approach to the problem of computing the S matrix in the presence of bound states. The types of wave functions used in this and the previous letter arise naturally in this formalism.

[5]The infinite-momentum frame used is defined by

$$p_B = p = \left(P + \frac{m_B{}^2}{2P}\ ,\ \vec{0}_\perp\ ,\ P\right),$$

$$q = \left(\frac{p\cdot q}{P}\ ,\ \vec{q}_\perp\ ,\ 0\right),$$

$$r = \left(\frac{p\cdot r}{P}\ ,\ \vec{r}_\perp,\ 0\right),$$

$$p_a = \left(xP + \frac{\vec{k}_\perp{}^2 + m_a{}^2}{2xP}\ ,\ \vec{k}_\perp\ ,\ xP\right)\ .$$

The other vectors follow from 3-momentum conservation.

[6]S. D. Drell and T.-M. Yan, Phys. Rev. Letters 24, 181 (1970).

[7]P. Landshoff, J. C. Polkinghorne, and R. Short, Nucl. Phys. B28, 225 (1971). See also S. J. Brodsky, F. E. Close, and J. F. Gunion, Phys. Rev. D 5, 1384 (1972).

[8]We assume that the parton-B amplitude at large momentum transfer reduces to a product of the two wave functions describing the breakup of the particle B into its parton and core constituents, just as in the form-factor calculations. We also make this assumption in the case of the forward amplitude when the relative momentum of the parton and core is large – this corresponds to the assumption of a strong off-shell damping of the coherent Regge terms. (See Ref. 7). It is due to this last assumption that the contribution of the alternate core routing of Fig. 2(b) is not given in terms of a structure function, but rather is related only to the simple breakup wave functions.

[9]E. D. Bloom and F. J. Gilman, Phys. Rev. Letters 25, 1140 (1970).

[10]As in the exclusive case, one should use $l_p = 2$ only when the wave-function arguments are $\gtrsim 10\ \mathrm{GeV}^2$, as it is only there that the form factor falls as t^{-2}. Between 5 and 10 GeV^2, $t^{-1.7}$ is a better description of the form-factor data, and the results given should be modified accordingly.

[11]T. T. Wu and C. N. Yang, Phys. Rev. 137, B708 (1965); H. D. I. Abarbanel, S. D. Drell, and F. J. Gilman, *ibid.* 177, 2458 (1969); J. V. Allaby *et al.*, CERN Report No. 70-12 (unpublished). J. Kogut and L. Susskind, IAS report, 1972 (unpublished).

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Multiparticle Spectra in Simple Theoretical Models: A Cluster-Decomposition Analysis*

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We present a detailed investigation of inclusive and exclusive multiparticle spectra in several simple models of high-energy hadronic interactions. To clarify the origins of particular structure in these spectra we divide the study into two sections. First, we discuss models designed to isolate the features arising from kinematic constraints. Specifically, we treat both pure longitudinal phase space and a longitudinal phase space modified to produce a "leading-particle" effect. In the second section we consider examples of specific dynamical models. Here we investigate the simplified Chew-Pignotti model, in both the strong-ordered and general versions, and a "φ^3-type" multiperipheral model. Our analysis is based on a cluster-decomposition approach analogous to that used in statistical mechanics. In particular, we apply a set of "cluster-function sum rules" to relate the correlations found in inclusive spectra to those observed in exclusive processes. Within the context of the models, we establish that these sum rules provide both an attractive qualitative picture of correlations in multiparticle spectra and a useful quantitative framework in which to calculate these correlations. We conclude with a discussion of the possible extensions of these cluster techniques to phenomenological analyses of high-energy interactions.

I. INTRODUCTION

Attempts to describe the behavior of strong interactions at very high energy have recently stimulated considerable interest in multiparticle production reactions. The obvious complexities involved in analyzing the many-body final states, however, have led theorists to seek first a general conceptual framework in which to discuss the gross features of these reactions and experimentalists to study quantities which are, hopefully, sensitive only to these broad features. Hence, in particular, interest in both theoretical calculations and experimental investigations has shifted from the detailed examination of specific exclusive processes toward studies of inclusive reactions. By summing over many exclusive channels, one seeks to isolate in inclusive reactions salient general characteristics of multiparticle processes which, in any given exclusive channel, may be obscured by more specific features. Eventually, of course, one would hope to understand in detail the manner in which the individual exclusive reactions combine to yield the observed inclusive cross sections.[1]

At a general qualitative level a very attractive picture of high-energy multiparticle processes follows from the Wilson-Feynman proposal[2] that these interactions should be analogous to the behavior of a real gas contained in a finite volume. From this "gas analogy" many of the observed properties of one-particle spectra, for example, can be derived. Further, this framework provides strong motivation for the current interest in studying correlations among the momenta of final-state particles.

In two recent articles[3] – hereafter referred to as I and II – we have developed a systematic approach to the analysis of multiparticle reactions which combines the intuitive appeal of the "gas analogy" with an explicit calculational algorithm for determining spectra, correlations, and all other features of these processes. In a sense, the approach provides a quantitative realization of the "gas-analogy" framework. Based on techniques used in statistical-mechanical treatments of real gases, the approach was derived in an explicit field-theory model; however, as we shall see presently, it can be applied to a variety of other models. By analogy to a similar technique in statistical mechanics, we call this approach a "cluster decomposition."[4] In the simplest terms, the cluster decomposition represents a systematic treatment of correlations in multiparticle reactions. Indeed, one of the most appealing features of the method is the emergence of direct and simple relations between correlations in inclusive and in exclusive spectra; the cluster decomposition provides a natural link between the "microscopic" and "macroscopic" perspectives on multiparticle processes.

The purpose of the present investigation is twofold:

(1) to examine possible qualitative features of high-energy exclusive and inclusive multiparticle spectra and correlations; and

(2) to illustrate the use of the cluster-decomposition techniques in several simple models.

This dual purpose leads us naturally to a survey of several models rather than to an exhaustive analysis of a single model. Further, in view of their simplified character,[5] more detailed studies of these specific models may not be warranted. We shall discover, however, that despite their relative simplicity, the models can offer considerable insight into the possible structure of many particle reactions at high energy.

In Sec. II we first define the notation and kinematic variables. We then recapitulate those general results of the cluster-decomposition approach which we shall apply to the analysis of inclusive and exclusive correlations. This summary of previous work is rather condensed, but, to render the presentation reasonably self-contained, brief physical motivation is given for most of the results. The technical details are, of course, accessible in I and II.

To clarify the role of kinematic constraints in shaping the multiparticle spectra, we consider in Sec. III those correlations which are induced solely by energy-momentum conservation. We therefore analyze briefly the inclusive kinematic correlations arising purely from longitudinal phase space. In addition, we discuss a "modified" phase-space model, which is designed to illustrate the kinematic correlations present when the observed existence of "leading" particles is incorporated.

Possible forms of dynamical correlation effects in multiparticle spectra are analyzed in Sec. IV. We treat two specific models for the exclusive cross sections and, using the cluster-decomposition techniques, study correlations in both inclusive and exclusive spectra.

The first dynamical model is a simplified version of the Chew-Pignotti multi-Regge model[6] that has previously been studied in the context of single-particle inclusive spectra.[7] We examine this model in both its "strong-ordered" and general[8] forms. In the former case, we clarify the relation between the absence of correlations in inclusive and exclusive spectra and the Poisson distribution of the exclusive partial cross sections. In the latter case we find nonvanishing dynamical correlations and discuss these within the cluster-decomposition framework.

The second model is closely related to the $\lambda\varphi^3$ model treated in I and II; in fact, its explicit form is chosen to reproduce the general structure of the dynamical correlations of this model. Since it is mathematically much simpler than the full $\lambda\varphi^3$ models, however, it is amenable to analytical as opposed to numerical analysis[9] and therefore is more suitable to our present purposes.

Finally, in Sec. V, we present a critical discussion of the cluster-decomposition approach in its current form and consider the possible extension of the technique to quantitative phenomenology.

II. KINEMATICS, DEFINITIONS, AND SUMMARY OF PREVIOUS RESULTS

Consider the exclusive processes

$$A+B \to A'+B'+1+\cdots+n \tag{2.1}$$

and their inclusive analogs

$$A+B\to A'+B'+1+\cdots+n+X, \tag{2.2}$$

where X represents any number of undetected particles. For simplicity, we assume that all particles are spinless and, for the present, identical. To specify the kinematics we use the momentum variables

$$q_i^{\pm}=q_i^0 \pm q_i^3, \tag{2.3}$$

and $\vec{q}_i=(0, q_i^1, q_i^2, 0)\equiv q_{\perp i}$; the mass-shell restriction requires

$$q_i^+ q_i^- = m^2+\vec{q}_i^{\,2} \equiv m_{Ti}^{\,2}. \tag{2.4}$$

The $q_i^{\pm}$ are immediately related to the "rapidity" variable,

$$y_i \equiv \tfrac{1}{2}\ln(q_i^+/q_i^-)=\ln(q_i^+/m_{Ti})=\ln(m_{Ti}/q_i^-). \tag{2.5}$$

Hence translating expressions from y_i to $q_i^{\pm}$ is particularly simple. Further, in terms of these variables, the single-particle phase space is just

$$d\Phi \equiv d^2\vec{q}\,\frac{dq^+}{2q^+}.$$

Since at fixed q^+ the differential dq^+/q^+ is independent of $\vec{q}$, one can integrate over $\vec{q}$ without altering the longitudinal (dq^+/q^+) part of phase space. Note that this is not true for the conventional phase space,

$$d\Phi_c \equiv d^2\vec{q}\,\frac{dq^3}{2q^0}.$$

We choose to work in the center-of-mass frame with p_A representing the beam, incident from $z=-\infty$, and p_B the target. Hence at large s we have

$$\begin{aligned} p_A^+ &\simeq \sqrt{s}, \quad p_A^- \simeq \frac{m^2}{\sqrt{s}}, \\ p_B^+ &\simeq \frac{m^2}{\sqrt{s}}, \quad p_B^- \simeq \sqrt{s}. \end{aligned} \tag{2.6}$$

The requirement of conservation of energy momentum becomes, in the center-of-mass system,

$$\sqrt{s}=p_A^{\pm}+p_B^{\pm}=p_A'^{\pm}+p_B'^{\pm}+\sum_{i=1}^{n} q_i^{\pm} \tag{2.7}$$

and

$$0=\vec{p}_A'+\vec{p}_B'+\sum_{i=1}^{n}\vec{q}_i. \tag{2.8}$$

Since we shall integrate over all transverse momenta, only Eqs. (2.7) will concern us explicitly. Notice further that if we do not observe the particles A' and B' – that is, if we integrate over their longitudinal as well as transverse momenta – then the longitudinal energy-momentum constraints on the $q_i^{\pm}$ become

$$\sqrt{s}-\frac{2m^2}{\sqrt{s}} \sim \sqrt{s} > \sum_{i=1}^{n} q_i^{\pm}. \tag{2.9}$$

To complete the kinematic preliminaries let us introduce some useful definitions. We shall refer to a produced particle, characterized by its momentum q_i, as a "beam fragment" if

$$\sqrt{s} \simeq p_A^+ > q_i^+ \geq \eta p_A^+ \simeq \eta\sqrt{s}, \tag{2.10}$$

where $0<\eta \ll 1$, and η is fixed and s-independent. Similarly, the term "target fragment" will refer to a particle for which

$$\sqrt{s} \simeq p_B^- > q_i^- > \eta' p_B^- \simeq \eta'\sqrt{s}, \tag{2.11}$$

with η' like η; notice that (2.11) is equivalent to

$$\frac{m_{T_i}^{\,2}}{\sqrt{s}} < q_i^+ < \frac{m_{T_i}^{\,2}}{\eta'\sqrt{s}}. \tag{2.12}$$

For our purposes it will be sufficient to replace $m_{T_i}^{\,2}=m^2+\vec{q}_i^{\,2}$ in (2.12) by $\langle m_T^{\,2}\rangle \equiv m^2+\langle \vec{q}^{\,2}\rangle$. Finally, we define "pionization products" as those particles with momentum such that, in the center-of-mass system,

$$\sqrt{s} \gg \eta\sqrt{s} > q_i^+ > \frac{\langle m_T^{\,2}\rangle}{\eta'\sqrt{s}} \gg \frac{m^2}{\sqrt{s}}. \tag{2.13}$$

Referring to Fig. 1, we see that in terms of rapidity, fragments are produced within a fixed distance from the ends of the plot, whereas pionization products emanate from the central region, the length of which grows as $\ln s$.

Let us try to convey briefly the purpose of distinguishing among these kinematic regions. First we observe that if the distinction between "pioniza-

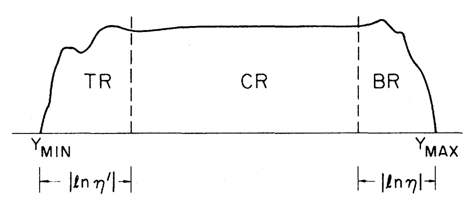


FIG. 1. A schematic representation of an inclusive one-particle spectrum illustrating the definitions of the beam and target "fragmentation" regions – labeled BR and TR, respectively – and the "central" or "pionization" region – labeled CR.

tion" and "fragmentation" is to be meaningful, it should not depend crucially on η and η', provided they are small and s is large. This means, more specifically, that the one-particle inclusive spectrum for "soft" beam fragments, $q_i^+ \lesssim \eta\sqrt{s}$, should join smoothly with that spectrum for "hard" pionization products, $q_i^+ \gtrsim \eta\sqrt{s}$. In I this result was shown to be valid for models which satisfy the cluster-decomposition approach; further, in Sec. III we shall see explicit examples of this smooth transition. Second, we note that the crucial motivation for the distinction is that, crudely speaking, the beam fragments take away "all" of the plus momentum, and the target fragments "all" the minus momentum, leaving the pionization products nearly unconstrained by energy-momentum conservation.[10] We shall see that this means that the underlying dynamical effects, as opposed to the purely kinematical constraints, can most clearly be isolated in the pionization region. Finally, in view of the observed phenomenon of "leading particles" and the consequent small inelasticities, the distinction seems to have true physical significance.

To summarize the results of our previous investigations of field-theory models, we begin by considering the form of the exclusive differential cross sections.[11]

After integration over all transverse momenta and over the longitudinal momenta of the particles A' and B', the exclusive differential cross section for the production of n additional particles can be written at large s as

$$\begin{aligned}\frac{d^n\sigma_n}{\sigma_0} &\equiv \bar{f}_n(q_1^+, \ldots, q_n^+)\prod_{i=1}^n \frac{dq_i^+}{q_i^+} \\ &\equiv f_n(q_1^+, \ldots, q_n^+)\theta\left(\sqrt{s} - \sum_{i=1}^n q_i^+\right) \\ &\quad\times \theta_-(n)\prod_{i=1}^n \frac{dq_i^+}{q_i^+} . \end{aligned} \tag{2.14}$$

Here $\bar{f}_n(q_1^+, \ldots, q_n^+)$ is a completely symmetric function of its arguments and will be called the "normalized n-particle exclusive spectrum." We have chosen to exhibit the energy-momentum constraints explicitly in the threshold functions $\theta(\sqrt{s} - \sum_{i=1}^n q_i^+)$ and $\theta_-(n)$. The function $\theta_-(n)$, which arises from the constraint $\sqrt{s} > \sum_{i=1}^n q_i^-$, becomes a complicated function of the q_i^+ after the transverse-momenta integrations; this is because at fixed q_i^+, q_i^- depends on $\vec{q}$. If we simply ignored the transverse momenta, then

$$\theta_-(n) \equiv \theta\left(\sqrt{s} - \sum_{i=1}^n \text{``}q_i^-\text{''}\right) = \theta\left(\sqrt{s} - \sum_{i=1}^n \frac{m^2}{q_i^+}\right) .$$

In our later study of simple models we shall in fact ignore transverse momenta and thus be able to use this explicit form of $\theta_-(n)$. Notice further that we have chosen to normalize this differential cross section to σ_0, the cross section for the process $A + B \to A' + B'$. Finally, we observe that the partial cross section for producing n additional particles is just

$$\begin{aligned}\frac{\sigma_n}{\sigma_0} &= \frac{1}{n!}\prod_{i=1}^n \int_{m^2/\sqrt{s}}^{\sqrt{s}} \frac{dq_i^+}{q_i^+}\bar{f}_n(q_1^+, \ldots, q_n^+) \\ &= \frac{1}{n!}\int_{\theta_-(n)}^{\theta(\sqrt{s} - \Sigma q_i^+)} \prod_{i=1}^n \frac{dq_i^+}{q_i^+} f_n(q_1^+, \ldots, q_n^+) . \end{aligned} \tag{2.15}$$

In II we established that, in the simple $\lambda\varphi^3$ ladder model, the f_n as defined by (2.14) obey a crucial factorization property; namely, for

$$\begin{aligned}&(q_1^+, \ldots, q_m^+) \gg (q_{m+1}^+, \ldots, q_n^+), \\ &f_n(q_1^+, \ldots, q_n^+) \to f_m(q_1^+, \ldots, q_m^+)f_{n-m}(q_{m+1}^+, \ldots, q_n^+) \\ &\qquad + O([q_j^+/q_i^+]), \end{aligned} \tag{2.16}$$

where $i \in (1, \ldots, m)$ and $j \in (m+1, \ldots, n)$. This fundamental factorization property is basic to the cluster-decomposition approach; we shall establish it explicitly for the simple models discussed in Secs. III and IV. We continue the present discussion by recalling the consequences of (2.16) derived in our previous work.

The remarkable similarity between (2.16) and the "short-range order" or "cluster" properties of real gases led us to investigate the extent to which a quantitative treatment of high-energy interactions could be based on techniques borrowed from statistical mechanics. In particular, we began by introducing "cluster functions," $g_n(q_1^+, \ldots, q_n^+)$, to describe the correlations that could exist among particle momenta in exclusive processes when $q_1^+ \simeq q_2^+ \approx \cdots \simeq q_n^+$. The $g_n(q_1^+, \ldots, q_n^+)$ are defined in the canonical manner:

$$\begin{aligned}&f_1(q^+) \equiv g_1(q^+), \\ &f_2(q_1^+, q_2^+) \equiv g_1(q_1^+)g_1(q_2^+) + g_2(q_1^+, q_2^+), \\ &f_3(q_1^+, q_2^+, q_3^+) \equiv g_1(q_1^+)g_1(q_2^+)g_1(q_3^+) + g_1(q_1^+)g_2(q_2^+, q_3^+) \\ &\qquad + g_1(q_2^+)g_2(q_1^+, q_3^+) + g_1(q_3^+)g_2(q_1^+, q_2^+) \\ &\qquad + g_3(q_1^+, q_2^+, q_3^+), \\ &f_n(q_1^+, \ldots, q_n^+) \equiv \sum_{\substack{[n_i] \\ n_1+n_2+\cdots+n_r=n}} (g_{n_1}(\cdots)\cdots g_{n_r}(\cdots)), \end{aligned} \tag{2.17}$$

where the sum runs over all partitions of n. From (2.16) we find by induction that, for

$$\begin{aligned}&(q_1^+, \ldots, q_m^+) \gg (q_{m+1}^+, \ldots, q_n^+), \\ &g_n(q_1^+, \ldots, q_n^+) \to O([q_j^+/q_i^+]) \sim O(e^{-|y_i - y_j|}) \to 0 \end{aligned} \tag{2.18}$$

where $i \in (1, \ldots, m)$ and $j \in (m+1, \ldots, n)$. Hence, in rapidity space, the g_n have finite "correlation lengths." It is thus indeed reasonable to think of the g_n as representing short-range correlations among the final-state momenta in exclusive processes. In addition, from (2.18) we were able in I and II to establish that the integrated cluster functions, $G_n(s)$, satisfied

$$G_n(s) = \int_{\theta_-(n)}^{\theta(\sqrt{s}-\Sigma q_i^+)} \prod_{i=1}^{n} \frac{dq_i^+}{q_i^+} g_n(q_1^+, \ldots, q_n^+)$$
$$= \alpha_n \ln s + \beta_n + O((\ln s)/s). \quad (2.19)$$

As discussed in I and II, this result further strengthens the analogy between the functions g_n and the cluster functions of statistical mechanics.

Pursuing this analogy further, we were able to demonstrate that a cluster-decomposition technique, again similar to an approach used in statistical mechanics, can be used to relate in a very simple manner the exclusive cluster functions to inclusive processes. To establish these exclusive-inclusive relations, however, we found it necessary, in order to include properly the kinematic constraints on the q_i^+, to introduce "modified" exclusive cluster functions, $\bar{g}_n$, defined in terms of $\bar{f}_n$. Thus we have

$$\begin{aligned} &\bar{f}_1(q_1^+) \equiv f_1(q)\theta(\sqrt{s} - q_1^+)\theta_-(1) \equiv \bar{g}_1(q_1^+), \\ &\bar{f}_2(q_1^+, q_2^+) \equiv \bar{g}_1(q_1^+)\bar{g}_1(q_2^+) + \bar{g}_2(q_1^+, q_2^+), \end{aligned} \quad (2.20)$$

and similarly for the higher $\bar{g}_n$. The introduction of the $\bar{g}_n$ involves some important technical subtleties. It is obvious, for instance, that the $\bar{f}_n$ do not satisfy a simple factorization property analogous to (2.16), for the threshold functions themselves are not factorizable. Hence the important "short-range order" property of g_n expressed by (2.18) does not follow exactly for the $\bar{g}_n$. It develops, however, that one can show that the $\bar{g}_n$ can still correctly be interpreted as representing short-range correlations among momenta in exclusive spectra and that the analog of (2.19) for $\bar{g}_n$ is valid. In Sec. III the simple models will provide explicit verification of these assertions; readers interested in the general technical details are referred to II. For the present, we limit ourselves to a simple qualitative argument which clarifies the results of both these references. The f_n, which contain the fundamental dynamical information, are assumed to obey (2.16); thus the nonfactorizability of the $\bar{f}_n$ arises solely from the threshold functions, $\theta(\sqrt{s} - \sum_{i=1}^{n} q_i^+)$ and $\theta_-(n)$. But these functions reflect the kinematic constraints and therefore, by our previous arguments, should be significant only in the fragmentation regions. Indeed, in the pionization region, we see that $\bar{f}_n \to f_n$ and thus $\bar{g}_n \to g_n$.[12] Now by definition the fragmentation regions represent a fixed, s-independent amount of the relevant phase space, whereas the pionization region grows as $\ln s$. In terms of the gas analogy, the fragmentation regions reflect "surface" effects and the pionization regions reflect "volume" effects. Since at large s the $\bar{f}_n$ and $\bar{g}_n$ effectively "reduce" to f_n and g_n, respectively, throughout the bulk of phase space, it is plausible that the nonfactorizability of the $\bar{f}_n$ does not vitiate the interpretation of $\bar{g}_n$ as representing short-range correlations in exclusive processes.

To study the explicit relations between the exclusive cluster functions, $\bar{g}_n$, and inclusive quantities, we began with the total cross section. From the definition

$$\frac{\sigma_T}{\sigma_0} \equiv \sum_n \frac{1}{n!} \prod_{i=1}^{n} \int_{m^2/\sqrt{s}}^{\sqrt{s}} \frac{dq_i^+}{q_i^+} \bar{f}_n(q_1^+, \ldots, q_n^+) \quad (2.21)$$

and (2.20), the definition of $\bar{g}_n$, we were able to prove the "cluster-decomposition theorem."[13]

$$\frac{\sigma_T(s)}{\sigma_0(s)} = \exp\left(\sum_n \frac{1}{n!} \prod_{i=1}^{n} \int_{m^2/\sqrt{s}}^{\sqrt{s}} \frac{dq_i^+}{q_i^+} \bar{g}_n(q_1^+, \ldots, q_n^+)\right)$$
$$= \exp\left(\sum_n \frac{1}{n!} \bar{G}_n(s)\right). \quad (2.22)$$

Together with the analog of (2.19), this result implies that σ_T/σ_0 will exhibit a simple power dependence on s as $s \to \infty$,

$$\sigma_T(s)/\sigma_0(s) = e^{\beta} s^{\alpha}, \quad (2.23)$$

with $\alpha = \sum \alpha_n$ and $\beta = \sum \beta_n$.

Let us next consider the relation between the $\bar{g}_n$ and the multiparticle inclusive spectra. As in the case of exclusive spectra, we first integrate over all transverse momenta and over the longitudinal momenta of the particles A', B', and the undetected particles, X. The inclusive differential cross section for the production of n additional observed particles plus anything else can then be written as

$$\frac{d^n\sigma}{\sigma_T} \equiv \bar{\rho}_n(q_1^+, \ldots, q_n^+) \prod_{i=1}^{n} \frac{dq_i^+}{q_i^+}$$
$$\equiv \rho_n(q_1^+, \ldots, q_n^+)\theta(\sqrt{s} - \sum q_i^+)\theta_-(n) \prod_{i=1}^{n} \frac{dq_i^+}{q_i^+}. \quad (2.24)$$

The function $\bar{\rho}_n(q_1^+, \ldots, q_n^+)$ is a completely symmetric function of its arguments and will be called the "normalized n-particle inclusive spectrum." Again the threshold functions reflecting the kinematic constraints are exhibited explicitly in (2.24). In models in which (2.16) holds, it is possible to show that an analogous factorization property applies to ρ_n: namely, for

$$(q_1^+, \ldots, q_m^+) \gg (q_{m+1}^+, \ldots, q_n^+),$$
$$\rho_n(q_1^+, \ldots, q_n^+) \to \rho_m(q_1^+, \ldots, q_m^+)\rho_{n-m}(q_{m+1}^+, \ldots, q_n^+) + O([q_i^+/q_j^+]^r), \tag{2.25}$$

where $i \in (1, \ldots, m)$, $j \in (m+1, \ldots, n)$, and $r > 0$. Hence we introduce "inclusive cluster functions," τ_n, to describe the correlations among momenta in inclusive spectra. We have

$$\begin{aligned}\rho_1(q^+) &\equiv \tau_1(q_1^+),\\ \rho_2(q_1^+, q_2^+) &\equiv \tau_1(q_1^+)\tau_1(q_2^+) + \tau_2(q_1^+, q_2^+),\end{aligned} \tag{2.26}$$

and similarly for higher τ_n. A similar set of equations defines $\bar{\tau}_n$ in terms of $\bar{\rho}_n$. Expressing the n-particle inclusive spectrum as a sum over exclusive spectra,

$$\frac{d^n\sigma}{\sigma_0} = \sum_{m \geq n} \frac{d^n\sigma_m}{\sigma_0} = \sum_{m \geq n} \frac{1}{(m-n)!}\left(\prod_{i=1}^{m-n} \int_{m^2/\sqrt{s}}^{\sqrt{s}} \frac{dq_i^+}{q_i^+}\right) \bar{f}_m(q_1^+, \ldots, q_m^+), \tag{2.27}$$

and using the definitions (2.20) and (2.26), it is possible to establish the "cluster-function sum rules" relating inclusive correlations to sums over exclusive correlations. We find the general relation

$$\bar{\tau}_n(q_1^+, \ldots, q_n^+) = \sum_{l=0}^{\infty} \frac{1}{l!}\left(\prod_{i=1}^{l} \int_{m^2/\sqrt{s}}^{\sqrt{s}} \frac{dp_i^+}{p_i^+}\right) \bar{g}_{n+l}(q_1^+, \ldots, q_n^+, p_1^+, \ldots, p_l^+). \tag{2.28}$$

This equation expresses the physically intuitive result, essential for the consistency of the approach, that only particles within the same exclusive cluster (and therefore correlated in the exclusive process) can be correlated in the inclusive spectrum. Furthermore, (2.28) can be inverted formally to yield expressions for the exclusive cluster in terms of sums over inclusive cluster functions. The general form of these "cluster-functions inversion formulas" is

$$\bar{g}_n(q_1^+, \ldots, q_n^+) = \sum_{l} \frac{(-1)^l}{l!}\left(\prod_{i=1}^{l} \int_{m^2/\sqrt{s}}^{\sqrt{s}} \frac{dp_i^+}{p_i^+}\right) \bar{\tau}_{n+l}(q_1^+, \ldots, q_n^+, p_1^+, \ldots, p_l^+). \tag{2.29}$$

Several remarks on the significance of the general "cluster-decomposition" results are in order. First, given the definitions of the $\bar{g}_n$ and $\bar{\tau}_n$ and using (2.21) and (2.27), the results in (2.22), (2.28), and (2.29) are formal mathematical identities; that is, they follow formally independent of the properties of the $\bar{g}_n$ and $\bar{\tau}_n$. However, the usefulness of these results in any model calculation, or in potential applications to phenomenology, hinges crucially on the "short-range-order" property of the correlation functions; this, in turn, depends on the underlying factorization property of the f_n. We shall later illustrate and amplify these statements.

Further, although (2.28) and (2.29) represent systematic linear relations between correlations in inclusive and exclusive processes, it is not immediately clear from the present discussion that these relations should be preferred to the "direct inclusive-exclusive relations" of (2.21) and (2.27). To illustrate the potential superiority of the cluster-decomposition approach vis à vis the direct relations, let us consider the one-particle inclusive spectrum. In the cluster expression for $\bar{\tau}_1$ each term will be independent of s for large s; this follows from (2.18). Hence, the number of terms needed to saturate the sum rule will also be independent of s. Indeed, in Sec. IV we shall demonstrate that in certain cases the first few terms of (2.28) with $n=1$ are sufficient to give an accurate determination of $\bar{\tau}_1$. In the "direct relations" for $\bar{\tau}_1$ – given by (2.27) with $n=1$ – the contributions are clearly dependent on s. In fact, since (2.17) implies that

$$\bar{f}_n(q_1^+, \ldots, q_n^+) = \prod_{i=1}^{n} \bar{g}_1(q_i) + \cdots, \tag{2.30}$$

we find

$$\int\left(\prod_{i=2}^{n} \frac{dq_i^+}{q_i^+}\right) \bar{f}_n(q_1^+, \ldots, q_n^+) \propto (\ln s)^{n-1} + \cdots. \tag{2.31}$$

Thus the contributions from large n become increasingly important as s grows. Simple inspection suggests that the number of terms which are important at large s increases linearly with $\ln s$. To saturate the "direct relation," therefore, requires a number of terms which increases with s.

Finally, we remark that the total symmetry of the exclusive and inclusive differential cross sections, $\bar{f}_n$ and $\bar{\rho}_n$, in their arguments allows us to recast the cluster-decomposition results in a slightly different form by integrating over ordered regions of phase space only. As the resulting forms of these equations will be useful in Sec. IV, we shall quote them here. From the symmetry of $\bar{f}_n$ we see that

$$\frac{\sigma_n}{\sigma_0} \equiv \frac{1}{n!}\left(\prod_{i=1}^{n}\int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_i^+}{q_i^+}\right)\bar{f}_n(q_1^+,\ldots,q_n^+)$$
$$= \int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\int_{m^2/\sqrt{s}}^{q_1^+}\frac{dq_2^+}{q_2^+}\cdots\int_{m^2/\sqrt{s}}^{q_{n-1}^+}\frac{dq_n^+}{q_n^+}\,\bar{f}_n(q_1^+,\ldots,q_n^+) \equiv \int_{q_1^+>q_2^+>\cdots>q_n^+}\left(\prod_{i=1}^{n}\frac{dq_i^+}{q_i^+}\right)\bar{f}_n(q_1^+,\ldots,q_n^+). \tag{2.15'}$$

This leads to

$$\frac{\sigma_T(s)}{\sigma_0(s)} = \exp\left[\sum_n\int_{q_1^+>\cdots>q_n^+}\left(\prod_{i=1}^{n}\frac{dq_i^+}{q_i^+}\right)\bar{g}_n(q_1^+,\ldots,q_n^+)\right]. \tag{2.22'}$$

A similar result obtains for the general $\bar{\tau}_n$. For $q_1^+>q_2^+>\cdots>q_n^+$,

$$\bar{\tau}_n(q_1^+,\ldots,q_n^+) = \sum_{l=0}^{\infty}\sum_{\substack{\gamma \\ (\gamma\ni q_1^+>\cdots>q_n^+;p_1>\cdots>p_l^+)}}\int\left(\prod_{i=1}^{l}\frac{dp_i^+}{p_i^+}\right)\bar{g}_{n+l}(q_1^+,\ldots,q_n^+;p_1^+,\ldots,p_l^+). \tag{2.28'}$$

Here the symbol

$$\sum_{\substack{\gamma \\ (\gamma\ni q_1^+>,\ldots,q_n^+;p_1^+>\cdots>p_l^+)}}$$

indicates that one integrates over the phase space $\prod_{i=1}^{l}dp_i^+/p_i^+$ subject to $p_1^+>\cdots>p_l^+$ and $q_1^+>\cdots>q_n^+$ and sums over the $(n+l)!/n!l!$ distinct orderings of the ordered p_i relative to the ordered q_j. Similarly, the inversion formulas become, for $q_1^+>\cdots>q_n^+$,

$$\bar{g}_n(q_1^+,\ldots,q_n^+) = \sum_l(-1)^l\sum_{\substack{\gamma \\ (\gamma\ni q_1^+>\cdots>q_n^+;p_1^+>\cdots>p_l^+)}}\int\left(\prod_{i=1}^{l}\frac{dp_i^+}{p_i^+}\right)\bar{\tau}_{n+l}(q_1^+,\ldots,q_n^+;p_1^+,\ldots,p_l^+). \tag{2.29'}$$

In the ensuing sections we shall illustrate these general cluster-decomposition results by examining the relations between inclusive and exclusive processes in a series of simple models. These examples will provide both an intuitive basis for understanding the cluster-decomposition techniques and an insight into the possible structure of multiparticle spectra at high energy.

III. MODELS ILLUSTRATING KINEMATIC CONSTRAINTS

A. Introduction

At least two considerations suggest that the role of kinematic constraints within the cluster-decomposition framework requires thorough understanding. First, these constraints produce crucial technical complications, since they have no analogs in the standard statistical mechanical applications of cluster expansions.[14] Second, and perhaps more important, at presently available energies, it is clear [15,16] that kinematic constraints will represent a significant factor in determining these correlations.

In this section we shall examine in simple models the correlations which arise from the constraints of energy-momentum conservation; we shall refer to these as "kinematic correlations." To isolate these kinematic effects it is natural to begin by replacing all dynamical matrix elements by one and hence by studying pure phase space. We shall not, however, treat pure phase space in (3 + 1) dimensions, as it is unsuitable to our purposes for two reasons. First, its consequences – particularly those for the average multiplicity and transverse-momentum distributions [17] – would bear no resemblance to present experimental observations. Second, we wish to study only longitudinal momentum spectra. Hence we shall first examine pure longitudinal – mathematically, (1 + 1)-dimensional – phase space.[18] The results of this model may be viewed as reflecting the kinematic constraints imposed on longitudinal spectra by the full (3 + 1)-dimensional phase space when the observed rapid cutoff in transverse momentum is included.[18,19]

We shall augment our investigation of kinematic correlations by studying as a second model a "modified" phase space, designed to illustrate the kinematic constraints among secondary particles when the observed existence of "leading" particles is taken into account.

We remark at the outset that whereas n-particle exclusive phase space is well defined, the concept of "inclusive phase space" is ambiguous. To construct an inclusive cross section one must prescribe the relative weights associated with each contributing exclusive process. We shall adopt the prescription of associating with each final-state particle a "coupling constant," $\bar{\lambda}$.[20] Hence the "matrix element" for the production of n particles is given by the constant $|M_n|^2 \equiv \bar{\lambda}^n$. Varying $\bar{\lambda}$ will thus alter the relative importance of the various exclusive contributions to the inclusive reaction.

B. Longitudinal Phase Space

The exclusive cross section for the process $A+B \to A'+B'+1+\cdots+n$ in the "longitudinal phase-space model" is given at large s by[21]

$$\frac{\sigma_n}{\sigma_0} \equiv \frac{2s\bar{\lambda}^n}{(n+2)!}\int_0^\infty dp_A'^+ dp_A'^- \delta(p_A'^2 - m^2) dp_B'^+ dp_B'^- \delta(p_B'^2 - m^2)\left(\prod_{i=1}^n dq_i^+ dq_i^- \delta(q_i^2 - m^2)\right)$$
$$\times 2\delta\left(\sqrt{s} - p_A'^+ - p_B'^+ - \sum_{i=1}^n q_i^+\right)\delta\left(\sqrt{s} - p_A'^- - p_B'^- - \sum_{i=1}^n q_i^-\right)$$
$$= s\left(\tfrac{1}{2}\bar{\lambda}\right)^n \int_{p_A'^+ > q_1^+ > \cdots > q_n^+ > p_B'^+} \frac{dp_A'^+}{p_A'^+}\frac{dp_B'^-}{p_B'^-}\left(\prod_{i=1}^n \frac{dq_i^+}{q_i^+}\right)$$
$$\times \delta\left(\sqrt{s} - p_A'^+ - p_B'^+ - \sum_{i=1}^n q_i^+\right)\delta\left(\sqrt{s} - p_A'^- - p_B'^- - \sum_{i=1}^n q_i^-\right). \tag{3.1}$$

Here $\sigma_0 = \bar{\lambda}^2/2s^2$.[22] We continue to assume that all particles are identical. Further, we define $\lambda \equiv \frac{1}{2}\bar{\lambda}$ for compactness of notation.

To introduce a cluster decomposition for the exclusive differential cross sections, one might first proceed by defining $\bar{h}_{n+2}$ by

$$\frac{\sigma_n}{\sigma_0} \equiv s\lambda^n \int_{p_A'^+ > q_1^+ > \cdots > p_B'^+} \frac{dp_A'^+}{p_A'^+}\frac{dp_B'^-}{p_B'^-}\left(\prod_{i=1}^n \frac{dq_i^+}{q_i^+}\right)\bar{h}_{n+2}(p_A'^+, \{q_i^+\}, p_B'^-) \tag{3.2}$$

and attempting to define cluster functions, $\bar{k}_n$, for the $\bar{h}_n$. These "cluster functions," however, could not be interpreted as representing short-range correlations. Mathematically, this result is made obvious by examination of the lowest-order $\bar{k}_n$. Physically, it is clear that the crucial independence of the clusters is destroyed if their arguments are themselves not independent. Thus, as indicated in Sec. II, to introduce "cluster functions" in this context it is necessary to integrate over two momenta in order to replace the δ functions by less restrictive θ function constraints. Our current results, as well as the later more detailed discussion and the prior general treatment,[3] verify that the cluster-decomposition technique can then indeed be applied.

To establish a well-defined prescription, we choose to integrate over the largest plus and largest minus momenta in (3.1), with the result[23]

$$\frac{\sigma_n}{\sigma_0} \simeq \lambda^n \int_{x_1 > \cdots > x_n}\left(\prod_{i=1}^n \frac{dx_i}{x_i}\right)\frac{\theta\left(1 - x_1 - \sum_{i=1}^n x_i\right)}{1 - \sum x_i}\,\frac{\theta\left(1 - \frac{m^2}{sx_n} - \sum_{i=1}^n \frac{m^2}{sx_i}\right)}{1 - \sum \frac{m^2}{sx_i}} \equiv \int_{x_1 > \cdots > x_n}\left(\prod_{i=1}^n \frac{dx_i}{x_i}\right)\bar{f}_n(x_1, \ldots, x_n), \tag{3.3}$$

where $x_i = q_i^+/\sqrt{s}$.

Introducing cluster functions for the $\bar{f}_n$ of (3.3) in the standard manner, we find

$$\bar{g}_1(x_1) = \lambda\frac{\theta(1 - 2x_1)\theta(1 - 2m^2/sx_1)}{(1 - x_1)(1 - m^2/sx_1)} \tag{3.4}$$

and, for $x_1 > x_2$,

$$\bar{g}_2(x_1, x_2) = \lambda^2\frac{\theta(1 - 2x_1 - x_2)\theta\left(1 - \frac{2m^2}{sx_2} - \frac{m^2}{sx_1}\right)}{(1 - x_1 - x_2)\left(1 - \frac{m^2}{sx_1} - \frac{m^2}{sx_2}\right)} - \frac{\theta(1 - 2x_1)\theta\left(1 - \frac{2m^2}{sx_1}\right)}{(1 - x_1)\left(1 - \frac{m^2}{sx_1}\right)}\,\frac{\theta(1 - 2x_2)\theta\left(1 - \frac{2m^2}{sx_2}\right)}{(1 - x_2)\left(1 - \frac{m^2}{sx_2}\right)}.$$

Defining

$$\bar{G}_n(s) = \left(\prod_{i=1}^n \int_{m^2/s}^1 \frac{dx_i}{x_i}\right)\bar{g}_n(x_1, \ldots, x_n), \tag{3.5}$$

we obtain

$$\bar{G}_1(s) \equiv \alpha_1 \ln s + \beta_1 = \lambda \ln\frac{s}{m^2}$$

and

$$\bar{G}_2(s) \equiv \alpha_2 \ln s + \beta_2 = -\tfrac{1}{6}\pi^2\lambda^2 . \tag{3.6}$$

Similarly one can show that higher $\bar{G}_n(s)$ are also independent of s; that is, $\alpha_n = 0$ for all $n \geq 2$. By the "cluster-decomposition theorem" as formulated in (2.22), this suggests that the total cross section (normalized by σ_0) should have s dependence given by s^λ.

To verify this result, we must evaluate the total cross section by direct calculation. Calling $p_A' = q_0$ and $p_B' = q_{n+1}$, we may write $\sigma_n(s)$ as

$$\sigma_n(s) = \frac{\lambda^{n+2}}{(n+2)!}\int_0^\infty \left(\prod_{i=0}^{n+1}\frac{dq_i^+}{q_i^+}\right) 2\delta(P^+ - \textstyle\sum q_i^+)\delta(P^- - \sum q_i^+), \tag{3.7}$$

where $P^+ = P^- = \sqrt{s}$. Using standard Laplace-transform techniques previously applied to analyses of $(3+1)$-dimensional phase space[24] and summing over all n, we find the leading behavior of the total cross section as $s \to \infty$ to be

$$\sigma_T(s) = \frac{2e^{-2\gamma\lambda}}{m^4}\frac{(s/m^2)^{\lambda-2}}{[\Gamma(\lambda)]^2} , \tag{3.8}$$

where $\gamma = +0.5772\ldots$ is Euler's constant.

With $\sigma_0 = 2\lambda^2/s^2$, we obtain

$$\frac{\sigma_T(s)}{\sigma_0(s)} = \frac{e^{-2\gamma\lambda}(s/m^2)^\lambda}{[\Gamma(\lambda+1)]^2} . \tag{3.9}$$

As anticipated, the s dependence of the ratio σ_T/σ_0 is simply s^λ.

Notice that in addition to providing a mathematical framework in which to calculate this result, the cluster decomposition approach offers an intuitive understanding of the s^λ behavior. In particular, the analogy to statistical mechanics suggests that the dependence of all physical quantities on the volume should be independent of the surface effects. Since the volume of phase space is $\ln s$, this argument implies that the dependence of the total cross section on $\ln s$ (and hence on s) is independent of the fragmentation regions and can be determined from the form of the $\bar{g}_n$ in the pionization region alone. But in the pionization region, (3.3) implies immediately that $\bar{f}_n \to f_n = \lambda^n$ and thus $\bar{g}_n \to g_n \to 0$, $n \geq 2$, and $\bar{g}_1 \to \lambda$. Thus the s dependence in (3.9) is understood trivially. In addition, by writing (3.9) in the form suggested by (2.22),

$$\frac{\sigma_T}{\sigma_0} \equiv e^{\Sigma\beta_n\lambda^n} s^{\Sigma\alpha_n\lambda^n} = \frac{e^{-2\gamma\lambda}(s/m^2)^\lambda}{[\Gamma(\lambda+1)]^2} , \tag{3.10}$$

and expanding $\beta(\lambda)$ in a power series, one can verify directly that

$$\beta(\lambda) \equiv \beta_1\lambda + \beta_2\lambda^2 + \cdots = -\tfrac{1}{6}\pi^2\lambda^2 + \cdots .$$

This is in agreement with the cluster-decomposition results as shown in (3.6).

By similar manipulations we have calculated the first few terms of the cluster sum rules for τ_1 and τ_2 and verified that they agree with the expansions of the exact forms of these inclusive quantities. Since the calculations become somewhat involved technically, we shall not present the details here.[25] Instead, we shall concentrate on the exact forms of τ_1 and τ_2 to illustrate the effects of purely kinematic constraints in inclusive processes.

Applying the methods used in the derivation of (3.8) to the one- and two-particle inclusive spectra leads to the expressions

$$\frac{d\sigma}{\sigma_T} \equiv \rho_1(x)\frac{dx}{x} = \lambda\frac{dx}{x}(1-x)^{\lambda-1}\left(1-\frac{m^2}{sx}\right)^{\lambda-1} \tag{3.11}$$

and

$$\begin{aligned}\frac{d^2\sigma}{\sigma_T} &\equiv \rho(x_1, x_2)\frac{dx_1}{x_1}\frac{dx_2}{x_2}\\ &= \lambda^2\frac{dx_1}{x_1}\frac{dx_2}{x_2}\left[(1-x_1-x_2)^{\lambda-1}\left(1-\frac{m^2}{sx_1}-\frac{m^2}{sx_2}\right)^{\lambda-1}\theta(1-x_1-x_2)\theta\left(1-\frac{m^2}{sx_1}-\frac{m^2}{sx_2}\right)\right] ,\end{aligned} \tag{3.12}$$

where $x_i = q_i^+/\sqrt{s}$. Since these are leading-order asymptotic results, they are valid for $(1-x)(1-m^2/sx) \gg m^2/s$ and $(1-x_1-x_2)(1-m^2/sx_1 - m^2/sx_2) \gg m^2/s$; that is, the observed particles cannot be too near the boundary of phase space. Physically, these restrictions mean simply that the "missing mass" must be large enough to allow many exclusive channels to contribute to the inclusive processes.

Notice that the presence of factors such as $(1-x)^{\lambda-1}$ implies that the behavior of the spectra depends crucially on λ. In the one-particle spectrum, for example, for $\lambda<1$ we obtain a spectrum which, as a function of $y=\ln(q^+/m)$, is as shown in Fig. 2(a); for $\lambda>1$, however, the shape of the spectrum is qualitatively as in Fig. 2(b). From (3.1) it is clear that increasing λ at fixed s will increase the average multiplicity; the functional form of $\bar{n}(\lambda)$ given in (3.14) illustrates this explicitly. Thus we expect relatively smaller $n(\lambda<1)$ to yield spectra as in Fig. 2(a), whereas relatively larger $\bar{n}(\lambda>1)$ should yield spectra of the form of Fig. 2(b). It is interesting to note that recent numerical studies of longitudinal phase space illustrate this qualitative behavior even at low s.[16] We note also that for all λ the one-particle inclusive spectrum becomes constant in the "central region," as suggested by the gas analogy. Further, the explicit form of the spectrum in this region, $d\sigma/\sigma_T \to \lambda dy$, is a manifestation of a general result discussed in more detail in Sec. IV C.

For the two-particle inclusive correlation function

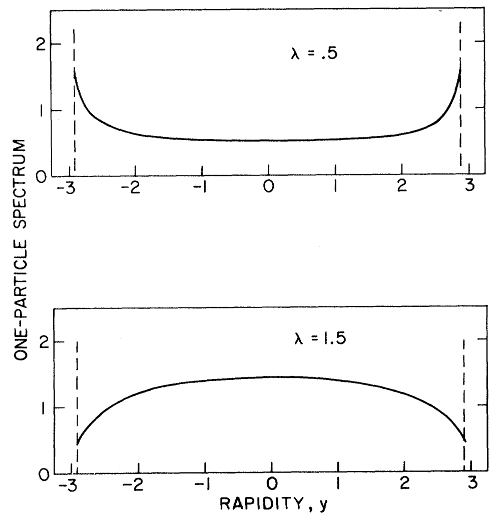


FIG. 2. (a) and (b) The asymptotic form – valid for $(1-x)(1-m^2/sx) \gg m^2/s$ – for the inclusive one-particle spectrum predicted by longitudinal phase space for $\frac{1}{2}\ln(s/m^2) = Y_{\max} = 3$. The solid curve in the central region is the asymptotic spectrum. The vertical dashed lines indicate that, near the ends of phase space, this asymptotic form is no longer valid.

$$\tau_2(x_1, x_2) \equiv \rho_2(x_1, x_2) - \rho_1(x_1)\rho_1(x_2), \tag{3.13}$$

we can observe a similar qualitative distinction between the cases $\lambda<1$ and $\lambda>1$. First, however, notice that for any λ when $x_1+x_2>1$ or $m^2/sx_1+m^2/sx_2>1$, τ_2 is negative definite since[26] $\rho_2=0$. For $x_1+x_2<1$ and $m^2/sx_1+m^2/sx_2<1$, we see that for $\lambda<1$, $\tau_2(x_1, x_2)>0$, whereas for $\lambda>1$, τ_2 remains negative.[27]

For all values of λ, however, the correlation function approaches zero when the momenta are in the central region. This is an illustration of our earlier remarks that kinematic correlations are important only in the fragmentation region.

Finally, we remark that from the form of the total cross section, it is possible to evaluate the moments of the multiplicity distribution predicted by this "pure phase-space" model.[28] Using the definition (3.1) and the result (3.8) we see that

$$\begin{aligned}\bar{n} \equiv \langle n \rangle &= \frac{\lambda}{\sigma_T}\frac{d}{d\lambda}(\sigma_T)\\ &= \lambda \ln\left(\frac{s}{m^2}\right) - 2\gamma\lambda - \lambda\psi(\lambda),\end{aligned} \tag{3.14}$$

where

$$\psi(\lambda) = \frac{d}{d\lambda}[\ln\Gamma(\lambda)].$$

Notice that (3.14) holds for fixed λ as $s\to\infty$.[29] Similarly, since

$$\langle n(n-1)\rangle = \lambda^2 \frac{1}{\sigma_T}\frac{d^2}{d\lambda^2}(\sigma_T), \tag{3.15}$$

we have that

$$\langle n(n-1)\rangle - \langle n\rangle^2 = -\lambda^2 \frac{d\psi(\lambda)}{d\lambda} \neq 0. \tag{3.16}$$

This result establishes that the multiplicity distribution is not Poisson.

C. "Modified" Phase-Space Model

To motivate the "modified" phase-space model let us consider the specific processes in which A, B, A', and B' are protons and all remaining particles are pions. Thus the exclusive processes we treat are

$$p+p \to p+p+n\pi$$

and their inclusive analogs are

$$p+p \to p+p+n\pi+X.$$

The protons and pions are, of course, distinguishable. Experimental results show that the final-state protons in these interactions tend to be "leading" particles; that is, the momentum spectra of the final-state protons are relatively damped when the final-state momenta $p_A'^+$ and $p_B'^-$ are very different from the initial-state momenta p_A^+ and p_B^-. This means, in particular, that the effects of the phase-space factors $1/p_A'^+$ and $1/p_B'^-$, which would tend to enhance the spectrum for small $p_A'^+$ and $p_B'^-$, are canceled by some dynamical mechanism. We can examine the kinematic correlations that arise when this quite general dynamical feature is incorporated by considering an *ad hoc* modification of phase space in which these factors are explicitly removed. Thus we propose to investigate the model[30]

$$\frac{\sigma_n}{\sigma_0}=\frac{\lambda^n}{n!}\int\prod_{i=1}^{n}\frac{dq_i^+}{q_i^+}\frac{dp_A'^+}{p_A'^+}\frac{dp_B'^-}{p_B'^-}(p_A'^+p_B'^-)$$
$$\times\delta\left(\sqrt{s}-p_A'^+-p_B'^+-\sum_{i=1}^{n}q_i^+\right)\delta\left(\sqrt{s}-p_A'^--p_B'^--\sum_{i=1}^{n}\frac{m^2}{q_i^+}\right) \tag{3.17}$$

$$\simeq\frac{\lambda^n}{n!}\int\prod_{i=1}^{n}\frac{dq_i^+}{q_i^+}\theta(\sqrt{s}-\sum q_i^+)\,\theta\left(\sqrt{s}-\sum\frac{m^2}{q_i^+}\right)\equiv\frac{1}{n!}\int\prod_{i=1}^{n}\frac{dq_i^+}{q_i^+}\bar{f}_n(q_1^+,\ldots,q_n)\quad\text{(see Ref. 31)}. \tag{3.18}$$

Here σ_0 is given by $\sigma_0=2\lambda^2/s$.[32]

From (3.18) the lowest-order exclusive cluster functions follow immediately:

$$\bar{g}_1(x)=\lambda\theta(1-x)\theta\left(1-\frac{m^2}{sx}\right),$$
$$\bar{g}_2(x_1,x_2)=\lambda^2\left[\theta(1-x_1-x_2)\theta\left(1-\frac{m^2}{sx_1}-\frac{m^2}{sx_2}\right)-\theta(1-x_1)\theta\left(1-\frac{m^2}{sx_1}\right)\theta(1-x_2)\theta\left(1-\frac{m^2}{sx_2}\right)\right], \tag{3.19}$$

and similarly for higher $\bar{g}_n$.

To discuss the general cluster-decomposition results in a comparative manner we must as usual evaluate the inclusive cross sections directly. Proceeding as in Sec. III B, we find that σ_T/σ_0 is still given by

$$\frac{\sigma_T}{\sigma_0}\underset{s/m^2\to\infty}{\sim}\frac{e^{-2\gamma\lambda}(s/m^2)^\lambda}{[\Gamma(\lambda+1)]^2}. \tag{3.20}$$

It is straightforward to verify that, to lowest orders, the contributions of the $\bar{g}_n$ to the cluster-decomposition theorem for σ_T/σ_0 are equal to the expansion of (3.20) in powers of λ. Note again that the vanishing of the g_n, $n\geq 2$, in the pionization region implies that the s dependence of the total cross section is determined solely by g_1.

From (3.20) we see that the average multiplicity in the "modified" phase-space model is also identical to that in the pure phase-space model and is thus given by (3.14). In treating the inclusive spectra, however, we must recall that p_A' and p_B' are distinguished from the "additional" particles. Hence in the "modified" model the inclusive spectra of the "protons" will differ from those of the other secondary particles.

Considering first the spectrum of the "pions," we find that

$$\tau_1(x)\frac{dx}{x}=\frac{d\sigma}{\sigma_T}=\lambda\frac{dx}{x}(1-x)^\lambda\left(1-\frac{m^2}{sx}\right)^\lambda, \tag{3.21}$$

for $(1-x)(1-m^2/sx)\gg m^2/s$. Notice that this "pion" spectrum, although similar in structure to the spectrum in the pure phase-space model, is of the form shown in Fig. 2(b) for all values of λ. It cannot exhibit the "peaking" effect illustrated in Fig. 2(a).

Extension of this approach to the two-particle spectrum establishes that the correlation function for the production of two pions is

$$\tau_2(x_1, x_2) = \lambda^2\left[(1-x_1-x_2)^\lambda\left(1-\frac{m^2}{sx_1}-\frac{m^2}{sx_2}\right)^\lambda \theta(1-x_1-x_2)\theta\left(1-\frac{m^2}{sx_1}-\frac{m^2}{sx_2}\right) - (1-x_1)^\lambda\left(1-\frac{m^2}{sx_1}\right)^\lambda(1-x_2)^\lambda\left(1-\frac{m^2}{sx_2}\right)^\lambda\right]. \tag{3.22}$$

For explicit form of the first term to be valid, we require $(1-x_1-x_2)(1-m^2/sx_1-m^2/sx_2) \gg m^2/s$. We note that, since $\lambda > 0$, this correlation function is negative definite. Further, when both momenta are in the central region, τ_2 approaches zero.

To facilitate the illustration of the cluster-function sum rules in this model, we shall restrict our considerations to the form of the spectra in the fragmentation region of one of the incident particles; let us choose the beam fragmentation region. In this case the θ function arising from the constraint on the minus components of momenta can be ignored, and the model cross sections reduce to

$$\frac{\sigma_n}{\sigma_0} = \frac{\lambda^n}{n!}\int \frac{dp_A'^+}{p_A'^+} p_A'^+ \prod_{i=1}^n \frac{dq_i^+}{q_i^+}\delta(\sqrt{s} - p_A'^+ - \sum q_i^+)$$

$$\simeq \frac{\lambda^n}{n!}\int \prod_{i=1}^n \frac{dq_i^+}{q_i^+}\theta(\sqrt{s} - \sum q_i^+). \tag{3.23}$$

With this simplification the calculational details will be more transparent and a more thorough study of the sum rules will be possible.

From (3.23) we find that the "total" cross section in this "single θ-function model" is

$$\frac{\sigma_T}{\sigma_0} \underset{s\to\infty}{\sim} \frac{e^{-\gamma\lambda}(s/m^2)^\lambda}{\Gamma(\lambda+1)} . \tag{3.24}$$

Further, the one- and two-particle inclusive correlation functions become

$$\tau_1(x) = \lambda(1-x)^\lambda \tag{3.25}$$

and

$$\tau_2(x_1, x_2) = \lambda^2[(1-x_1-x_2)^\lambda\theta(1-x_1-x_2) - (1-x_1)^\lambda(1-x_2)^\lambda]. \tag{3.26}$$

Notice that for large s the difference between the spectra predicted by the "single θ-function model" and by the full "two θ-function model" is indeed negligible in the beam fragmentation region. In Fig. 3, for example, we plot the two different forms for the one-particle spectra at an energy appropriate to conventional accelerators; the curves are virtually identical over a wide range in the rapidity plot. This provides another illustration that kinematic constraints produce important effects only in the fragmentation regions.

From (3.23) we can evaluate the first few exclusive cluster functions for the production of pions. We obtain

$$\bar{g}_1(x) = \lambda\theta(1-x),$$

$$\bar{g}_2(x_1, x_2) = \lambda^2[\theta(1-x_1-x_2) - \theta(1-x_1)\theta(1-x_2)],$$

and

$$\bar{g}_3(x_1, x_2, x_3) = \lambda^3[\theta(1-x_1-x_2-x_3) - \theta(1-x_1-x_2) - \theta(1-x_1-x_3) - \theta(1-x_2-x_3) + 2]. \tag{3.27}$$

The contributions of these $\bar{g}_n$ to the sum rules can now be evaluated explicitly. We obtain

$$\bar{g}_1(x) + \int_{m^2/s}^1 \frac{dx_1}{x_1}\bar{g}_2(x, x_1) + \frac{1}{2!}\int_{m^2/s}^1\int_{m^2/s}^1 \frac{dx_1}{x_1}\frac{dx_2}{x_2}\bar{g}_3(x, x_1, x_2) + \cdots = \lambda + \lambda^2\ln(1-x) + \frac{\lambda^3}{2!}\ln^2(1-x) + \cdots . \tag{3.28}$$

From (3.25), we see that to this order the inclusive one-particle spectrum for the production of pions is

$$\bar{\tau}_1(x) = \lambda(1-x)^\lambda = \lambda + \lambda^2\ln(1-x) + \frac{\lambda^3}{2!}\ln^2(1-x) + \cdots$$

and hence the sum rule for τ_1 is verified. Similarly, the sum rule for τ_2 requires

$$\bar{\tau}_2(x_1, x_2) = \bar{g}_2(x_1, x_2) + \int_{m^2/s}^{1} \bar{g}_3(x_1, x_2, x_3)\frac{dx_3}{x_3} + \cdots$$
$$= \lambda^2[\theta(1-x_1-x_2) - \theta(1-x_1)\theta(1-x_2)]$$
$$+\lambda^3[\theta(1-x_1-x_2)\ln(1-x_1-x_2) - \theta(1-x_1)\ln(1-x_1) - \theta(1-x_2)\ln(1-x_2)] + \cdots \quad . \tag{3.29}$$

A direct expansion of τ_2 as given by (3.26) verifies this result.

At this juncture there are several important comments to be made. First, in both (3.28) and (3.29) the contributions of the individual exclusive cluster functions appear to be logarithmically singular near the phase-space boundary. This is a consequence of the asymptotic approximations in (3.24) through (3.27) which are valid to $O(m^2/s)$. Very near the phase-space boundary – that is, for $x \simeq m^2/s$ or $(1-x) \simeq m^2/s$ – these approximations break down, and the explicit forms of both the inclusive spectra in (3.25) and (3.26) and the exclusive contributions in (3.28) and (3.29) are modified. Thus although they may be strongly peaked near the end of phase space, the actual exclusive contributions to the sum rules for τ_1 and τ_2 are not singular. Further, the dynamics can reduce this strong peaking effect by "softening" the abrupt phase-space cutoff.[33] In II, for example, we established that in a multiperipheral model the propagators of the exchanged particles provided damping such that the individual exclusive contributions to the cluster sum rules were smooth – that is, neither apparently singular nor sharply peaked – near the phase-space boundary.

Second, both the spectra calculated in (3.25) and (3.26), and the exclusive clusters, given by (3.27), apply, as previously noted, only to the pions. There is a particularly informative way of seeing that (3.25) cannot represent the full one-particle spectrum. Several authors have recently emphasized that the general constraints of energy-momentum conservation together with the definition of the inclusive one-particle spectrum require that[34-36]

$$\sum_i \int_{m^2/s}^{1} \frac{1}{\sigma_T}\frac{d\sigma}{(dx_i/x_i)} x_i \frac{dx_i}{x_i} = 1, \tag{3.30}$$

where the sum over i is over all types of particles produced. Since we have calculated inclusive cross sections explicitly as sums over exclusive ones, (3.30) becomes a consistency check.[35] From (3.25) we see that the contribution of the pion spectrum to (3.30) is

$$\lambda\int_{m^2/s}^{1}(1-x)^{\lambda}dx = \frac{\lambda}{\lambda+1} < 1, \tag{3.31}$$

and thus there is an additional contribution necessary to yield (3.30). This is, of course, just the term arising from the spectrum of the "proton." To calculate this spectrum in the single θ-function model we must return to (3.23). From this result we see that

$$\frac{d\sigma^n}{\sigma_0} = dp_A'^{+}\frac{\lambda^n}{n!}\int\prod_{i=1}^{n}\frac{dq_i^+}{q_i^+}\delta(\sqrt{s} - p_A'^{+} - \textstyle\sum q_i^+). \tag{3.32}$$

By manipulations similar to those used in the treatment of the previously discussed inclusive spectra, we find

$$\left(\frac{(d\sigma)}{\sigma_T}\right)_{\text{"proton"}} = \lambda\frac{dx_A}{x_A}x_A(1-x_A)^{\lambda-1}, \tag{3.33}$$

where $x_A \equiv p_A'^{+}/\sqrt{s}$. Notice that, as anticipated, this spectrum is suppressed for small x_A relative to (3.25). The contribution to (3.30) from (3.33) is

$$\lambda\int_{m^2/s}^{1}x_A(1-x_A)^{\lambda-1}dx_A = \frac{1}{\lambda+1}, \tag{3.34}$$

and thus the consistency requirement is satisfied.

Notice further that the average multiplicity of the "protons" is

$$(\bar{n})_{\text{"proton"}} = \lambda\int_{m^2/s}^{1}\frac{dx}{x}x(1-x)^{\lambda-1}$$
$$= 1 \tag{3.35}$$

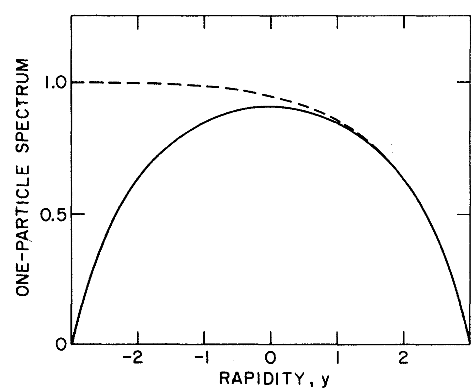


FIG. 3. The one-particle inclusive spectra predicted by the "single-function" model – dashed curve – and the "two θ-function" model – solid curve – as a function of rapidity for $\lambda = 1$ and $Y_{\max} = \frac{1}{2}\ln(s/m^2) = 3$. Notice that in the beam fragmentation and central regions the curves are indistinguishable.

independent of λ. This is, of course, the expected result since in the single θ-function model there is only one "distinguished" particle in any exclusive cross section. In addition, from (3.33), we see that

$$\bar{n}_{\text{secondaries}} = \lambda \int_{m^2/s}^{1} \frac{dx}{x} (1-x)^{\lambda}$$
$$= \lambda \ln(s/m^2) - \gamma\lambda - \lambda\psi(\lambda) - 1. \qquad (3.36)$$

Combining (3.35) and (3.36), we find that the total average multiplicity is $\bar{n} = \lambda \ln(s/m^2) - \gamma\lambda - \lambda\psi(\lambda)$, which also follows by calculating $\bar{n}$ from

$$\bar{n} = \frac{1}{\sigma_T} \lambda \frac{d}{d\lambda} \sigma_T$$

using σ_T as given by (3.24).

Finally, it is of interest to compare the predictions of the "modified" phase-space model with recent experimental results on inclusive spectra. For this purpose we return to the full two θ-function modified phase-space model, as its results can be applied over virtually the whole range of phase space. Consider the one-particle inclusive spectrum, which the model predicts should be given by (3.21). The experimental data shown in Fig. 4 are taken from an analysis of the reaction

$$\pi^+ p \to \pi^- + X$$

with incident π^+ momentum = 18.5 GeV/c.[37] Since the observed π^- is not present in the initial state, the "leading" particle effect is removed. Hence (3.21) may be applied. To compare the simple model with the data, however, we must recall that the experimental longitudinal-momentum spectra include proper integrations over the transverse momenta, whereas the model is formulated in (1+1) dimensions. For our present purposes, it will be sufficient to replace m^2 in (3.21) by $\langle m_T^2 \rangle = m^2 + \langle q^2 \rangle$, where $\langle q^2 \rangle$ is the average value of the square of the transverse momentum.[38] With this replacement one obtains the curve shown in Fig. 4 for the model spectrum. Since the comparison is intended solely to illustrate qualitative features, we have taken $\lambda = 1$ for simplicity; this gives, by (3.14), a total average multiplicity $\bar{n} = 6$, which is roughly correct at this energy when one allows for unobserved neutrals. We observe that the data fall off more rapidly at large $|y|$ than the pure kinematic constraints require. Indeed, whereas the model spectrum shows the nearly flat central region expected by the gas analogy, this effect is not present in the data. Although increasing λ would make the model curve narrower and hence more like the data, it would also lead to an average multiplicity higher than that observed experimentally.

In Fig. 5 we plot the two-particle correlations which follow from the modified phase space model and compare these to a sketch of the data for the process $\pi^+ p \to \pi^-\pi^- X$ with incident π^+momentum = 18.5 GeV/c.[37] The data clearly show structure quite different from that required by the simple kinematic constraints. In particular, τ_2 in the model, as anticipated, approaches zero in the central region[39]; the data, however, exhibit strong positive correlations for $y_1 \approx y_2 \approx 0$. Thus even our rough considerations suggest that kinematic constraints alone do not control the one-particle

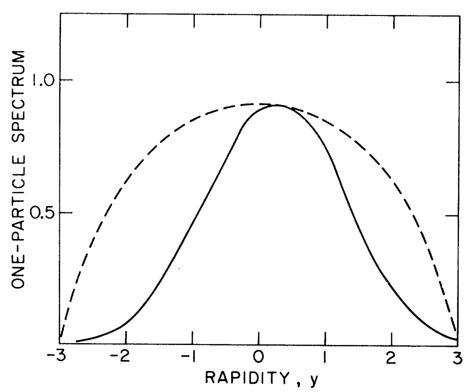


FIG. 4. A rough comparison of the experimental one-particle inclusive spectrum – solid curve – for $\pi^+ p \to \pi^- X$ at $p_L = 18.5$ GeV/c as reported in Ref. 37 with the spectrum calculated from the "modified" phase-space model – dashed curve – with $\lambda = 1$ and $Y_{\max} = \frac{1}{2}\ln(s/m^2) = 3$. The curves are normalized to the same height; hence the vertical scale is arbitrary.

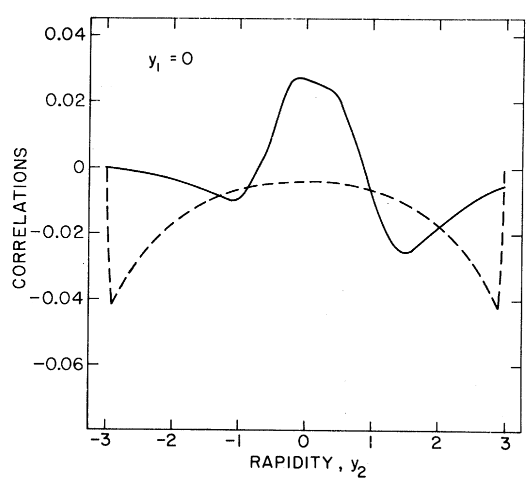


FIG. 5. A rough comparison of the experimental $\pi^-\pi^-$ correlations – solid curve – in the process $\pi^+ p \to \pi^-\pi^- + X$ at $p_L = 18.5$ GeV/c as reported in Ref. 37 with the correlations predicted from the "modified" phase-space model – dashed curve – with $\lambda = 1$ and $Y_{\max} = \frac{1}{2}\ln(s/m^2) = 3$. The curves shown are plotted as functions of y_2 at fixed $y_1 = 0$.

inclusive spectrum; there must be significant dynamical effects.

To extract in a quantitative manner the dynamical effects from present data one clearly requires a more detailed knowledge of the structure imposed by kinematic constraints in specific inclusive processes. Given such knowledge, one could analyze the experimental results in terms of a kinematic "background" and dynamical contributions. However, the thorough, quantitative analysis of kinematic constraints required for this approach is beyond the scope of the present paper. We intend to treat this problem elsewhere.[33]

Happily, to study dynamical effects in simple theoretical models there is an alternative to the above approach. Since the kinematic constraints vanish in the pionization region,[12] for which $1 \gg x_i \gg m^2/s$, we can isolate dynamical effects by analyzing the inclusive spectra in this region. In the next section we shall use this second approach to investigate possible simple forms for the dynamical effects in multiparticle spectra.

IV. DYNAMICAL MODELS

A. General Considerations

From the preceding section we can extract two immediate conclusions. First, the kinematical constraints on final-state momenta arising from energy-momentum conservation can indeed be incorporated within the framework of the cluster decomposition. Second, these constraints induce important correlations among particles only in the fragmentation regions; at very high energies particles produced in the pionization region are not correlated kinematically. Mathematically, this result is expressed by considering the form of the exclusive spectra in the pure kinematic models in the limit in which $x_i = q_i^+/\sqrt{s} \to 0$. One finds immediately that

$$\bar{f}_n \to f_n \to \lambda^n, \tag{4.1}$$

so that

$$\bar{g}_n \to g_n \to 0, \quad n > 1. \tag{4.2}$$

Hence, from the sum rules,

$$\bar{\tau}_n \to \tau_n \to 0, \quad n > 1. \tag{4.3}$$

These results suggest that, at least in preliminary theoretical investigations, the correlations arising from the underlying dynamical interaction can and should be isolated by considering dynamical models in the pionization region only. This restriction, although it provides a considerable simplification, is not a limitation of the approach, for the behavior of any specific dynamical model in the fragmentation region, in which kinematical and dynamical correlations coexist, can be studied by the same techniques introduced here.

To illustrate the use of the cluster-decomposition analysis and of the sum rules relating inclusive and exclusive spectra, we shall study two simple dynamical models for the exclusive differential cross sections. The first model is a version of the Chew-Pignotti model,[6] which has been discussed elsewhere in the literature.[7] This model is ideally suited to our purpose, for, although it is quite simple, many of its features are shared by more sophisticated multi-Regge models. The second model is *ad hoc* in the sense that it cannot be derived from an underlying theoretical framework. However, in its qualitative features, it is very similar to the $\lambda\varphi^3$ ladder models which have been studied previously in this context.[3,9,40]

As in Sec. III, we shall formulate specific models for the exclusive differential cross sections in terms of a longitudinal phase space only. We shall further integrate over two of the momenta in the final state; this will replace the δ functions of energy-momenta conservation by θ functions restricting the sum of the $q_i^\pm$. In the pionization region, these θ functions can then be ignored. In calculating cross sections, therefore, each of the momenta can be integrated independently over the region $\epsilon\sqrt{s} > q_i^+ > m^2/\epsilon'\sqrt{s}$ in the center-of-mass system.[12]

Finally, to motivate the explicit form of the *ad hoc* model, we recall that the property crucial to the "cluster-decomposition" approach is "short-range order" in momentum space, realized explicitly by factorizable differential exclusive cross sections. In its simplest nontrivial form, "short-range order" suggests that the differential exclusive cross sections depend explicitly only on the separation between adjacent or "nearest-neighbor" momenta. Indeed, in I we established that in a certain approximation the exclusive cross sections corresponding to φ^3 ladder diagrams did become dependent only on "nearest-neighbor" separations. Thus as a second example of the applications of the cluster technique, we shall study a "φ^3-type" model in which the n-particle exclusive differential spectrum in the pionization region is given by

$$\frac{d^n\sigma^n/\sigma_0}{d\Phi_n} \equiv f_n(q_1^+, \ldots, q_n^+) = \lambda^n\left(1 - \frac{q_2^+}{q_1^+}\right)\left(1 - \frac{q_3^+}{q_2^+}\right)\cdots\left(1 - \frac{q_n^+}{q_{n-1}^+}\right) \tag{4.4}$$

in the region in which $q_1^+ \geq q_2^+ \geq \cdots \geq q_n^+$. Since $d^n\sigma^n/\sigma_0$ is a symmetric function in all its arguments, the extensions of (4.4) to other regions of phase space are immediate.

B. Quantitative Discussion

For clarity of exposition we shall divide the discussion of the dynamical models into two parts. In the present section, we examine the basic quantitative aspects of the models and calculate the exclusive and inclusive correlation functions we wish to study. In addition, we verify explicitly the cluster function sum rules through the first few orders. Thus the thrust of this section is primarily quantitative. In Sec. IV C, we discuss and compare the qualitative features of the models and interpret these in physical terms.

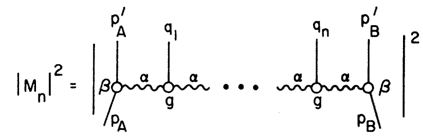


FIG. 6. A schematic representation of the simplified Chew-Pignotti model showing the particle-particle-Reggeon coupling constant β and the Reggeon-Reggeon-particle coupling constant g.

1. *The Chew-Pignotti Model*

The simplified version of the Chew-Pignotti model, formulated directly in terms of longitudinal phase space only, asserts that the cross section for producing n additional particles can be written as

$$\sigma_n\left(\frac{s}{m^2}\right)=\frac{1}{2s}\int |M_n|^2 d\Phi_{n+2}$$
$$=\frac{1}{2s}\,\frac{1}{2^{n+1}}\int_{p_A'^+\ge q_i^+\cdots\ge q_n^+\ge p_B'^+}|M_n|^2\,\frac{dp_A'^+}{p_A'^+}\,\frac{dp_B'^-}{p_B'^-}\left(\prod_{i=1}^n \frac{dq_i^+}{q_i^+}\right)$$
$$\times\,\delta\left(\sqrt{s}-p_A'^+-p_B'^+-\sum_{i=1}^n q_i^+\right)\delta\left(\sqrt{s}-p_A'^--p_B'^--\sum_{i=1}^n q_i^-\right). \quad (4.5)$$

As in Secs. II and III, we have chosen to exploit the symmetry of $|M_n|^2$ under interchange of the identical final-state particles to write the integral in (4.5) over an ordered phase space.[41] In the region of phase space in which $p_A'^+\geqslant q_i^+\geqslant\cdots\geqslant q_n^+\geqslant p_B'^+$, the matrix element M_n is given by

$$|M_n|^2=\beta^4 g^{2n}\prod_{i=0}^n\left(\frac{s_{i,i+1}}{m^2}\right)^{2\alpha}, \quad (4.6)$$

where $q_0\equiv p'_A$, $q_{n+1}\equiv p'_B$, and

$$s_{i,i+1}\equiv(q_i+q_{i+1})^2=m^2\left(\frac{q_i^+}{q_{i+1}^+}\right)\left(1+\frac{q_{i+1}^+}{q_i^+}\right)^2. \quad (4.7)$$

The coupling constants β and g are defined in Fig. 6. Using (4.7), we may recast (4.6) as

$$|M_n|^2=\beta^4 g^{2n}\left(\frac{p_A'^+p_B'^-}{m^2}\right)^{2\alpha}\left(1+\frac{q_i^+}{p_A'^+}\right)^{4\alpha}\left(1+\frac{m^2}{p_B'^-q_n^+}\right)^{4\alpha} f_n(q_1^+,\ldots,q_n^+). \quad (4.8)$$

Here

$$f_n(q_1^+,\ldots,q_n^+)=\lambda^n\left(1+\frac{q_2^+}{q_1^+}\right)^{4\alpha}\cdots\left(1+\frac{q_n^+}{q_{n-1}^+}\right)^{4\alpha} \quad (4.9)$$

and $\lambda=g^2/2$.

In (4.8) we see explicitly that the factor $(p_A'^+p_B'^-)^{2\alpha}$ provides the type of "dynamical damping" which we incorporated into the "modified" phase-space model of Sec. III. Hence this dynamical result offers *a posteriori* support for our previously *ad hoc* modification.

Applying the arguments detailed in Sec. II to integrate over the "leading-particle" momenta $p_A'^+$ and $p_B'^+$, we find immediately that

$$\sigma_n\left(\frac{s}{m^2}\right)\simeq\frac{\beta^4}{4s}\int_{q_1^+\ge\cdots\ge q_n^+}\left(\prod_{i=1}^n\frac{dq_i^+}{q_i^+}\right)\frac{\theta(\sqrt{s}-q_1^+-\sum_{i=1}^n q_i^+)}{\bar{p}_A'^+\bar{p}_B'^-}\,\theta\left(\sqrt{s}-q_n^--\sum_{i=1}^n q_i^-\right)\frac{(\bar{p}_A'^+\bar{p}_B'^-)^{2\alpha}}{m^2}$$
$$\times\left(1+\frac{q_1^+}{\bar{p}_A'^+}\right)^{4\alpha}\left(1+\frac{m^2}{\bar{p}_B'^-q_n^+}\right)^{4\alpha} f_n(q_1^+,\ldots,q_n^+). \quad (4.10)$$

Here

$$\bar{p}_A'^{+} \simeq \sqrt{s} - \sum_{i=1}^{n} q_i^+ \text{ and } \bar{p}_B'^{-} \simeq \sqrt{s} - \sum_{i=1}^{n} q_i^- .$$

To isolate the dynamical from the kinematical correlations, we shall study (4.10) in the "pionization region." Then

$$\bar{p}_A'^{+} \to \sqrt{s}\,,\quad \bar{p}_B'^{-} \to \sqrt{s}\,,\quad \left(1+\frac{q_1^+}{\bar{p}_A'^{+}}\right)\to 1, \quad\text{and}\quad \left(1+\frac{m^2}{\bar{p}_B'^{-}q_n^+}\right)\to 1.$$

Further, the θ functions reflecting the kinematic constraints on the q_i may be ignored; that is, each q_i^+ may be integrated independently over the pionization region. In terms of rapidity, the length of the pionization region is

$$L = Y_{\max} + \ln(\epsilon\epsilon')$$

where $Y_{\max} = \ln(s/m^2)$ and ϵ and ϵ' are constants independent of s. In I we established that in models satisfying (2.16) the actual total cross section has the same dependence on $Y_{\max}$ that the cross section calculated in the pionization region does on L; that is, after combining the contributions from the fragmentation and pionization regions, the $\epsilon\epsilon'$ dependence vanishes. We have further seen an explicit example of this result in the behavior of the total cross sections in the kinematical models of Sec. III. Thus for simplicity we shall henceforth ignore the distinction between L and $Y_{\max}$. We may then write

$$\sigma_n\left(\frac{s}{m^2}\right) = \frac{\beta^4}{4}\left(\frac{s}{m^2}\right)^{2\alpha-2}\int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\int_{m^2/\sqrt{s}}^{q_1^+}\frac{dq_2^+}{q_2^+}\cdots\int_{m^2/\sqrt{s}}^{q_{n-1}^+}\frac{dq_n^+}{q_n^+}\, f_n(q_1^+,\ldots,q_n^+)\,. \tag{4.11}$$

Finally, using $\sigma_0(s/m^2) = \frac{1}{4}\beta^4(s/m^2)^{2\alpha-2}$, we find

$$\frac{\sigma_n}{\sigma_0}\left(\frac{s}{m^2}\right) = \int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\int_{m^2/\sqrt{s}}^{q_1^+}\frac{dq_2^+}{q_2^+}\cdots\int_{m^2/\sqrt{s}}^{q_{n-1}^+}\frac{dq_n^+}{q_n^+}\, f_n(q_1^+,\ldots,q_n^+)\,. \tag{4.12}$$

Given the form of f_n in (4.9), the fundamental factorization requirement (2.16) is manifest. Hence this model is directly amenable to the cluster-decomposition approach.

Before presenting the details of this analysis, however, we should point out that the multi-Regge form of the matrix element M_n shown in (4.6) allows two different physical interpretations. One interpretation argues that since the dominance of a single Regge exchange is valid only for large energy, the matrix element M_n must for consistency be considered in the "strong-ordered" limit, which, in terms of our variables, requires $q_i^+ \gg q_{i+1}^+$. This means that (4.7) should read

$$s_{i,i+1} \simeq m^2\left(\frac{q_i^+}{q_{i+1}^+}\right) \tag{4.13}$$

and the subsequent equations should be altered accordingly. The second interpretation takes the form of M_n given in (4.6) as an ansatz valid over all phase space.[42] Since, as we shall shortly see, each of these versions of the model helps to clarify certain aspects of correlations in inclusive and exclusive spectra, we shall treat them both.

a. Strong-ordered limit. In the "strong-ordered" limit, the matrix element M_n becomes, using (4.13) and canceling common factors,

$$|M_n|^2 = \beta^4 g^{2n}\left(\frac{p_A'^{+}p_B'^{-}}{m^2}\right)^{2\alpha}. \tag{4.14}$$

Hence, after integrating over the "leading" particle momenta and restricting consideration to the pionization region, we find that

$$\frac{\sigma_n}{\sigma_0}\left(\frac{s}{m^2}\right) = \lambda^n\int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\int_{m^2/\sqrt{s}}^{q_1^+}\frac{dq_2^+}{q_2^+}\cdots\int_{m^2/\sqrt{s}}^{q_{n-1}^+}\frac{dq_n^+}{q_n^+}\times 1\,. \tag{4.15}$$

This equation can be integrated immediately to give a Poisson distribution for the partial cross sections,

$$\frac{\sigma_n}{\sigma_0}\left(\frac{s}{m^2}\right) = \frac{[\lambda\ln(s/m^2)]^n}{n!}\,. \tag{4.16}$$

But since "a Poisson distribution arises from uncorrelated emission," we anticipate that in this simplified limit all the correlation functions, both inclusive and exclusive, should vanish. From the cluster-decomposition analysis it is trivial to verify that this is indeed the case. First, we observe that the expression for the partial cross section in (4.15) is in the form of (2.15) with all the $f_n = \lambda^n$.[43] Hence from (2.17), all $g_n = 0$, trivially, for $n \geq 2$, and $g_1 = \lambda$. Thus the sum rules relating inclusive to exclusive correlations show that $\tau_n = 0$, $n \geq 2$, and $\tau_1 = \lambda$, and the cluster decomposition instantly verifies our intuition.

To establish directly that there are no correlations in the inclusive spectra and therefore to verify the validity of the cluster sum rules, we can evaluate these spectra explicitly. We find for the inclusive one-particle spectra

$$\begin{aligned}\frac{d\sigma}{\sigma_0} &= \lambda \frac{dq}{q} \sum_{m,n} \left(\lambda^{m-1} \int_q^{\sqrt{s}} \frac{dq_1^+}{q_1^+} \int_q^{q_1^+} \frac{dq_2^+}{q_2^+} \cdots \int_q^{q_{m-2}} \frac{dq_{m-1}^+}{q_{m-1}^+} \right) \left(\lambda^{n-m} \int_{m^2/\sqrt{s}}^{q} \frac{dq_{m+1}^+}{q_{m+1}^+} \cdots \int_{m^2/\sqrt{s}}^{q} \frac{dq_n^+}{q_n^+} \right) \\ &= \lambda \frac{dq}{q} \sum_{1 \leq m \leq n} \frac{[\lambda \ln(\sqrt{s}/q)]^{m-1}}{(m-1)!} \frac{[\lambda \ln(\sqrt{s}\, q/m^2)]^{n-m}}{(n-m)!} \\ &= \lambda \frac{dq}{q} s^{\lambda} . \end{aligned} \tag{4.17}$$

Using the result that $\sigma_T/\sigma_0 = s^\lambda$, which follows immediately from (4.16), we have

$$\tau_1 \frac{dq}{q} \equiv \rho_1 \frac{dq}{q} = \frac{d\sigma}{\sigma_T} = \lambda \frac{dq}{q} , \tag{4.18}$$

as asserted. Similarly, the two-particle inclusive spectrum for $p_1^+ \geq p_2^+$ follows from

$$\begin{aligned}\frac{d^2\sigma}{\sigma_0} &= \lambda^2 \frac{dp_1}{p_1} \frac{dp_2}{p_2} \sum_{l,m,n} \left(\lambda^{l-1} \int_{p_1}^{\sqrt{s}} \frac{dq_1^+}{q_1^+} \cdots \int_{p_1}^{q_{l-2}^+} \frac{dq_{l-1}^+}{q_{l-1}^+} \right) \\ &\quad \times \left(\lambda^{m-l-1} \int_{p_2}^{p_1} \frac{dq_{l+1}^+}{q_{l+1}^+} \int_{p_2}^{q_{l+1}^+} \frac{dq_{l+2}^+}{q_{l+2}^+} \cdots \int_{p_2}^{q_{m-2}^+} \frac{dq_{m-1}^+}{q_{m-1}^+} \right) \\ &\quad \times \left(\lambda^{n-m} \int_{m^2/\sqrt{s}}^{p_2} \frac{dq_{m+1}^+}{q_{m+1}^+} \int_{m^2/\sqrt{s}}^{q_{m+1}^+} \frac{dq_{m+2}^+}{q_{m+2}^+} \cdots \int_{m^2/\sqrt{s}}^{q_{n-1}^+} \frac{dq_n^+}{q_n^+} \right) \\ &= \lambda^2 \frac{dp_1}{p_1} \frac{dp_2}{p_2} \sum_{1 \leq l < m < n} \frac{[\lambda \ln(\sqrt{s}/p_1)]^{l-1}}{(l-1)!} \frac{[\lambda \ln(p_1/p_2)]^{m-l-1}}{(m-l-1)!} \frac{[\lambda \ln(\sqrt{s}\, p_2/m^2)]^{n-m}}{(n-m)!} \\ &= \lambda^2 \frac{dp_1}{p_1} \frac{dp_2}{p_2} \sum_{m<n} \frac{[\lambda \ln(\sqrt{s}/p_2)]^{m-1}}{(n-m)!} \frac{[\lambda \ln(\sqrt{s}\, p_2/m^2)]^{n-m}}{(n-m)!} \\ &= \lambda^2 \frac{dp_1}{p_1} \frac{dp_2}{p_2} \sum_{n=2}^{\infty} \frac{(\lambda \ln s)^{n-2}}{(n-2)!} \\ &= \lambda^2 s^{\lambda} \frac{dp_1}{p_1} \frac{dp_2}{p_2} . \end{aligned} \tag{4.19}$$

Hence, with $x_i = p_i^+/s$,

$$\begin{aligned}\rho_2(x_1, x_2) \frac{dx_1}{x_1} \frac{dx_2}{x_2} &\equiv \frac{d^2\sigma}{\sigma_T} = \lambda^2 \frac{dx_1}{x_1} \frac{dx_2}{x_2} \\ &= \tau_1(x_1) \tau_1(x_2) \frac{dx_1}{x_1} \frac{dx_2}{x_2} , \end{aligned} \tag{4.20}$$

and therefore $\tau_2 = 0$ by explicit calculation. Thus the strong-ordered version of the Chew-Pignotti model yields no dynamical correlations among the final-state momenta.[44] To investigate nontrivial correlations we must turn to the second version of the model.

Finally, although we have restricted our considerations to the form of the inclusive spectra in the pionization region, it is clear that the similarity of $|M_n|^2$ given by (4.14) to the matrix element in the "modified" phase-space model allows us immediately to determine the spectra in the fragmentation regions as well. Since the resulting spectra are essentially identical to those discussed in Sec. III C, we shall not treat them explicitly here.

b. General Case. If we interpret the f_n given by (4.9) as an ansatz for the form of the normalized exclusive spectra in all regions of phase space, then the total cross section is given by

$$\frac{\sigma_T(s/m^2)}{\sigma_0} = \sum_n \int_{m^2/\sqrt{s}}^{\sqrt{s}} \frac{dq_1^+}{q_1^+} \int_{m^2/\sqrt{s}}^{q_1^+} \frac{dq_2^+}{q_2^+} \cdots \int_{m^2/\sqrt{s}}^{q_{n-1}^+} \frac{dq_n^+}{q_n^+} f_n(q_1^+, \ldots, q_n^+)$$
$$= \sum_n \lambda^n \int_{m^2/\sqrt{s}}^{\sqrt{s}} \frac{dq_1^+}{q_1^+} \int_{m^2/\sqrt{s}}^{q_1^+} \frac{dq_2^+}{q_2^+} \cdots \int_{m^2/\sqrt{s}}^{q_{n-1}^+} \frac{dq_n^+}{q_n^+} \left(1+\frac{q_2^+}{q_1^+}\right)^a \cdots \left(1+\frac{q_n^+}{q_{n-1}^+}\right)^a , \tag{4.21}$$

where $a = 4\alpha$.

The sum in (4.21) can be evaluated by standard integral equation and Laplace-transform techniques; we treat the details in the Appendix. The result is that the total cross section has a leading asymptotic behavior of the Regge form

$$\frac{\sigma_T}{\sigma_0}\left(\frac{s}{m^2}\right) = c(\lambda)\left(\frac{s}{m^2}\right)^{\xi_0(\lambda)} , \tag{4.22}$$

where $\xi_0(\lambda)$ is the leading eigenvalue of the eigenvalue equation

$$1 = \lambda \int_0^1 dy\, y^{\xi-1}(1+y)^a \equiv \lambda f(\xi, a). \tag{4.23}$$

As shown in the Appendix, the constant $c(\lambda)$ can be written as

$$c(\lambda) = \frac{1}{\xi_0^2(\lambda)|f_0'|} , \tag{4.24}$$

where

$$f_0' \equiv \left.\frac{\partial f(\xi, a)}{\partial \xi}\right|_{\xi=\xi_0} .$$

Notice that the "total cross section" given in (4.22) has in fact been calculated from the behavior of the exclusive cross sections in the pionization region. Both the general cluster-decomposition results of I and II and the preceding specific examples in Sec. III establish that the actual total cross section will be modified by multiplicative constants dependent on λ which arise from the contributions of the fragmentation regions. The s dependence, of course, is correctly given by (4.22).

A further consequence of our restriction to the pionization region is that we can determine $\bar{n}$ only to within a constant dependent on λ. From (4.22) we have, using (3.14),

$$\bar{n} = \lambda \frac{d\xi_0}{d\lambda} \ln\left(\frac{s}{m^2}\right) + \varphi(\lambda), \tag{4.25}$$

where $\varphi(\lambda)$ is essentially determined by the behavior in the fragmentation region.

Evaluating the inclusive one-particle spectrum, we find using techniques discussed in the Appendix, that

$$\frac{d\sigma}{\sigma_T} \equiv \tau_1(x)\frac{dx}{x} \equiv \rho_1(x)\frac{dx}{x}$$
$$= \frac{dx}{x}\frac{1}{\lambda|f_0'|}$$
$$= \lambda \frac{d\xi_0(\lambda)}{d\lambda}\frac{dx}{x} . \tag{4.26}$$

The final equality follows directly by differentiating (4.23).[45] As discussed in I, the result that the one-particle spectrum in the pionization region is related to the derivative of the leading Regge trajectory function with respect to the coupling constant is common to all models in which the cluster decomposition is valid and in which an nth-order cluster contains n particles.

It is straightforward to extend the discussion to multiparticle inclusive spectra. In particular, we find that the n-particle inclusive spectrum can be written, in the region of phase space in which $x_1 > x_2 \cdots > x_n$, as

$$\frac{d^n\sigma}{\sigma_T} = \rho_n(x_1, \ldots, x_n)\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$
$$= \left(\frac{1}{\lambda|f_0'|}\right)^n \frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}$$
$$\times C\left(\frac{x_1}{x_2}\right)C\left(\frac{x_2}{x_3}\right)\cdots C\left(\frac{x_{n-1}}{x_n}\right). \tag{4.27}$$

The function $C(z)$, which is discussed in detail in the Appendix, is given by

$$C(z) = 1 + \sum_{N=1}^{\infty} \frac{f_0'}{f_N'} z^{\xi_N - \xi_0} \equiv 1 + \Sigma(z), \tag{4.28}$$

where the sum runs over all the eigenvalues, ξ_N, given by (4.23) and for convenience we have introduced the notation $f_N' \equiv \partial f/\partial\xi|_{\xi=\xi_N}$. The structure of the n-particle inclusive spectrum as a product of functions reflects the underlying "short-range order" or factorization property of the exclusive cross sections. From the general definition (2.26) of the inclusive correlation functions, we can use (4.27) to find

$$\tau_1(x) = \lambda\frac{d\xi_0}{d\lambda} \equiv \gamma(\lambda)$$

and (4.29)

$$\tau_2(x_1, x_2) = \gamma^2(\lambda)\Sigma(x_1/x_2),$$

for $x_1 > x_2$. Higher τ_n can be calculated similarly.

To illustrate the cluster-decomposition sum rules explicitly, it is convenient to chose for simplicity a specific value for $a = 4\alpha_{\text{in}}$. A natural choice is $\alpha_{\text{in}} = \frac{1}{2}$, the approximate average meson-trajectory intercept, which leads to $a = 2$. For comparative purposes we can also consider the case $\alpha_{\text{in}} = \frac{1}{4}$, so that $a = 1$. Further, to avoid excessive detail, we shall restrict our present explicit verification of the cluster techniques to the cluster-decomposition theorem for σ_T/σ_0 and the sum rules for τ_1 and τ_2 for $\alpha_{\rm in}=\frac{1}{2}$. We have also verified the inversion formulas for $\alpha_{\rm in}=\frac{1}{2}$ and both the sum rules and inversion formulas for $\alpha_{\rm in}=\frac{1}{4}$; we shall discuss these results in the qualitative discussion of Sec. IV C.

For $\alpha_{\rm in}=\frac{1}{2}$ the first four exclusive cluster functions become

$$g_1(q)=\lambda\,,$$

$$g_2(q_1,q_2)=\lambda^2\frac{q_2^+}{q_1^+}\left(2+\frac{q_2^+}{q_1^+}\right),\quad q_1^+>q_2^+\,,$$

$$g_3(q_1,q_2,q_3)=\lambda^3\,2\left(\frac{q_3^+}{q_1^+}\right)\left(1+\frac{q_3^+}{q_2^+}+\frac{q_2^+}{q_1^+}\right), \tag{4.30}$$

for $q_1^+>q_2^+>q_3^+$, and

$$g_4(q_1,q_2,q_3,q_4)=\lambda^4\,2\left\{\frac{q_4^+}{q_1^+}\left[1+\frac{q_2^+}{q_1^+}+\frac{q_4^+}{q_3^+}-2\frac{q_3^+}{q_2^+}-\left(\frac{q_3^+}{q_2^+}\right)^2-\frac{q_3^+q_4^+}{q_2^{+2}}-\frac{q_3^{+2}}{q_1^+q_2^+}+\frac{q_4^+q_2^+}{q_3^+q_1^+}\right]+\frac{q_4^{+2}}{q_1^+}\left(-\frac{q_3^{+2}}{q_2^{+2}}-\frac{q_3^+}{q_2^+}\right)\right\}.$$

Hence the lowest-order terms in the cluster-function sum rules for σ_T/σ_0, τ_1, and τ_2 are

$$\begin{aligned}\ln\frac{\sigma_T}{\sigma_0}&=\int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}g_1(q_1^+)+\int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\int_{m^2/\sqrt{s}}^{q_1^+}\frac{dq_2^+}{q_2^+}g_2(q_1^+,q_2^+)\\&\quad+\int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\int_{m^2/\sqrt{s}}^{q_1^+}\frac{dq_2^+}{q_2^+}\int_{m^2/\sqrt{s}}^{q_2^+}\frac{dq_3^+}{q_3^+}g_3(q_1^+,q_2^+,q_3^+)+\cdots\\&=\lambda\ln s+\lambda^2(\tfrac{5}{2}\ln s-\tfrac{9}{4})+\lambda^3(4\ln s-7)+\cdots,\end{aligned}\tag{4.31a}$$

$$\begin{aligned}\tau_1(q)&=g_1(q)+\sum_{[x,q]}\int\frac{dx}{x}g_2(q,x)+\sum_{[x_1,x_2,q]}\int\int\frac{dx_1}{x_1}\frac{dx_2}{x_2}g_3(q,x_1,x_2)+\cdots\\&=\lambda+5\lambda^2+12\lambda^3+\cdots\end{aligned}\tag{4.31b}$$

and finally, for $q_1^+>q_2^+$,

$$\begin{aligned}\tau_2(q_1,q_2)&=g_2(q_1,q_2)+\int_{[q_1,q_2,x]}\frac{dx}{x}g_3(q_1,q_2,x)+\sum_{[q_1,q_2,x_1,x_2]}\int\int\frac{dx_1}{x_1}\frac{dx_2}{x_2}g_4(q_1,q_2,x_1,x_2)+\cdots\\&=\lambda^2\frac{q_2^+}{q_1^+}\left(2+\frac{q_2^+}{q_1^+}\right)+\lambda^3\left[10\frac{q_2^+}{q_1^+}-2\frac{q_2^+}{q_1^+}\ln\left(\frac{q_2^+}{q_1^+}\right)\right]\\&\quad-\lambda^4\left[-\left(\frac{q_2^+}{q_1^+}\right)^2-5\frac{q_2^+}{q_1^+}\ln\left(\frac{q_2^+}{q_1^+}\right)+\frac{q_2^+}{q_1^+}\ln^2\left(\frac{q_2^+}{q_1^+}\right)+5\left(\frac{q_2^+}{q_1^+}\right)^2\ln\left(\frac{q_2^+}{q_1^+}\right)\right]+\cdots.\end{aligned}\tag{4.31c}$$

Referring to the Appendix, in which the expansions of the exact forms of σ_T/σ_0, τ_1, and τ_2 for $\alpha_{\rm in}=\frac{1}{2}$ are calculated, we see that the cluster sum rules, as given in (4.31), do indeed agree order by order with the exact forms of the inclusive quantities. The details of this agreement, while they provide explicit verification of the cluster expansion, are primarily of technical interest. The less technical, and therefore perhaps physically more interesting, results of this model are treated in the qualitative discussion of Sec. IV C.

2. φ^3-Type Model

The calculational techniques used to analyze the Chew-Pignotti model are directly applicable to model cross sections in the φ^3-type model. For brevity we shall simply quote the results.

The partial cross sections in the pionization region are given by

$$\frac{\sigma_n}{\sigma_0}\left(\frac{s}{m^2}\right)=\lambda^n\int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\int_{m^2/\sqrt{s}}^{q_1^+}\frac{dq_2^+}{q_2^+}\cdots\int_{m^2/\sqrt{s}}^{q_{n-1}^+}\frac{dq_n^+}{q_n^+}\left(1-\frac{q_2^+}{q_1^+}\right)\cdots\left(1-\frac{q_n^+}{q_{n-1}^+}\right).\tag{4.32}$$

Thus the first few exclusive cluster functions are given by

$$g_1(q^+)=\lambda,$$

$$g_2(q_1^+,q_2^+)=-\lambda^2\left(\frac{q_2^+}{q_1^+}\right) \text{ for } q_1^+>q_2^+,$$

$$g_3(q_1^+,q_2^+,q_3^+)=2\lambda^3\left(\frac{q_3^+}{q_1^+}\right) \text{ for } q_1^+>q_2^+>q_3^+, \tag{4.33}$$

and

$$g_4(q_1^+,q_2^+,q_3^+,q_4^+)=-2\lambda^4\left(2\frac{q_4^+}{q_1^+}+\frac{q_3^+q_4^+}{q_1^+q_2^+}\right) \text{ for } q_1^+>q_2^+>q_3^+>q_4^+.$$

The structural similarity of this model to that treated in the previous section allows us to read off the results for the simplest inclusive cross sections. In particular, we see that the eigenvalue equation determining the Regge-pole spectrum becomes

$$1=\lambda\int_0^1 dy\, y^{\xi-1}(1-y). \tag{4.34}$$

Thus

$$\xi_{0,1}=\frac{-1\pm(1+4\lambda)^{1/2}}{2}, \tag{4.35}$$

and the total cross section at high energy is given by

$$\frac{\sigma_T}{\sigma_0}\left(\frac{s}{m^2}\right)=\frac{1}{4}\,\frac{1+(1+4\lambda^2)^{1/2}}{(1+4\lambda)^{1/2}}\left(\frac{s}{m^2}\right)^{\xi_0(\lambda)}. \tag{4.36}$$

Similarly, we find that

$$\tau_1(x)=\frac{\lambda}{(1+4\lambda)^{1/2}}=\lambda\frac{d\xi_0}{d\lambda} \tag{4.37}$$

and

$$\tau_2(x_1,x_2)=-\left(\frac{\lambda^2}{1+4\lambda}\right)\left(\frac{x_2}{x_1}\right)^{(1+4\lambda)^{1/2}}, \text{ for } x_1>x_2. \tag{4.38}$$

Verification of the cluster-decomposition sum rules is now completely straightforward; we shall therefore simply remark that the sum rules do indeed agree with the expansions of the exact results and proceed to the discussion of the qualitative features of the models.

C. Qualitative Discussion

Let us begin with some general remarks on the model results for inclusive cross sections. First, the total cross sections exhibit Regge asymptotic behavior,

$$\frac{\sigma_T}{\sigma_0}\simeq c(\lambda)\left(\frac{s}{m^2}\right)^{\xi_0(\lambda)}. \tag{4.39}$$

From the cluster-decomposition theorem (2.22) we have both an intuitive understanding of this power dependence on s and an explicit calculational technique for determining $\xi_0(\lambda)$ from the exclusive clusters. Note, however, that this calculational technique leads to a power-series expansion for $\xi_0(\lambda)$; since the trajectory function will in general have singularities as a function of λ – see, for example, (4.35) – the question of the convergence of this series must be confronted. We shall discuss this point in detail in Sec. V.

In the pionization region the one-particle longitudinal inclusive spectra in the models are constants, independent of the rapidity $y=\ln(q_+/m)$. This result is common to all models exhibiting short-range order in the sense of (2.16) and is a specific illustration of the "central plateau" hypothesis [1,2,7] which maintains that in the pionization region the one-particle spectrum should depend only on $\vec{q}$. Further, the explicit form of the spectra is in each case

$$\frac{d\sigma}{\sigma_T}=\lambda\frac{d\xi_0(\lambda)}{d\lambda}\,\frac{dx}{x}, \tag{4.40}$$

where $\xi_0(\lambda)$ is the Regge trajectory that determines the leading behavior of the total cross section in the model. In this context we should remark that the kinematic correlation models treated in Sec. III also give this result; there, since $\xi_0(\lambda)=\lambda$, the one-particle spectra in the pionization region simply reduce to $\tau_1=\lambda$.

The two-particle correlation functions in these dynamical models also exhibit several general features. When both particles are in the pionization region, we have found that the correlations depend only on the ratio of the momenta of the two observed particles; in terms of the rapidities of the particle, y_1 and y_2, this means that the correlations depend only on the difference y_1-y_2. Further, the range of the correlations – that is, the effective "correlation length" – as a function of rapidity depends on the separation between the leading Regge trajectory and the secondary trajectories. Both these features are common to models in which short-range order is present.[1,2,7] As Figs. 7(a)–7(c) illustrate, however, the shape of the correlation function is strongly model-dependent. But this is exactly as one would hope, since it implies that the structure of τ_2 determined from experiment will provide dynamical information which will permit one to discriminate among various possible models. In this sense, the two-particle correlations are potentially considerably more interesting than are the one-particle spectra.

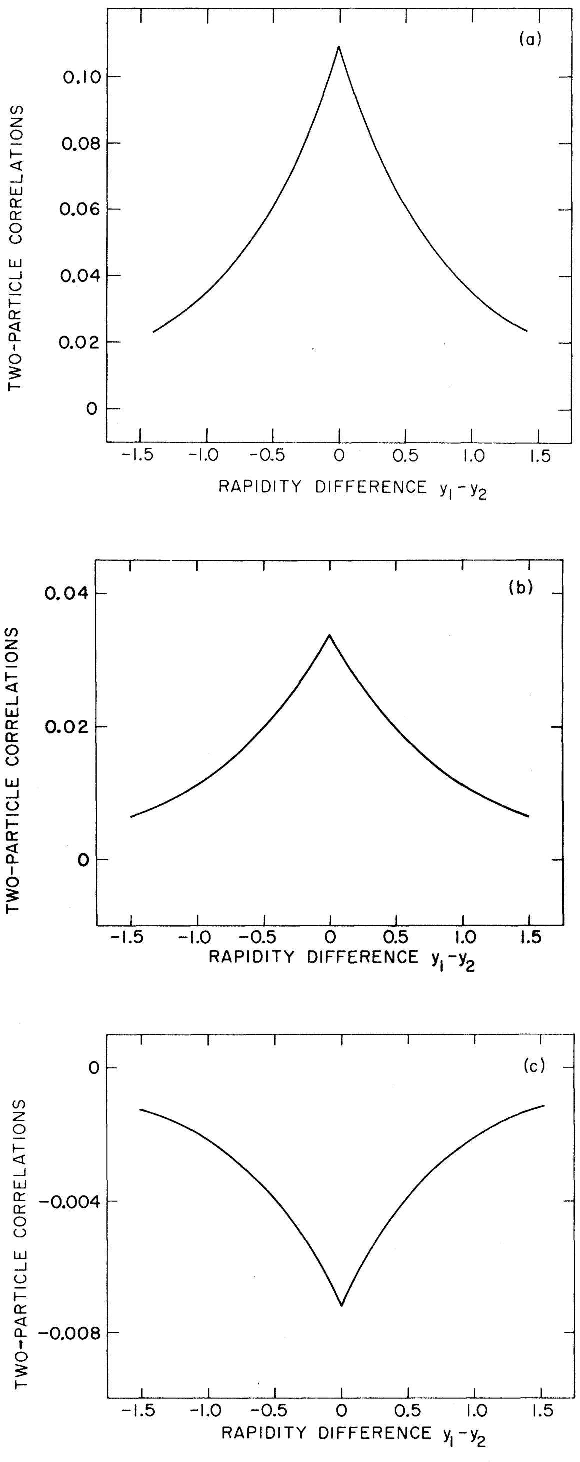


FIG. 7. The two-particle inclusive correlation functions, τ_2, in the pionization region for (a) the Chew-Pignotti model with $\alpha_{in} = \frac{1}{2}$ and $\lambda = 0.165$ so that $\xi_0(\lambda) = \frac{1}{4}$; (b) the Chew-Pignotti model with $\alpha_{in} = \frac{1}{4}$ and $\lambda = 0.2$ so that $\xi_0(\lambda) \cong 0.2$; and (c) the "φ^3-type" model with $\lambda = 0.1$ so that $\xi_0(\lambda) \cong 0.1$. Notice that in each case τ_2 depends only on $y_1 - y_2$ and, in particular, is independent of s.

To discuss qualitatively the cluster-decomposition sum rules relating inclusive and exclusive correlation functions, it is illuminating to compare graphically the exact and sum rule results. In Figs. 8(a)–8(d) we plot τ_2, the full two-particle correlation functions, and $a\tau_2$, the approximations to these functions obtained from the first three terms in the cluster-function sum rule. The three figures refer to the three different examples treated in Sec. IV B, that is, the Chew-Pignotti model with $\alpha_{in} = \frac{1}{2}$ and with $\alpha_{in} = \frac{1}{4}$ and the "φ^3-type" model. In each instance we see that the approximation reproduces both the magnitude and shape of the exact result remarkably well.

We can also interpret the difference in the structure of the contributions of the various exclusive correlation functions to the sum rules in an intuitively appealing manner. Notice that in each case the contributions of the higher-order exclusive clusters to $a\tau_2$ are wider than those of the lower order g_n. But this is exactly as we should expect, for the undetected particles in the higher-order clusters can lie "between" the two observed particles and thereby correlate their motion over a range in rapidity that is greater than the correlation lengths found in g_2. Figure 9 illustrates this effect schematically.

In contrast to this general result, we note that the width of τ_2 relative to g_2 – that is, the relation of the inclusive and exclusive "correlation lengths" – is model-dependent. Indeed, in Fig. 8(a) we see that $w_{\tau_2} > w_{g_2}$, whereas in Figs. 8(b)–8(d) the reverse is true.

From the results of Sec. IV B it is obvious that the accuracy of the approximations given by the cluster sum rules depends on λ; a comparison between Figs. 8(b) and 8(c) illustrates this result explicitly. Further, we have already mentioned that the question of the convergence of the cluster-decomposition results, viewed as power series in λ, must be discussed. In view of these observations, it might be suspected that the accuracy of the approximation in Figs. 8(a)–8(d) arises solely from the relative smallness of λ. To dispell this suspicion and to underscore the significance of the cluster decomposition, we have plotted in Figs. 10(a)–10(c) the approximations to τ_2 resulting from the "direct inclusive/exclusive relations" in the sense of Sec. II. Recall that, schematically,

$$\frac{d\sigma}{\sigma_0} = \frac{dq}{q} \sum_n \int d\Phi_{n-1} \bar{f}_n \tag{4.41}$$

and

$$\frac{d^2\sigma}{\sigma_0} = \frac{dq_1}{q_1} \frac{dq_2}{q_2} \sum_n \int d\Phi_{n-2} \bar{f}_n \, , \tag{4.42}$$

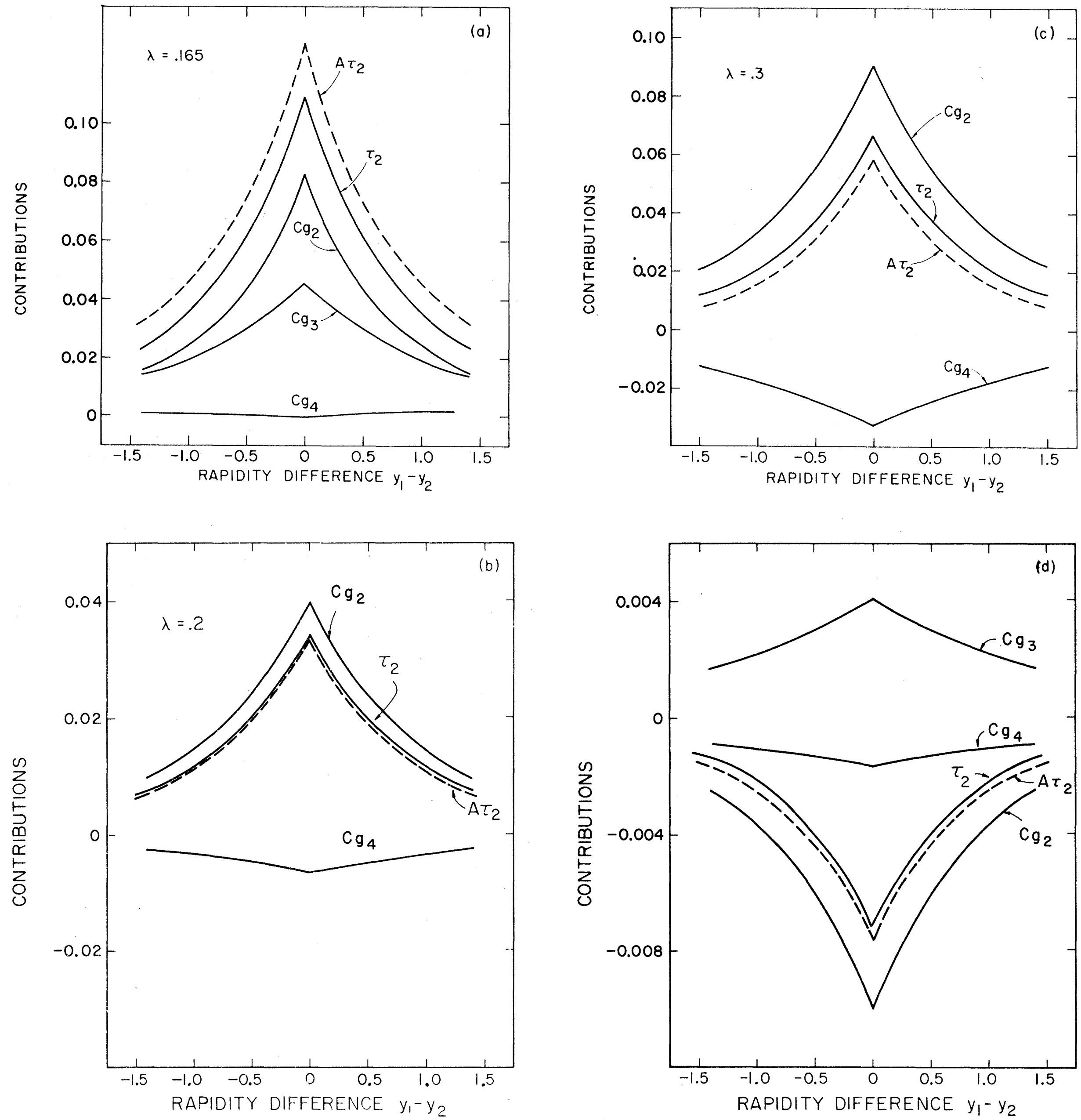


FIG. 8. The exact two-particle inclusive correlation functions compared to the approximations obtained by keeping only three terms in the cluster-function sum rule for (a) the Chew-Pignotti model with $\alpha_{in} = \frac{1}{2}$ and $\lambda = 0.165$ so that $\xi_0(\lambda) = \frac{1}{4}$; (b) the Chew-Pignotti model with $\alpha_{in} = \frac{1}{4}$ and $\lambda = 0.2$ so that $\xi_0(\lambda) \cong 0.2$; (c) the Chew-Pignotti model with $\alpha_{in} = \frac{1}{4}$ and $\lambda = 0.3$ so that $\xi_0(\lambda) \simeq 0.38$; and (d) the "$\varphi^3$-type" model for $\lambda = 0.1$ so that $\xi_0(\lambda) \simeq 0.1$. In each case the contributions of the three exclusive terms are shown individually. In (b) and (c) notice that $Cg_3 = 0$ since $g_3 = 0$ for these cases.

so that τ_2 can be calculated directly from the f_n by

$$\tau_2(x_1, x_2)\frac{dx_1}{x_1}\frac{dx_2}{x_2} = \frac{\sigma_0}{\sigma_T}\left(\frac{d^2\sigma}{\sigma_0}\right) - \left(\frac{\sigma_0}{\sigma_T}\frac{d\sigma}{\sigma_0}\right)\left(\frac{\sigma_0}{\sigma_T}\frac{d\sigma}{\sigma_0}\right)$$

$$= \frac{dx_1}{x_1}\frac{dx_2}{x_2}\left[\frac{\sigma_0}{\sigma_T}\sum_n d\Phi_{n-2}\bar{f}_n - \frac{\sigma_0}{\sigma_T}\left(\sum_n \int d\Phi_{n-1}\bar{f}_n\right)\frac{\sigma_0}{\sigma_T}\left(\sum_n \int d\Phi_{n-1}\bar{f}_n\right)\right] . \qquad (4.43)$$

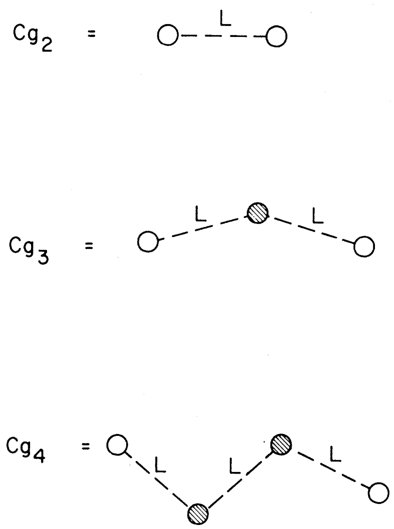


FIG. 9. A schematic illustration of the result that the contributions to τ_2 arising from exclusive cluster functions g_n should have widths that increase as n increases. The shaded circles represent unobserved particles.

Since the cluster functions g_2 to g_4 are determined by the exclusive spectra f_1 to f_4, to compare (4.43) to the results shown in Figs. 8(a)–8(d) we should keep only the terms through f_4. The resulting approximations are shown in Figs. 10(a)–10(c); as anticipated, the results depend strongly on $\ln(s/m^2)=2Y_{\max}$ and the approximations to τ_2 become increasingly poor as s grows. Recalling that these "direct relation" approximations contain the same dynamical input as the cluster approximations shown in Figs. 8(a)–8(d), we see clearly that the cluster decomposition, in addition to offering an appealing conceptual framework, can provide a useful and efficient calculational procedure to relate inclusive and exclusive correlations.

V. CRITICAL DISCUSSION AND CONCLUSIONS

To introduce our final remarks we shall summarize several general observations that follow from the simple model results of the preceding sections.

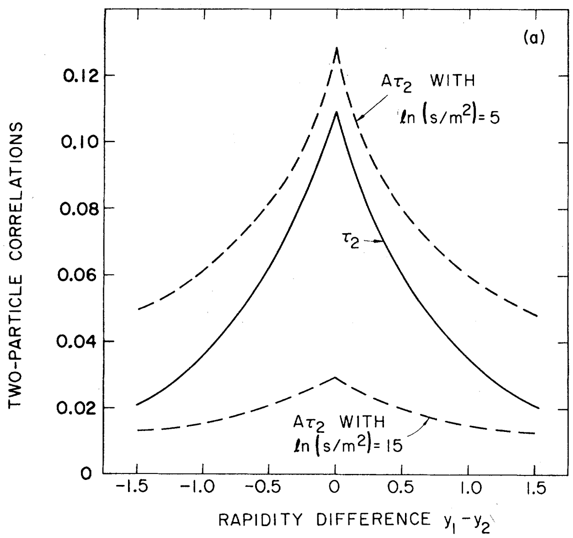
In the discussion of kinematic models we found that the constraints of energy-momentum conservation led to one-particle inclusive spectra which exhibit a relatively flat central region; this behavior is expected from the "gas analogy" but is not yet confirmed by experimental results at current energies.

In the pure longitudinal phase-space model, the shape of the inclusive spectrum in the fragmentation regions depends crucially on the relative weights of the contributing exclusive processes. For small $\bar{n}$, the spectrum tends to peak in these regions; for large $\bar{n}$, the spectrum rises smoothly to the central plateau. In the "modified" phase-space model, in which the "leading" particles in the final state are distinguished from the "secondaries," the one-particle inclusive spectrum for the secondaries shows a smooth rise to the central plateau for all $\bar{n}$. Further, in both kinematic models, the correlations in the two-particle inclusive spectra vanish when both particles are in the central region. From these remarks and the more specific results of Sec. III we conclude that kinematic constraints can produce considerable structure in the fragmentation regions in multiparticle spectra but should have small effects in the central region at large s. Clearly, the structure resulting from kinematic constraints must be understood in more detail before one can hope to isolate the dynamical information contained in inclusive spectra.

The simple dynamical models treated in Sec. IV are all characterized by "short-range order" among the final-state momenta in multiparticle processes. Hence their specific predictions for inclusive spectra exhibit the general features common to "multiperipheral" models. In particular, the one-particle inclusive spectra are flat in the central region. Further, the correlations in two-particle inclusive spectra depend, in the central region, only on the difference between the rapidities of the two particles. In addition, the widths of these correlations – that is, the correlation lengths – are determined by the differences between the leading and secondary Regge poles in the models. The detailed structure of the two-particle dynamical correlations, however, is quite model-dependent; for instance, whereas the general Chew-Pignotti model predicts positive two-particle correlations, the φ^3-type model leads to negative correlations.

In addition to providing illustrations of possible structure in inclusive spectra, these simple models offer detailed examples of the application of the cluster-decomposition technique to the study of multiparticle spectra. To complement their specific results and to place them in proper perspective, we shall now present a general, critical evaluation of the utility of the cluster-decomposition approach in this context and discuss the prospects for applying the techniques to phenomenological analyses of multiparticle spectra.

First, we observe that the cluster decomposition provides a conceptual framework in which important qualitative features of the model cross sections and spectra can be understood in simple intuitive ways. Hence, for example, the s^λ behavior of the total cross section in the longitudinal



phase-space models emerges naturally by considering the form of the exclusive cluster functions in the pionization region. Further, in the strong-ordered limit of the Chew-Pignotti model, the direct relation between the Poisson distribution of partial cross sections and the lack of correlations in the multiparticle spectra is immediately clear in the cluster framework.

Second, we note that the cluster decomposition leads naturally to quantitative relations among inclusive and exclusive processes. Of these quantitative relations perhaps the most interesting are the "sum rules" expressing correlations in inclusive spectra as sums over those in exclusive processes. As our specific calculations illustrate, these sum rules have the very attractive feature that the number of exclusive contributions necessary to give an approximation of any required accuracy to an inclusive correlation function is independent of s. Indeed, the model calculations establish that in certain instances the first few exclusive contributions provide a very accurate approximation to the two-particle inclusive correlation function.[46]

Thus from both qualitative and quantitative points of view the cluster decomposition offers valuable insight into the structure of the model multiparticle spectra. Even within the context of these simple models, however, there remain certain questions to be resolved. We shall concentrate here on two: the formal extension of the sums over partial cross sections from the kinematic limit of $N_{\max} = \sqrt{s}$ to infinity, and the convergence of the cluster-decomposition sum rules considered as power series in λ.

It is relatively straightforward to demonstrate that the formal extension of the sums from $\sqrt{s}$ to infinity introduces totally negligible errors for

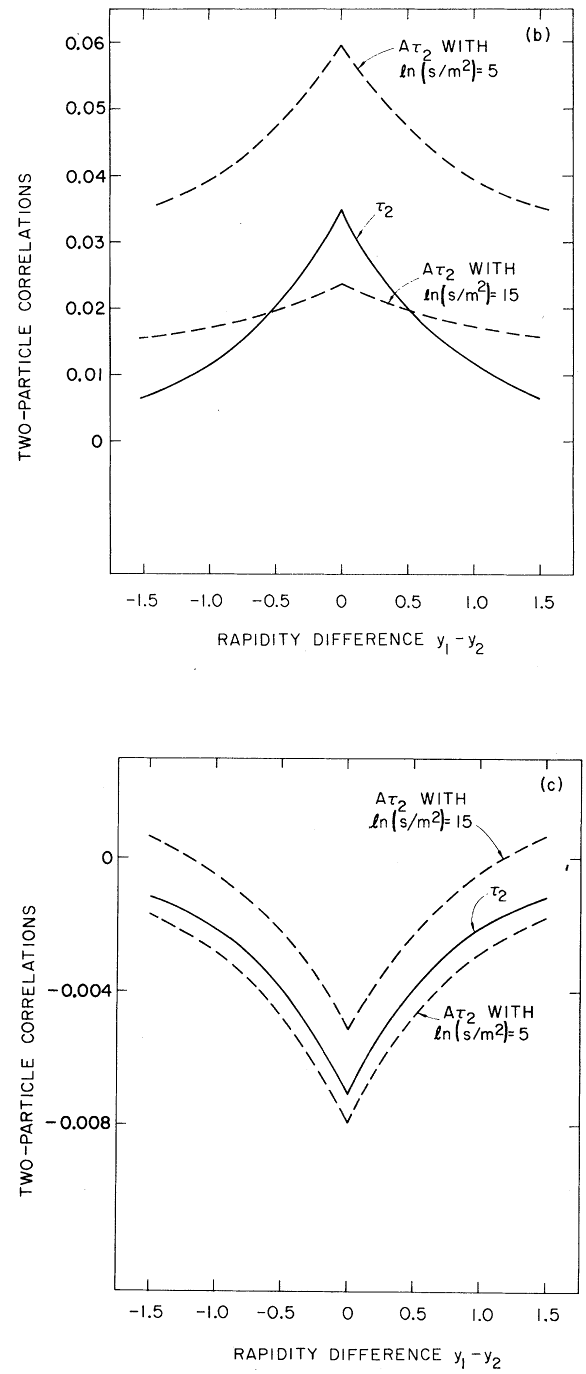


FIG. 10. A comparison of the approximation obtained by keeping the first three terms in the "direct relation" for τ_2 with the exact result at the two energies $\ln(s/m^2) = 2Y_{\max} = 5$ and $\ln(s/m^2) = 2Y_{\max} = 15$ for (a) the Chew-Pignotti model with $\alpha_{in} = \frac{1}{2}$ and $\lambda = 0.165$ so that $\xi_0(\lambda) = \frac{1}{4}$; (b) the Chew-Pignotti model with $\alpha_{in} = \frac{1}{4}$ and $\lambda = 0.2$ so that $\xi_0(\lambda) = 0.2$; (c) the "φ^3-type" model with $\lambda = 0.1$ so that $\xi_0(\lambda) \simeq 0.1$. Notice that the approximations become poorer as s increases.

large s in the models treated in Secs. III and IV. Indeed, since in all cases the average multiplicity is proportional to $\ln s$, one can show that the sum of all terms from $C\ln(s/m^2)$ – here C is a constant – to infinity is negligible. Consider, as an illustration of this result, the case in which the partial cross sections obey a Poisson distribution

$$\frac{\sigma_n}{\sigma_T}=\frac{\lambda^n[\ln(s/m^2)]^n}{n!}\quad. \tag{5.1}$$

Formally, (5.1) implies

$$\frac{\sigma_T}{\sigma_0}\left(\frac{s}{m^2}\right)=\sum_{n=0}^{\infty}\frac{\lambda^n[\ln(s/m^2)]^n}{n!}=\left(\frac{s}{m^2}\right)^{\lambda}. \tag{5.2}$$

In fact, we know that the sum in (5.2) should be truncated at $N_{\max}=(s/m^2)^{1/2}$. However, we can readily see that the error term resulting from truncating the series at $N=C\ln(s/m^2)$ satisfies

$$E\left(\frac{s}{m^2}\right)\equiv\sum_{n=C\ln(s/m^2)}^{\infty}\frac{\lambda^n[\ln(s/m^2)]^n}{n!}\simeq\int_{C\ln(s/m^2)}^{\infty}\frac{\exp\{z[\ln\lambda+\ln\ln(s/m^2)]\}}{\Gamma(z+1)}dz \tag{5.3}$$

$$\leqslant\frac{1}{[2\pi(C\ln s)^{1/2}]^{1/2}}\,\frac{s^{-C(\ln C-\ln\lambda-1)}}{\ln C-\ln\lambda-1}\ , \tag{5.4}$$

provided

$$\ln C>\ln\lambda+1. \tag{5.5}$$

That (5.5) is a natural condition is seen by noting that $\bar{n}=\lambda\ln s$; to obtain a good approximation to the formal result (5.2) we must obviously include those partial cross sections near the average multiplicity. Similar proofs would apply to all the models discussed here.

The question regarding the convergence of the cluster sum rules forces us to confront a more serious problem. As a specific illustration of the possible convergence difficulties, we shall study the φ^3-type model introduced in Sec. IV. We recall that, in the pionization region,

$$f_n(q_1^+,q_2^+,\ldots,q_n^+)=\lambda^n\left(1-\frac{q_2^+}{q_1^+}\right)\cdots\left(1-\frac{q_n^+}{q_{n-1}^+}\right) \tag{5.6}$$

for $q_1^+>\cdots>q_n^+$,

$$\frac{\sigma_n}{\sigma_0}=\lambda^n\int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\int_{m^2/\sqrt{s}}^{q_1^+}\frac{dq_2^+}{q_2^+}\cdots\times\int_{m^2/\sqrt{s}}^{q_{n-1}^+}\frac{dq_n^+}{q_n^+}f_n(q_1^+,\ldots,q_n^+) \tag{5.7}$$

and

$$\frac{\sigma_T}{\sigma_0}\equiv\sum_{n=0}^{\infty}\frac{\sigma_n}{\sigma_0}=c(\lambda)s^{\alpha(\lambda)}\ , \tag{5.8}$$

where

$$\alpha(\lambda)=\frac{-1+(1+4\lambda)^{1/2}}{2} \tag{5.9}$$

and

$$c(\lambda)=\frac{1}{4}\,\frac{[1+(1+4\lambda)^{1/2}]^2}{(1+4\lambda)^{1/2}}\quad. \tag{5.10}$$

From (5.9) and (5.10) we observe that both $c(\lambda)$ and $\alpha(\lambda)$ have power-series expansions which converge only for $\lambda<\frac{1}{4}$. But the general cluster-decomposition formalism establishes that

$$\alpha(\lambda)=\sum_{n=1}^{\infty}\alpha_n(\lambda), \tag{5.11}$$

where

$$G_n\left(\frac{s}{m^2}\right)\equiv\int_{m^2/\sqrt{s}}^{\sqrt{s}}\frac{dq_1^+}{q_1^+}\cdots\int_{m^2/\sqrt{s}}^{q_{n-1}^+}\frac{dq_n^+}{q_n^+}g_n(q_1^+,\ldots,q_n^+)=\alpha_n(\lambda)\ln\left(\frac{s}{m^2}\right)+\beta_n(\lambda)+O(m^2/s). \tag{5.12}$$

Since the nth-order exclusive cluster, g_n, is proportional to λ^n, we see that (5.11) is just a power series in λ for $\alpha(\lambda)$ and hence will diverge for $\lambda>\frac{1}{4}$. Thus although $\alpha(\lambda)$ is a well-defined analytic function of λ, with no singularities for $\lambda>0$, the cluster-decomposition expression for $\alpha(\lambda)$ converges only for $\lambda<\frac{1}{4}$. Similar convergence problems will beset the cluster-function sum rules for inclusive spectra; the one-particle spectrum, for example, is

$$\tau_1(x)\frac{dx}{x}\equiv\frac{\lambda}{(1+4\lambda)^{1/2}}\,\frac{dx}{x}\ , \tag{5.13}$$

and thus clearly has a cluster expansion which diverges for $\lambda>\frac{1}{4}$.

It is apparent that these convergence difficulties do not prevent our using the cluster decomposition to study simple theoretical models, for we have alternative ways of calculating the inclusive quantities explicitly and hence of knowing the region in which the cluster expansion converges. It is equally clear, however, that one must understand how to proceed when one does not have alternative calculational methods or when one wishes to use the sum rules to verify the assumptions underlying the cluster decomposition. In particular, in potential applications to phenomenology one would have neither freedom to vary λ nor knowledge of the validity of the cluster decomposition.

One possible approach to this problem is the use of Padé approximants[47] or other summability

TABLE I. The exact trajectory function, the two lowest-order Padé approximants, $f^{(1,1)}$ and $f^{(2,1)}$, and the corresponding number of terms in the cluster-decomposition expression for $\alpha(\lambda)$ for several values of $\lambda > \frac{1}{4}$.

λ	$\alpha(\lambda)$	$f^{(1,1)}$	$\alpha_1(\lambda)+\alpha_2(\lambda)$	$f^{(2,1)}$	$\alpha_1(\lambda)+\alpha_2(\lambda)+\alpha_3(\lambda)$
0.5	0.37	0.33	0.25	0.38	0.5
1	0.62	0.50	0	0.67	2
2	1	0.67	−2	1	14
5	1.8	0.83	−20	2.7	230

methods to evaluate those cluster expansions which diverge. As illustrations of these techniques, we present in Table I the results of applying lower-order Padé approximants to the first few terms in the cluster expansion of $\alpha(\lambda)$ and in Fig. 11 a Padé approximation to the two-particle correlation function, τ_2. Notice in each case that well beyond the region of convergence of the power series the Padé approximants still give reasonable values for the results.

Finally, let us consider the prospects for applying the cluster-decomposition approach to analysis of experimental data. Clearly the cluster approach has very attractive features, particularly in the relations it yields between inclusive and exclusive correlations. If the results of the simple model calculations proved indicative, the implications

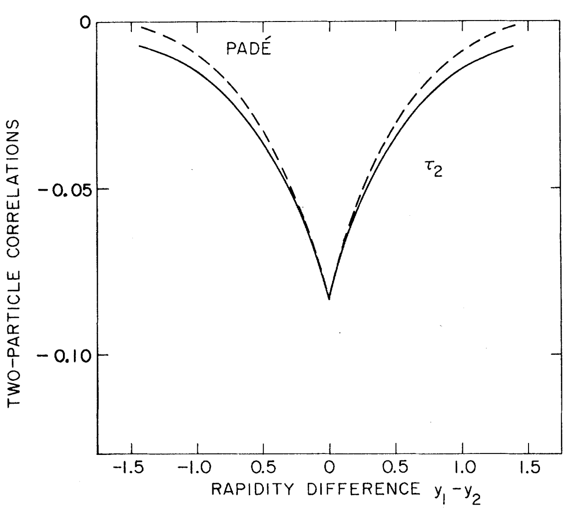


FIG. 11. An illustration of the use of Padé approximants to approximate the cluster-function sum rules beyond their regions of convergence. Shown for the "φ^3-type" model with $\lambda = 0.5$ are the exact two-particle correlation function, τ_2, and the $f^{(1,1)}$ Padé approximant to τ_2/λ^2 as determined from the first three terms in the cluster sum rule. The cluster sum rule itself diverges for this value of λ; keeping only three terms in the sum rule leads to an approximation for τ_2 which is roughly an order of magnitude too large and hence cannot be drawn on this scale.

for phenomenology would be profound; specifically, these results would suggest that by analyzing a fixed and possibly small number of exclusive spectra at large energy, one could understand in detail inclusive spectra at all higher energies.

Before one can hope to develop a "phenomenological" cluster decomposition, however, several vital questions must be answered:

(1) What is the proper manner to incorporate charged particles into the approach? In particular, should one also seek to include isospin – or higher internal symmetries – by assuming a "matrix factorization"[48] of the exclusive cross sections?

(2) To what extent is the factorizability of exclusive differential cross sections empirically valid? Further, should this be a simple factorization – as in (2.16) – or a "matrix" factorization?

(3) Beyond what energy should one expect the asymptotic picture to apply? Can one meaningfully attempt to relate exclusive and inclusive correlation functions from present data?

Possible resolutions to these and related questions are currently under investigation.

Note Added

Although our "cluster-decomposition" approach was originally motivated[3] by an analogy to statistical mechanics, we have not in the present article dealt explicitly with possible relations between the simple models for multiparticle spectra and known, specific statistical mechanical systems. Since it develops that in some instances exact statistical analogs of the simple models can be found, it seems appropriate to remark briefly on the general relation of our discussion to statistical mechanics.

First, we note that the dynamical models treated in Sec. IV are exact analogs of one-dimensional, nearest-neighbor interaction models for fluids or imperfect gases. A general discussion of such systems is found in Z. W. Salsburg, R. W. Zwanzig, and J. G. Kirkwood, J. Chem. Phys. 21, 1098 (1953). The conceptual link between multiparticle hadronic systems and imperfect gases provided by this reference may prove very useful in refining and clarifying the "Feynman-Wilson" gas analogy.[2] I thank Dr. Richard Arnold for stressing this point to me.

Second, it develops that more complicated – and hence, more realistic – models for multiparticle hadronic spectra also have exact statistical mechanical analogs. In particular, Professor T. D. Lee (private communication, and paper in preparation) has shown that ladder diagrams in a (3 + 1)-dimensional ϕ^3 theory are in exact mathematical correspondence with a particular one-dimensional gas system. This equivalence permits the use of the full formalism of statistical mechanics in discussions of the multiperipheral model and further suggests possible approximation procedures to be used in the analysis of experimental data. I am grateful to Professor Lee for informing me of his results prior to their publication.

ACKNOWLEDGMENTS

It is a pleasure to thank Shau-Jin Chang for many valuable discussions and suggestions and for a critical reading of the manuscript. I wish also to thank John Stack for several stimulating discussions.

APPENDIX

In this Appendix we provide the technical details necessary to complete the discussion of dynamical correlations in Sec. IV. Specifically, we derive the total cross section and the one- and two-particle inclusive spectra in the general case of the Chew-Pignotti model. The methods are relatively standard and apply directly to the "φ^3-type" model as well.

From (4.9) and (4.12) we see that the exclusive partial cross section for the production of n additional particles in the Chew-Pignotti model is

$$\frac{\sigma_n}{\sigma_0}\left(\frac{s}{m^2}\right)=\lambda^n\int_{m^2/s}^{1}\frac{dx_1}{x_1}\int_{m^2/s}^{x_1}\frac{dx_2}{x_2}\cdots\int_{m^2/s}^{x_{n-1}}\frac{dx_n}{x_n}\left[\left(1+\frac{x_2}{x_1}\right)^a\cdots\left(1+\frac{x_n}{x_{n-1}}\right)^a\right], \tag{A1}$$

where $x_i=q_i^+/\sqrt{s}$. Introducing $y_i=x_i/x_i-1$, $x_0\equiv 1$, we find that

$$\frac{\sigma_n}{\sigma_0}=\lambda^n\int_{m^2/s}^{1}\frac{dy_1}{y_1}\int_{m^2/sy_1}^{1}\frac{dy_2}{y_2}\cdots\int_{m^2/sy_1\cdots y_{n-1}}^{1}\frac{dy_n}{y_n}\left[(1+y_2)^a\cdots(1+y_n)^a\right]$$

$$=\lambda^n\int_0^1\frac{dy_1}{y_1}\left(\prod_{i=2}^{n}\int_0^1\frac{dy_i}{y_i}(1+y_i)^a\right)\theta\left(y_1\cdots y_n-\frac{m^2}{s}\right)\equiv\lambda\int_0^1\frac{dy}{y}D_{n-1}\left(\frac{m^2}{ys}\right). \tag{A2}$$

Here

$$D_{n-1}\left(\frac{m^2}{y_1 s}\right)=\lambda^{n-1}\left(\prod_{i=2}^{n}\int_0^1\frac{dy_i}{y_i}(1+y_i)^a\right)\theta\left(y_2\cdots y_n-\frac{m^2}{y_1 s}\right) \tag{A3}$$

with $D_0(x)=\theta(1-x)$. Defining $D(x)\equiv\sum_{n=0}^{\infty}D_n(x)$, we see that the total cross section satisfies

$$\frac{\sigma_T}{\sigma_0}\left(\frac{s}{m^2}\right)\equiv\sum_{n=0}^{\infty}\frac{\sigma_n}{\sigma_0}\left(\frac{s}{m^2}\right)$$

$$=1+\lambda\int_0^1\frac{dy}{y}D\left(\frac{m^2}{ys}\right). \tag{A4}$$

From (A3) we observe that

$$D_n(x)=\lambda\int_0^1\frac{dy}{y}(1+y)^a D\left(\frac{x}{y}\right), \tag{A5}$$

and hence that $D(x)$ satisfies the integral equation

$$D(x)=\theta(1-x)+\lambda\int_0^1\frac{dy}{y}(1+y)^a D\left(\frac{x}{y}\right). \tag{A6}$$

Introducing a Laplace-type transform [49]

$$\tilde{D}(\nu)\equiv\int_0^1 dx\,x^{\nu-1}D(x), \tag{A7}$$

we can diagonalize (A6) via

$$\tilde{D}(\nu)=\frac{1}{\nu}+\lambda\int_0^1\frac{dy}{y}(1+y)^a\left[y^\nu\int_0^{1/y}d\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{\nu-1}D\left(\frac{x}{y}\right)\right]$$

$$=\frac{1}{\nu}+\lambda\int_0^1\frac{dy}{y}(1+y)^a y^\nu\int_0^1 d\left(\frac{x}{y}\right)\left(\frac{x}{y}\right)^{\nu-1}D\left(\frac{x}{y}\right)$$

$$=\frac{1}{\nu}+\lambda\int_0^1 dy\,(1+y)^a y^{\nu-1}\tilde{D}(\nu). \tag{A8}$$

In (A8) the second equality follows because $D(t)=0$ for $t>1$. Thus

$$\tilde{D}(\nu)=\frac{1/\nu}{1-\lambda\int_0^1 dy\,(1+y)^a y^{\nu-1}}, \tag{A9}$$

and the solutions of the eigenvalue equation

$$1=\lambda\int_0^1 dy\,y^{\nu-1}(1+y)^a\equiv\lambda f(\nu,a) \tag{A10}$$

determine the spectrum of $\tilde{D}(\nu)$. Further, the rightmost pole of $\tilde{D}(\nu)$ determines the leading behavior as $s\to\infty$ of $D(m^2/ys)$ and hence of σ_T/σ_0. Near this pole, which occurs at $\nu\equiv\xi_0(\lambda)>0$,

$$\tilde{D}(\nu)\simeq\frac{1}{\xi_0(\lambda)(-\partial f/\partial\nu|_{\nu=\xi_0(\lambda)})[\nu-\xi_0(\lambda)]} \tag{A11}$$

and hence, using the inversion formula

$$D(e^{-x}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp(\nu x)\tilde{D}(\nu)d\nu\,, \tag{A12}$$

we find

$$D\left(\frac{m^2}{sy}\right) \underset{s\to\infty}{\sim} \frac{(ys/m^2)^{\xi_0(\lambda)}}{\lambda\xi_0(\lambda)|f_0'|}\ . \tag{A13}$$

Here we have introduced the notation $f_0' \equiv \partial f/\partial\nu|_{\nu=\xi_0}$ and used the result that $f_0' < 0$ to write $-\partial f/\partial\nu|_{\nu=\xi_0} = |f_0'|$. Using (A4) we see that as $s\to\infty$,

$$\frac{\sigma_T}{\sigma_0}\left(\frac{s}{m^2}\right) \underset{s\to\infty}{\sim} \frac{1}{\xi_0^2(\lambda)}\frac{1}{|f_0'|}\left(\frac{s}{m^2}\right)^{\xi_0(\lambda)}\ . \tag{A14}$$

The differential spectrum for producing a single particle at the mth vertex in an n-particle exclusive cross section can be written as

$$\frac{(d\sigma^n)_m}{\sigma_0} \equiv \lambda\frac{dx_m}{x_m} A_{m-1}\left(\frac{1}{x_m}\right)A_{n-m}\left(\frac{x_m s}{m^2}\right)\,, \tag{A15}$$

where

$$A_n\left(\frac{1}{x}\right) = \lambda^n\int_x^1 \frac{dx_1}{x_1}\cdots\int_x^{x_{n-1}}\frac{dx_n}{x_n}\left(1+\frac{x_2}{x_1}\right)^a\cdots \times\left(1+\frac{x}{x_n}\right)^a . \tag{A16}$$

Changing variables to $y_i = x_i/x_i - 1$, $x_0 = 1$, we obtain

$$A_n\left(\frac{1}{x}\right) \equiv \lambda\int_0^1 \frac{dy}{y}F_{n-1}\left(\frac{x}{y}\right), \tag{A17}$$

where

$$F_{n-1}\left(\frac{x}{y}\right) = \lambda^{n-1}\left(\prod_{i=1}^{n-1}\int_0^1\frac{dy_i}{y_i}(1+y_i)^a\right)\left(1+\frac{x}{y}\frac{1}{y_1\cdots y_{n-1}}\right) \times\theta\left(y_1\cdots y_{n-1}-\frac{x}{y}\right) \tag{A18}$$

and

$$F_0\left(\frac{x}{y}\right) = \theta\left(1-\frac{x}{y}\right)\left(1+\frac{x}{y}\right)^a .$$

Introducing

$$A(x) \equiv \sum_{n=1}^{\infty} A_n(x)$$

and

$$F(x) \equiv \sum_{n=0}^{\infty} F_n(x)$$

we see that (A17) leads to

$$A(x) = \lambda\int_0^1 \frac{dy}{y}F\left(\frac{x}{y}\right), \tag{A19}$$

where $F(u)$ satisfies the integral equation

$$F(u) = \theta(1-u)(1+u)^a + \lambda\int_0^1\frac{dy}{y}F\left(\frac{u}{y}\right)(1+y)^a . \tag{A20}$$

The diagonalization procedure outlined previously yields

$$\tilde{F}(\nu) = \frac{f(\nu,a)}{1-\lambda f(\nu,a)}\,, \tag{A21}$$

where $f(\nu, a)$ is defined by (A10). Again the rightmost pole of $\tilde{F}(\nu)$ will determine the leading asymptotic behavior of $F(x)$ and thence of $A(x)$; since we are evaluating the spectrum in the pionization region, $1 \gg x \gg m^2/s$, and hence the leading asymptotic behavior of $A(x)$ is sufficient to determine the spectrum. Inverting the Laplace transform for $F(x)$ and substituting into the equation for A, we find

$$A\left(\frac{1}{x}\right) \underset{1\gg x}{\approx} \frac{(1/x)^{\xi_0(\lambda)}}{\lambda\xi_0(\lambda)|f_0'|}\ . \tag{A22}$$

Hence the inclusive one-particle spectrum, which follows from (A15) by summing over all n and m, becomes, after normalizing by σ_T,

$$\tau_1(x)\frac{dx}{x} \equiv \frac{d\sigma}{\sigma_T} = \frac{dx}{x}\frac{1}{\lambda|f_0'|} = \lambda\frac{d\xi_0}{d\lambda}\frac{dx}{x}\ . \tag{A23}$$

Similarly, we can readily establish that the differential cross section for the production of two particles at positions l and m in an n-particle exclusive process can be written, for $x_l > x_m$, as

$$\frac{(d^2\sigma)^n_{l,m}}{\sigma_0} \equiv \lambda^2\frac{dx_l}{x_l}\frac{dx_m}{x_m}A_{l-1}\left(\frac{1}{x_l}\right)B_{m-l-1}\left(\frac{x_l}{x_m}\right)A_{n-m}\left(\frac{sx_m}{m^2}\right)\,, \tag{A24}$$

where A is defined by (A16) and

$$B_n\left(\frac{z}{x}\right) \equiv \lambda^n\int_x^z\frac{dx_1}{x_1}\int_x^{x_1}\frac{dx_2}{x_2}\cdots\int_x^{x_{n-1}}\frac{dx_n}{x_n}\left(1+\frac{x_1}{z}\right)^a\left(1+\frac{x_2}{x_1}\right)^a\cdots\left(1+\frac{x_n}{x_{n-1}}\right)^a\left(1+\frac{x}{x_n}\right)^a . \tag{A25}$$

Changing variables to $y_1 = x_1/z$, $y_i = x_i/(x_i - 1)$, $i \geq 2$, and calling $z/x = u$ we find

$$B_n(u) = \lambda^n\left(\prod_{i=1}^n\int_0^1\frac{dy_i}{y_i}(1+y_i)^a\right)\left(1+\frac{1}{u}\frac{1}{y_1\cdots y_n}\right)^a\theta\left(y_1\cdots y_n - \frac{1}{u}\right)\ . \tag{A26}$$

Comparing (A26) to (A18), we see that $B_n(u) = F_n(1/u)$ and we can apply the previous results to $B(z/x) \equiv \sum_{n=0}^{\infty} B_n(z/x)$. Note, however, that although $1 \gg z > x \gg m^2/s$, we cannot assume that $z \gg x$. Hence, in particular, we cannot claim that the rightmost singularity of $\tilde{B}(\nu)$ will dominate; we must in fact retain all poles in the expansion of

$$\tilde{B}(\nu) = \frac{f(\nu, a)}{1 - \lambda f(\nu, a)} \tag{A27}$$

in order to determine the full behavior of $B(z/x)$ for $z/x \approx 1$. From the eigenvalue condition (A10) we see that

$$\begin{aligned} 1 &= \lambda \int_0^1 dy\,(1+y)^a y^{\nu-1} \\ &= \lambda \sum_{n=0}^{\infty} \frac{1}{\nu+n}\binom{a}{n}. \end{aligned} \tag{A28}$$

As $\lambda \to 0$ for general a we get an infinite sequence of eigenvalues

$$\xi_N \cong -N + \lambda\left[\binom{a}{N}\right]^{-1}. \tag{A29}$$

For arbitrary a and λ the ξ_N become complicated functions of λ. For $a = n_0$, there are $n_0 + 1$ eigenvalues and the infinite sum in (A28) truncates at $n = n_0 + 1$. Proceeding formally in the general case we note that near the Nth pole,

$$\tilde{B}(\nu) \simeq \frac{1}{\lambda^2(-f_N')[\nu - \xi_N(\lambda)]}\ , \tag{A30}$$

where $f_N' = \partial f/\partial\nu|_{\nu=\xi_N}$. Thus upon applying the inverse Laplace transform to $\tilde{B}(\nu)$ and evaluating all the pole contributions, we obtain

$$B\left(\frac{z}{x}\right) = \sum_{N=0}^{\infty} \frac{(z/x)^{\xi_N}}{\lambda^2(-f_N')}\ . \tag{A31}$$

Thus the normalized two-particle inclusive spectrum, obtained by summing over n, l, and m in (A24) and using (A14), (A22), and (A31), assumes the form, for $x_1 > x_2$,

$$\begin{aligned} \frac{d^2\sigma(x_1, x_2)}{\sigma_T} &= \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{1}{\lambda^2|f_0'|^2} \\ &\quad\times\left[1 + \sum_{N=1}^{\infty}\frac{f_0'}{f_N'}\left(\frac{x_1}{x_2}\right)^{\xi_N - \xi_0}\right] \\ &\equiv \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{1}{\lambda^2|f_0'|^2}\,C\left(\frac{x_1}{x_2}\right). \end{aligned} \tag{A32}$$

Recall here that the eigenvalues are ordered such that $\xi_0 > \xi_1 > \cdots > \xi_N > \cdots$.

To compare the exact forms of the inclusive cross sections with the results of the cluster-function sum rules for $\alpha_{\text{in}} = \frac{1}{2}$, we note that in this case the eigenvalue condition (4.23) reduces to the cubic equation

$$\xi^3 - \xi^2(3 - 4\lambda) + \xi(2 - 8\lambda) - 2\lambda = 0. \tag{A33}$$

This equation yields three (real) roots ξ_0, ξ_1, and ξ_2. The explicit forms of these roots as functions of λ are complicated and not of particular interest here; we note only that from (A33) we can establish the power-series expansions

$$\begin{aligned} &\xi_0(\lambda) = \lambda + \tfrac{5}{2}\lambda^2 + 4\lambda^3 + \cdots, \\ &\xi_1(\lambda) = -1 + 2\lambda - 8\lambda^3 + \cdots, \\ \text{and} \\ &\xi_2(\lambda) = -2 + \lambda - \tfrac{5}{2}\lambda^2 + 4\lambda^3 + \cdots. \end{aligned} \tag{A34}$$

From (A10) we see that for $a = 4\alpha_{\text{in}} = 2$

$$\frac{\partial f}{\partial \xi}(\xi, 2) = -\frac{1}{\xi^2} - \frac{2}{(\xi+1)^2} - \frac{1}{(\xi+2)^2}\ . \tag{A35}$$

Further, (A31) then implies that

$$\Sigma(x) = \frac{f_0'}{f_1'}(x)^{\xi_1 - \xi_0} + \frac{f_0'}{f_2'}(x)^{\xi_2 - \xi_0}. \tag{A36}$$

Combining these results with (4.22) and (4.29) we find that the lowest-order terms in the expansions of the exact forms of σ_T/σ_0, τ_1, and τ_2 are given by

$$\begin{aligned} \ln\left(\frac{\sigma_T}{\sigma_0}\right) &= \xi_0(\lambda)\ln\left(\frac{s}{m^2}\right) + \ln C(\lambda) \\ &= (\lambda + \tfrac{5}{2}\lambda^2 + 4\lambda^3 + \cdots)\ln\left(\frac{s}{m^2}\right) \\ &\quad + (-\tfrac{9}{4}\lambda^2 - 7\lambda^3 + \cdots), \end{aligned} \tag{A37a}$$

$$\tau_1(q) = \gamma(\lambda) \equiv \lambda\frac{d\xi_0}{d\lambda} = \lambda + 5\lambda^2 + 12\lambda^3 + \cdots, \tag{A37b}$$

and for $q_1^+ > q_2^+$,

$$\begin{aligned} \tau_2(q_1^+, q_2^+) &= \gamma^2(\lambda)\Sigma\left(\frac{q_1^+}{q_2^+}\right) \\ &= \lambda^2\left[\frac{q_2^+}{q_1^+}\left(2 + \frac{q_2^+}{q_1^+}\right)\right] + \lambda^3\left[10\frac{q_2^+}{q_1^+} - 2\frac{q_2^+}{q_1^+}\ln\left(\frac{q_2^+}{q_1^+}\right)\right] \\ &\quad + \lambda^4\left[-\left(\frac{q_2^+}{q_1^+}\right)^2 - 5\frac{q_2^+}{q_1^+}\ln\left(\frac{q_2^+}{q_1^+}\right)\right. \\ &\quad \left. + \frac{q_2^+}{q_1^+}\ln^2\left(\frac{q_2^+}{q_1^+}\right) + 5\left(\frac{q_2^+}{q_1^+}\right)^2\ln\left(\frac{q_2^+}{q_1^+}\right)\right] + \cdots. \end{aligned}$$

Comparing these equations with (4.31) establishes the anticipated term-by-term agreement between the cluster sum rules and the expansions of the exact results.

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[1]For discussions of several different aspects of the shifting emphasis in both theoretical and experimental investigations see W. R. Frazer *et al*., Rev. Mod. Phys. 44, 284 (1972); E. L. Berger, in *Proceedings of International Colloquium on Multiparticle Dynamics*, edited by E. Byckling *et al*. (Research Institute for Theoretical Physics, Helsinki, Finland, 1972); Richard C. Arnold, Argonne National Laboratory Report No. ANL/HEP 7139, 1971 (unpublished).

[2]R. P. Feynman, Phys. Rev. Letters 23, 1415 (1969); in *High Energy Collisions*, Third International Conference held at State University of New York, Stony Brook, 1969, edited by C. N. Yang *et al*. (Gordon and Breach, New York, 1969); K. Wilson, Cornell University Report No. CLNS-131, 1970 (unpublished).

[3]D. K. Campbell and S.-J. Chang, Phys. Rev. D 4, 1151 (1971); 4, 3658 (1971). These articles will be referred to as I and II, respectively.

[4]Several authors have discussed related approaches in recent articles. See A. H. Mueller, Phys. Rev. D 4, 150 (1971); Lowell S. Brown, *ibid*. 5, 748 (1972); Sun-Sheng Shei and Tung-Mow Yan, Phys. Rev. D 6, 1744 (1972).

[5]We shall, for example, ignore internal quantum numbers and not treat transverse momentum explicitly.

[6]G. F. Chew and A. P. Pignotti, Phys. Rev. 176, 2112 (1968); Phys. Rev. Letters 19, 614 (1967).

[7]C. E. DeTar, Phys. Rev. D 3, 128 (1971).

[8]For an explanation of this terminology see Sec. IV.

[9]A numerical study of the inclusive cross sections in the $\lambda\varphi^3$ model is reported in F. Bopp, Phys. Rev. D 5, 2297 (1972).

[10]Since we are integrating over the transverse momenta, the constraint on these variables does not concern us explicitly.

[11]For a detailed derivation and presentation of the following results, the reader should refer to I and II.

[12]Strictly speaking, if each of the $\bar{n} \propto \ln s$ particles in the pionization region had momentum $\epsilon\sqrt{s}$ – the maximum allowed by our definition of the pionization region – then for fixed ϵ eventually $\epsilon\sqrt{s}\,\ln s > \sqrt{s}$. Hence, when we speak of a group of particles being "in the pionization region" we shall imply that their total plus and minus momenta are small relative to $\sqrt{s}$ in the lab frame.

[13]K. Huang, *Statistical Mechanics* (Wiley, New York, 1963); J. E. Mayer and M. G. Mayer, *Statistical Mechanics* (Wiley, New York, 1940).

[14]Transforming the cross sections to an analog of "temperature space," in which the constraints among the q^+ disappear, is a useful device theoretically but may be of limited value phenomenologically. Hence we shall concentrate on developing a cluster approach which explicitly includes the constraints.

[15]S. Stone, T. Ferbel, P. Slattery, and B. Werner, Phys. Rev. D 5, 1621 (1972).

[16]M. Kugler and R. G. Roberts, Phys. Rev. D (to be published).

[17]For a detailed discussion of the properties of (3 + 1)-dimensional phase space for fixed final-state multiplicity, see, for example, G. H. Campbell, J. V. Lepore, and R. J. Riddell, Jr., J. Math. Phys. 8, 687 (1967).

[18]For a discussion of longitudinal phase space for fixed final-state multiplicity see K. Kajantie and V. Karimäki, Research Institute for Theoretical Physics report, University of Helsinki, 1971 (unpublished).

[19]For a detailed treatment of (3 + 1)-dimensional phase space with a transverse-momentum cutoff, see K. Huang, Phys. Rev. 156, 1555 (1967).

[20]This prescription seems a reasonable one for at least three reasons: its simplicity, its natural association of a "coupling" with the production of a particle, and its similarity to the prescription for forming the grand canonical ensemble in statistical mechanics.

[21]Several authors have recently considered these "phase space" or "uncorrelated jet" models in the context of inclusive reactions. See, e.g., E. H. de Groot and Th. W. Ruijgrok, Nucl. Phys. B27, 45 (1971); B. R. Webber, Nucl. Phys. B43, 541 (1972); and D. Sivers and G. H. Thomas, Phys. Rev. D 6, 1961 (1972).

[22]Notice that we have included the asymptotic form of the flux factor $1/s$ in this expression.

[23]In deriving (3.3) from (3.1) we have made use of the approximation of "linearizing" the δ functions, that is, of ignoring $p_B'^{+}$ in the "plus component" δ function of (3.1) and $p_A'^{-}$ in the "minus component" δ function. As discussed in II, these approximations are valid to $O(m^2/s)$.

[24]F. Lurçat and P. Mazur, Nuovo Cimento 31, 140 (1964).

[25]Those readers interested in the technical details will find that the complications arise in part from the integration over the momenta of the "leading" particles. Whereas (3.21) represents the one-particle spectrum for all particles, the exclusive clusters of (3.4) describe only the secondary-particle spectra; hence one must explicitly calculate the spectrum of the leading particles and subtract it from (3.21) to verify the cluster-function sum rules.

[26]This result, of course, merely reflects the constraint that the final-state particles cannot have more energy than the initial-state particles.

[27]The general results of Ref. 34 imply that in both cases the integral over $\tau_2(x_1, x_2)$ will be negative definite. Indeed, this will always be the case for a correlation function defined canonically as in (3.13). Some authors (see Ref. 15) use a definition in which

$$\hat{\tau}_2(x_1, x_2) \equiv \frac{\rho_2(x_1, x_2)}{\langle n(n-1)\rangle} - \frac{\rho_1(x_1)\,\rho_1(x_2)}{\langle n\rangle^2}$$

so that the integral over $\hat{\tau}_2$ vanishes. We believe that this definition can obscure certain dynamical effects; in particular, one sees that if at large separation $\rho_2(x_1, x_2) \rightarrow \rho_1(x_1)\rho_1(x_2)$, $\hat{\tau}_2 \not\rightarrow 0$ in general. The extent to which $\hat{\tau}$ represents short-range correlations is thus unclear, and we therefore prefer the canonical definition (3.13).

[28]For a general discussion of the properties of the multiplicity distributions in high-energy scattering, see A. H. Mueller, Phys. Rev. D 4, 150 (1971).

[29]Since one can show that $\partial\bar{n}/\partial\lambda > 0$, it is clear that (3.14) can be used at finite s only when $\lambda \ll \ln s$.

[30]Notice that the factor $(p_A'^{+} p_B'^{-})$ does not guarantee that the particle with $p_A'^{+}$ $(p_B'^{-})$ will have the largest plus (minus) momentum in the final state but does strongly enhance the spectra of these distinguishable particles for large plus (minus) momenta.

[31]We have again used the approximation of "linearizing" the δ functions. See Ref. 23.

[32]The difference in s dependence between σ_0 in this model and σ_0 in the "pure" phase-space model is explained by the factor $(p_A'^+ p_B'^-)$.

[33]In a more detailed study, currently in preparation, of the kinematic constraints on inclusive processes in a (3 + 1)-dimensional phase space with a transverse cutoff, we establish that the existence of the (strongly damped) transverse momenta also smooths this abrupt cutoff: D. K. Campbell (unpublished).

[34]E. Predazzi and G. Veneziano, Lett. Nuovo Cimento 2, 749 (1971).

[35]C. E. DeTar, D. Z. Freedman, and G. Veneziano, Phys. Rev. D 4, 904 (1971).

[36]K. Biebl and J. Wolf, Phys. Letters 37B, 197 (1971).

[37]W. D. Shephard *et al.*, Phys. Rev. Letters 28, 703 (1972).

[38]This replacement is, of course, a crude approximation and in particular cannot reflect the "smoothing" effects of proper integration over transverse momenta.

[39]The result that $\tau_2 \to 0$ in the central region is, of course, an asymptotic one. But even for this relatively low s we see that the model τ_2 is quite small when $y_2 \approx y_1 \approx 0$.

[40]N. Nakanishi, Phys. Rev. 135, B1430 (1964).

[41]Notice that (4.5) does *not* represent a "strong-ordering" assumption but is rather a general consequence of the symmetry of $|M_n|^2$.

[42]G. F. Chew, in *Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, 1968,* edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

[43]Recall that we have restricted our considerations to the pionization region and thus all kinematic correlations have been ignored.

[44]In Ref. 7 a quantity called a "correlation fraction" is introduced to describe correlations among particles at given links on a multi-Regge chain. This "correlation fraction" is found to be nonzero in the strong-ordered Chew-Pignotti model. It is important to recognize, however, that the canonically defined inclusive and exclusive correlation functions *do* vanish identically in this model for $n \geq 2$.

[45]We note that $\partial f(\xi, a)/\partial \xi \,|_{\xi=\xi_0} < 0$.

[46]We shall shortly see that this result could have profound implications for phenomenology.

[47]For a general introduction to and survey of the applications of Padé approximants in physics see G. A. Baker and J. L. Gammel, *The Padé Approximant in Theoretical Physics* (Academic, New York, 1970).

[48]By "matrix factorization" we mean factorization in the internal symmetry indices which would permit the exchange of other than vacuum quantum numbers.

[49]The equivalence of this transform to a Laplace transform is established by changing variables to $y = \ln x$.