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[1]C. Fronsdal and R. W. Huff, preceding paper, Phys. Rev. D 6, 2755 (1972), hereafter referred to as Paper I.

[2]It may be possible to account for some self-energy effects by introducing finite nucleon size factors. Here we have merely replaced the bare coupling constant g_0 by the renormalized coupling constant g.

[3]There is an ambiguity associated with the last term in (11). The field operator $\varphi_\alpha \varphi_\alpha$ commutes with the current density operator $u^* \tau_\alpha \sigma u$; hence this term does not contribute to the expression for the time derivative of the current operator. Nevertheless, if we retain this (vanishing) term, and then introduce the approximation of neglecting all but two-nucleon states in the operator product, then a nonvanishing contribution to Eq. (13) results. A more detailed investigation of the nature and reliability of our approximations is indicated.

[4]Strictly speaking, H_{st} is defined by Eq. (13) up to a multiple of the unit matrix only. This suggests a less rigid test of (24) in which the singlet-isosinglet component is treated phenomenologically.

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Rigorous Bounds on Coupling Constants in Two-Dimensional Field Theories*

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We show that renormalized three-particle coupling constants in a field theory with one space and one time dimension are bounded. This bound depends on the particle spectrum and assumes only analyticity, crossing, unitarity, and polynomial boundedness of the S matrix at infinity.

The idea that coupling constants in a field theory might be bounded has an intuitive appeal. Once the spectrum of states is given, it may not be possible to make any particular coupling constant too large without introducing new "bound" states. Some time ago Geshkenbein and Ioffe[1] gave such a restriction. Recently Okubo[2] rederived this condition on slightly weaker assumptions. The problem with both these derivations is the assumption of no zero points in a particle propagator when continued into its domain of analyticity. That this assumption is hard to justify was discussed by Goebel and Sakita.[3]

We will show here that in a world with only one space and one time dimension there are rigorous bounds on renormalized coupling constants. Our only assumptions are analyticity, crossing, unitarity, and polynomial boundedness at infinite complex energy for scattering amplitudes. Despite our weaker assumptions, the bound we find is stronger than that of Geshkenbein and Ioffe.[1]

We can get such a strong result in two-dimensional space because the constraint of unitarity on crossing-symmetric amplitudes assumes a simpler form than in four dimensions. This point has been emphasized by Schlitt.[4] In four dimensions, unitarity is simple when applied to partial waves, but under crossing we know that a partial-wave amplitude has complicated "left-hand cut" singularities involving all other partial waves. In the two-dimensional world all elastic scattering is either forward or backward, giving only two possible final states. Because of this, unitarity is simple when applied directly to crossing-symmetric amplitudes.[5]

Ruderman[6] discussed the possibility of obtaining bounds on coupling constants from unitarity in four dimensions. He concluded that little can be said without some knowledge of the left-hand cut of a partial-wave amplitude. Using techniques due to Martin,[7] Müller[8] has derived a bound on the coupling constant for three identical self-conjugate scalar particles in four-dimensional space-time. This bound is rather weak, and its derivation relies heavily on full crossing symmetry. It does, however, show that some rigorous bounds on coupling constants do exist in four dimensions. We find that by going to two dimensions, the left-hand cut becomes known, and restrictions on coupling constants are easy to find.

For simplicity we derive our result in a theory containing a stable self-conjugate boson of mass m and possessing a three-particle self-coupling. The generalization to nonidentical particles will be obvious. We assume that other than the boson itself there are no states in the theory with mass less than $2m$ possessing a coupling to two of our bosons. Other than this, we make no assumption on the spectrum of states in the theory.

We consider the elastic scattering of two of these bosons. Since the particles are identical, there is only forward scattering. We define the function $s(\nu)$ by

$$_{\rm out}\langle p_3 p_4 | p_1 p_2\rangle_{\rm in} = s(\nu)\, _{\rm in}\langle p_3 p_4 | p_1 p_2\rangle_{\rm in}\ , \tag{1}$$

where $\nu = p_1 \cdot p_2$. By standard reduction techniques, $s(\nu)$ can be continued as an analytic function of ν^2 into the entire complex ν^2 plane except for a cut from ν_0^2 to $+\infty$ and a pole at ν_B^2 where $\nu_0 = m^2$ and $\nu_B = -m^2/2$. The value of $s(\nu)$ defined by Eq. (1) is equal to the value of the continued $s(\nu)$ just above the cut. Crossing symmetry is embodied in the fact that $s(\nu)$ can be continued as a function of ν^2. The pole at $\nu^2 = \nu_B^2$ is given in terms of the renormalized coupling constant g^2 by

$$s(\nu) = \frac{g^2 \nu_B}{4(\nu^2 - \nu_B^2)(\nu_0^2 - \nu_B^2)^{1/2}} + (\text{regular at } \nu^2 = \nu_B^2)\,. \tag{2}$$

To derive our restriction on g^2, we observe from Eq. (1) that for all $\nu \geq \nu_0$

$$|s(\nu)| \leq 1\,. \tag{3}$$

This is a direct consequence of unitarity. We now introduce the function

$$f(\nu^2) = s(\nu)\frac{\nu^2 - \nu_B^2}{[(\nu_0^2 - \nu_B^2)^{1/2} + (\nu_0^2 - \nu^2)^{1/2}]^2} \tag{4}$$

where here, and throughout this paper, we take the branch of the square root function with positive real part. Clearly $f(\nu^2)$ is analytic in the ν^2 plane except for a cut from ν_0^2 to $+\infty$. Furthermore, $f(\nu^2)$ has the properties

$$f(\nu_B^2) = \frac{g^2 \nu_B}{16(\nu_0^2 - \nu_B^2)^{3/2}}\ , \tag{5}$$

$$|f(\nu^2)| = |s(\nu)| \leq 1 \quad \text{for } \nu^2 > \nu_0^2\,. \tag{6}$$

If we now assume that $s(\nu^2)$ is polynomially bounded at infinity in the entire ν^2 plane, it immediately follows that everywhere in the ν^2 plane

$$|f(\nu^2)| \leq 1 \tag{7}$$

and thus we get our result

$$g^2 \leq \frac{16(\nu_0^2 - \nu_B^2)^{3/2}}{|\nu_B|} = 12\sqrt{3}\, m^4\,. \tag{8}$$

For comparison we give the result of Geshkenbein and Ioffe for the situation discussed here:

$$g^2 \leq 48(3 + 2\sqrt{3})m^4\,. \tag{9}$$

Note that this is weaker than our result even though it was derived from stronger assumptions.

Our bound in inequality (8) is in some sense the best possible. This is because the function

$$\frac{(\nu_0^2 - \nu^2)^{1/2} + (\nu_0^2 - \nu_B^2)^{1/2}}{(\nu_0^2 - \nu^2)^{1/2} - (\nu_0^2 - \nu_B^2)^{1/2}} \tag{10}$$

satisfies our assumptions on $s(\nu)$ while it gives a g^2 that saturates our bound.

The presence of an additional stable particle pole in $s(\nu)$ will allow the coupling constant to exceed our bound. Indeed, a pole in $s(\nu)$ at $\nu_{B'}^2$ permits g^2 to be larger than the bound in Eq. (8) by the factor

$$\left|\frac{(\nu_0^2 - \nu_{B'}^2)^{1/2} + (\nu_0^2 - \nu_B^2)^{1/2}}{(\nu_0^2 - \nu_{B'}^2)^{1/2} - (\nu_0^2 - \nu_B^2)^{1/2}}\right| \geq 1\,. \tag{11}$$

This raising of the bound when additional particles are around is consistent with the idea that a large coupling will create "bound states."

If we know more about $s(\nu)$ we can further restrict g^2. For instance if we know

$$|s(\nu)| \leq \eta(\nu) \leq 1 \quad \text{for } \nu > \nu_0\,, \tag{12}$$

then we can reduce our bound of inequality (8) by the factor

$$\exp\left[\frac{(\nu_0^2 - \nu_B^2)^{1/2}}{\pi}\int_{\nu_0^2}^{\infty}\frac{d\nu'^2 \ln\eta(\nu')}{(\nu'^2 - \nu_B^2)(\nu'^2 - \nu_0^2)^{1/2}}\right] \leq 1. \tag{13}$$

If we know that $s(\nu)$ has a zero at some complex value $\nu^2 = \nu_z^2$, then the bound is reduced further by the factor

$$\left|\frac{(\nu_0^2 - \nu_z^2)^{1/2} - (\nu_0^2 - \nu_B^2)^{1/2}}{(\nu_0^2 - \nu_z^2)^{1/2} + (\nu_0^2 - \nu_B^2)^{1/2}}\right| \leq 1\,. \tag{14}$$

The factors in expressions (11), (13), and (14) are all independent and may occur simultaneously.

If there is a finite range of $\nu^2 \geq \nu_0^2$ along which the two-identical-boson state is the only allowed final state in the boson-boson scattering, then in this range $|s(\nu)| = 1$ and we can analytically continue $s(\nu)$ through this portion of the cut onto another sheet. On this sheet the only singularities of $s(\nu)$ in the variable ν^2 apart from the right-hand cut will be poles at the positions of zeros of $s(\nu)$ on the original sheet. If such a pole is near the cut, it is a resonance. Thus resonances correspond to zeros in $s(\nu)$ on the first sheet and lower our bound by factors like expression (14). In fact there are two such factors for each resonance since a zero at ν_z^2 must be accompanied by another at ν_z^{*2} whenever ν_z^2 is not real. Note the sharp contrast between the effect of resonances and stable particles on our bound. A stable bound state allows g^2 to be larger while a resonance forces it to be smaller.

Because unitarity assumes a particularly simple form in two dimensions, we have been able to demonstrate rigorous upper bounds on renormalized coupling constants in two-dimensional field theories. These constraints depend on the dynamics of the theory only through the physical mass spectrum. Such rigorous bounds are unlikely to be unique to two dimensions; however, due to the complexity of combining unitarity and crossing, only weak constraints in special situations have yet been found in higher dimensions.

Note added. The main result of this paper [Eq. (8)] and the method of deriving it are contained in a remark in a paper of some time ago by K. Symanzik [in *Lectures on Field Theory and the Many-Body Problem,* edited by E. R. Caianiello (Academic, New York, 1961), p. 92]. The discussion of this result in the present paper is more extensive. I thank Professor Symanzik for bringing his interesting paper to my attention.

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Triple-Regge Vertex in a Dual Resonance Model with Nonlinear Trajectories*

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Recently, DeTar and Weis studied the analytic structure of the triple-Regge vertex in the Veneziano model. We extend their conclusions to a wider class of dual resonance models by deriving the triple-Regge vertex in the Baker-Coon theory, which allows both linear and logarithmic trajectories. In particular, we find that for unit trajectory intercepts, the triple-Regge coupling vanishes at zero momentum transfer.

I. INTRODUCTION

The properties of the triple-Regge vertex[1,2] are of interest because the vertex is directly useful in the study of single-particle inclusive reactions.[3] Recently, DeTar and Weis[1] examined the analytic structure of the triple-Regge vertex in the Veneziano model. Here we generalize their results to a wider class of dual resonance models by deriving the triple-Regge vertex in the Baker-Coon model, which is believed to be the most general possible dual model satisfying the requirements of meromorphy, crossing symmetry, polynomial residues, and no ancestors.[4,5] The model contains three parameters a, b, q $(0<q<1)$, and in particular, when $q\to 1$, it reduces to the conventional Veneziano model.[6] The Regge trajectories are logarithmic and are given by

$$\alpha(s)=-\frac{\ln\sigma}{\ln q}, \tag{1}$$

where $\sigma\equiv 1+(1-q)(as+b)$. The trajectories become linear when $q\to 1$.

The Baker-Coon N-point function B_N is a multiple sum, which converges only when its arguments σ_{ij} lie between 0 and 1.[4] Therefore, in its original form, B_N is unsuitable for going to any Regge limits. We make the desired analytic continuation to large σ_{ij} with the help of contour integrals in the complex plane around rectangular strip contours. This is described in detail in Sec. II, in which the double-Regge-particle vertex is derived from B_5.