# Quasisecular Perturbative Method for Calculating Bound States in Quantum Field Theory

S. Ø. Aks and B. B. Varga
*Physics Department, University of Illinois, Chicago, Illinois 60680*
(Received 21 April 1972)

A calculational procedure is given for determining bound states in quantum field theory using an approximation to the field determined from the equations of motion by the method of quasisecular perturbation theory. The method is illustrated using the $\phi^4$ model of quantum field theory. Although the procedure is based on a perturbation theory, it leads to binding energies which are nonanalytic in the coupling constant. The bound state occurs for arbitrarily weak coupling in the cases of one and two space dimensions, but not for three. Comparison is made with the Bethe-Salpeter equation in the ladder approximation. The result agrees for one dimension, but differs for two and three.

## I. INTRODUCTION

One of the more important problems in elementary particle physics is to develop perturbative methods based on field-theoretic models for calculating bound states, because of the large number of observed elementary particles and resonances. The binding of particles like the self-energies are persistent effects arising from repeated interactions over an infinite (or at least very long) time scale. In contrast with the case of ordinary collisions which take place over a limited time, the persistent effects modify the motion of arbitrary wave packets at all times.[1] In previous work[2] it was shown that persistent effects arise from the presence of secular and quasisecular terms in a perturbative solution of the Heisenberg field equations. Secular terms are not periodic, and blow up for large times. These are the terms associated with small energy denominators and after modification to restore periodic behavior in time, they lead to $q$-number frequency (mass) and amplitude (wave-function) renormalization. The modified perturbation theory was called quasisecular perturbation theory. The renormalized amplitude operator was used to construct the Hilbert space containing the physical states.[3] For the $\phi^4$ interaction the physical Hilbert space was shown to be different from the auxiliary Fock space (which is associated with the field at time $t=0$) in the cases of two and three space dimensions (i.e., strange representations of the commutation relations occur). In contrast to ordinary perturbative theory which is formulated entirely within Fock space, the quasisecular perturbative method picked out the new Hilbert space.

In this paper it is shown that the renormalized frequency operator can be used to find eigenstates and eigenvalues in the physical Hilbert space. It is shown to lowest order in the coupling constant for the cases of one and two space dimensions that a two-particle bound state occurs for the $\phi^4$ interaction if this term appears in the Hamiltonian with a negative coefficient.

The $\phi^4$ coupling has been chosen because it has nontrivial consequences in first order and allows a treatment of the essential points with a minimum of calculational complications.[4]

The Heisenberg equation of motion for the real scalar field $\phi(t, x) = \phi^*(t, x)$ in the case of one dimension is

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2\right)\phi = \lambda : \phi^3 : - \mu : \phi^5 : , \qquad (1.1)$$

where $\lambda > 0$ and $\mu > 0$ and the initial condition is $[\phi(0, x), \dot{\phi}(0, y)] = i\delta(x - y)$. To ensure that the Hamiltonian is bounded from below, the term $\frac{1}{6}\mu\phi^6$ with $\mu > 0$ has been added to the Hamiltonian.[5] This is the source of the additional term on the right-hand side of Eq. (1.1). In what follows, $\mu$ is taken to be proportional to $\lambda^2$, and is thus of second order in $\lambda$, as is explained in Sec. II. The field is studied on the interval $-\frac{1}{2}L \leq x \leq +\frac{1}{2}L$ with periodic boundary conditions, and we choose $\hbar = c = 1$.

Results are given for one, two, and three space dimensions; however, we choose to express most equations in one space dimension to simplify notation. The other cases are obtained by the appropriate natural notational changes.

In Sec. II the general dynamical properties of the model are discussed in terms of the Lagrangian. The model has dynamical stability for $\mu > \lambda^2/4m^2$, and the states of the system correspond to oscillations of the field about a mean value of zero. We choose $\mu$ proportional to $\lambda^2$ so that it will not affect the first-order renormalized Heisenberg field and states. The previously obtained results for these quantities are recapitulated in Sec. III. It is shown that the commutator of the renormalized amplitude with the Hamiltonian involves the renormalized frequency operator, and this relation is used to derive eigenvalue equations for the one- and two-

particle states. The one-particle states lie at the unperturbed energy $\omega_k = (k^2 + m^2)^{1/2}$ to lowest order in $\lambda$.

At the end of Sec. III the quasisecular eigenvalue equation for the two-body amplitude is constructed. In Sec. IV an approximate solution of the two-particle eigenvalue equation (accurate in the weak coupling limit) is given and is used to obtain the binding energy of an $S$-wave bound state. The results are compared in Sec. V with results obtained using the ladder-approximation calculation (involving the elementary four-particle vertex) in the Bethe-Salpeter equation.[6-9]

In the case of one space dimension the result is the same as that obtained from the Bethe-Salpeter equation in the weak-coupling limit. This is not entirely surprising since the ladder approximation was originally justified by consideration of persistent effects.[9] It was argued that although the probability for the exchange of one quantum (across the legs of the ladder) was small during a small time interval (for weak coupling), nevertheless, during the infinite time of existence of a bound state an indefinitely large number of quanta could be exchanged successively.[9]

For two dimensions the bound-state energy obtained from the quasisecular eigenvalue equation differs from the result of the Bethe-Salpeter equation in that the binding obtained here is much weaker for small coupling. In the case of three space dimensions the Feynman integrals in the Bethe-Salpeter equation are divergent and counterterms have to be introduced to obtain finite results. Since the counterterms are usually fitted to the assumed position of a bound state (or resonance), the value of the Bethe-Salpeter equation is limited to predicting the locations of other bound states (or resonances) should they exist. In contrast the quasisecular two-body equation still provides a well-defined bound state for sufficiently strong coupling. However the critical value for which binding occurs is well above the values for which a perturbative approximation should be valid.

In Appendix A the mathematical nature of the spectrum of the quasisecular eigenvalue equation is discussed. The accuracy of the approximate binding energies found in Sec. IV is demonstrated in Appendix B by calculating a lower bound on the lowest eigenvalue.

## II. THE MODEL AND DYNAMICAL STABILITY

The Lagrangian density of the model field theory is given in terms of the real scalar field $\phi$ by

$$\mathfrak{L} = :\sum_{i=0}^{1} \left(\frac{\partial \phi}{\partial x^i}\right)^2 - V(\phi):, \tag{2.1}$$

$$V(\phi) = \tfrac{1}{2} m^2 \phi^2 - \tfrac{1}{4}\lambda \phi^4 + \tfrac{1}{6}\mu \phi^6, \tag{2.2}$$

where :: is the normal-ordering symbol, $m$ is the mass of the bare particles, and $\lambda$ and $\mu$ are coupling constants. As a result of work on conditions for dynamical stability in certain model field theories,[10,11] it is usually assumed that the behavior of a model is controlled by the polynomial $V(\xi)$, where $\xi$ is a real variable. If the minimum value of $V(\xi)$ is taken on for a single value $\xi = \xi_1$, then the theory is dynamically stable and has a unique vacuum state. If the minimum value of $V(\xi)$ occurs at several distinct points $\xi_1, \ldots, \xi_n$, the theory is dynamically unstable. There will be a pure theory for each such value $\xi_j$ and the states corresponding to oscillations around $\phi = \xi_j$ are all orthogonal for different values of $j$, in the limit of infinite volume.[10] Since we find that $\lambda > 0$ is necessary for the occurrence of a bound state, the $\phi^6$ term is needed in Eq. (2.2) with $\mu > 0$ in order that $V(\xi)$ shall even have a minimum value. If the $\phi^6$ term were absent, then $V(\xi)$ has no minimum and the resulting quantum field theory is pathological. It can be shown that in a box with periodic boundary conditions the expectation values $(\Psi, \phi(x_1, t_1) \cdots \phi(x_n, t_n)\Psi)$ in any reasonable state $\Psi$ go to infinity in a finite time (i.e., when any $t_j - t_k$ becomes large enough).[12]

Since the calculation is perturbative, it is based on the assumption of continuity of the solution in the coupling parameters starting from the values $\lambda = 0$ and $\mu = 0$, which correspond in this case to continuing from the free field of mass $m$. To ensure dynamical stability we choose $\mu$ sufficiently large to keep the minimum unique and at $\xi = 0$. If $V(\xi)$ had minima away from $\xi = 0$, the model would be dynamically unstable and a perturbative method beginning from the free field would not be sensible. For $\mu > \frac{1}{4}\lambda^2 m^{-2}$, the only stationary point of $V(\xi)$ occurs at $\xi = 0$ and corresponds to the minimum. Therefore, we choose $\mu = \nu\lambda^2 m^{-2}$ with $\nu > \frac{1}{4}$. This gives a $\phi^6$ term of second order in the (small) coupling constant $\lambda$, so this term will not enter into a first-order calculation.

In summarizing this section, we note that the $\phi^6$ term is needed for dynamical stability without which the following calculation of a bound state is meaningless, yet the bound-state energy is not dependent in lowest (first) order on the magnitude of this term. In second order, however, the term will contribute to the dynamics.

## III. THE FIRST-ORDER RENORMALIZED FIELD AND STATES

In a previous paper[2] the method of quasisecular perturbation theory was introduced and applied to

the $\phi^4$ model of quantum field theory. The Heisenberg field can be written in the form

$$\phi(t, x) = L^{-1/2}\sum_l a_l(t)e^{ilx}, \tag{3.1}$$

with $l$ denoting the momenta allowed by the periodic boundary conditions. The amplitude $a_l(t)$ has a first-order renormalized solution given by

$$a_l(t) = U_l e^{-i\Omega_l t} + e^{i\Omega^*_{-l}t}U^*_{-l} + W_l(t). \tag{3.2}$$

The operator $W_l$ plays no role in what follows because its Fourier transform vanishes in intervals of width $2|\lambda|m^{-1}$ centered at $\pm\omega_l$, where $\omega_l^2 = m^2 + l^2$. The renormalized wave-function operator $U_l$ and the renormalized frequency operator $\Omega_l$ can be expressed as[3]

$$U_l = (2\omega_l)^{-1/2}e^{\lambda S}\alpha_l e^{-\lambda S} + (2\omega_l)^{-1}V_l, \tag{3.3}$$

$$\begin{aligned}\Omega_l &= \omega_l - U_l^{-1}V_l \\ &= \omega_l - (2\omega_l)^{1/2}\alpha_l^*(\alpha_l^*\alpha_l + 1)^{-1}V_l.\end{aligned} \tag{3.4}$$

Here $\alpha_l^*$ and $\alpha_l$ are the boson creation and destruction operators, $S$ is an anti-Hermitian operator,[3] and

$$V_l = 3\lambda(2\omega_l L)^{-1}\sum_{pqr} D_{lpqr}(8\omega_p\omega_q\omega_r)^{-1/2}\alpha^*_{-p}\alpha_q\alpha_r. \tag{3.5}$$

The $D$ function takes the value 1 if both $l = p+q+r$ and

$$|\omega_l + \omega_p - \omega_q - \omega_r| \leq |\lambda|m^{-1},$$

and it vanishes otherwise. The last inequality defines the quasisecular region.[2]

The gaps in the Fourier transform of $W_l$ make it possible to extract the time dependence of the first term of Eq. (3.2),

$$\begin{aligned}U_l(t) &= e^{iHt}U_l e^{-iHt} \\ &= U_l e^{-i\Omega_l t},\end{aligned} \tag{3.6}$$

from which we obtain

$$[H, U_l] = -U_l\Omega_l. \tag{3.7}$$

To demonstrate this assertion in detail we return to the first-order solution for the field before frequency renormalization[2] with the secular term displayed:

$$\begin{aligned}a_l(t) &= (U_l + itV_l)e^{-i\omega_l t} \\ &\quad + e^{i\omega_l t}(U^*_{-l} - itV^*_{-l}) + W_l(t).\end{aligned} \tag{3.8}$$

By smearing $a_l(t)$ with a test function $f(t)$ such that the support of its Fourier transform lies within an interval of width $z|\lambda|m^{-1}$ centered at $\omega_l$, we obtain

$$\begin{aligned}a_l(f) &= (2\pi)^{-1}\int_{-\infty}^{\infty} f(t)a_l(t)dt \\ &= U_l\tilde{f}(\omega_l) - V_l\tilde{f}'(\omega_l),\end{aligned} \tag{3.9}$$

where $\tilde{f}(\omega)$ denotes the Fourier transform of $f$. From the equation $[H, a_l(t)] = -i\dot{a}_l(t)$, we obtain

$$\begin{aligned}[H, a_l(f)] &= [H, U_l]\tilde{f}(\omega_l) - [H, V_l]\tilde{f}'(\omega_l) \\ &= (2\pi i)^{-1}\int_{-\infty}^{\infty} f(t)\dot{a}_l(t)dt \\ &= (-\omega_l U_l + V_l)\tilde{f}(\omega_l) + \omega_l V_l\tilde{f}'(\omega_l).\end{aligned} \tag{3.10}$$

Choosing $\tilde{f}(\omega_l) = 1$ and $\tilde{f}'(\omega_l) = 0$ gives Eq. (3.7) since $U_l$ possesses a right inverse.[2]

The physical vacuum $\Phi_0$ is taken to be the unique state[3] which is annihilated by all of the $U_l$. This state is an eigenstate of $H$ because

$$U_l H\Phi_0 = [U_l, H]\Phi_0 = U_l\Omega_l\Phi_0 = -V_l\Phi_0 = 0. \tag{3.11}$$

The last term vanishes because, to lowest order in $\lambda$, we can replace $\alpha_r$ in Eq. (3.5) by $(2\omega_r)^{1/2}U_r$. Thus $H\Phi_0$ is annihilated by all of the $U_l$ and is therefore a multiple of $\Phi_0$. It follows that $\Phi_0$ is an eigenvector of $H$. We take the associated energy eigenvalue to be zero, $H\Phi_0 = 0$. This may involve a displacement of the energy scale.

The Hilbert space of physical states is generated by applying polynomials in the $U_l^*$, for various values of $l$, to the state $\Phi_0$.[3] In the following, the energies of one- and two-particle states are evaluated on the chosen energy scale.

The Hermitian conjugate of Eq. (3.7) implies that

$$\begin{aligned}HU_l^*\Phi_0 &= \Omega_l^*U_l^*\Phi_0 \\ &= (\omega_l U_l^* - V_l^*)\Phi_0 \\ &= \omega_l U_l^*\Phi_0.\end{aligned} \tag{3.12}$$

This follows because $\alpha_{-p}$ in $V_l^*$ in Eq. (3.5) can be replaced by $(2\omega_p)^{1/2}U_{-p}$. The states $U_l^*\Phi_0$ are eigenstates of energy $\omega_l$ and momentum $l$. These are the one-particle states. To lowest order in $\lambda$ there is no shift in the one-particle energy, because the first self-energy contribution is of order $\lambda^2$.

The subspace of two-particle states is generated by the vectors $U_l^*U_k^*\Phi_0$. The conjugate of Eq. (3.7) implies that

$$\begin{aligned}HU_l^*U_k^*\Phi_0 &= U_l^*HU_k^*\Phi_0 + (\omega_l U_l^* - V_l^*)U_k^*\Phi_0 \\ &= (\omega_k + \omega_l)U_l^*U_k^*\Phi_0 \\ &\quad -3\lambda(4\omega_l\omega_k L)^{-1}\sum_{qr} D_{lkqr}U_r^*U_q^*\Phi_0,\end{aligned} \tag{3.13}$$

which is valid again to lowest order in $\lambda$. This equation is considered in Sec. IV.

## IV. BOUND-STATE ENERGY CALCULATION

We look for eigenstates $\Psi$ of total momentum zero in the two-particle subspace having the form

$$\Psi = \sum_s f_s U_s^* U_{-s}^* \Phi_0 ,$$

where $s$ ranges over the allowed momenta. We take $f_s = f_{-s}$ since $U_s^* U_{-s}^* \Phi_0 = U_{-s}^* U_s^* \Phi_0$, and assume $\lambda > 0$. The quasisecular eigenvalue equation becomes from Eq. (3.13)

$$Ef_s = \sum_r \left[ 2\omega_r \delta_{r,s} - \frac{3\lambda}{4L} \omega_r^{-2} \theta(\lambda m^{-1} - 2|\omega_r - \omega_s|) \right] f_r , \tag{4.1}$$

where $E$ is the energy eigenvalue and $\theta(x) = 1$ when $x \geq 0$ and vanishes otherwise. Note that the matrix elements appearing on the right-hand side of Eq. (4.1) are not those of a Hermitian matrix because, although the states $U_s^* U_{-s}^* \Phi_0$ are orthogonal for different values of $s$, their norm is $(2\omega_s)^{-1}$ as can be seen from Eq. (3.3). This suggests the modified expansion

$$\Psi = \sum_s g_s (2\omega_s) U_s^* U_{-s}^* \Phi_0 ;$$

so that $g_s = (2\omega_s)^{-1} f_s$ and

$$Eg_s = \sum_r \left[ 2\omega_r \delta_{r,s} - \frac{3\lambda}{4L\omega_s\omega_r} \theta(\lambda m^{-1} - 2|\omega_r - \omega_s|) \right] g_r . \tag{4.2}$$

The matrix is now Hermitian.

It is shown in Appendix A that in the limit of infinite spatial volume the integral operator corresponding to the matrix of Eq. (4.2) has a continuous spectrum (the scattering continuum) extending from $2m$ to $\infty$. The Rayleigh-Ritz variational principle, which gives an upper bound on the lowest eigenvalue, can be used to prove the occurrence of a bound state below the scattering continuum.

For weak coupling the amplitude $g_s$ for a bound state may be expected to be large only when $\omega_s - m < \frac{1}{2}\lambda m^{-1}$ or $|r| < \lambda^{1/2}$. For further discussion see Appendix B. This suggests choosing a variational trial function which vanishes for $|s| > \lambda^{1/2}$. The best such trial function (which gives the lowest variational energy) is the eigenvector $g_s$ corresponding to the lowest eigenvalue of the reduced matrix $2\omega_r \delta_{r,s} - 3\lambda/4L\omega_s\omega_r$, where $|r| < \lambda^{1/2}$ and $|s| < \lambda^{1/2}$. The variational energy (of the complete matrix) obtained using the eigenvector $g_s$ is then the lowest eigenvalue of the reduced matrix.

The eigenvalue equation for the reduced matrix is

$$Eg_s = 2\omega_s g_s - (3\lambda/4L\omega_s) \sum_{|r|<\lambda^{1/2}} \omega_r^{-1} g_r . \tag{4.3}$$

Letting

$$L^{-1} \sum_{|r|<\lambda^{1/2}} \omega_r^{-1} g_r = \chi$$

gives

$$g_s = (2\omega_s - E)^{-1} (3\lambda/4\omega_s)\chi ,$$

which leads to the eigenvalue condition

$$L^{-1} \sum_{|s|<\lambda^{1/2}} \omega_s^{-1} (2\omega_s - E)^{-1} (3\lambda/4\omega_s) = 1 . \tag{4.4}$$

Since $\omega_s = m + (2m)^{-1} s^2$ in the nonrelativistic approximation, which is clearly valid in the restricted region, the equation simplifies to

$$L^{-1} \sum_{|s|<\lambda^{1/2}} (m^{-1} s^2 + E_B)^{-1} = 4m^2/3\lambda , \tag{4.5}$$

where $E_B = 2m - E$ is the binding energy. The equation only has a solution if $\lambda > 0$ which corresponds to a negative sign for the $\phi^4$ term in the Hamiltonian, as can be seen from Eq. (2.2).

Passing to the limit of infinite volume in the case of one space dimension

$$\int_0^{\lambda^{1/2}} ds (s^2 + mE_B)^{-1} = 4\pi m/3\lambda , \tag{4.6}$$

which leads to the transcendental equation

$$(mE_B)^{-1/2} \tan^{-1}(\lambda/mE_B)^{1/2} = 4\pi m/3\lambda . \tag{4.7}$$

As $\lambda \to 0$ the right-hand side becomes very large so that $E_B \to 0$ as $\lambda^2$ and $\tan^{-1}(\lambda/mE_B)^{1/2} \to \frac{1}{2}\pi$. For weak coupling the asymptotic solution of Eq. (4.7) is

$$E_B = (3\lambda/8m^2)^2 m . \tag{4.8}$$

The binding energy is not an analytic function of the coupling constant because Eq. (4.7) is transcendental and Eq. (4.8) is an asymptotic approximation which is accurate for the dimensionless parameter $\lambda m^{-2}$ close to zero. The next term in the asymptotic expansion is $O(\lambda^{5/2})$.

In the case of two space dimensions and in the limit of infinite volume, the analog of Eq. (4.6) is

$$\int_0^{\lambda'^{1/2}} s\,ds (s^2 + mE_B)^{-1} = 8\pi m/3\lambda , \tag{4.9}$$

which has the asymptotic solution ($mE_B \ll \lambda'$)

$$E_B = \lambda e^{-16\pi m/3\lambda} . \tag{4.10}$$

Here $\lambda' = m\lambda$ for dimensional reasons because, for the case of two space dimensions, $\lambda$ has the same units as $m$ and $\lambda m^{-1}$ is the dimensionless coupling constant. This follows from the dimension of the field which is related to the commutation relations of the field and the dimensional units carried by the Dirac $\delta$ function. The nonanalytic nature of the bound state is again explicit in Eq. (4.10).

For three space dimensions the analog of Eq. (4.6) is

$$\int_0^{\lambda'^{1/2}} s^2 ds (s^2 + mE_B)^{-1} = 8\pi^2 m/3\lambda , \tag{4.11}$$

which leads to the condition

$$m\lambda^{1/2} - (mE_B)^{1/2} \tan^{-1}(\lambda m/E_B) = 8\pi^2 m/3\lambda , \tag{4.12}$$

since $\lambda' = \lambda m^2$, where $\lambda$ is now the dimensionless coupling constant. Again, for weak coupling, $E_B \ll m\lambda$ and $\tan^{-1}(\lambda m/E_B) \to \frac{1}{2}\pi$, so that the equation does not have a solution. The bound state appears (at zero energy) when the coupling constant reaches the critical value $\lambda_c = (\frac{8}{3}\pi^2)^{2/3} \approx 8.5$. This value is well beyond the range where a perturbative method can be expected to be valid.

## V. DISCUSSION

It is interesting to compare the results of Sec. IV with the results of the Bethe-Salpeter equation in the simplest ladder approximation. The scattering amplitude is obtained by summing the "chain of bubbles" corresponding to the Feynman diagrams shown in Fig. 1.

In one space dimension the scattering amplitude is proportional to

$$\{(6\lambda)^{-1} - (2\pi)^{-1} s^{-1/2} (4m^2 - s)^{-1/2} \tan^{-1}[s(4m^2 - s)^{-1}]\}^{-1} , \tag{5.1}$$

where $s^{1/2}$ is the center-of-mass energy of the incoming particles. Note that the scattering is entirely $S$ wave. For weak coupling it follows that the scattering amplitude has a pole determined by the condition

$$(m/E_B)^{1/2} \tan^{-1}(m/E_B)^{1/2} = 4\pi m^2/3\lambda , \tag{5.2}$$

where $E_B = 2m - \sqrt{s}$ is the binding energy. This pole condition differs from the quasisecular eigenvalue condition in Eq. (4.7), nevertheless, for weak coupling the asymptotic solution of Eq. (5.2) is $E_B = (3\lambda/8m^2)^2 m$, identical to Eq. (4.8).

For the case of two space dimensions the scattering amplitude is

$$\{(6\lambda)^{-1} - (8\pi)^{-1} s^{-1/2} \ln[(2m + s^{1/2})(2m - s^{1/2})^{-1}]\}^{-1} , \tag{5.3}$$

which has a pole at

$$E_B = 4m\, e^{-8\pi m/3\lambda} \tag{5.4}$$

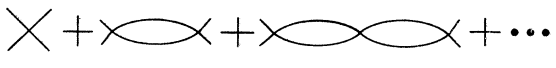

FIG. 1. The diagrams summed in the Bethe-Salpeter scattering amplitude.

for weak coupling. This is similar to the expression (4.10) in form but differs significantly by a factor of 2 in the exponent and by the coefficient in front of the exponential. This expression for the binding energy is much larger for weak coupling than the quasisecular expression Eq. (4.10). Since the latter is only an upper-bound value the question arises whether the difference in the two expressions is due to a poor variational estimate. That this is not the case is shown in Appendix B where a lower bound to the energy of the bound state is computed. This implies an upper bound on $E_B$ which is $e$ times the value given by Eq. (4.10) (which is a lower bound on $E_B$).

For three space dimensions the scattering amplitude calculated from the Feynman diagrams diverges, and a comparison cannot be made. We note that no divergence occurs in the quasisecular scheme where a bound state occurs for strong enough coupling.

It is interesting that the two calculation schemes agree in one space dimension, but not for two or three. It is quite possible that this is related to the fact that the Hilbert space of physical states coincides with the (bare particle) Fock space in one space dimension but not in two or three.[3] The Bethe-Salpeter equation is formulated in terms of Feynman diagrams[9] which are based on the Fock representation.[13] In the cases of two and three space dimensions the physical Hilbert space is completely distinct from Fock space[3] and carries a so-called strange representation of the commutation relations.[14] The connection between the bare particles and the physical particles is given by a dressing transformation[3] which substantially alters the form of the Hamiltonian as expressed in terms of the physical particle operators. This is evident in Eq. (3.13) which involves the quasisecularity condition. The interaction between the dressed particles, which is quite different from the interaction of the bare particles, is the basis for the quasisecular scheme of calculation.

We conclude this section with an additional qualitative comparison of the two calculational procedures. In the Bethe-Salpeter approach the diagrams are first renormalized to include self-energy parts. The scattering amplitude is then computed. This is often justified by the assertion that self-energy effects occur for all times (i.e., persistent effect on the infinite time scale) whereas scattering occurs on a time scale of the order of $m^{-1}$. The resulting scattering amplitude is then used to locate poles which are associated with bound states. We note however that the binding of particles also occurs on the long time scale (and not on the scattering time scale). Although an approximation to the scattering amplitude may be appropriate for

phenomena occurring on the scattering time scale it may be quite inaccurate for persistent phenomena such as binding. The quasisecular method treats all persistent effects (self-energy, binding) on the same footing. No distinction is made between virtual quanta which form the cloud around a dressed particle and those which are exchanged to account for binding. Two comoving particles form a collective cloud of quanta in which the above distinction is not physically meaningful. The nonlinearity of the field equations in terms of $\phi$ destroys the additivity of the two effects of the virtual quanta.


## ACKNOWLEDGMENT

The authors would like to acknowledge several helpful comments by Professor A. S. Wightman concerning the matter of dynamical stability.


## APPENDIX A

We consider the mathematical properties of the spectrum of the quasisecular operator of Eq. (4.2). It is convenient to consider the case of two space dimensions in the limit of infinite volume, because the density of states as a function of energy is a constant. The other cases can be treated in a similar way. The integral equation corresponding to Eq. (4.2) becomes (for $\lambda > 0$)

$$Eg(\epsilon) = \epsilon g(\epsilon) - (3\lambda/8\pi\epsilon) \int_{[\lambda-\epsilon,\, 2m]}^{\lambda+\epsilon} g(\epsilon')d\epsilon', \qquad \text{(A1)}$$

where the lower limit is the greater of $\lambda - \epsilon$ and $2m$. Here the energy scale for $\epsilon$ runs from $2m$ to $\infty$ and a bound state corresponds to a value for $E$ slightly below $2m$. If the integral operator is cut off at some (arbitrarily high) energy, it has a finite Hilbert-Schmidt norm and therefore is a compact (or completely continuous) operator.[15] The remaining part of the integral operator converges to zero in the norm because of the factor $\epsilon^{-1}$ before the last term of Eq. (A1). Thus increasing the cutoff defines a sequence of compact operators converging in norm to the full integral operator of Eq. (A1). This latter operator is therefore compact.[16] By a theorem of Weyl[16] the limit points of the spectrum remain invariant under such a compact perturbation. Therefore the continuous spectrum remains as the scattering continuum, which extends from $2m$ to $\infty$.

It follows from Weyl's theorem that any parts of the spectrum which lie below $2m$ belong to the discrete spectrum and correspond to bound states.

Since the integral operator has a bound of $b = 3\lambda^2/8\pi m$ it follows that the spectrum lies above $2m - b$ and bound states (if they exist) lie in the interval $(2m - b, 2m)$. The variational calculation of Sec. IV shows that an eigenvalue (i.e., a member of the discrete spectrum) lies between $2m - b$ and $2m - \lambda e^{-16\pi m/3\lambda}$. In order to make a decisive comparison with the Bethe-Salpeter result of Eq. (5.4) a much sharper lower bound is needed (closer to the upper bound). This lower bound is derived in Appendix B.

## APPENDIX B

For the eigenvalue of Eq. (A1) a lower bound is obtained by splitting the integral operator into three parts. In the nonrelativistic limit ($\epsilon - 2m \ll m$) we write

$$Eg(\epsilon) = \int_0^\infty K(\epsilon, \epsilon')g(\epsilon')d\epsilon', \qquad \text{(B1)}$$

$$K(\epsilon, \epsilon') = \epsilon\delta(\epsilon - \epsilon') - (3\lambda/16m\pi)\hat{\theta}(\epsilon, \epsilon'), \qquad \text{(B2)}$$

where $\hat{\theta}$ is the characteristic function of the region $|\epsilon - \epsilon'| < \lambda$ taking the value 1 inside the region and 0 outside. It is convenient to redefine the scale of energies choosing $2m$ as the new zero, as in Eq. (B1).

Let $K = L + M + N$ be a Hermitian partitioning of the integral operator into three regions; for $L$ the region of support is $\epsilon < \lambda$, $\epsilon' < \lambda$, for $M$ the region of support is $\epsilon < \lambda$, $\epsilon' > \lambda$, and $\epsilon > \lambda$, $\epsilon' < \lambda$, and for $N$ the region of support is $\epsilon > \lambda$, $\epsilon' > \lambda$. Since $N$ is a displaced copy of $K$ with the diagonal shifted by $\lambda$, it is greater than the operator $(\lambda - E_B)J$, where $E_B$ is the binding energy of the lowest discrete state of $K$ and $J$ is a projection operator $J = \theta(\epsilon - \lambda)\delta(\epsilon - \epsilon')$. Then $N \geq \lambda - E_B$, hence $K \geq L + M + (\lambda - E_B)J$, where $\geq$ denotes the usual ordering[17] of self-adjoint operators in terms of the expectation values [i.e., $A \geq B$ means $(\Psi, A\Psi) \geq (\Psi, B\Psi)$ for all reasonable $\Psi$]. Then the lowest eigenvalue of $K$ lies above the lowest eigenvalue of $L + M + (\lambda - E_B)J = P$.

The eigenvalue equation of $P$ is written as

$$\epsilon f(\epsilon) - \frac{3\lambda}{16\pi m} \int_0^{\lambda+\epsilon} f(\epsilon')d\epsilon' = Ef(\epsilon), \qquad 0 < \epsilon < \lambda, \qquad \text{(B3)}$$

$$(\lambda - E_B)f(\epsilon) - \frac{3\lambda}{16\pi m} \int_{\epsilon-\lambda}^{\lambda} f(\epsilon')d\epsilon' = Ef(\epsilon), \qquad \lambda < \epsilon < 2\lambda. \qquad \text{(B4)}$$

Solving Eq. (B4) for $f(\epsilon)$ in the interval $\lambda < \epsilon < 2\lambda$ and substituting into Eq. (B3) gives

$$\epsilon f(\epsilon) - \frac{3\lambda}{16\pi m} \int_0^{\lambda} f(\epsilon')d\epsilon' - \left(\frac{3\lambda}{16\pi m}\right)^2 (\lambda + E - E_B)^{-1} \times \int_0^{\lambda} Q(\epsilon, \epsilon')f(\epsilon')d\epsilon' = Ef(\epsilon), \qquad 0 < \epsilon < \lambda, \qquad \text{(B5)}$$

where $Q(\epsilon, \epsilon') = \min(\epsilon, \epsilon')$ is a Hermitian kernel. Note that this equation involves only $f(\epsilon)$ in the interval $0 < \epsilon < \lambda$. In the last term on the left-hand side of Eq. (B5) we may neglect $E$ and $E_B$ which are very small compared to $\lambda$ for weak coupling. If the kernel $Q(\epsilon, \epsilon')$ which is positive definite is replaced by its maximum value $\lambda$ the eigenvalue of Eq. (B5) is lowered, preserving the lower bound character. The modified equation is

$$\epsilon f(\epsilon) - \frac{3\tilde{\lambda}}{16\pi m}\int_0^{\lambda} f(\epsilon')d\epsilon' = Ef(\epsilon), \quad 0 < \epsilon < \lambda, \quad \text{(B6)}$$

where $\tilde{\lambda} = \lambda + (3\lambda^2/16m\pi)$. This is the integral equation of the truncated matrix of Sec. IV with $\lambda$ replaced by $\tilde{\lambda}$, so, by Eq. (4.10), the solution is

$$E = -\tilde{\lambda}e^{-16m\pi/3\tilde{\lambda}}. \quad \text{(B7)}$$

Since $\lambda m^{-1}$ is very small, it is useful to expand Eq. (B7) to second order, giving

$$E = -e\lambda e^{-16m\pi/3\lambda}[1 + O(\lambda^2)]. \quad \text{(B8)}$$

Thus the binding energy of the quasisecular equation for two space dimensions lies between the value of Eq. (4.10) and $e$ times that value.

The same splitting scheme works to give an accurate lower bound on the spectrum in the case of one space dimension. Details differ since the density of states is proportional to $\epsilon^{-1/2}$. The result is a lower bound of the form $E = -(3\lambda/8m^2)^2 m - \alpha(\lambda m^{-2})^{5/2} m$, where $\alpha$ is a positive constant. The binding energy as given by Eq. (4.8) is therefore accurate to order $(\lambda/m^2)^2$.

[1]L. Van Hove, Physica 21, 901 (1955); 22, 343 (1956).
[2]S. Aks, J. Sienicki, and B. Varga, Phys. Rev. D 6, 520 (1972).
[3]B. Varga and S. Aks, Phys. Rev. D 6, 529 (1972).
[4]A. M. Jaffe, Rev. Mod. Phys. 41, 576 (1969).
[5]A. M. Jaffe, O. Lanford III, and A. S. Wightman, Commun. Math. Phys. 15, 47 (1969).
[6]Y. Nambu, Progr. Theoret. Phys. (Kyoto) 5, 614 (1950).
[7]J. Schwinger, Proc. Natl. Acad. Sci. U. S. 37, 452 (1951); 37, 455 (1951).
[8]M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).
[9]E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).
[10]J. Goldstone, Nuovo Cimento 19, 154 (1961).
[11]Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122, 345 (1961).
[12]A. Jaffe, Ph. D. thesis, Princeton University, 1965 (unpublished), p. 53.
[13]G. Barton, *Introduction to Advanced Field Theory* (Interscience, New York, 1963), p. 125.
[14]A. S. Wightman and S. Schweber, Phys. Rev. 98, 812 (1955).
[15]T. Kato, *Perturbation Theory for Linear Operators* (Springer, New York, 1966), p. 264.
[16]F. Riesz and B. Sz-Nagy, *Functional Analysis* (Ungar, New York, 1955), pp. 178, 367.
[17]Reference 16, p. 262.