{(\cosh\frac{1}{2}\Theta)^{n+m}(\sinh\frac{1}{2}\Theta)^{|n-m|}} , \tag{54}$$

where $n=\alpha(t_b)+\alpha(t_b')$ and $m=\alpha(t_a)+\alpha(t_a')$, and we note that this is the residue of a fixed pole in the $t$-channel partial-wave amplitude for the Reggeon-Reggeon process.

### C. Reggeon-Reggeon "Scattering"

Finally we turn to the decomposition of $C(s,\vec{p},\vec{q},\vec{k})$ into its irreducible and reducible parts. Essentially no further argument leads one to

$$C^{(\pm)}=K^{(\pm)}+K^{(\pm)}DC^{(\pm)} \tag{55}$$

(see Fig. 6), and doing the integral in (52) in the standard fashion yields

$$H(l,\vec{p},\vec{q},\vec{k})=K(l,\vec{p},\vec{q},\vec{k})+\int d^2p'K(l,\vec{p},\vec{q},\vec{p}')\frac{H(l,\vec{p}',\vec{q},\vec{k})}{l-\alpha(t_1)-\alpha(t_2)+1} . \tag{56}$$

The discontinuity across the two-Reggeon cut in $H$ also follows from our previous arguments:

$$\text{Im}H(E,\vec{p},\vec{q},\vec{k})=-\pi\int d^2p'H(E,\vec{p},\vec{q},\vec{p}')H^*(E,\vec{p}',\vec{q},\vec{k})\delta(E-[\alpha(t_1)-1]-[\alpha(t_2)-1]) ; \tag{57}$$

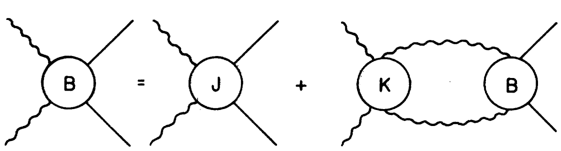


FIG. 5. The equation for the Reggeon-particle absorptive part $B$. $B$ is decomposed into its two-Reggeon irreducible piece, $J$, and its reducible piece. The latter is composed of the two-Reggeon irreducible piece of the Reggeon-Reggeon absorptive part, called $K$, and $B$ again.

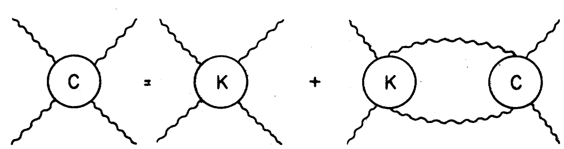


FIG. 6. The equation for the full Reggeon-Reggeon absorptive part.

we will indicate in the next section how one may actually measure the function $C(s,\vec{p},\vec{q},\vec{k})$ in inclusive processes so that (57) will find its relation to physics.

In this long section (Sec. II) we have established a coupled set of integral equations for the transition amplitudes of particles and Reggeons. Each of our equations is the appropriate matrix element of the operator equation

$$T = V + VD_0T$$

in the two-dimensional space where the particle and Reggeon "states" of two-momentum $\vec{p}$ are defined. The various discontinuity equations are also just matrix elements of the basic formula (46). The value of these equations is that they emphasize the dynamical role played by the two-Reggeon cut while lumping the rest of the physics into the potential operator $V$. In a situation where that cut may be expected to be important these equations will be a useful framework in which to cast the dynamics. In the last section we will work out an approximate solution to this set of equations when the $l$-plane physics is primarily dictated by a pole and a close-by two-Reggeon cut. One can easily contemplate that his knowledge and intuition about nonrelativistic quantum mechanics will be valuable in suggesting approximation techniques for the cut dynamics. Our next task, however, will be to show how one may, via inclusive reactions, get a direct handle on quantities such as our Reggeon-particle absorptive part $B$.

### III. CONNECTION WITH INCLUSIVE REACTIONS

We want to examine now the manner in which we may extract information from experiments on the functions $B$ and $C$. By referring to the definition of $B$ via the nonforward unitarity relation Eq. (12) we see that in principle detailed knowledge of the $T_{2\to N}$ matrix elements would allow us to construct $B$. In particular we could take all events in which $J$ consists of a single particle $c$ of mass $m_c$ so that

$$J(p,Q,p') \propto \beta_{ac}((p'+\tfrac{1}{2}Q)^2)\beta_{ac}((p'-\tfrac{1}{2}Q)^2)\delta(m_c^{\,2}-(p-p')^2)\,, \tag{58}$$

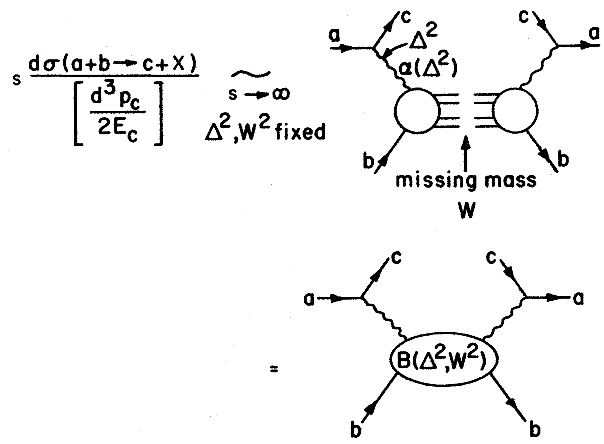


FIG. 7. The single-particle inclusive reaction $a+b \to c+$ anything (called $X$) when $s\to\infty$; $\Delta^2$ and $W^2$ are held fixed. This exposes the Reggeon-particle *forward* absorptive part.

where the particle-$a$–particle-$c$–Reggeon coupling is determined from two-body physics. Such contributions to $T_{2\to N}$ are those at large $s$ where one particle called $c$ moves along the direction of particle $a$ with its longitudinal momentum a significant fraction of $a$'s momentum. Clearly the subenergy between $c$ and any other produced particle is large in this class of events. Were one able to have in detail $T_{2\to N}$ for such processes, he could now find $B$ knowing $\beta_{ac}((p'\pm\tfrac{1}{2}Q)^2)$.

It is much more feasible, however, to contemplate evaluating the function $B$ at $Q=0$, since then it is proportional to the differential cross section for $a+b\to c+$anything else in the region where $c$ is in the fragmentation region of $a$.

Let us label momenta as in Fig. 7 for the contribution to the inclusive cross section $a+b\to c+X$ coming from the exchange of a trajectory $\alpha(\Delta^2)$. In our previous notation we would have

$$u'=\Delta^2\,,\qquad u=m_a^{\,2}\,,\qquad v=m_b^{\,2}\,, \tag{59}$$

$$s=s\,,\quad s'=W^2=(\text{missing mass})^2\,,\quad s_0=m_c^{\,2}\,. \tag{60}$$

The irreducible part $J$ is

$$J(m_a^{\,2},\Delta^2,m_c^{\,2}) = \left[\frac{\Delta^{1/2}(m_a^{\,2},\Delta^2,m_c^{\,2})}{2(-\Delta^2)^{1/2}}\right]^{2\alpha(\Delta^2)}\beta_{ac}(\Delta^2)^2\delta(s_0-m_c^{\,2})\,. \tag{61}$$

The reduced residue functions are normalized so that if the trajectory $\alpha(\Delta^2)$ is exchanged in the process $a+a\to c+c$ at momentum transfer $\Delta^2$, the differential cross section for that reaction coming from that exchange is

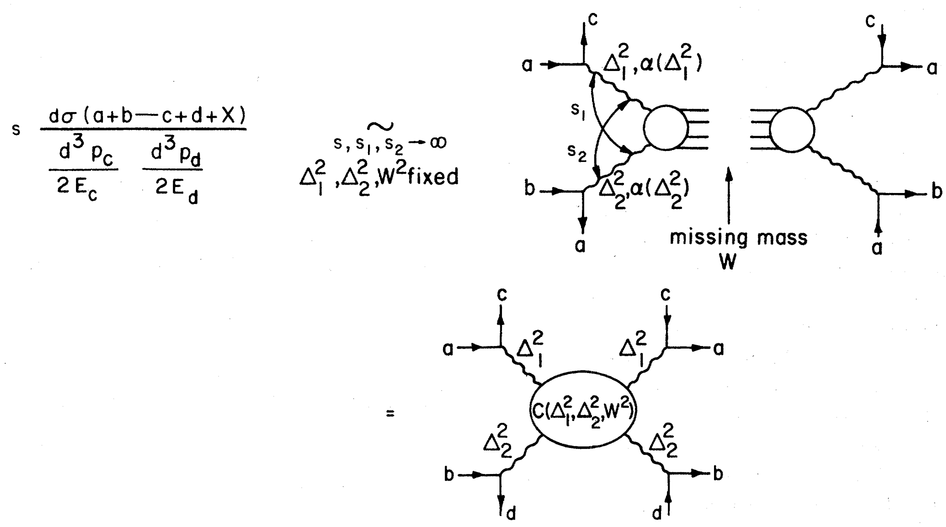


FIG. 8. The two-particle inclusive process $a+b\to c+d+X$ when $s, s_1, s_2\to\infty$ while $\Delta_1^{\,2}$, $\Delta_2^{\,2}$, and $W^2$ are held fixed. This reveals the Reggeon-Reggeon *forward* absorptive part.

$$\frac{s^2 d\sigma(a+a\to c+c)}{-d\Delta^2} \underset{s\to\infty;\ \Delta^2\,\text{fixed}}{\sim} \frac{1}{16\pi}\beta_{ac}(\Delta^2)^4 s^{2\alpha(\Delta^2)}\frac{|e^{-i\pi\alpha(\Delta^2)}+\tau|^2}{\sin^2\pi\alpha(\Delta^2)}\,, \tag{62}$$

with $\tau=\pm1$ the signature of the trajectory $\alpha(\Delta^2)$. Note that $\beta_{ac}(\Delta^2)$ is real for $\Delta^2$ spacelike.

The inclusive cross section is now read off from Eq. (12) and Eq. (17) to be

$$\Delta^{1/2}(s, m_a^{\,2}, m_b^{\,2})\,\frac{d\sigma(a+b\to c+\text{anything})}{d^3p_c/2E_c} \underset{s\to\infty;\ \Delta^2\text{ and }W^2\text{ fixed}}{\sim} \beta_{ac}(\Delta^2)^2 s^{2\alpha(\Delta^2)}\left[\frac{\Delta^{1/2}(\Delta^2, W^2, m_b^{\,2})}{2(-\Delta^2)^{1/2}}\right]^{-2\alpha(\Delta^2)} B(\Delta^2, W^2)\,. \tag{63}$$

The triangle function is just a relative momentum factor in the $\Delta^2$ channel which comes from our using $e^{\psi}$ instead of $s$ for arithmetic convenience.

So we see that in a single-particle inclusive experiment one may evaluate the function $B(p, 0, k)$, which is the particle-Reggeon forward absorptive part,[14] and that, in principle, one may extract $B(p, Q, k)$ from the unitarity equation using only physical amplitudes $T_{2\to N}$. The former is surely feasible while the latter, since it requires detailed phase information on $T_{2\to N}$, may be nigh on intractable.

To find the function $C$ from experiment it is necessary to do a two-particle inclusive reaction $a+b\to c+d+\text{anything}$ in the region where $c$ is at the edge of the fragmentation region of $a$, and $d$ at the edge of the fragmentation region of $b$. Then we can have the double-Regge exchange shown in Fig. 8. The resulting two-particle inclusive cross section is for large $s$

$$\begin{aligned}\Delta^{1/2}(s, m_a^{\,2}, m_b^{\,2})\frac{d\sigma(a+b\to c+d+\text{anything})}{(d^3p_c/2E_c)(d^3p_d/2E_d)} &\underset{\substack{s,s_1,s_2\to\infty;\\ \Delta_1^{\,2},\,\Delta_2^{\,2},\,W^2\,\text{fixed}}}{\sim} \beta_{ac}(\Delta_1^{\,2})^2\beta_{bd}(\Delta_2^{\,2})^2(s_1)^{2\alpha(\Delta_1^{\,2})}(s_2)^{2\alpha(\Delta_2^{\,2})}\\ &\qquad\times\left[\frac{W^2-[(-\Delta_1^{\,2})^{1/2}-(-\Delta_2^{\,2})^{1/2}]^2}{4(\Delta_1^{\,2}\Delta_2^{\,2})^{1/2}}\right]^{-[\alpha(\Delta_1^{\,2})+\alpha(\Delta_2^{\,2})]}\\ &\qquad\times C(\Delta_1^{\,2}, \Delta_2^{\,2}, W^2)\,.\end{aligned} \tag{64}$$

Again, one is measuring here only the *forward* Reggeon-Reggeon absorptive part. We would have to work enormously harder to extract $C$ for $Q\neq0$; in principle, it could be done.

## IV. A MODEL CALCULATION

In this section we want to build a little model involving an $l$-plane pole and the associated two-Reggeon cut based on the unitarity relations given before. We will imagine that these two singularities represent the only important structure in some neighborhood of the $l$ plane. In particular we have in mind here the Pomeranchukon pole at $l=1+\alpha' t$ and the two-Pomeranchukon cut at $\alpha_c(t) = 1+\frac{1}{2}\alpha' t$ generated by it. We will write down dispersion relations in the energy ($=l-1$) plane for our functions $F$, $G$, and $H$ which reflect this simple analytic structure, and using the elastic unitarity relation (46) to give the discontinuity across the branch lines we will solve the resulting integral equations by quite standard methods.

We shall ignore, for reasons of obvious simplicity, three-, four-, etc. Reggeon cuts. One can make a plausibility argument that they are not as important as two-Reggeon cuts, at least in the vicinity of the tips of the cuts, because their discontinuities are known to vanish more rapidly there[15]; that is, multi-Reggeon phase space is vanishingly small near threshold. For the moment, however, I have no real way to assess the importance of multi-Reggeon cuts and will frankly ignore them. One may include their effect in a systematic fashion by writing $T$-matrix equations to take into account two- and three-Reggeon states and then 2, 3, and 4, etc. This little exercise we leave to the reader.

We shall proceed by imagining that there is an interesting region of $E$, $t$ space (or $l$, $t$ if you like) where a pole at $E_0(t)=\alpha(t)-1$ and a two-Reggeon branch point at $E_c(t)=\alpha_c(t)-1=2E_0(\frac{1}{4}t)$ arising from two Reggeons $\alpha(t)$ are nearby. (Again, the problem of colliding Pomeranchukons and the associated branch cut is what we have in mind.) Using standard dispersion-theory lore, we write

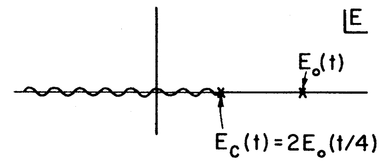


FIG. 9. The $E=l-1$ plane. This shows the pole at $E_0(t)=\alpha(t)-1$ and the two-Reggeon branch cut, all for $t>0$. This analytic structure is put together with "elastic unitarity" in the integral equations for $F$, $G$, and $H$.

a dispersion relation for $F(E,\vec{p},\vec{q},\vec{k})$ in $E$ for fixed $\vec{p},\vec{q},\vec{k}$ exhibiting the pole and branch cut (see Fig. 9):

$$F(E,\vec{p},\vec{q},\vec{k})=\frac{g(\vec{p},\vec{q})g(\vec{k},\vec{q})}{E-E_0(t)} + \frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\,\mathrm{Im}F(E',\vec{p},\vec{q},\vec{k}), \tag{65}$$

where we intend to evaluate the discontinuity across the cut by employing Eq. (47). The residue at the pole has been written in factorized form and each factor $g$ is a two-particle Reggeon vertex.

This dispersion relation is not enormously useful unless we make some simplification. The complication comes from the presence of the vector $\vec{q}$, which provides a fixed direction in the scattering and, in a sense, acts like an "external potential." Our modification of (65) will be to neglect all dependence on angles with respect to $\vec{q}$. This is like taking only $s$ waves in conventional quantum mechanics. With this accepted, $F$ is a function only of $E$, $p=(|\vec{p}|^2)^{1/2}$, $k=(|\vec{k}|^2)^{1/2}$, and $q=(|\vec{q}|^2)^{1/2}$. This is also true for $G$ and $H$, and we may carry out the angular integration in (47) to write

$$\mathrm{Im}F(E,p,q,k)=-\pi^2\int_0^\infty dp'^2 G(E,p,q,p')G^*(E,p',q,k)\delta(E-2E_0(-p'^2-\tfrac{1}{4}q^2)) \tag{66}$$

$$=\rho(E,t)G(E,p,q,p_0)G^*(E,p_0,q,k), \tag{67}$$

where $p_0$ is the solution to

$$E=2E_0(-p_0{}^2-\tfrac{1}{4}q^2), \tag{68}$$

and

$$\rho^{-1}(E,t)=-\frac{2}{\pi^2}\left|\frac{d}{dp'^2}E_0(-p'^2-\tfrac{1}{4}q^2)\right|_{p'=p_0}. \tag{69}$$

In the same approximation of neglecting angles with respect to $\vec{q}$ we find for Im$G$ and Im$H$

$$\mathrm{Im}G(E,p,q,k)=\rho(E,t)G(E,p,q,p_0)H^*(E,p_0,q,k) \tag{70}$$

$$=\rho(E,t)H(E,p,q,p_0)G^*(E,p_0,q,k) \tag{71}$$

and

$$\mathrm{Im}H(E,p,q,k)=\rho(E,t)H(E,p,q,p_0)H^*(E,p_0,q,k). \tag{72}$$

Since one's experience is that $E_c(t)>E_0(t)$ for $t<0$, it is convenient for calculation to simply continue by hand all our formulas to $t=-q^2>0$, solve there, and then evaluate anywhere we like. Let us do that and write for $F(E,p,q,k)$, then,

$$F(E,p,t,k=\frac{g(p,t)g(k,t)}{E-E_0(t)}+\frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\rho(E',t)G(E',p,t,p_0)G(E',p_0,t,k). \tag{73}$$

This equation and the similar ones for $G$ and $H$ are, in conventional potential-scattering language, off-shell equations. By examining the form of the discontinuities we see that to reconstruct $F(E,p,t,k)$ we need the half-on-shell matrix elements $G(E,p,t,p_0)$ and $G(E,p_0,t,k)$. To evaluate those matrix elements it is sufficient to know $H$ completely "on shell"; that is, $H(E,p_0,t,p_0)$. So we write for our coupled set of equations (73), together with

$$G(E,p,t,p_0)=\frac{g(p,t)\tilde{g}(t)}{E-E_0(t)}+\frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\rho(E',t)G(E',p,t,p_0)H^*(E',p_0,t,p_0), \tag{74}$$

$$G(E,p_0,t,k)=\frac{\tilde{g}(t)g(k,t)}{E-E_0(t)}+\frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\rho(E',t)H(E',p_0,t,p_0)G^*(E',p_0,t,k), \tag{75}$$

and

$$H(E,p_0,t,p_0)=\frac{\tilde{g}(t)^2}{E-E_0(t)}+\frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\rho(E',t)|H(E',p_0,t,p_0)|^2. \tag{76}$$

In these equations we have designated the three-Reggeon coupling $\tilde{g}$ and noted it is a function of $t$ only.

This set of equations can be solved by standard techniques.[16] We will proceed by solving (76) by the usual $N/D$ method. Next (74) and (75) can be solved by the Omnès-Muskhelishvili method, writing $H=(1/\rho)e^{i\phi}\times\sin\phi$. At the end $F$ is constructed by quadratures over $GG^*$. What we are doing, in more standard language, is taking a coupled-channel situation where only the elastic unitarity cut of one channel (here the Reggeons) is significant and solving the problem of making the Born (or pole) terms consistent with elastic unitarity. (If one likes analogies, he may think of the problem of neutron-proton scattering near threshold in the partial wave containing the deuteron.)

To solve (76) we write

$$H(E,t)=\frac{\tilde{g}(t)^2}{[E-E_0(t)]f(E,t)}, \tag{77}$$

where we have abbreviated $H(E,p_0,t,p_0)$ in an obvious fashion. The function $f(E,t)$ shares the branch point at $E=E_c(t)$ and is equal to unity $E=E_0(t)$. We may write a dispersion relation for it:

$$f(E,t)=1+\frac{E-E_0(t)}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\frac{\mathrm{Im}f(E',t)}{E'-E_0(t)}, \tag{78}$$

and from (76) we learn

$$\mathrm{Im}f(E',t)=-\frac{\tilde{g}(t)^2\rho(E',t)}{E'-E_0(t)}, \tag{79}$$

and thus

$$f(E,t)=1-\frac{\tilde{g}(t)^2[E-E_0(t)]}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\frac{\rho(E',t)}{[E'-E_0(t)]^2}. \tag{80}$$

Of course, there is the canonical ambiguity in solving equations like (76) having to do with CDD (Castillejo-Dalitz-Dyson) poles.[17] We have presented the solution assuming there are none. More general solutions may be exhibited but probably without illuminating consequences.

To carry out the integral in (80) it is necessary to know $\rho(E,t)$, and thus we must now say something about the input trajectory or, in the present language, the energy-momentum relation of the free Reggeons. For want of a reason to make life more complicated we will take a linear trajectory $\alpha(t)=1+\alpha't$. Choosing $\alpha(0)=1$ really does tie us to the Pomeranchukon, but dropping the cloak of generality we now concentrate on that problem. With this choice,

$$E_0(-p^2-\tfrac{1}{4}q^2)=-\alpha'(p^2+\tfrac{1}{4}q^2), \tag{81}$$

$$\rho(E,t)=-\pi^2/2\alpha'. \tag{82}$$

Then we may integrate to find for $E>E_c(t)=\frac{1}{2}\alpha' t$

$$f(E,t)=1+\frac{\pi\tilde{g}(t)^2}{2\alpha'}\left[\frac{1}{E_c(t)-E_0(t)}+\frac{1}{E-E_0(t)}\ln\frac{E-E_c(t)}{E_0(t)-E_c(t)}\right]. \tag{83}$$

It is convenient before continuing to note the meaning of $\tilde{g}(t)$. It is the three-Reggeon vertex with one leg at $(\text{mass})^2=t=-|\vec{q}|^2$, while the other legs are "on shell." That is, each one has $(\text{mass})^2=-p^2-\frac{1}{4}q^2$, but since we are dealing with the on-shell function $H(E,t)$,

$$p^2=p_0{}^2=\frac{E-\frac{1}{2}\alpha' t}{-2\alpha'}$$

for the linear trajectory. Further, since $\tilde{g}$ is the residue of a pole at $E=E_0(t)=\alpha' t$, we have $p_0{}^2=-\frac{1}{4}t$, so that the $(\text{mass})^2$ of the legs of the triple-Reggeon vertex are $t$, $\frac{1}{2}t$, and $\frac{1}{2}t$. This is shown in Fig. 10.

The Reggeon-Reggeon on-shell amplitude which results from our solution of (76) is, for $t>0$, $E>E_c(t)$,

$$H(E,t)=\frac{\tilde{g}(t)^2}{E-E_0(t)}\left[1+\frac{\pi\tilde{g}(t)^2}{2\alpha'}\left(\frac{1}{E_c(t)-E_0(t)}+\frac{1}{E-E_0(t)}\ln\frac{E-E_c(t)}{E_0(t)-E_c(t)}\right)\right]^{-1}. \tag{84}$$

From the unitarity relation

$$\operatorname{Im}H(E,t)=\rho\,|H(E,t)|^2\Theta(E_c(t)-E), \tag{85}$$

we know we can write $H(E,t)$ in the form

$$H(E,t)=e^{i\phi}\sin\phi/\rho \tag{86}$$

$$=\frac{1}{\rho\cot\phi-i\rho} \tag{87}$$

for $E_c>E$. Indeed we find for the phase shift $\phi(E,t)$

$$-\frac{\pi^2}{2\alpha'}\cot\phi(E,t)=\frac{E-E_0(t)}{\tilde{g}(t)^2}+\frac{\pi}{2\alpha'}\left[\frac{E-E_0(t)}{E_c(t)-E_0(t)}+\ln\frac{E_c(t)-E}{E_0(t)-E_c(t)}\right]. \tag{88}$$

One should view the solution (84) as the functional form for $H(E,t)$ which is required by "elastic unitarity" in the $t$ channel. By construction it has a simple pole at $E=E_0(t)$ with residue $\tilde{g}(t)^2$ and a branch cut running left from $E_c(t)$ to $-\infty$. It is the generalization of the form of the Reggeon-Reggeon partial-wave amplitude given by Gribov *et al.* in Ref. 1 to the case where both a pole and a cut are present. Those authors considered the structure of $H(E,t)$ when only a cut was present and isolated the important $\ln[E_c(t)-E]$ factor which arises because the dynamical space is two-dimensional.

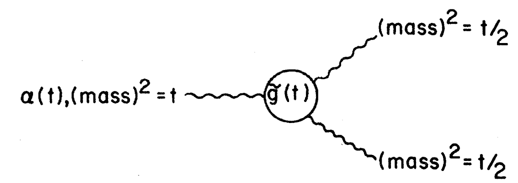


FIG. 10. The "on-shell" triple-Reggeon vertex $\tilde{g}(t)$ which enters the dispersion relation Eq. (76) as the factorized residue of the pole at $E=\alpha(t)-1$. For a linear trajectory on the on-shell condition constrains the external legs to have $(\text{mass})^2=\frac{1}{2}t$.

A very amusing consequence of the linearity of $\alpha(t)$ and the choice $\alpha(0)=1$ comes when we consider $H(E,t)$ as given by (84) near $t=0$ for fixed $E$. Because of the $E_c(t)-E_0(t)=-\frac{1}{2}\alpha' t$ appearing in the denominator we find that $H(E,0)$ seems to vanish at $t=0$. In fact if we arrange $\tilde{g}(t)$ also to vanish so as to cancel the $t^{-1}$ behavior in the denominator, its presence in the numerator causes $H(E,0)$ to vanish nevertheless. So in the model we have constructed the *on-shell Reggeon-Reggeon partial-wave amplitude vanishes identically at $t=0$.*

The general form (80) for the denominator of $H(E,t)$ is also employed by Bronzan[18] in an interesting program which seeks a "self-consistent" pole trajectory $E_0(t)$. He recognizes, as we have ignored in our little model, that $E_0(t)$ cannot strictly be $\alpha' t$ and maintain the real analyticity of $H$. In detail his program is much the same as our model here, but he seeks the trajectory as an *output*, and by requiring it to be the same as the input trajectory entering disc$H(E, t)$ he determines the parameters of possible CDD ambiguities His trajectories are complex for $t<0$ and, as pointed out by Bronzan,[18] thus avoid spurious fixed-$t$ cuts. Because of the $\ln[E_0(t)-E_c(t)]$ present, say in (84), our solution has this disease. For our purposes here, the important feature we wish to emphasize is the rule played by $\tilde{g}(t)$ in setting the scale of the two-Reggeon cut even in particle scattering. It should be straightforward to repair this defect of fixed-$t$ cuts in a more complicated model. The role of $\tilde{g}(t)$ will clearly be the same.

If we look at the denominator of (84) to make quite sure that we have no poles beyond the one at $E=E_0(t)$ which we built in, we find a condition on $\tilde{g}(t)$. Rewrite (84) as

$$H(E, t)=\frac{2\alpha'/\pi}{\ln\lambda+\{[\alpha'^2 t/\pi\tilde{g}(t)^2]-1\}(\lambda-1)}, \tag{89}$$

where $\lambda=[E-E_c(t)]/[E_0(t)-E_c(t)]$ and we have used $E_0(t)-E_c(t)=\frac{1}{2}\alpha' t$. The condition that the denominator of (89) vanish only for $\lambda=1$, that is, $E=E_0(t)$, is that the coefficient of $\lambda-1$ be positive. This requires

$$\tilde{g}(t)<\frac{\alpha'\sqrt{t}}{\pi}. \tag{90}$$

Thus, the requirement that indeed we have a Pomeranchukon pole leads to the vanishing of the triple-Pomeranchukon vertex at $t=0$. For a trajectory analytic at $t=0$, this is a known requirement of $\tilde{g}(t)$.[19]

We are now prepared to evaluate the half-on-shell Reggeon-particle partial-wave amplitude $G(E, p, t, p_0)$ which satisfies the integral equation

$$G(E, p, t, p_0)=\frac{g(p, t)\tilde{g}(t)}{E-E_0(t)}+\frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}e^{-i\phi}\sin\phi\, G(E', p, t, p_0), \tag{91}$$

which is the unsubtracted dispersion relation (74), noting $H=e^{i\phi}\sin\phi/\rho$.

The solution of this equation is known to involve the phase function

$$\Phi(E)=\frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\phi(E', t), \tag{92}$$

so we ought to enumerate some of the properties of $\phi$. From the standard arguments which go into Levinson's theorem for potential-scattering phase shifts we learn

$$\phi(-\infty)-\phi(E_c(t))=\pi, \tag{93}$$

since there is only one bound state at $E=E_0(t)$. Specifying $\phi(-\infty)=0$, we find that $\phi(E_c)=-\pi$. One may do even better by examining the form of $\phi(E)$ near $E_c$ from (88) to see

$$\phi(E)\underset{E\to E_c(t)}{\sim}-\pi+\pi\Big/\ln\left[\frac{E_0(t)-E_c(t)}{E_c(t)-E}\right]. \tag{94}$$

In a real sense the value of $\phi(E)$ at $E=-\infty$ is outside the realm of validity of our equations since we have restricted ourselves to the neighborhood of $E_c(t)$ or $E_0(t)$, and for $t$ not too large this means $E\approx 0$. However, mathematics forces us to make some announcement about $\phi(-\infty)$, so we set it equal to zero. (See Fig. 11.)

Since $\phi(E_c)\neq 0$ there is a $\ln(E_c-E)$ behavior in $\Phi$ near $E_c$ which we must account for in the solution of (91). We now write that solution in the form

$$G(E, p, t, p_0)=\frac{g(p, t)\tilde{g}(t)}{E-E_0(t)}\left\{1+\frac{[E-E_0(t)][E_c(t)-E]}{\pi}\times e^{\Phi(E)}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\frac{e^{-\delta(E')}\sin\phi(E')}{[E_c(t)-E'][E'-E_0(t)]}\right\} \tag{95}$$

for all $E$, and

$$G(E, p, t, p_0)=\frac{g(p, t)\tilde{g}(t)}{E-E_0(t)}e^{i\phi(E)}\left\{\cos\phi(E)+\frac{[E-E_0(t)](E_c-E)}{\pi}e^{\delta(E)}\mathrm{P}\int_{-\infty}^{E_c(t)}\frac{dE'}{(E'-E)}\frac{e^{-\delta(E')}\sin\phi(E')}{[E_c(t)-E'][E'-E_0(t)]}\right\} \tag{96}$$

for $E<E_c(t)$.

In these formulas we have written the principal-part phase function

$$\delta(E)=\frac{\mathrm{P}}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\phi(E'), \tag{97}$$

and in (96) we have explicitly noted that the phase of $G$ is that of $H$, a fact which is known in potential scattering[20] and follows directly from the linear nature of the unitarity relation for the half-on-shell amplitude.

If we are to be consistent with the original unsubtracted dispersion relation it is necessary

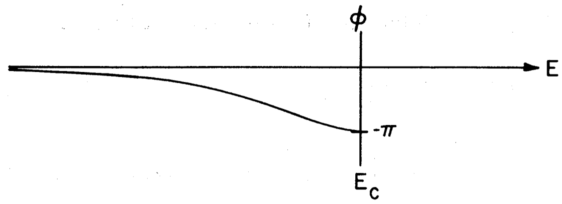


FIG. 11. The Reggeon-Reggeon phase shift $\phi$ as a function of $E$. It has been assumed that $\phi(-\infty)=0$, and then it follows that $\phi(E_c)=-\pi$ by Levinson's theorem. Further, it has been assumed that $\phi$ has no further zeros.

that the constraint

$$\int_{-\infty}^{E_c(t)} \frac{dE' e^{-\delta(E')}\sin\phi(E')}{[E_c(t)-E'][E'-E_0(t)]}=0 \tag{98}$$

hold. This may be of some value in nuclear physics applications,[21] but here, because of our disclaimer about real knowledge of the partial-wave amplitudes as $|E|\to\infty$, it has no particular content.

The significant feature of (95) is that at $E=E_c(t)$, $G$ vanishes. When $E=E_c$ and we have a linear trajectory, then the external legs are at $\frac{1}{4}t$, and since $G(E_c)$ is proportional to a certain fixed-pole residue, as noted in Eq. (48), we learn that that fixed-pole residue vanishes when two legs are at $\frac{1}{4}t$ and one is at $(\text{mass})^2=t$. As noted in the last paper of Ref. 14, this result, taken to $t=0$ by continuity, implies in its turn that $\tilde{g}(0)=0$. It is pleasing to see all these features appear in this little model calculation. Furthermore, because we know $\tilde{g}(t=0)=0$ from before and given our assumption that this zero is linear, we can see from (95) that $G(E, p, 0, p_0)$ also vanishes. So $G$ is zero at $t=0$ in whatever manner one approaches that point. The rate of going to zero may change if one sets $E=0$ first, but the vanishing still obtains.

Having the half-on-shell $G$ in hand one can go on to construct $F(l, t)$. Nothing immediately striking comes forth from putting Eq. (95) or Eq. (96) into the dispersion relation (73) for $F$. There are some features one can readily deduce, however. First, because $G$ is proportional to the triple-Reggeon coupling $\tilde{g}(t)$, the contribution of the cut to $F(l, t)$ vanishes at $t=0$. This is somewhat surprising and is likely due to the rather simplified model we have constructed. If it is true, then the total cross section arising from $F(l, 0)$ would not show secondary logarithmic contributions. Away from $t=0$ the elastic differential cross section would show such $\ln s$ factors but considerably weakened because $\tilde{g}(t)$ is small for small $t$.

The next feature is the sign of the cut contribution relative to the pole term. One can easily see that they are relatively positive in their contribution to $A(s, t)$, but because of our construction, proceeding as it does from the unitarity relation, one should not be terribly surprised by this.[22]

Finally we note that at $l=l_c(t)$ [or $E=E_c(t)$] the cut contribution to $F(l, t)$ vanishes as $[\ln(E_c(t)-E)]^{-1}$ since each of the $G$'s entering it behaves that way. This is completely consistent with the behavior found by Gribov *et al.*[1] But what is perhaps surprising is that since the cut and pole enter the dispersion relation for $F(l, t)$ in an additive fashion, the pole term survives at $E=E_c(t)$ and is

$$F(E_c(t), t)=\frac{g(p, t)g(k, t)}{E_c(t)-E_0(t)}. \tag{99}$$

As a further use of the dispersion relations we have constructed, one may evaluate $G$ and $H$ off shell in a straightforward manner. The value of $G$ off shell is of particular interest since at $t=0$ it is the object directly reconstructed through Eq. (25) from inclusive experiments. If we wish $G(E, p, t, k)$ with no "on-shell" restriction on the momenta $p$ or $k$ we need the imaginary part in $E$

$$\operatorname{Im}G(E, p, t, k)=\rho(E, t)G(E, p, t, p_0)\times H^*(E, p_0, t, k)\Theta(E_c(t)-E), \tag{100}$$

which involves the half-on-shell $G$ given by Eq. (95) or (96) and the half-on-shell $H$. We can compute the latter by noting that its discontinuity across the two-Reggeon cut is

$$\operatorname{Im}H(E, p_0, t, k)=\rho(E, t)H(E, p_0, t, p_0)\times H^*(E, p_0, t, k)\Theta(E_c(t)-E). \tag{101}$$

This involves $H(E, p_0, t, p_0)=H(E, t)$, which we know to be $e^{i\phi}\sin\phi/\rho$, so we need once again to solve an Omnès equation. Namely, if we designate the triple-Pomeranchukon coupling with legs of $(\text{mass})^2=t$, $-(k^2+\frac{1}{4}q^2)$, and $-(k^2+\frac{1}{4}q^2)$ by $\tilde{g}(t, k)$, we will write

$$H(E, p_0, t, k)=\frac{\tilde{g}(t)\tilde{g}(t, k)}{E-E_0(t)}+\frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}e^{-i\phi(E')}\sin\phi(E')H(E', p_0, t, k). \tag{102}$$

This is just (91) with a new inhomogeneous term. The solution is

$$H(E,p_0,t,k)=\frac{\tilde{g}(t)\tilde{g}(t,k)}{E-E_0(t)}\left\{1+\frac{[E-E_0(t)][E_c(t)-E]}{\pi}e^{\Phi(E)}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\,\frac{e^{-\delta(E')}\sin\phi(E')}{[E'-E_0(t)][E_c(t)-E']}\right\}. \quad (103)$$

From this function the Reggeon-particle partial-wave amplitude of interest is given as

$$G(E,p,t,k)=\frac{g(p,t)\tilde{g}(k,t)}{E-E_0(t)}+\frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\rho(E',t)G(E',p,t,p_0)H^*(E',p_0,t,k). \quad (104)$$

One could use this at $t=0$ to estimate the Regge-cut corrections to the leading triple-Regge behavior[14] coming from the pole term. By the same procedure one may construct $H(E,p,t,k)$ from

$$H(E,p,t,k)=\frac{\tilde{g}(p,t)\tilde{g}(k,t)}{E-E_0(t)}+\frac{1}{\pi}\int_{-\infty}^{E_c(t)}\frac{dE'}{E'-E}\rho(E',t)H(E',p,t,p_0)H^*(E',p_0,t,k). \quad (105)$$

and use it to evaluate the cut corrections to the missing-mass behavior of the two-particle inclusive process described in Sec. III.

## V. DISCUSSION

The greater portion of this paper has been devoted to discussing the general structure of the discontinuity in particle-particle scattering partial-wave amplitudes across the branch cuts in the $l$ plane generated by two moving $l$-plane poles. Also, the connection between the functions which set the scale of those discontinuities and Reggeon-particle absorptive parts directly measurable in inclusive reactions was emphasized. These give one a rather clean handle, presumably significant from a phenomenological viewpoint, on the size of Regge-cut contributions to scattering amplitudes. Hopefully such a handle will give us also a firmer understanding of the role Regge cuts must play as secondary corrections to simple power behavior in $s$.

Let us repeat the basic formula. We found, on the assumption that in the production matrix element $T_{2\to N}$ there is factorized Regge behavior in subenergies, that through the unitarity relation one could decompose the contributions to the signatured $t$-channel partial-wave amplitude $F(l,\vec{p},\vec{q},\vec{k})$ into a piece containing the two-Reggeon cut (the two-Reggeon reducible piece) and a piece without that cut (the two-Reggeon irreducible piece). The discontinuity across the two-Reggeon cut was then given by

$$\operatorname{disc}F(l,\vec{p},\vec{q},\vec{k})=-2\pi i\int d^2p'\;G(l,\vec{p},\vec{q},\vec{p}\,')G^*(l,\vec{p}\,',\vec{q},\vec{k})\,\delta(l-\alpha(-(\vec{p}\,'+\tfrac{1}{2}\vec{q})^2)-\alpha(-(\vec{p}\,'-\tfrac{1}{2}\vec{q})^2)+1), \quad (106)$$

where, at $\vec{q}=0$, $G$ is given by a definite integral [Eq. (25)] over a function measurable in inclusive reactions. The latter demonstration was given in Sec. III.

This last formula is in its essence given in the paper by Gribov *et al.*[1] by approximating the four-particle contribution to unitarity. Basically they build the two Reggeons $\alpha(t_1)$ and $\alpha(t_2)$ out of pairs of particles. Our derivation is a generalization, albeit a minor one, of their result in the sense that the Regge poles exchanged in the $T_{2\to N}$ relation used in the $s$-channel unitarity relation are built out of two, three, ... particles since they are the physical Reggeons. No doubt if one were to approximate the six-particle unitarity relation by having two clumps of three particles give the moving poles, the form (106) would follow.

It is clear from the manner in which we argued that one can progressively separate out and isolate the discontinuity across the three-, four-, etc. Reggeon cut and can relate such discontinuities to quantities measurable (with unspeakable difficulty) in single-particle inclusive processes. A more profitable approach to understanding such multi-Reggeon cuts is probably to take the route followed by Gribov and collaborators.[23] They effectively take the Lippmann-Schwinger equation and second-quantize it. Then they examine solutions to the resulting field theory by looking in detail at the Dyson equations for that theory.

With discontinuity formulas like (106) we then constructed a little model of a coupled (by analyticity) $l$-plane pole and associated two-Reggeon cut. By solving the resulting elastic unitarity equations we were able to extract detailed forms for the Reggeon-Reggeon and particle-Reggeon amplitudes in the case where the trajectory functions $\alpha(t)$ were taken to be linear. Even though one should be reluctant to take seriously the solutions for the various particle and Reggeon matrix elements in all their aspects, certain general features are probably correct. For example, the necessity of the vanishing of the triple-Pomeranchukon vertex at $t=0$ is surely independent of the model.[14] Also, the $[\ln(E_c-E)]^{-1}$ which keeps appearing in the denominator of the partial-wave amplitudes is generally true.[1] It is, after all, just the reflection of the manner in which two-dimensional phase space vanishes near threshold. A bolder statement is that the pervasive zero associated with the triple Pomeranchukon vertex $\tilde{g}(t)$ persists to cut down or even eliminate the contribution of two-Reggeon cuts for vanishing momentum transfers. Less bold, and less interesting, is the phase relation between the Reggeon-particle and Reggeon-Reggeon partial-wave amplitudes exhibited in Eq. (96) and more generally in Eq. (53).

### *Notes Added*

It may be useful to emphasize here that the method of construction of the two-Reggeon reducible and irreducible parts, which plays a key role in the derivation of the Lippmann-Schwinger–like equations (38), is perhaps open to some question. The construction given is certainly suggested by one's experience with multiperipheral models[7] and is at variance with what might be expected from the examination of selected sets of Feynman graphs.[5] The discussion herein has not tried to resolve the now ancient debate between these views of Regge cuts. Our results may turn out to be only a general property that all multiperipheral-like models must satisfy, rather than a property all models must satisfy. That it is a model of the two-Reggeon cut is, of course, true. The generality of the present construction has been discussed in a recent paper by Chew.[24]

Since this paper was written two significant references which have been omitted have come to my attention. First, Kaidalov[25] has recognized the connection of our function $B$, Eq. (63), with single-particle inclusive reactions and also with $J$-plane cuts. The important point that it is only the discontinuity across the cut which is set by $B$ [or $G$, Eq. (25)] is not, however, brought out. I wish to thank M. B. Einhorn for bringing this work to my attention.

Second, White[26] has reconsidered the work of Gribov *et al.*,[1] with very careful attention paid to questions of signature and sign. He finds our Eq. (47) for the two-Reggeon cut discontinuity; however, he finds the opposite sign. That sign agrees with Mandelstam[1] and Gribov[5] and disagrees with Amati *et al.*[1] As Chew has emphasized in the report just referred to, this sign has physical content, and the reconciliation of White's work with the present paper would seem to be a useful task.

## ACKNOWLEDGMENTS

I have profited by many conversations with J. B. Bronzan, M. B. Green, L. M. Saunders, R. L. Sugar, S. B. Treiman, and J. H. Weis on the subject matter in this paper. Much of this work was carried out while I enjoyed the hospitality of the Institute for Advanced Study.

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[1]The classical papers on this are S. Mandelstam, Nuovo Cimento 30, 1113 (1963); 30, 1127 (1963); 30, 1143 (1963); J. C. Polkinghorne, J. Math Phys. 4, 1396 (1963); D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento 26, 896 (1962); V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan, Yad. Fiz. 2, 361 (1965) [Sov. J. Nucl. Phys. 2, 258 (1966)].

[2]J. B. Bronzan and C. E. Jones, Phys. Rev. 160, 1494 (1967).

[3]The possibility of such irreducibility has probably been known for some time. The first discussion I could find in the literature is by P. V. Landshoff [Nucl. Phys. B15, 284 (1970)].

[4]Since we will be discussing leading behavior in energy or rightmost singularities in $l$, that we should encounter two-dimensional nonrelativistic physics is suggested by the general discussions of L. Susskind [Phys. Rev. 165, 1535 (1968)] and K. Bardacki and M. B. Halpern [*ibid.* 176, 1686 (1968)].

[5]These identifications have been made earlier by Gribov and his co-workers. In particular, see V. N. Gribov, Zh. Eksp. Teor. Fiz. 53, 654 (1967) [Sov. Phys. JETP 26, 414 (1968)].

[6]For three-particle systematics one may consult the lectures of C. Lovelace, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorehouse (Plenum, New York, 1964). For more particles yet, see S. Weinberg, in *Lectures on Particles and Field Theory*, 1964 Brandeis Summer Institute in Theoretical Physics, edited by S. Deser and K. W. Ford (Prentice Hall, Englewood Cliffs, N. J., 1965).

[7]A modern reference on this subject is H. D. I. Abarbanel, G. F. Chew, M. L. Goldberger, and L. M. Saunders, Princeton report, 1971 [to be published in Ann. Phys. (N.Y.)].

[8]V. N. Gribov, Ref. 5.

[9]A detailed account of the physics and mathematics behind this remark is given by N. Bali, G. F. Chew, and A. Pignotti, Phys. Rev. 163, 1572 (1967). Further exposition is found in lectures by G. F. Chew, in *Summer School in Elementary Particle Physics*, held at Brookhaven National Laboratory, Upton, 1969, edited by R. F. Peierls (Brookhaven National Laboratory, Upton, N. Y., 1970).

[10]H. D. I. Abarbanel and L. M. Saunders, Ann. Phys. (N.Y.) 64, 254 (1971).

[11]The identification of these analytically continued helicity amplitudes has also been made by P. Goddard and A. R. White [Nuovo Cimento 1A, 645 (1971)] and by

C. E. Jones, F. E. Low, and J. E. Young [Phys. Rev. D 4, 2358 (1971)]. See also V.N. Gribov, E. M. Levin, and A. A. Migdal, Yad. Fiz. 12, 173 (1970) [Sov. J. Nucl. Phys. 12, 93 (1971)].

[12]M. Andrews and J. Gunson, J. Math. Phys. 5, 1391 (1964).

[13]These models are reviewed by C. Lovelace [Phys. Letters 36B, 127 (1971)].

[14]This connection has been made in the past by C. E. De Tar *et al*. [Phys. Rev. Letters 26, 675 (1971)]; H. D. I. Abarbanel *et al*. [*ibid*. 26, 937 (1971)]; C. De Tar and J. Weis [Phys. Rev. D 4, 3141 (1971)]; and by H. D. I. Abarbanel and M. B. Green [Phys. Letters 38B, 1972].

[15]V. N. Gribov *et al*., Ref. 1.

[16]N. I. Muskhelishvili, *Singular Integral Equations* (Nordhoff, Gronigen, 1953); R. Omnès, Nuovo Cimento 8, 316 (1958).

[17]L. Castillejo, R. H. Dalitz, and F. Dyson, Phys. Rev. 101, 543 (1956).

[18]J. B. Bronzan, Phys. Rev. D 4, 1097 (1971); J. B. Bronzan and C. S. Hui, *ibid*. 5, 964 (1972). I am indebted to J. B. Bronzan for several informative discussions on these papers.

[19]See the second and third papers in Ref. 14.

[20]A clean statement of this is given in the lectures by R. Amado, in *Elementary Particle Physics*, edited by M. Chrétien and S. S. Schweber (Gordon and Breach, New York, 1970). Also see K. L. Kowalski, Phys. Rev. 144, 1239 (1966). Particle physicists have known this result under the guise of Watson's theorem for some time.

[21]See the recent work by G. Alberi, B. Mosconi, and P. J. R. Soper [Nuovo Cimento 9A, 107 (1972)]. I would like to thank M. L. Goldberger for bringing this paper to my attention.

[22]See the discussion of J. Finkelstein and M. Jacob [Nuovo Cimento 56A, 681 (1968)] on the question of this sign.

[23]The work of Gribov and his co-workers can be traced from V. N. Gribov, E. M. Levin, and A. A. Migdal, Yad. Fiz. 12, 173 (1970) [Sov. J. Nucl. Phys. 12, 93 (1971)].

[24]G. F. Chew, LBL report (unpublished).

[25]A. B. Kaidalov, Yad. Fiz. 13, 401 (1971) [Sov. J. Nucl. Phys. 13, 226 (1971)].

[26]A. R. White, Cambridge Univ. Report No. DAMTP 72/18 (unpublished).

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# Spectral Forms for Three-Point Functions*

Richard J. Ivanetich
*Department of Physics, Harvard University, Cambridge, Massachusetts 02138*
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Single- and double-spectral forms for three-point functions are studied, in a detailed manner, within the framework of source theory. The methods developed, which are applicable beyond the present problem, are based on causal considerations and appear to provide some simplifications and advances over the conventional analytic methods. The spectral variables are any one or two of the momentum scalar products ($p_\alpha^2$, $p_\beta^2$, $p_\gamma^2$) on which the three-point function depends, $p_\alpha^2$ and $p_\alpha^2-p_\gamma^2$, to be specific. After studying the lowest-order nontrivial contributions, a source-theoretic calculational scheme for contributions of arbitrary order is qualitatively developed, and it is used in establishing the spectral forms for general order. In lowest order the spectral weight functions are explicitly given, while in general order the main concern is the existence of the spectral forms. The spectral forms considered here are only ones with normal thresholds, and the methods give regions of those variables not in spectral form ($p_\beta^2-p_\gamma^2$ and $p_\beta^2$) for which such spectral forms of general order occur, with all particles being allowed different masses. For the single-spectral studies the region is in agreement with that obtained conventionally. The region for the double-spectral form is all spacelike values of $p_\beta^2$; double-spectral forms for three-point functions of general order do not appear to have been investigated previously.

## I. INTRODUCTION

Working mainly within the context of electrodynamics, Schwinger[1-3] has illustrated the fundamental and natural way in which spectral forms arise in source theory. In this present work the establishment of spectral forms for three-source couplings (or three-point functions) is restudied from a more general and more systematic standpoint.[4] Generality means two things here: First, owing to the importance of kinematics in this work, all particles are allowed to have different masses. And second, considerations for contributions of any order are presented, whereas Schwinger, being mindful of the importance of quantitative predictions in electrodynamics, has