PHYSICAL REVIEW D VOLUME 6, NUMBER 1 1 JULY 1972

Linear-Trajectory Fits of Pion-Nucleon Scattering at High Energies*

R. J. Gleiser†
Instituto Matematica, Astronomia y Fisica, Universidad Nacional de Cordoba, Cordoba, Argentina

and

A. M. Gleeson
Center for Particle Theory, University of Texas, Austin, Texas 78712
(Received 6 December 1971)

In this paper we investigate a simple Regge model for high-energy pion-nucleon scattering. The trajectory parameters of this model are completely determined by the spins and masses of the particles lying on them. We avoid the necessity for bending the trajectories in order to reproduce the observed high-energy dependence of the differential cross sections by not assuming that the continued partial-wave amplitudes satisfy Mandelstam symmetry. The absence of this symmetry gives rise to extra terms in the asymptotic form of a given amplitude. These extra terms are unambiguously determined by the model, and do not require further parametrization or physical interpretation. We give an explicit example showing that in relativistic scattering, Mandelstam symmetry is not a necessary consequence of the Froissart-Gribov representation. This model is tested by fitting all the available high-energy pion-nucleon scattering data with experiment. An extended version of this model predicts, for forward pion-nucleon charge-exchange scattering, a nonzero polarization with only ρ exchange.

I. INTRODUCTION

During the last few years there have been some indications that the particle states have fallen onto trajectory families that within the region of the mass spectrum observed are linear-trajectory groupings.[1] In this paper we will give a description of all the forward high-energy pion-nucleon scattering data in terms of these observed linear-trajectory fits. There are several features of these fits that are unique to this analysis. We emphasize that it is a characteristic of all Regge analysis that the spectrum observed in the physical-mass region determines the angular distributions in the crossed channel. Since the trajectories show so little tendency to bend in the mass region, it is natural to question the bending of trajectories in the crossed-channel scattering processes. It is found in this analysis that a model with linear trajectories can fit all the angular scattering data. It is an added feature of this model that terms which are required to reduce the "shrinkage" of the secondary peaks (a fact usually requiring trajectory bending) arise naturally from the background integral if there is no Mandelstam symmetry in the Regge amplitude. Since it is also natural that a model with few trajectories, all of which are linear, will not have Mandelstam symmetry, it is natural to include these factors. They arise automatically and involve no new free parameters in the fit. In order to clarify the unique features of this program, in Sec. II we describe a spinless Reggeization and identify the source and substance of these terms. In this section we also describe models with and without Mandelstam symmetry and discuss their features.

In the third and fourth sections we fit a model to the data for pion-nucleon scattering in the forward direction. By analyzing the charge-exchange data separately in Sec. III with only the ρ trajectory, we can completely determine the residue function associated with the ρ family. In the usual analysis of these data there is no possibility of having polarization with single-trajectory exchange. In our case by absorbing the $2j+1$ kinematic factors into the trajectory residue function, the background terms arising from the pole at $j=-\frac{1}{2}$ are found to lead to polarization consistent with the data. With the ρ residue function determined, all the elastic cross sections are fit with the Pomeranchukon and f_0 trajectories in Sec. IV. In this case the polarization is again predicted in accord with the available data.

Recently it has become more common to attribute deviations from the simply formulated Regge-pole model to the existence of cuts in the angular momentum plane. The existence of these cuts is generally implied on the basis of both theoretical considerations and fits to recent data. The model proposed in this paper can be extended to include cuts and their presence might even be expected. In this paper we find that we can establish a reasonable phenomenology using only the simplest Regge trajectories, and therefore have omitted cut contributions from this computation. If the analysis of further reactions requires cuts they can easily be added.

The analysis of pion-nucleon scattering presented here can be extended to include backward scattering. The problems of unequal-mass kinematics and a suitable large-angle parametrization are much more stringent, and the simple application of this method has not been completely satisfactory. A new analysis of the backward data using a more complete model, not using only leading terms, is being developed and the results of this analysis will be presented elsewhere.

II. THE KINEMATICS OF LARGE-ANGLE REGGE ANALYSIS

As is well known, the usual form of the Watson-Sommerfeld transformation used by Regge in potential scattering[2] has a background integral running parallel to the imaginary axis at $j=-\frac{1}{2}$. This term contributes a fixed asymptotic contribution of $s^{-1/2}$ which will dominate any Regge-pole terms of lower order.[3] Mandelstam developed a modification of the Regge program which allowed the background integral to be moved arbitrarily far to the left and, more importantly, to have an arbitrarily small asymptotic contribution.

In order to describe all the features of this analysis, we shall review briefly this approach to Regge theory. Starting with the usual partial-wave expansion

$$A(t, z_t)=\sum_{l=0}^{\infty}(2l+1)a_l(t)P_l(z_t), \qquad (1)$$

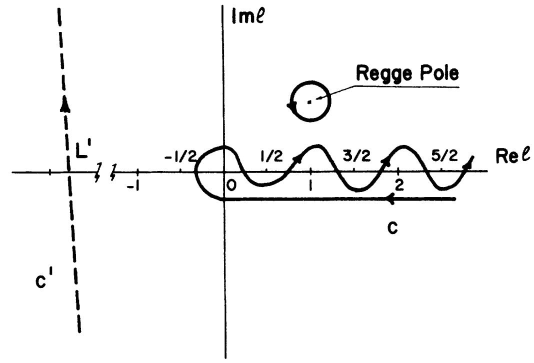

FIG. 1. The contours c and c' for Eqs. (2) and (3).

we can convert this sum to an integral in the complex l plane in the form

$$A(t, z_t)=\frac{1}{2\pi i}\int_c (2l+1)\frac{Q_{-l-1}(-z_t)}{\cos\pi l}\,a(l, t)\,dt, \qquad (2)$$

where c is the contour shown in Fig. 1. The function $a(l, t)$ is the complex extension of $a_l(t)$ from the integer values. The properties of $a(l, t)$ are assumed to be consistent with Carlson's theorem so that this extension is unique. This question will be discussed in more detail later when various forms of $a(l, t)$ are described. Opening out the contour c to c', and assuming that $a(l, t)$ is analytic except for poles at positions α_i in the half-sheet, we obtain

$$\begin{aligned}A(t, z_t)=&\frac{1}{2\pi i}\int_{L'-i\infty}^{L'+i\infty} dl\, a(l, t)\frac{2l+1}{\cos\pi l}\,Q_{-l-1}(-z_t)+\sum_{\mathrm{Re}\alpha_i>L'}\beta_i(t)\frac{2\alpha_i+1}{\cos\pi\alpha_i}\,Q_{-\alpha_i-1}(-z_t)\\ &-\sum_{m=0}^{n(-L''-3/2)}\frac{1}{\pi}(-1)^m(2m+2)Q_{m+1/2}(-z_t)[a(m+\tfrac{1}{2}, t)-a(-m-\tfrac{3}{2}, t)]\\ &-\sum_{m=n(-L'-1/2)}^{\infty}\frac{1}{\pi}(-1)^m(2m+2)Q_{m+1/2}(-z_t)a(m+\tfrac{1}{2}, t),\end{aligned} \qquad (3)$$

where α_i depends on t and is the Regge-pole trajectory, β_i the associated residue function, and $n(L')$ the largest positive integer less than L'. The first term is the new background integral and has asymptote $z_t^{L'}$. Since L' can be made as negative as desired, the contribution from this integral can be neglected at high energies. The second term is the Regge-pole contribution and contains all the dynamics of the theory. The third and fourth terms arise from the zeros of $\cos\pi l$ when l is a half-integer. The third term contains contributions which have asymptotic parts greater than the background, and the fourth term has those contributions less than background. We point out that the contribution to the amplitude from the term where $l=-\frac{1}{2}$ is not present since it is canceled by the kinematic factor $2l+1$ occurring in the partial-wave series. In our second model, we will absorb this kinematic factor into the amplitude $a(l, t)$, and thus have the additional contribution from $\cos\pi l$ when $l=-\frac{1}{2}$. As is usual, at high enough energies, all background terms are neglected. This leaves only the second and third terms.

The resulting amplitude will be completely determined by the Regge terms if the third term in

Eq. (1) can now be eliminated. It is the usual assumption of Regge analysis that the amplitudes $a(l, t)$ satisfy Mandelstam symmetry,

$$a(l+\tfrac{1}{2}, t) = a(-l-\tfrac{3}{2}, t), \quad l = 0, 1, 2, \ldots . \tag{4}$$

This symmetry, which is present for the $a(l, t)$'s definable in potential scattering for a large class of potentials, will remove these terms.

In the case of particle physics where there is no potential, the only inputs are the perturbation theory of Feynman diagrams or its outgrowth, the analytic structure of $A(t, z_t)$ as contained in the Mandelstam representation. Those last two means of investigation are very difficult and only weaker conclusions are possible. In this case besides both moving and fixed poles there are probably cuts.[4] Almost all conclusions about the structure of $a(l, t)$ are contained in the Froissart-Gribov representation. If the amplitude $A(t, z_t)$ satisfies a dispersion relation of the form[5]

$$A(t, z_t) = \frac{1}{\pi}\int_{z_0}^{\infty} \frac{\operatorname{Im}A(t, z')}{z'-z_t}\, dz' , \tag{5}$$

then, using the relation

$$Q_l(z) = \tfrac{1}{2}\int_{-1}^{1} \frac{P_l(z')}{z'-z}\, dz', \quad l = \text{integer} \tag{6}$$

we get

$$a(l, t) = \frac{1}{\pi}\int_{z_0}^{\infty} \operatorname{Im}A(t, z')\, Q_l(z')\, dz' , \tag{7}$$

which has the right asymptotic behavior in l to be extended to complex values of l. Equation (7) is the Froissart-Gribov continuation of the partial-wave amplitude.[6] It is often argued from Eq. (6) that, since the Q_l functions satisfy the required symmetry at l equal to a half-integer, the $a(l, t)$ should also satisfy the same symmetry. This assumption is of course valid only if the functions $a(l, t)$ and $a(-l-1, t)$ given by the representation of Eq. (7) have a common region of definition.

Since the range of validity of Eq. (7) is determined by the upper limit of the integral, we can see the difficulty of proving directly the Mandelstam symmetry by utilizing the asymptotic form of Q_l. In this case

$$a(l, t) \sim \int^{\infty} \operatorname{Im}A(t, z') z'^{-l-1}\, dz' , \tag{8}$$

and this form is generally not defined for $\operatorname{Im}A(t, z')$ for both l and $-l-1$. For this reason any further progress on this proof will require more definite statements about the form of $\operatorname{Im}A(t, z')$. One such attempt was the investigation of Roy.[7] By assuming that

$$A(t, z_t) = \frac{1}{\pi}\int_{z_0}^{\infty} \frac{\beta(t)(2z')^{\alpha(t)}}{z'-z_t}\, dz' \tag{9}$$

or

$$\operatorname{Im}A(t, z_t) = \theta(z_t - z_0)\beta(2z_t)^{\alpha(t)} \tag{10}$$

for $l > \alpha$, Eq. (7) can be integrated to yield

$$a(l, t) = \sqrt{\pi}\,\frac{\beta(t)}{2}\sum_{n=0}^{\infty} \frac{\Gamma(l+1+2n)}{\Gamma(l+\frac{3}{2}+n)n!} \times \frac{1}{l-\alpha+2n}(2z_0)^{\alpha-l-2n} . \tag{11}$$

This form provides a direct extension of $a(l, t)$ to the complex l plane, and the Mandelstam symmetry can be tested. The usual properties of Regge theory are manifest in this extension along with some more unusual features. $a(l, t)$ has a pole at $\alpha(t)$ directly associated with the asymptote z^{α} but, in addition, an infinite tower of daughter poles spaced two units behind. When $\alpha(t)$ is considered a moving point on a trajectory, these daughters are necessary for the cancellation of singularities required by Mandelstam symmetry. In addition $a(l, t)$ has poles at negative integers. These poles are identified directly in Eq. (7) as the poles of Q_l at the negative integers.

The Mandelstam symmetry of the continuation of $a(l, t)$ given by Eq. (11) can be verified by direct substitution. For all α not a half-integer

$$a(l, t) = a(-l-1, t) \text{ for } l = \text{half-integer}.$$

For $\alpha(s)$ equal to a half-integer, $a(l, t)$ in Eq. (11) is not defined for l at the half-integer values and therefore the Mandelstam symmetry cannot be directly tested. To study the question of Mandelstam symmetry for α near a half-integer, we will have to take a closer look at this model. It is clear that for the trajectory function at α equal to a half-integer, the relation of Mandelstam symmetry can be satisfied by one of two means of compensation. If there is no pole in $a(-l-1, t)$ to match the pole in $a(l, t)$, the residue of the Regge pole must vanish (i.e., not be a pole) or both $a(-l-1, t)$ and $a(l, t)$ must have poles with equal residues. This model is interesting because it displays both types of compensation. For $\alpha < -\frac{1}{2}$ but half-integer, the residue of the Regge pole vanishes. For α a positive half-integer there are compensating poles among the daughters. These features are summarized in Fig. 2.

Besides the two methods of compensation illustrated above, there is an obvious third method. In this case, we require Mandelstam symmetry everywhere and, with simple trajectory fits, this requires trajectories rising for $l < -\frac{1}{2}$. This case

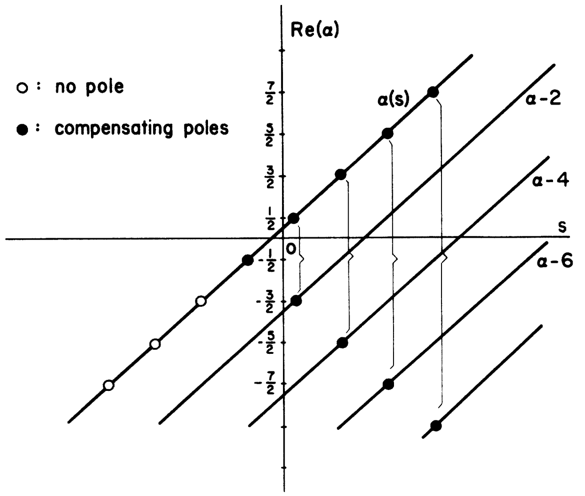

FIG. 2. Compensation in a simple Regge model.

would of course lead to increasing widths of diffraction-side lobes at larger t and can be disregarded.

From the above analysis we see that a model with a simple single trajectory cannot satisfy Mandelstam symmetry unless the residue vanishes at all the half-integer trajectory positions. This infinite set of zeros in the residue function would lead to an $a(l, s)$ which violates the asymptote required for the uniqueness of $a(l, s)$ in Carlson's theorem.

These statements lead us to directly display a model with a simple single trajectory which violates Mandelstam symmetry and none of the other requirements of Carlson's theorem. This model does not have the complete requirements of the usual analyticity, but then again neither do any of the models based on Regge trajectories only.

The approach to the model is built along the same lines as the phenomenology of the rest of this paper. By postulating a form for the structure of $a(l, s)$ which is reasonable for $l>0$ and $t>0$ and using this definition of $a(l, t)$ in the crossed-channel region for $t<0$, the structure of $A(t, \cos\theta_t)$ can be analyzed. We thus set

$$a(l, t) = \frac{\beta(t) e^{-x_0[l-\alpha(t)]}}{(2l+1)[l-\alpha(t)]}, \tag{12}$$

where x_0 is a (positive) parameter that could depend on t. $a(l, t)$ therefore contains a single trajectory and obviously does not satisfy Mandelstam symmetry. The $2l+1$ in the denominator was added for the sake of convenience. With this form of $a(l, t)$, the amplitude $A(t, z_t)$ is

$$\begin{aligned} A(t, z_t) &= \sum_{l=0}^{\infty} (2l+1) a(l, t) P_l(z_t) \\ &= \beta(t) \int_{x_0}^{\infty} e^{x\alpha} \sum_{l=0}^{\infty} e^{-xl} P_l(z_t)\, dx \\ &= \beta(t) \int_{x_0}^{\infty} e^{x\alpha} (1 - 2e^{-x} z_t + e^{-2x})^{-1/2}\, dx, \end{aligned} \tag{13}$$

where in the second step the identity

$$\frac{e^{-x_0 a}}{a} = \int_{x_0}^{\infty} e^{-xa}\, dx \tag{14}$$

was used, and in the third the well-known form of the generating function for the Legendre polynomials was substituted. A close look at this last form shows that for $\mathrm{Re}(\alpha(t)) < 0$, $A(t, z_t)$ is an analytic function of z_t in the complex z_t-plane cut from $z_{t_0} = \cosh x_0$ to $+\infty$. $A(t, z_t)$ therefore satisfies a dispersion relation of the right form. Carlson's theorem will ensure that we recover $a(l, t)$ when we apply the Froissart-Gribov formula. Finally, the asymptotic behavior of $A(t, z_t)$ in z_t can be obtained by noticing that in the limit $|z_t| \to \infty$

$$1 - 2e^{-x} z_t + e^{-2x} \sim 1 - 2e^{-x} z_t \quad (x_0 \le x \le \infty) \tag{15}$$

and

$$1 - 2e^{-x} z_t + e^{-2x} \sim -2e^{-x} z_t \qquad (0 \le x \le x_0), \quad |z| \gg e^{-x_0}. \tag{16}$$

All these terms lead to an $A(t, z_t)$ (Ref. 8)

$$\begin{aligned} A(t, z_t) &= \beta(t) \int_0^{\infty} e^{x\alpha} (1 - 2z_t e^{-x} + e^{-2x})^{-1/2}\, dx - \beta(t) \int_0^{x_0} e^{x\alpha} (1 - 2z_t e^{-x} + e^{-2x})^{-1/2}\, dx \\ &\underset{z\to\infty}{\sim} \beta(s) \frac{1}{\alpha} F(\tfrac{1}{2}, -\alpha; 1-\alpha, 2z_t) + iO(|z_t|^{-1/2}), \end{aligned} \tag{17}$$

where the first integral uses the approximation (15) and the second uses (16). Finally the relations[9]

$$F(a, b; c; z) = \frac{\Gamma(c)\Gamma(b-z)}{\Gamma(b)\Gamma(c-a)} F(a, 1-c+a; 1-b+a; z^{-1})(-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} F(b, 1-c+b; 1-a+b; z^{-1})(-z)^{-b} \tag{18}$$

yield

$$A(t, z_t) \underset{z_t\to\infty}{\sim} \beta(t) \frac{1}{\alpha} \frac{\Gamma(1-\alpha)}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}+\alpha)(-2z_t)^{\alpha} + i\,O(|z_t|^{-1/2}), \tag{19}$$

which explicitly shows the poles at $\alpha = 0, 1, 2, \ldots$ and the usual asymptotic behavior z_t^{α}. In addition, there is a contribution with fixed asymptote $z_t^{-1/2}$. This is the term characteristic of the violation of Mandelstam symmetry. The poles of $\Gamma(\frac{1}{2}+\alpha)$ are canceled by terms with asymptotic behavior $z_t^{-1/2-n}$ $(n = 0, 1, 2, \ldots)$ of which only terms of the leading order have been given. Alternatively notice that the hypergeometric function in Eq. (17) is finite for $\alpha = -m - \frac{1}{2}$ $(m = 0, 1, \ldots)$.

With this model in hand, it is now clear how to identify one of the features of an amplitude $A(t, z_t)$ which can be described by a simple Regge-trajectory model without Mandelstam symmetry. The amplitude contains an asymptotic contribution which goes like $z^{(-2n+1)/2}$ for n a positive integer.

The above model shows that the requirement of Mandelstam symmetry in the Regge models is certainly not necessary and, since the usual trajectories are known to have values near $\alpha = -\frac{1}{2}$ in the scattering region, the question can be tested with the data. In this paper we propose a phenomenology based on observations of the structure of $a(l, t)$ as seen in the physical t region and use this phenomenology for scattering at negative t. This is the essence of crossing symmetry. We shall assume that $a(l, t)$ has only a few trajectories and that the trajectories are linear. These two requirements almost require that there be no Mandelstam symmetry. There are no daughters to compensate and the rising trajectories are excluded. It is interesting to point out that in fits of effective Regge trajectories, there are indications that the ρ trajectory function bends from linearity around $\alpha = -\frac{1}{2}$.[10] In a model without Mandelstam symmetry this phenomenon is accounted for by the fixed-asymptote background terms. This effect is displayed in Fig. 3.

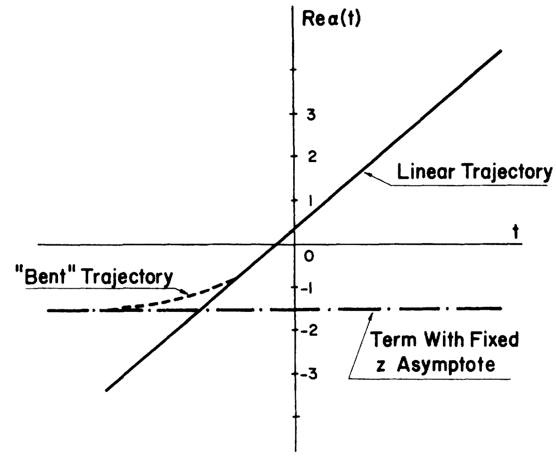

FIG. 3. The "effective" trajectory.

III. ANALYSIS OF PION-NUCLEON CHARGE-EXCHANGE SCATTERING

The problem of pion-nucleon scattering off the forward direction is a very well-studied one, and the kinematic conventions are fairly standard.[11] We follow Singh and use the amplitude

$$A'(s, t) = A(s, t) + \frac{\nu + t/4m}{1 - t/4m^2} B(s, t), \tag{20}$$

where $\nu = (s - m^2 - \mu^2)/2m$ = laboratory energy of the pion, and A and B are the usual invariant amplitudes. With those amplitudes, the differential and total cross sections are

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{4\pi W}\right)^2 \left[\left(1 - \frac{t}{4m^2}\right) |A'|^2 + \frac{t}{4m^2}\left(s - \frac{(m+\nu)^2}{1 - t/4m^2}\right) |B|^2\right] \tag{21}$$

and

$$\sigma(\text{total}) = \frac{1}{p_{\text{lab}}} \operatorname{Im} A'(s, t=0). \tag{22}$$

In terms of the t-channel partial-wave amplitudes, A' and B have the partial-wave expansions

$$A'^{(\pm)}(s, t) = \frac{8\pi}{p^2} \sum_{J=0}^{\infty} \left(\frac{pq}{N_A}\right)^J (J + \tfrac{1}{2}) f_+^{(\pm)J}(t) \tfrac{1}{2}[1 \pm (-1)^J] P_J(\cos\theta_t), \tag{23}$$

$$B^{(\pm)}(s, t) = 8\pi \sum_{J=0}^{\infty} \left(\frac{pq}{N_B}\right)^{J-1} (J + \tfrac{1}{2}) f_-^{(\pm)J}(t) \tfrac{1}{2}[1 \pm (-1)^J] P_J'(\cos\theta_t), \tag{24}$$

where (+) and (−) correspond, respectively, to isospin zero and one.

The Watson-Sommerfeld technique, as modified above, utilizing functions $f^{(\pm)}(J, t)$ which are the continuations in the complex J plane of the corresponding partial-wave amplitudes, produces the following Regge form:

$$A'(s,t) = -\frac{8\pi}{p^2}\left[\alpha(t)+\tfrac{1}{2}\right]\left(\frac{pq}{N_A}\right)^{\alpha(t)}\beta(t)\,\frac{Q^{\pm}_{-\alpha-1}(z)}{\cos\pi\alpha}$$
$$-\frac{8}{p^2}\left\{Q^{\pm}_{1/2}(z)\left[\left(\frac{pq}{N_A}\right)^{-3/2} f_+(-\tfrac{3}{2},t)-\left(\frac{pq}{N_A}\right)^{1/2} f_+(\tfrac{1}{2},t)\right]-2Q^{\pm}_{3/2}(z)\left[\left(\frac{pq}{N_A}\right)^{-5/2} f_+(-\tfrac{5}{2},t)-\left(\frac{pq}{N_A}\right)^{3/2} f_+(\tfrac{3}{2},t)\right]\right\} \tag{25}$$

and

$$B(s,t) = -8\pi\left[\alpha(t)+\tfrac{1}{2}\right]\left(\frac{pq}{N_B}\right)^{\alpha}\beta(t)\,\frac{Q^{\mp}_{-\alpha-1}{}'(z)}{\cos\pi\alpha}$$
$$-8\left\{Q^{\mp}_{1/2}{}'(z)\left[\left(\frac{pq}{N_B}\right)^{-3/2} f_-(-\tfrac{3}{2},t)-\left(\frac{pq}{N_B}\right)^{1/2} f_-(\tfrac{1}{2},t)\right]-2Q^{\mp}_{3/2}{}'(z)\left[\left(\frac{pq}{N_B}\right)^{-5/2} f_-(-\tfrac{5}{2},t)-\left(\frac{pq}{N_B}\right)^{3/2} f_-(\tfrac{3}{2},t)\right]\right\}, \tag{26}$$

where $Q_l^{\pm}$ are the suitably signatured Q functions and we have neglected terms of higher order.

These expressions, together with the assumption of single-Regge-pole dominance, will be used in the next sections in trying to fit the experimental data on forward pion-nucleon scattering at high energies. Because of its simplicity, the case of pion-nucleon charge-exchange scattering is treated first. In this case only the ρ trajectory is exchanged. As the model is developed, we will point out several differences with the usual Regge-pole treatment.

In this case, the complex extension of $f_{\pm}^{(-)}(J,t)$ is simply

$$f_{\pm}^{(-)}(J,t) = \frac{\beta_{\pm}(J,t)}{J-\alpha_\rho(t)}. \tag{27}$$

Substituting this into the previous expressions for A' and B, we have

$$A'^{(-)}(s,t) = -\frac{8\pi}{p^2}(\alpha+\tfrac{1}{2})\left(\frac{pq}{N_A}\right)^{\alpha}\beta_+(t)\,\frac{Q^{-}_{-\alpha-1}(z)}{\cos\pi\alpha}$$
$$-\frac{8}{p^2}\left\{Q^{-}_{1/2}(z)\left[\left(\frac{pq}{N_A}\right)^{-3/2}\frac{\beta_+(t)}{-\frac{3}{2}-\alpha}-\left(\frac{pq}{N_A}\right)^{1/2}\frac{\beta_+(t)}{\frac{1}{2}-\alpha}\right]-Q^{-}_{3/2}(z)\left[2\left(\frac{pq}{N_A}\right)^{-5/2}\frac{\beta_+(t)}{-\frac{5}{2}-\alpha}-2\left(\frac{pq}{N_A}\right)^{3/2}\frac{\beta_+(t)}{\frac{3}{2}-\alpha}\right]\right\} \tag{28}$$

and

$$B^{(-)}(s,t) = -8\pi(\alpha+\tfrac{1}{2})\left(\frac{pq}{N_B}\right)^{\alpha}\beta_-(t)\,\frac{Q^{+}_{-\alpha-1}{}'(z)}{\cos\pi\alpha}$$
$$-8\left\{Q^{+}_{1/2}{}'(z)\left[\left(\frac{pq}{N_B}\right)^{-3/2}\frac{\beta_-(t)}{-\frac{3}{2}-\alpha}-\left(\frac{pq}{N_B}\right)^{1/2}\frac{\beta_-(t)}{\frac{1}{2}-\alpha}\right]-Q^{+}_{3/2}{}'(z)\left[\left(\frac{pq}{N_B}\right)^{-5/2}\frac{\beta_-(t)}{-\frac{5}{2}-\alpha}-\left(\frac{pq}{N_B}\right)^{3/2}\frac{\beta_-(t)}{\frac{3}{2}-\alpha}\right]\right\}. \tag{29}$$

The following points are pertinent to a complete understanding of this model.

(i) The rather complicated-looking form of the amplitudes A' and B is only a mathematical consequence of the assumption of single-Regge-pole dominance. The extra terms therefore do not require further physical interpretation.

(ii) The parameters N_A and N_B appear naturally and are required if the equations are to be dimensionally sound.

(iii) The well-known crossover phenomenon[12] requires that the residue function $\beta_+(t)$ change sign between $t=0$ and $t=-0.5$ $(\mathrm{BeV}/c)^2$. To allow for this phenomena, $\beta_+(t)$ is replaced by a "smoother" $\beta_+(t)$ in the form

$$\beta_+(t) \to (1+c_\rho t)\beta_+'(t). \tag{30}$$

The value of c_ρ is taken to be[13] 9 $(\mathrm{BeV}/c)^{-2}$.

(iv) In general, there are two aspects of any Regge-pole model that are unique to this approach. They depend on the partial factorization of the s- and t-dependent parts. The most well known of these properties is the power-law behavior of the s variable. This leads to the well-known requirement of "shrinkage of diffraction peaks" in the differential cross section. The other property is the separation of the phase of the amplitude into the signature factor. This phase, although a small effect in cross sections, is an important contribution to polarization predictions.

The extra simplifying assumption is usually made that after the threshold factors are separated, the functions $\beta(t)$ are "smooth." To a large extent, the degree of success of any model in accounting for the structure of the cross section (dips, bumps, etc.) is usually directly correlated to the simplicity [i.e., the number of explicit parameters of the functional form of $\beta(t)$]. This along with the number of parameters required to allow for bending the trajectory determine the usefulness of the model. In our model, the trajectories are strictly linear and therefore contain much less freedom for a data fit.

The linear trajectory for the ρ is parametrized as

$$\alpha_\rho(t) = a_\rho + b_\rho t \ . \tag{31}$$

The values of the parameters a_ρ and b_ρ are completely determined by the mass of the ρ meson and the trajectory intercept at $t=0$. The value of a_ρ is taken to be 0.5 and $b_\rho = 0.9\ (\mathrm{BeV}/c)^{-2}$, which places the ρ pole $(\alpha=1)$ at $t=(0.75\ \mathrm{BeV}/c)^2$.

If the data on forward elastic scattering of pions on nucleons are utilized, a fairly high-precision test of this value of a_ρ [or $\alpha_\rho(t=0)$] is possible. A suitable combination of the elastic cross sections is determined entirely by isospin-one exchange. This fact follows from the usual isospin decomposition

$$\begin{aligned} &A(\pi^- + p \to \pi^- + p) = A^{(+)} + A^{(-)} ,\\ &A(\pi^+ + p \to \pi^+ + p) = A^{(+)} - A^{(-)} ,\\ &A(\pi^- + p \to \pi^0 + n) = -\sqrt{2}\, A^{(-)} ,\end{aligned} \tag{32}$$

and similar relations for the B amplitudes. The optical theorem for the difference in the cross sections then yields

$$\sigma_{\mathrm{tot}}(\pi^- p) - \sigma_{\mathrm{tot}}(\pi^+ p) = \frac{2\,\mathrm{Im}A'^{(-)}(s, t=0)}{|p_{\mathrm{lab}}|} \tag{33}$$

In this formulation of pion-nucleon scattering, with the choice $\alpha_\rho = 0.5$, some terms in the expressions for $A'(s, t)$ are singular at $t=0$. $A'(s, t=0)$ must then be evaluated with care.

The limit that requires careful examination is

$$L = \lim_{t\to 0}\left[\pi[\alpha(t)+\tfrac{1}{2}]\left(\frac{pq}{N_A}\right)^{\alpha}\frac{Q^-_{-\alpha-1}(z)}{\cos\pi\alpha} - \left(\frac{pq}{N_A}\right)^{1/2}\frac{Q^-_{1/2}(z)}{\frac{1}{2}-\alpha}\right]. \tag{34}$$

This is evaluated by using

$$\frac{Q^-_{-\alpha-1}(z)}{\cos\pi\alpha} \simeq \frac{\sqrt{\pi}}{2}\left[(-2z)^\alpha - (2z)^\alpha\right]\frac{\Gamma(-\alpha)}{\Gamma(\frac{1}{2}-\alpha)}\left[1 + \frac{(-\alpha)(1-\alpha)}{(\frac{1}{2}-\alpha)(2z)^2} + \frac{(-\alpha)(1-\alpha)(2-\alpha)(3-\alpha)}{(\frac{1}{2}-\alpha)(\frac{3}{2}-\alpha)2(2z)^4}\right]\frac{1}{\cos\pi\alpha} \tag{35}$$

or

$$\frac{Q^-_{-\alpha-1}(z)}{\cos\pi\alpha} \simeq \frac{\sqrt{\pi}}{2}\left[(-2z)^\alpha - (2z)^\alpha\right]\frac{\Gamma(-\alpha)}{\Gamma(\frac{3}{2}-\alpha)}\left\{\frac{1}{\pi} + \left[\frac{(-\alpha)(1-\alpha)}{(2z)^2} + \frac{(-\alpha)(1-\alpha)(2-\alpha)(3-\alpha)}{(\frac{3}{2}-\alpha)2(2z)^4}\right]\frac{1}{\cos\pi\alpha}\right\}. \tag{36}$$

In this expansion, the usual Taylor series for $\cos\pi\alpha$ was used,

$$\cos\pi\alpha \simeq \pi(\tfrac{1}{2}-\alpha) \quad \text{for } \alpha \sim \tfrac{1}{2}. \tag{37}$$

The result is then

$$\begin{aligned} L = &-(pq/N_A)^{1/2}\pi[(-2z)^{1/2} - (2z)^{1/2}]\\ &+ \text{remainder } [O(z^{-3/2})]. \end{aligned} \tag{38}$$

The actual evaluation of the "remainder" is extremely tedious and we have only studied it by numerical methods. Placing this limit in A' and neglecting all higher (negative) powers in z, A' is

$$\begin{aligned} A'(s, t=0) = &(8\pi/p^2)\beta_+(pq/N_A)^{1/2}\\ &\times[(-2z)^{1/2} - (2z)^{1/2}]. \end{aligned} \tag{39}$$

Because of the presence of noninteger exponents, the phases of all kinematic variables have to be treated with care. The momenta and angles are

$$p^2 = \tfrac{1}{4}t - m^2, \quad pq = [(\tfrac{1}{4}t - m^2)(\tfrac{1}{4}t - \mu^2)]^{1/2} \tag{40}$$

and

$$z = \frac{s - m^2 - \mu^2 + \frac{1}{2}t}{2pq}. \tag{41}$$

The phase of pq cancels for all α and for large s. This is due to the z^α dependence of the $Q_{-\alpha-1}(z)$ for large z and

$$z \sim s/2pq, \tag{42}$$

and the threshold factor of the residue function being proportional to $(pq)^\alpha$. All these combine to remove any noninteger power of pq from A' and B. The physical amplitudes are defined for $s = |s|e^{i\pi\epsilon}$, $\epsilon \to 0^+$ from the Feynman rules, defining the z phase as

$$-z = |z|e^{-i\pi}. \tag{43}$$

Then the z dependence is given by

$$(-2z)^\alpha - (2z)^\alpha = (e^{-i\pi\alpha} - 1)(2z)^\alpha. \tag{44}$$

These definitions establish our phase conventions.

Combining all these results, the contribution of the trajectory with $\alpha_\rho(0)=\frac{1}{2}$ to the forward amplitude for isospin-one exchange is

$$A'^{(-)}(s,t=0)=\frac{8\pi}{-m^2}\beta_+(-i-1)\frac{1}{\sqrt{N_A}}(s-m^2-\mu^2)^{1/2}. \tag{45}$$

The difference between the two total elastic cross sections for charged-pion scattering off protons is then given by

$$\Delta_{\pi p}=\sigma_t(\pi^-p)-\sigma_t(\pi^+p)$$
$$=\frac{2}{|p_{\text{lab}}|}\frac{8\pi}{m^2}\frac{\beta_+}{\sqrt{N_A}}(s-m^2-\mu^2)^{1/2}. \tag{46}$$

Since the total cross sections are usually tabulated as functions of the beam momentum in the laboratory, it is useful to have $\Delta_{\pi p}$ as a function of p_{lab}. The relation between p_{lab} and s is

$$s=\mu^2+m^2+2m(p_{\text{lab}}{}^2+\mu^2)^{1/2}. \tag{47}$$

For the range of values of p_{lab} appropriate to this model, the approximation

$$s-m^2-\mu^2\cong 2mp_{\text{lab}} \tag{48}$$

is suitable. The final form of $\Delta_{\pi p}$ is

$$\Delta_{\pi p}\simeq\pi\left(\frac{4}{m}\right)^2\frac{\beta_+}{\sqrt{N_A}}\frac{\sqrt{2m}}{\sqrt{p_{\text{lab}}}}. \tag{49}$$

For $p_{\text{lab}}=4.56$ BeV/c, the experimental value of $\Delta_{\pi p}$ is

$$\Delta_{\pi p}=2.60\pm0.04 \text{ mb}.$$

Using this value to normalize the remaining constants, we have

$$\beta_+/\sqrt{N_A}=0.071\ (\text{BeV}/c)^2\text{ mb}.$$

For $p_{\text{lab}}=19.86$ BeV/c, this predicts $\Delta_{\pi p}$ to be

$$\Delta_{\pi p}=1.26 \text{ mb}.$$

This value is in good agreement with the rather imprecise experimental data. In Fig. 4 we have plotted $\Delta_{\pi p}$ as given by Eq. (49) as well as experimental data from Barashenkov's compilation.[14]

An independent test of the value of $\alpha_\rho(0)$ is obtained by studying the differential cross section for charge-exchange scattering at $t=0$, i.e.,

$$\left.\frac{d\sigma}{dt}\right|_{t=0}=\frac{\pi}{p_s{}^2}\left(\frac{m}{4\pi w}\right)^2\left|-\sqrt{2}\,A'^{(-)}\right|^2$$
$$=\frac{2\pi}{sp_s{}^2}\left(\frac{m}{4\pi}\right)^2\left(\frac{8\pi}{m^2}\right)^2 2\left|\frac{\beta_+}{\sqrt{N_A}}\right|^2(s-m^2-\mu^2)$$
$$+\frac{8\pi}{m^3}\frac{1}{p_{\text{lab}}}\left|\frac{\beta_+}{\sqrt{N_A}}\right|^2. \tag{50}$$

This last equation contains the same sequence of limits and approximations as were required in $\Delta_{\pi p}$. Although the same form of the $A'^{(-)}$ is used for both $\Delta_{\pi p}$ and $d\sigma/dt|_{t=0}$, the two experimental values test different aspects of $A'^{(-)}$. $d\sigma/dt|_{t=0}$

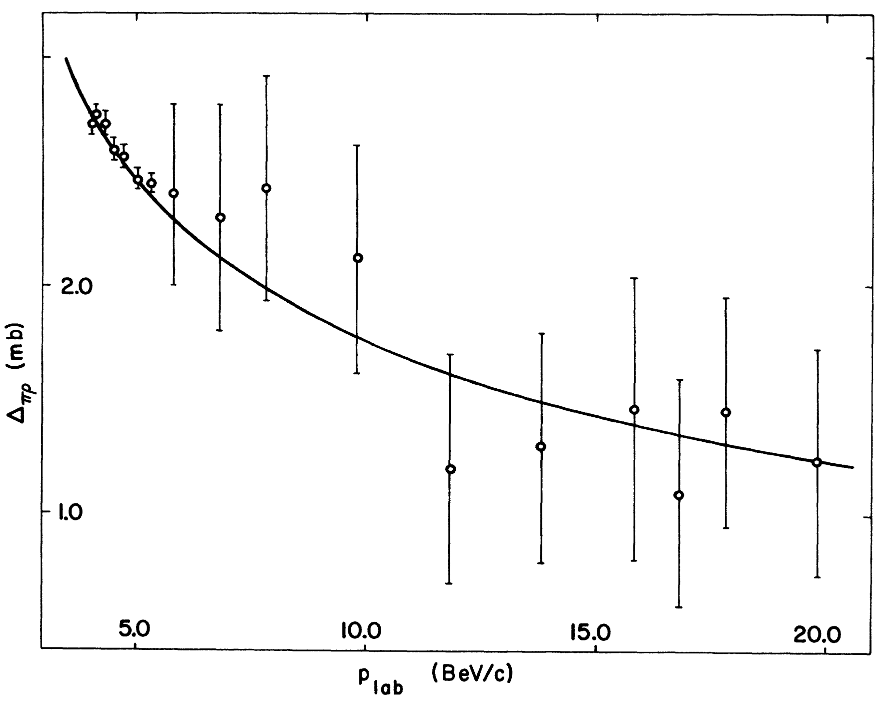

FIG. 4. Fit of $\Delta_{\pi p}$.

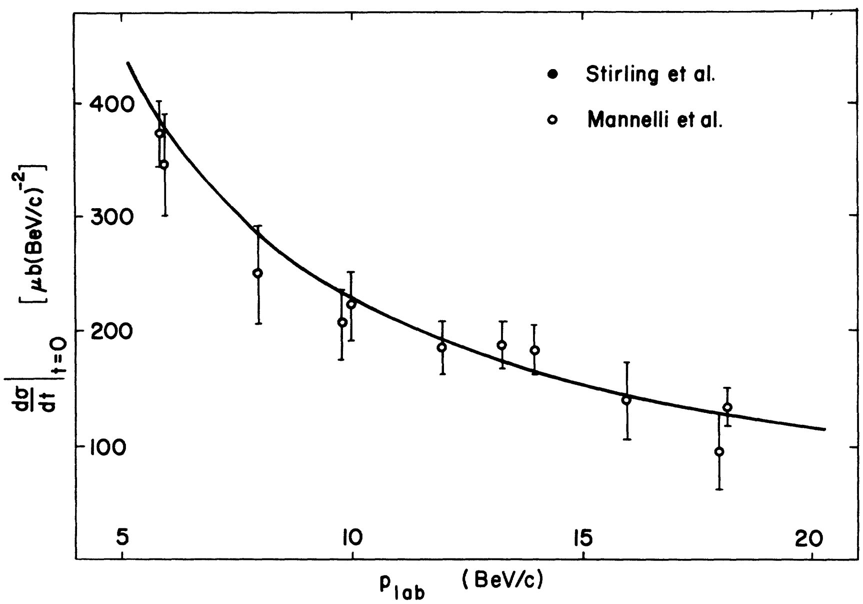

FIG. 5. Fit of charge-exchange differential cross sections at $t=0$.

depends on the modulus of $A'^{(-)}$, whereas $\Delta_{\pi p}$ is sensitive only to $\mathrm{Im}A'^{(-)}$. Since the Regge formalism predicts both phase and modulus simply these data are separate tests of $\alpha_\rho(0)$. The curve shown in Fig. 5 was obtained using the value

$$\beta_+/\sqrt{N_A} = 0.087\ (\mathrm{BeV}/c)^3\,\mathrm{mb}$$

in Eq. (50).

These results are in essentially good agreement with one another. The fit could perhaps be improved by changing $\alpha(0)$ slightly. Experimentally, however, the presence of a small but perfectly measurable amount of polarization, even at small values of $|t|$, shows that there must be other contributions to $A'^{(-)}$ of about the order of magnitude of the discrepancy found.

A unique feature of this treatment of single-ρ-exchange dominance is the prediction of a small amount of polarization. The effect, however, is too small in the low-$|t|$ region to account for the experimental data. At larger $|t|$, the model predicts much stronger polarization, but because of the lack of data no test is possible.

The main purpose of this section is to show that the hypothesis of a single Regge pole without Mandelstam symmetry can be successfully used to describe the high-energy charge-exchange scattering data. In this model a fit of the differential cross section for values of t such that $\alpha(t) < -\frac{1}{2}$ using only the ρ trajectory is very successful. The problem of the polarization for small $|t|$ exists in all Regge-pole fits and has always been corrected by adding new isospin-one exchanges (e.g., the ρ' trajectory[15] or perhaps J-plane cut contributions[16]). Any of these methods can be used in this model, but subsequently we will modify this single-Regge-pole model to improve this situation.

In order to fit the differential cross section, the functions $\beta_\pm(t)$ must be specified. The residues $\beta_+(t)$ and $\beta_-(t)$ can be disentangled from the data in two steps. One of the most prominent features of the experimental differential cross section is the dip at t about $-0.6\ (\mathrm{BeV}/c)^2$. This feature is explained most simply by assuming the dominance of the $B^{(-)}$ amplitude over the $A'^{(-)}$ amplitude at negative t (a nonsense-wrong-signature zero). It will be seen that this assumption is confirmed by a study of the elastic cross sections. The dip is naturally explained by the vanishing of $Q^{+}_{-\alpha-1}{}'$ at $\alpha=0$, which makes $B^{(-)}$ very small (notice that the background terms do not contain a signature factor and therefore do not vanish). The remaining amplitude is essentially pure $A'^{(-)}$. Therefore, at $\alpha=0$

$$\frac{d\sigma}{dt} = \left(\frac{m}{4\pi w}\right)^2 \frac{\pi}{p_s^{\,2}} \left(1 - \frac{t}{4m^2}\right) |A'|^2 . \tag{51}$$

This yields for $\beta_+(\alpha=0)$ (Ref.17)

$$\beta_+(\alpha=0) = 3.48\times 10^{-3}\ (\mathrm{BeV}/c)^2\,\mathrm{mb}.$$

Assuming that $\beta_+(t)$ can be expanded in a simple exponential

$$\beta_+(t) = a\exp(bt), \tag{52}$$

and using $N_A = 1.0\ \mathrm{BeV}/c$, the two experimental values derived above yield

$a = 0.087\,(\mathrm{BeV}/c)^2\,\mathrm{mb}$

and

$b = 1.65\ (\mathrm{BeV}/c)^2$.

Once $\beta_+(t)$ is completely known, it is a straightforward matter to obtain information on $\beta_-(t)$. The expression for $d\sigma/dt$ can be inverted to solve for $\beta_-(t)$. The resulting expression allows a direct computation of $\beta_-(t)$.

$$|\beta_-(t)|^2 = \frac{\dfrac{4\pi w}{m}\dfrac{p_s{}^2}{\pi}\dfrac{d\sigma}{dt}\bigg|_{\exp} - \left(1 - \dfrac{t}{4m^2}\right)|A'|^2}{\dfrac{t}{4m^2}\left(s - \dfrac{(m+\nu)^2}{1 - t/4m^2}\right)|B_0|^2}, \quad (53)$$

where B_0 is $B^{(-)}(s, t)$ for a constant residue equal to unity, and $d\sigma/dt|_{\exp}$ is the experimental cross section.

A plot of $\ln|\beta_-(t)|^2$ versus t, for several values of p_{lab}, is shown in Fig. 6. Only four values of p_{lab} have been included. All the others follow the same pattern. The points between $t = 0.0$ and $t \sim -0.4\ (\mathrm{BeV}/c)^2$ show little or no scatter, in good agreement with the model. It should be emphasized that ideally there is no p_{lab} dependence in $\beta_\pm(t)$. This is a very strict test of the separation of the dynamical variables s and t characteristic of all Regge theories. In the region $-1.0 \geq t > -3\ (\mathrm{BeV}/c)^2$, the agreement does not appear to be as striking. This effect is only apparent though since the data have very large error bars. These errors were not indicated for the sake of clarity. In general, their size is of the order of the dispersion in the points for the given value of t. This is further confirmed by the lack of any pattern in p_{lab} as t varies. Only in the region near the dip $[t \sim 0.6\ (\mathrm{BeV}/c)^2]$ the points show a p_{lab} dependence outside the errors. Experimentally, however, this region is particularly difficult, since the uncertainties in the values of t are of the order of the widths of the dips. In the rest of this section we will disregard this discrepancy, but in Sec. IV it will be shown that the model can be modified to account for the p_{lab} dependence of the dip.

The solid curve shown in Fig. 6 is an approximate fit to this data by a smooth function with a t dependence of the form

$$\ln[\beta_-(t)] = apq + bt + ct^2 + d. \quad (54)$$

The need for the pq variable in the t dependence reflects some residual threshold dependence in $\beta(t)$ even after the threshold's factors are removed. There are two rationales for this fact. In the first place, since the pion mass is so small, this threshold is nearby, and in the second the unitarity cuts of $\beta(t)$ are expected to be strong for pion-pion scattering. Since a straight polynomial fit in t would not have this cut, a pq term is required. This term is essential to fit the region $0 < t < -0.5$ which is the region in which our model and all others coincide. The parameters are determined to be

$a/2 = -6.25$, $b/2 = -1.1$,

$c/2 = -0.08$, $d/2 = 4.14$.

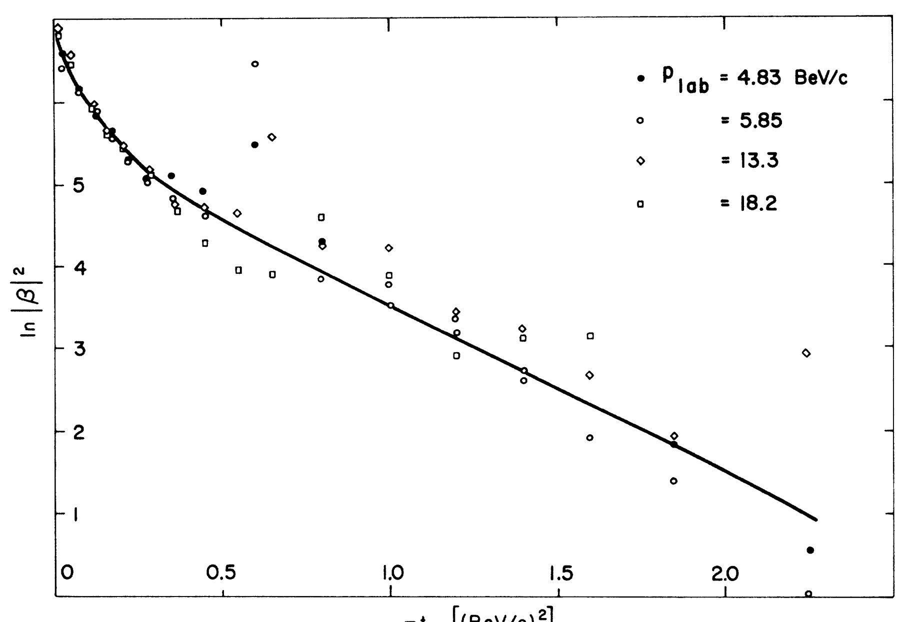

FIG. 6. Fit of the ρ spin-flip residue.

$\beta_-(t)$ is

$$\beta_-(t) = \exp[\tfrac{1}{2}(apq + bt + ct^2 + d)] . \quad (55)$$

The parameter d is only a scale factor. The t^2 dependence is small and is only used to obtain a better fit at very large t values.

As a final comparison with experiment, the solid curves in Figs. 7(a) and 7(b) were obtained using the choice of parameters indicated. From the figures it appears that the shrinkage predicted in this model for very large t is a little too fast. This conclusion, though, depends very much on a very benevolent interpretation of mean values and the sizes of the error bars.

It has been shown in the previous part of this section that it is possible to account for the general features of the π-p charge-exchange scattering in the forward direction by using only the concept of a single (linear) Regge trajectory as the dominant J-plane singularity. A closer study of the results shows several difficulties. These we can separate as follows:

(i) The experimental data seem to indicate that the dip at $t \sim -0.6$ $(\mathrm{BeV}/c)^2$ gets deeper as the energy increases. Although, as has been indicated, one should look at this part of the data with a certain degree of caution, this indicates a trend in complete disagreement with the model. In this model the reversed situation is predicted.

(ii) For $|t| \sim 1.0$ $(\mathrm{BeV}/c)^2$ the cross sections given in this model seem to decrease with energy at a somewhat faster rate than indicated by experiments.

(iii) At least in the region in which experimental data are available, there is no agreement between predicted and observed polarization.

(iv) Finally, it could be argued that we have not really tested Mandelstam symmetry. For values of t such that $\alpha(t) \approx -\frac{1}{2}$, the amplitude has a zero which removes the additional terms introduced by the lack of Mandelstam symmetry. At values of t such that $\alpha(t) = -\frac{3}{2}$ and $-\frac{5}{2}$, the fits are only qualitatively good. In fact, this last point is not very important since there are little or no data for $|t|$ larger than 2 $(\mathrm{BeV}/c)^2$.

At this point, then, one should perhaps stop and be satisfied with the results obtained thus far. The mounting theoretical evidence for the presence of cuts as important singularities in the J plane should be enough motivation for not trying to push this model too far. From the point of view of phenomenology, however, the single-Regge-pole picture has the definite advantage of its simplicity as regards the parametrization of the amplitudes. Furthermore, if by refining our model we can only claim to have an "effective"

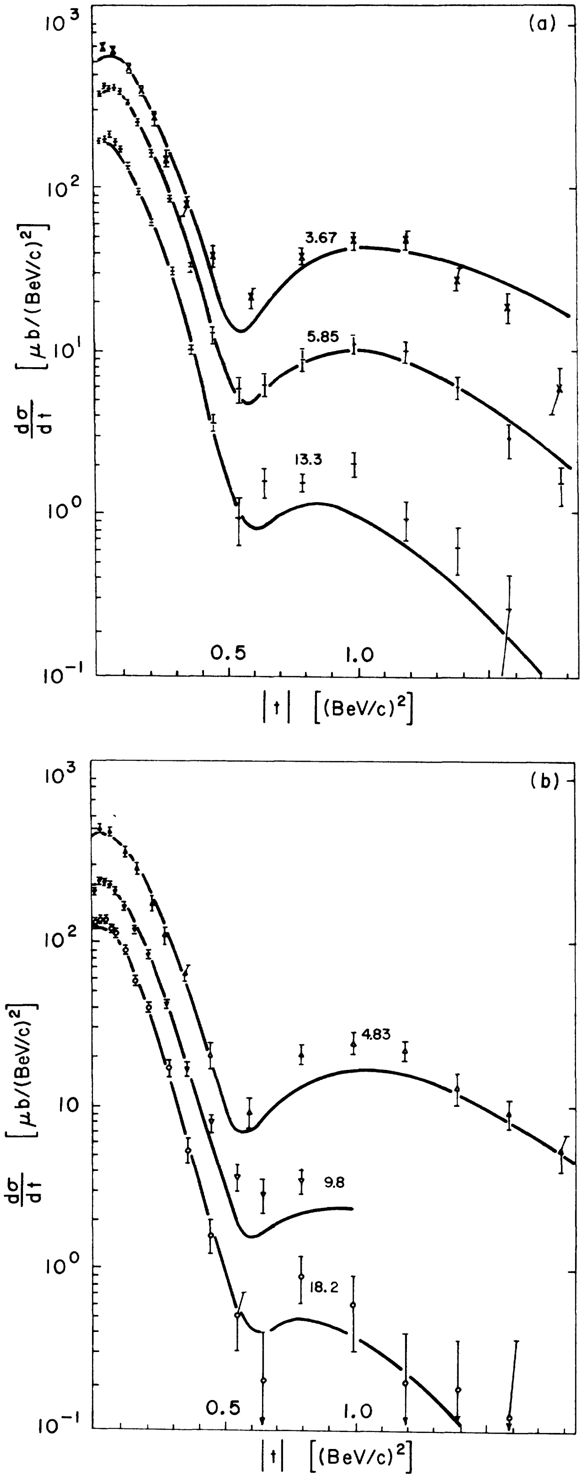

FIG. 7. Fits of the differential cross sections for πN charge-exchange scattering.

description in terms of trajectories and residues, the important correlation between high-energy scattering data in a given channel and particle and resonance parameters in the corresponding crossed channel is fully preserved. Therefore, we feel justified in trying to improve our results without resorting to the addition of other independent contributions.

It is a natural first step to try to isolate some features of the data that could serve as indicators of what changes should be involved to improve our model. We are after clues which can make quantitative the qualitative points of disagreement previously indicated. One of the clearest features of the data is the slow shrinkage of the second lobe of the differential cross section. A method for isolating this behavior of the differential cross section at high energy is to study the dominant terms in the expression for A' and B. The resulting differential cross section is of the form

$$\frac{d\sigma}{dt} \sim f(t) s^{2[\alpha(t)-1]}, \tag{56}$$

where $\alpha(t)$ stands for either the trajectory function or the highest of the exponents in the background terms depending on which one dominates. $f(t)$ is a function of t resulting from the product of the residue and threshold factors. In general, then

$$\ln\frac{d\sigma}{dt} \sim 2[\alpha(t)-1]\ln s + \ln f(t). \tag{57}$$

In Figs. 8(a), 8(b) the experimental data for $\ln(d\sigma/dt)$ are plotted against $\ln s$. The result we obtain is that both the points at the dip [Fig. 8(a)] and those for $|t|$ larger than about 1 $(\text{BeV}/c)^2$ [Fig. 8(b)] are compatible with Eq. (57), for $\alpha(t) \sim -0.5$. The dashed lines in Figs. 8(a), 8(b) correspond to $\alpha(t) = -0.5$ and were plotted only as a reference. These results are an important guide in solving or at least improving the difficulties (i) to (iv). The first two are now apparently solved if we can have a term with $\alpha = -0.5$ which dominates at the dip and at decreasing t. Such a term might also bring some improvement in the polarization. It will be seen that an approximately correct behavior is obtained by recovering the background term at $J = -\frac{1}{2}$ in the t-channel spin-flip amplitude. In the previous analysis this term was canceled by the $2J+1$ factor in the partial-wave expansion. The presence or absence of this factor can be discussed on a basis which is very similar to that used in the discussion of Mandelstam symmetry.

To make this point more explicit, consider the relation between the partial-wave amplitudes in the t channel and the invariant amplitudes $A(s, t)$ and $B(s, t)$. These are given by[18]

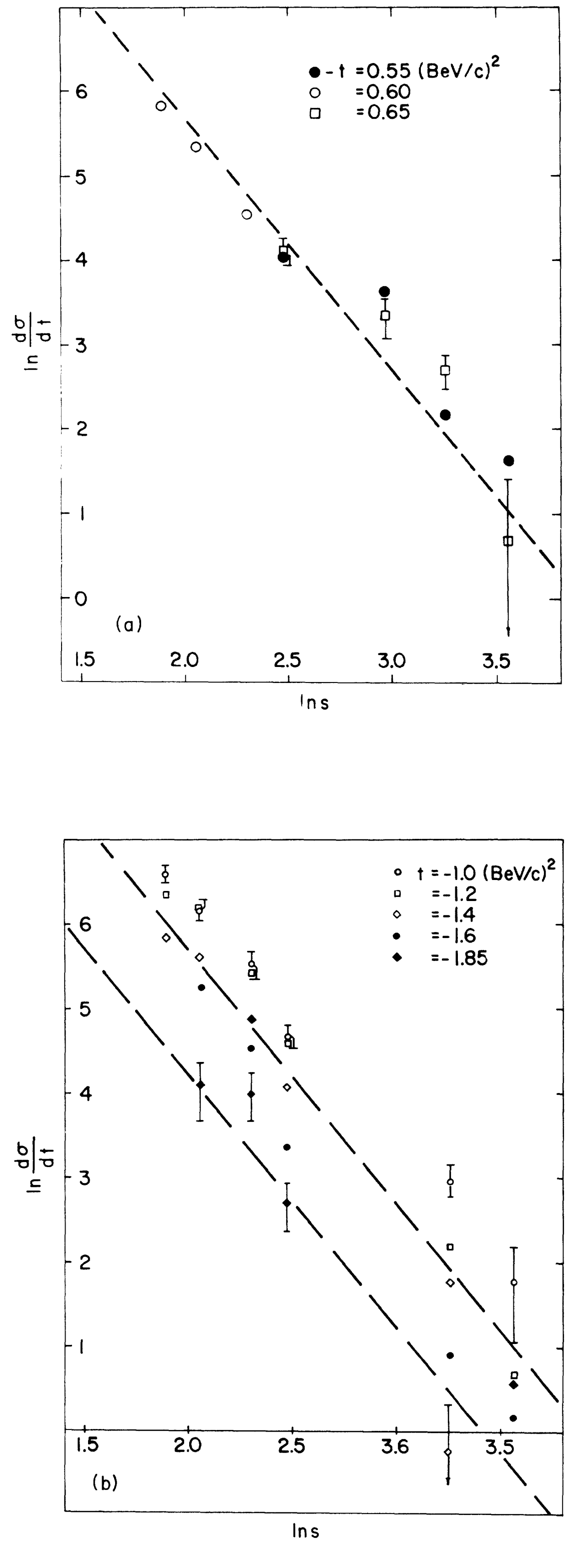

FIG. 8. Energy dependence of the charge-exchange differential cross sections at fixed t.

$$A(s,t)=(8\pi/p^2)\sum_J (J+\tfrac{1}{2})(pq)^J \frac{m z_t}{[J(J+1)]^{1/2}}[P_J'(z_t)f_-^J(t)-P_J(z_t)f_+^J(t)] \tag{58}$$

and

$$B(s,t)=8\pi\sum_J \frac{J+\frac{1}{2}}{[J(J+1)]^{1/2}}(pq)^{J-1}P_J'(z)f_-^J(t)\,. \tag{59}$$

Singh has shown that these expressions can be inverted to obtain the following Froissart-Gribov representation for $f_\pm^J(t)$:

$$f_+^J(t)=-\frac{p^2}{8\pi^2}\frac{1}{(pq)^{J+1}}\int ds'[A_s(s',t)-z'B_s(s',t)]Q_J(z') \tag{60}$$

and

$$f_-^J(t)=\frac{1}{16\pi^2}\frac{[J(J+1)]^{1/2}}{J+\frac{1}{2}}\frac{1}{(pq)^J}\int ds'\,B_s(s',t)[Q_{J-1}(z')-Q_{J+1}(z')]\,, \tag{61}$$

where

$$z'=(s+p^2+q^2)/2pq\,, \tag{62}$$

and A_s and B_s are the s discontinuities of $A(s,t)$ and $B(s,t)$ in their corresponding dispersion-relation representations. Since the factor $[J(J+1)]^{1/2}/(J+\frac{1}{2})$ in $f_-^J(t)$ is canceled in the expression for $B(s,t)$, it does not appear in the Watson-Sommerfeld transformation and produces no observable effects.[19] It is customary to define a new amplitude

$$(f)_-^J=\frac{J+\frac{1}{2}}{[J(J+1)]^{1/2}}f_-^J \tag{63}$$

such that

$$B(s,t)=8\pi\sum_J(pq)^{J-1}(f)_-^J P_J'(z_t)\,. \tag{64}$$

It is generally expected that $(f)_-^J$ will vanish at $J=-\frac{1}{2}$. This is not because of the arguments based on unitarity which imply the absence of fixed poles[20] of f^J, but because of the relation

$$Q_{J-1}(z)=Q_{J+1}(z)\quad \text{for } J=-\tfrac{1}{2}\,,$$

which suggests that the integrand in Eq. (61) vanishes at $J=-\frac{1}{2}$. After the discussions in the preceding sections, it should be clear that such heuristic arguments may not always be correct. In fact, the example given earlier shows that one can construct functions $B(s,t)$ such that $(f)_-^J$ is different from zero at $J=-\frac{1}{2}$ in spite of the form of Eq. (61). The effect of having nonvanishing $(f)_-^J$ at that value of J, on the other hand, would be to create a new background term in the Watson-Sommerfeld transformation for the spin-flip amplitude. This term would contribute an asymptotic behavior $z^{-0.5}$. Since this is exactly what seems to be indicated by the data as necessary, we shall make the very logical assumption as an alternative to the model described earlier that

$$(f)_-^J\simeq\frac{\beta_-(t)}{J-\alpha(t)}\,, \tag{65}$$

with $\beta_-(t)$ a "smooth" function of t. Since the f_+^J amplitude does not have the kinematic factors associated with this difficulty, there is no need to modify this amplitude. It then follows that $A'(s,t)$ is unmodified. Following the procedures developed in the previous sections, we find that the pole of $1/\cos\pi J$ at $J=-\frac{1}{2}$ is no longer canceled.

For the ρ trajectory (isospin = 1), the $B^{(-)}(s,t)$ amplitude is

$$\begin{aligned}B^{(-)}(s,t)=-8\pi\left(\frac{pq}{N_B}\right)^{\alpha}\beta_-(t)\frac{Q_{-\alpha-1}^{+}{}'(z)}{\cos\pi\alpha}-8\beta_-(t)\Bigg\{&-Q_{-1/2}^{+}{}'(z)\left(\frac{pq}{N_B}\right)^{-1/2}\frac{1}{\frac{1}{2}+\alpha}\\&+Q_{1/2}^{+}{}'(z)\left[\left(\frac{pq}{N_B}\right)^{-3/2}\frac{1}{\frac{3}{2}+\alpha}-\left(\frac{pq}{N_B}\right)^{1/2}\frac{1}{\frac{1}{2}-\alpha}\right]\\&-Q_{3/2}^{+}{}'(z)\left[\left(\frac{pq}{N_B}\right)^{-5/2}\frac{1}{\frac{5}{2}+\alpha}-\left(\frac{pq}{N_B}\right)^{3/2}\frac{1}{\frac{3}{2}-\alpha}\right]\Bigg\}\,.\end{aligned} \tag{66}$$

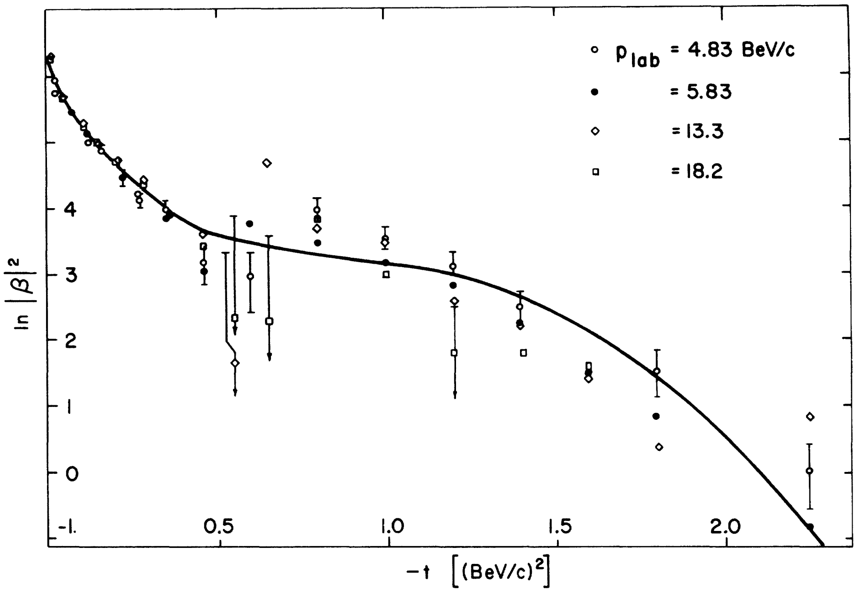

FIG. 9. Fit of $\beta_-(t)$ for the alternative model.

In the earlier model the fit was not sensitive to the value of N_B. With the addition of the new background term, N_B becomes an important factor in the behavior of $B^{(-)}(s, t)$. As is clearly seen in Eq. (66), N_B controls the relative magnitude of the "Regge" and leading "background" terms, and therefore to a considerable extent the phase and functional form of the amplitude.

Although the curves are plotted for only one particular choice of N_B, we have studied the predicted polarizations and differential cross sections for a range of its values. For large values of N_B, the large background terms dominate at small $|t|$. This improves considerably the polarization fit, but has the undesirable effect of wiping out the dip. For small values of N_B, there is a clear dip but no polarization. It was found, that, in any case, the bump predicted for a flat residue tends to be too small if the low-$|t|$ region is fit.

For a comparison with experiments, the value of N_B was fixed at 0.4 (BeV/c)^2. Using the same technique as Eq. (53) the residue $\beta_-(t)$ was factored out and is given by the data. Some of these points are plotted in Fig. 9. To avoid overcrowding the graph, the error bars have been indicated only partially. A look at that figure should now make our remarks clear; although the agreement, as far as consistency of the behavior with energy, is very good, and we emphasize that this is the principal feature of the model, the experimental situation seems to call for a certain amount of structure in $\ln|\beta|^2$. This in itself is not totally objectionable, especially since there are only vague ideas about the functional form of β_-. If this were true, it would unfortunately require a complicated parametrization of the residue. Rather than attempting such a perfect fit, we have decided that a reasonable approximation is still possible with a residue of the form

$$\beta_-(t) = \exp(a + bpq + ct + dt^3). \tag{67}$$

The only difference with $\beta_-(t)$ of Sec. II is that now the t^2 was eliminated in favor of a t^3. This allowed for a better fit with the same number of parameters. The solid curve in Fig. 10 was ob-

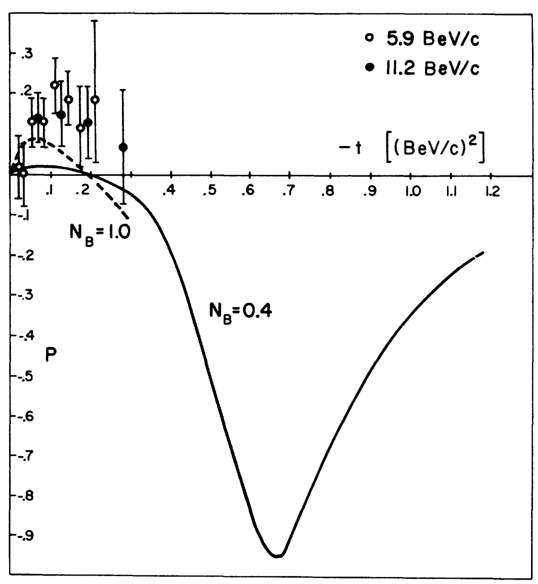

FIG. 10. Polarization predictions in charge-exchange scattering. The curves indicated are for p_{lab} = 5.9 BeV/c.

tained with

$$a=4, \quad b=-6.75,$$
$$c=2.05, \quad d=0.77.$$

This residue was used in the fit of Figs. 11(a) and 11(b). The residue for the spin-nonflip term was set to

$$\beta_+(t)=0.22\exp(1.8t). \tag{68}$$

Finally, the following comments can be made about the polarization:

(i) It is possible to fit the existing polarization data at small $|t|$ by taking N_B of the order of 2 $(\mathrm{BeV}/c)^2$. The price one has to pay is that now the dip-bump structure can only be reproduced by putting essentially all the structure in the residue functions.

(ii) Even with a small N_B, this model predicts a large (as much as -0.90 or more) negative polarization at $t\sim-0.6$ $(\mathrm{BeV}/c)^2$ (see Fig. 10). This is due to the large interference between spin-flip and spin-nonflip amplitudes.

Curiously enough, this prediction appears also in the eikonal model of Arnold and Blackmon.[16] No experimental evidence, either in favor of or against this prediction, is presently available. It would certainly be a remarkable fact if it is confirmed. As far as this model is concerned, the fit of the residue would have to be revised with a larger value of N_B. This would involve accepting more structure in $\beta_-(t)$. For the moment, the resolution of this difficulty will have to wait until more data are available.

IV. ELASTIC SCATTERING IN THE FORWARD DIRECTION

In this section, we extend the analysis with the previous model to pion-nucleon elastic scattering. In keeping with our program of a limited number of trajectories we add to the ρ, the Pomeranchukon and the f_0 or P' trajectory.

There is a considerable amount of experimental information on the processes[21]

$$\pi^+ + p \to \pi^+ + p \tag{69}$$

and

$$\pi^- + p \to \pi^- + p. \tag{70}$$

The differential cross sections for both processes show the characteristic forward "peaking" at high energies, and a clear break at $t\sim-0.8$ $(\mathrm{BeV}/c)^2$ which tends to disappear at larger values of s. An interesting feature is the near mirror symmetry of the measured polarization[22] at $p_{\mathrm{lab}}=5.15$ (BeV/c). This is very clear at $|t|$ less than about 0.8 $(\mathrm{BeV}/c)^2$, but at larger values the rather high uncertainties allow for a less symmetric structure.

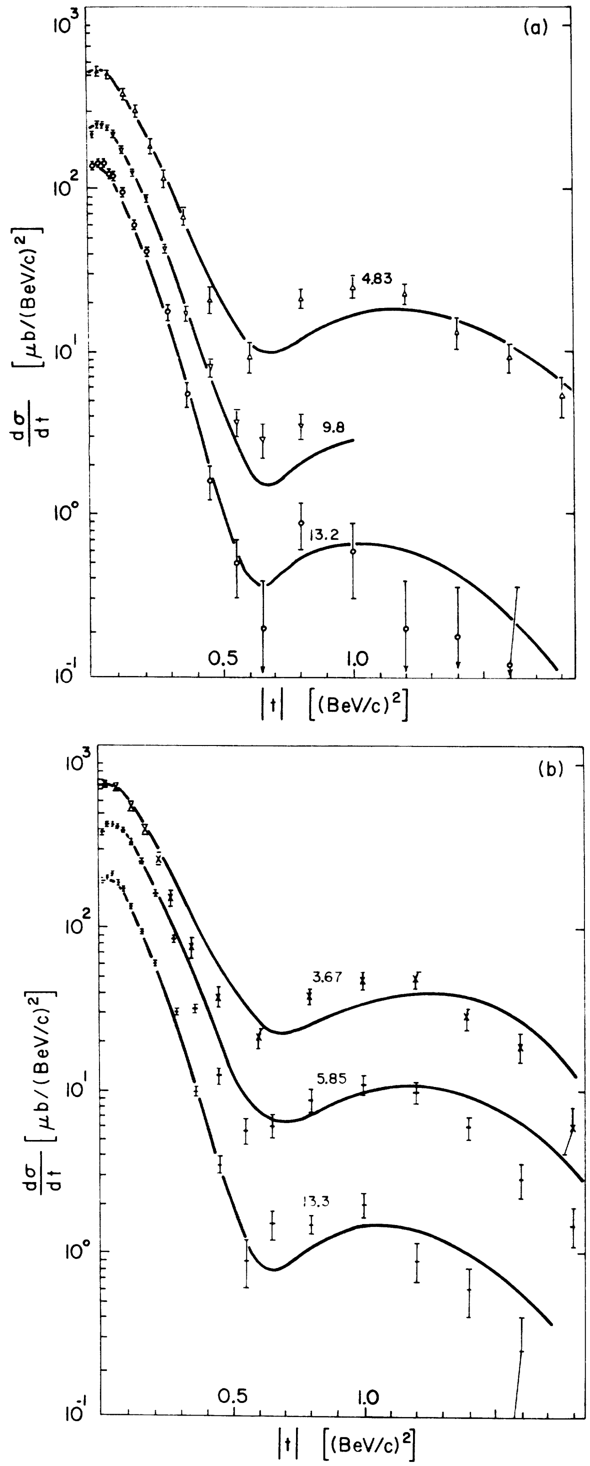

FIG. 11. Fits of the differential cross sections for $\pi^- p$ charge-exchange scattering using the alternative model.

The development of the model follows along the lines of Sec. III. There are some features of the

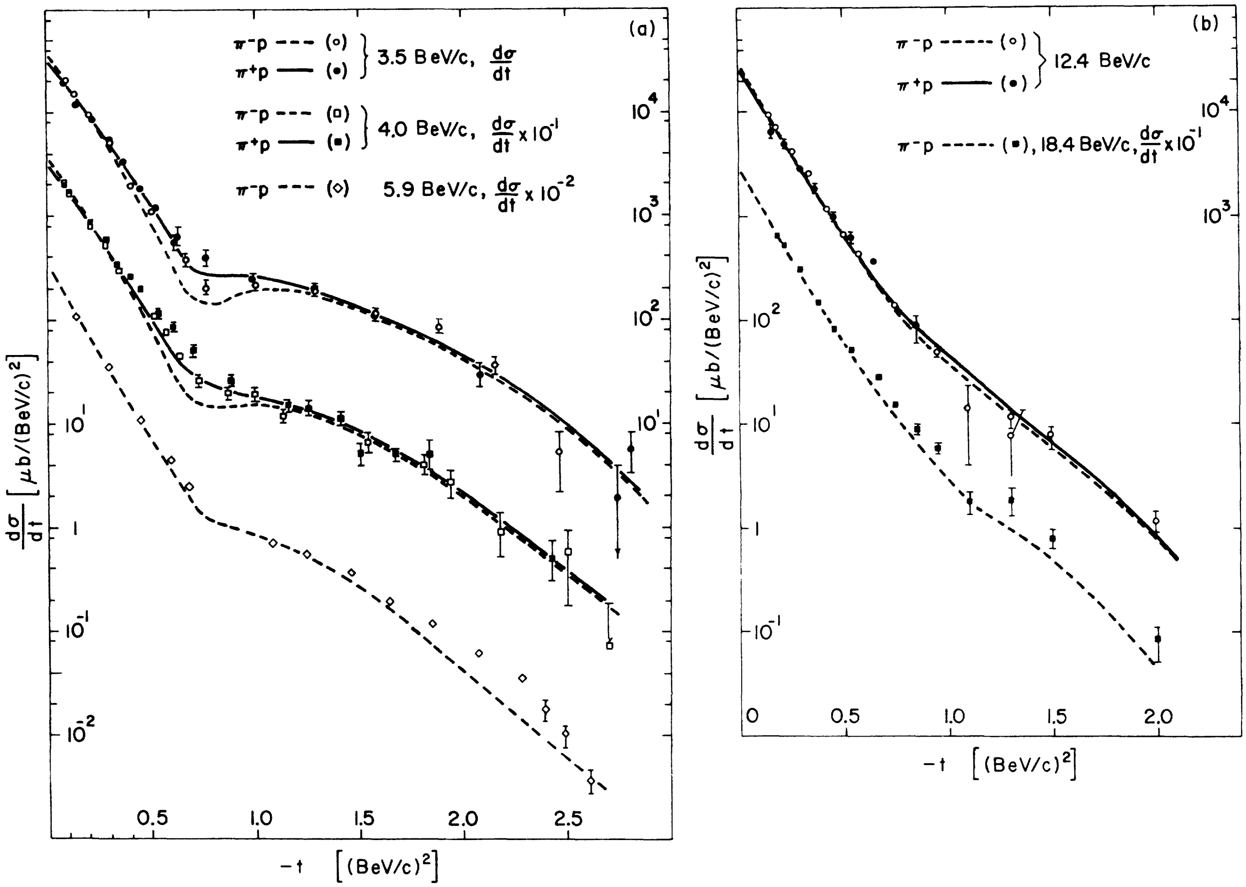

FIG. 12. Fit of the elastic differential cross sections near the forward direction.

elastic scattering that should be mentioned. The residue of the P' has to vanish for t such that $\alpha(t) = 0$. This is a right-signature point for this trajectory, but it corresponds to a spinless (chargeless) particle of negative mass squared (a tachyon?). Although the existence of such objects cannot be ruled out theoretically, we adopt the more conservative point of view that

$$\beta_{P'}(t)\,|_{\alpha(t)=0} = 0\,. \tag{71}$$

In the calculation we neglect the spin-flip amplitudes for both P and P'. This is justified, at least at small values of $-t$, since the presence of spin-flip terms would tend to give round peaks at $t=0$ in contrast to the sharp peaks observed experimentally. (See Fig. 12 and compare with the charge-exchange case.) This rather drastic simplification should be kept in mind when evaluating the agreement of the model with experiment. For the ρ trajectory and residues we used the results of Sec. III. The Pomeranchukon intercept is set equal to one. We have chosen the P' trajectory exchange degenerate with the ρ trajectory. We can now use the optical theorem and separate the contributions of P and P' at $t=0$. They are

$$\sigma_t(\pi^- p) + \sigma_t(\pi^+ p) = \frac{2}{p_{\rm lab}}\,\mathrm{Im}\,A'^{(+)}(s,\, t=0)\,. \tag{72}$$

The Pomeranchukon contribution is given by

$$A_P'^{(+)}(s,\, t=0) = 3\frac{i\pi^2}{m^2}\left(\frac{pq}{N_A}\right)\beta_P^+(t=0)(2z) + O\left((2z)^{-3/2}\right), \tag{73}$$

while the P' contribution is

$$A_{P'}'^{(+)}(s,\, t=0) = \frac{4\pi}{m^2}\beta_{P'}^+(t=0)(i-1)\frac{1}{\sqrt{N_A'}}(s-m^2-\mu^2)^{1/2}\,. \tag{74}$$

We take $N_A = N_A' = 1$ (BeV/c)^2, and measure all the remaining parameters in units of 1 (BeV/c)^2.

We can then write

$$\sigma_t(\pi^- p)+\sigma_t(\pi^+ p)=\left|\frac{2}{p_{\text{lab}}}\right|\left[\frac{4\pi^2}{m^2}\frac{3}{4}\beta_P^+(t=0)(s-m^2-\mu^2)+\beta_{P'}^+(t=0)(s-m^2-\mu^2)^{1/2}\right] \tag{75}$$

or with approximations similar to those used in Sec. II,

$$\Sigma \equiv \sigma_t(\pi^- p)+\sigma_t(\pi^+ p)\equiv 50.52\beta_P^+ +49.16\beta_{P'}^+\left(\frac{1}{p_{\text{lab}}}\right) \tag{76}$$

where the total cross sections are measured in millibarns and p_{lab} in BeV/c. We have plotted Σ as a function of the inverse square root of the pion laboratory momentum in Fig. 13. The solid line corresponds to a fit with

$$\beta_P^+(t=0)=0.8 \text{ BeV}/c$$

and

$$\beta_{P'}^+(t=0)=0.7 \text{ BeV}/c\,.$$

The slight discrepancy at the lower values of p_{lab} could very well be due to the "tails" of the resonances in the low-energy region.

We do not apply in this case the method of fixing the same parameters by fitting the differential elastic cross sections at $t=0$. The reason is that these cross sections are known with much larger errors than the total cross section. The values of β_P and $\beta_{P'}$ given above are consistent with these measurements.

Our next problem is to obtain a reasonable functional form for the residues. The presence of the break at $t\sim -0.8$ (BeV/c)^2 in the differential cross sections can be reproduced if the residue of the Pomeranchukon is of the form[23]

$$\beta_P(t)=\alpha_P(t)(b_1 e^{a_1 t}+b_2 e^{a_2 t})\,. \tag{77}$$

The P' residue is slightly more complicated. It

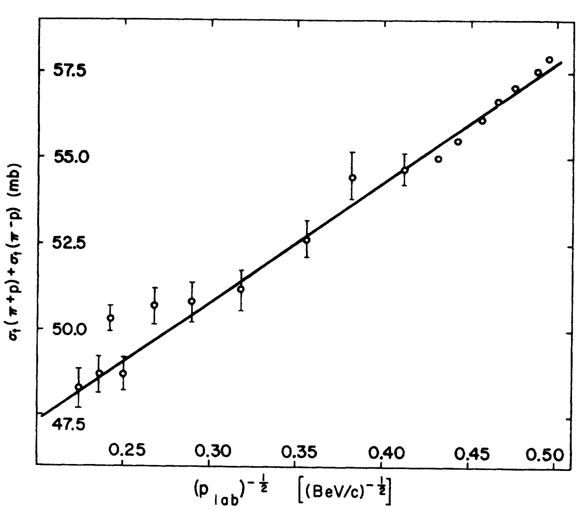

FIG. 13. Fit of the sum of the total elastic cross sections.

is well known[24] that an extra change in sign seems to be necessary in order to fit the data. In the noncompensation mechanism this is achieved by having a factor of α^2 in the residue. We have found that the agreement is better if we use instead a factor

$$(1+c_P t)\alpha_{P'}(t)\,. \tag{78}$$

In other words, the agreement is better if we assume that P' contains a "crossover factor" (similar to that present in the spin-nonflip amplitude for the ρ trajectory), rather than assuming a double zero at $\alpha_{P'}=0$. The form used for the P' residue is

$$\beta_{P'}(t)=(1+c_P t)\alpha_{P'}(t)(b_3 e^{a_3 t + a_4 t^2})\,. \tag{79}$$

Not all these parameters are independent since we required

$$b_1+b_2=0.8$$

and

$$\alpha_{P'}(t=0)b_3=0.7\,.$$

To fit the remaining parameters we used the relation

$$\frac{d\sigma}{dt}(I=0)=\frac{1}{2}\left(\frac{d\sigma}{dt}(\pi^+p\to\pi^+p)+\frac{d\sigma}{dt}(\pi^-p\to\pi^-p) - \frac{d\sigma}{dt}(\pi^-p\to\pi^0 n)\right)\,. \tag{80}$$

In the approximation of neglecting the spin-flip terms for P and P', we have

$$\frac{d\sigma}{dt}(I=0)=\frac{\pi}{p_s}\frac{m}{4\pi w}\left(1-\frac{t}{4m^2}\right)|A_P'+A_{P'}'|^2\,. \tag{81}$$

For simplicity we considered only the points indicated in Fig. 14 and used the fit of charge-exchange scattering of Sec. III instead of experimental points for $d\sigma/dt(\pi^-p\to\pi^0 n)$. The parameters were then adjusted by numerical methods.[25] The solid lines in Fig. 14 correspond to the choice of

$$b_1=0.752\,,\quad a_1=3.15\,,\quad a_4=-1.07\,,$$
$$b_2=0.048\,,\quad a_2=0.59\,,$$
$$b_3=1.40,\quad a_3=2.38\,,\quad c_P=1.5\,.$$

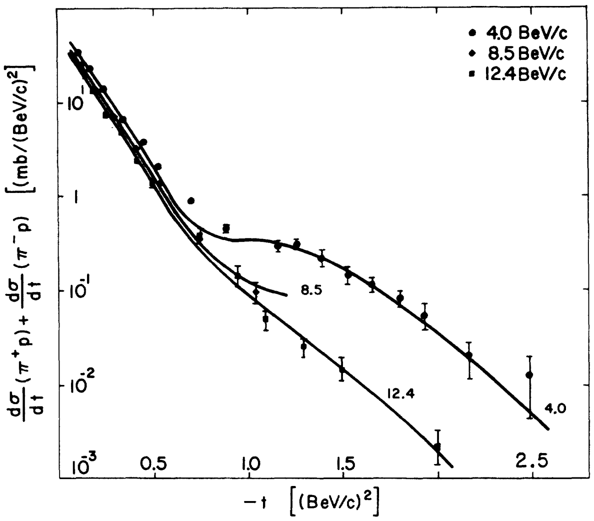

FIG. 14. Fit of the sum of the elastic differential cross sections.

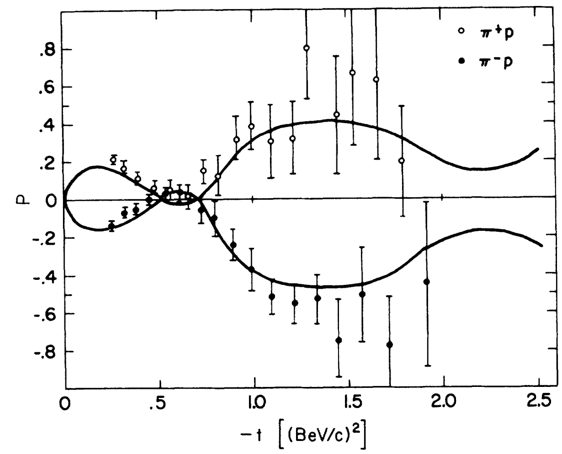

FIG. 15. Fit of the polarization in elastic scattering at $p_{\rm lab}$ = 5.15 BeV/c.

These parameters were then used to obtain the curves indicated in Figs. 12(a) and 12(b). Finally, and this is perhaps the most interesting result, we have compared in Fig. 15 the predicted polarization with the actual measurements for $p_{\rm lab}=5.15$ BeV/c. The agreement is even more striking from the fact that we have never used any knowledge of the polarization in fitting the parameters. In previous models, the large polarization at $-t\sim 1.5-2.0$ (BeV/c)^2 was found hard to reproduce. In the present model, the phases of the ρ spin-flip amplitude and P' spin-nonflip amplitude, which are the important factors in determining the polarization, are strongly influenced by the lack of Mandelstam symmetry. The result is the correct prediction for the polarization from a fit in which only cross-section data were used.

CONCLUSIONS

In this paper we have attempted to formulate a minimal Regge model that meets the requirements of the data by postulating a simple structure of the amplitude in the l plane that is consistent with the observed particle spectrum in the crossed channel. This model accounts directly for the observed angular distributions and polarizations in pion-nucleon forward scattering. The unique characteristic of this model is the violation of Mandelstam symmetry. A simple theoretical model shows that this violation of Mandelstam symmetry is possible, and the data analysis shows that the extra terms due to the lack of symmetry are just those necessary to bring about agreement with the data.

The addition of new features in the l plane such as cuts can be used to improve the fits, but were not found necessary for pion-nucleon forward scattering.

The analysis of pion-nucleon backward scattering is complicated by the unequal-mass kinematics and lack of data. A program of analysis for this reaction is presently underway.

---

*Work supported in part by the U. S. Atomic Energy Commission under Contract No. ORO-3992.

†Part of this work was submitted in partial fulfillment of the degree requirements for a Ph.D. at Syracuse University.

[1]Particle Data Group, Rev. Mod. Phys. 43, S1 (1971).

[2]T. Regge, Nuovo Cimento 8, 671 (1958); 14, 951 (1959).

[3]Several books have a complete discussion of the techniques utilized in Regge phenomenology. Among these are R. J. Eden, *High Energy Collisions of Elementary Particles* (Cambridge Univ. Press, London, 1967); E. J. Squires, *Complex Angular Momentum and Particle Physics* (Benjamin, New York, 1963).

[4]C. Jones and V. Teplitz, Phys. Rev. 159, 1271 (1967); S. Mandelstam and L. L. Wang, *ibid.* 160, 1490 (1967).

[5]We have dropped a similar term integrated from $-\infty$ to $-z$ (left-hand cut), and neglected subtractions, all of which are marginal to our discussion.

[6]M. Froissart, talk at La Jolla Conference, 1961 (unpublished); V. N. Gribov, Zh. Eksp. Teor. Fiz. 42, 1260 (1962) [Sov. Phys. JETP 15, 873 (1962)].

[7]S. M. Roy, Phys. Rev. 161, 1575 (1967).

[8]*Higher Transcendental Functions* (Bateman Manuscript

Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 2, p. 116, Eq. (15).

[9]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 2, p. 108, Eqs. (2) and (5).

[10]G. Höhler, H. Schaite, and P. Sonderegger, Phys. Letters 20, 79 (1966).

[11]V. Singh, Phys. Rev. 129, 1889 (1963).

[12]R. Phillips and W. Rarita, Phys. Rev. 139, B1336 (1965).

[13]The exact value of ρ is not critical and other choices are certainly possible, but to keep the number of parameters to be fit by numerical methods down to a minimum, we will assume this to be the correct one.

[14]V. S. Barashenkov, *Interaction Cross Sections of Elementary Particles* (Israel Program of Scientific Translations, Jerusalem, 1968).

[15]T. Gadjicar, R. Logan, and T. Moffat, Phys. Rev. 170, 1599 (1968).

[16]See, for example, R. C. Arnold and M. L. Blackmon, Phys. Rev. 175, 2082 (1968); M. Ross, F. Henyey, and G. Kane, Nucl. Phys. B23, 269 (1970).

[17]Whenever possible, we try to get approximate solutions at special points rather than attempting general optimizations (i.e., least squares, etc.), the significance of which is not always clear when dealing with a new model.

[18]W. Frazer and J. Fulco, Phys. Rev. 117, 1603 (1960).

[19]See Singh (Ref. 11), and for a more general treatment, A. H. Mueller and T. L. Trueman, Phys. Rev. 160, 1296 (1967).

[20]See R. Omnes, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Plenum, New York, 1963).

[21]A recent compilation of data on pion-nucleon scattering containing a wealth of references is G. Giacomelli, P. Pini, and S. Stagni, CERN/HERA Report No. 69-1 (unpublished).

[22]R. J. Esterling, N. E. Booth, G. Conforto, J. Parry, J. Scheid, and A. Yokosawa, in *Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968).

[23]The factor α_P is really not necessary since in the range considered, α_P is always close to unity.

[24]We are grateful to Professor M. T. Parkinson for making his optimizing subroutines available to us and for instructing us in their use.

[25]C. B. Chiu, S. Y. Chu, and L. L. Wang, Phys. Rev. 161, 1563 (1967).

PHYSICAL REVIEW D VOLUME 6, NUMBER 1 1 JULY 1972

ϕ^3 Analyticity and Finite-Energy Sum Rule for Inclusive Reactions

A. I. Sanda
National Accelerator Laboratory, Batavia, Illinois 60510
(Received 22 December 1971)

Using ϕ^3 theory as a model, the analytic structure of the six-point function is investigated. Specifically studied is the kinematical region appropriate to the single-particle inclusive reaction where the missing mass is much less than the incident energy. With some idea about the analyticity, a finite-energy sum rule is derived. This sum rule can be used to study the concept of generalized duality. The most striking feature of the sum rule is a possibility that the "triple-Regge vertex function" can be calculated by the data on the inclusive reaction with relatively low missing mass, i.e., the resonance-production region.

I. INTRODUCTION

It has been conjectured that the cross section for

$$a+b\to c+\text{anything} \tag{1}$$

is related to the absorptive part of a scattering amplitude for

$$a+b+\bar{c}\to a+b+\bar{c} \tag{2}$$

when the amplitude is analytically continued to the proper kinematical region.[1] Then various asymptotic behaviors of (1) can be obtained from that of (2). It is assumed that the asymptotic behaviors of (2) can be obtained by the O(2, 1) expansion.[2] Subsequently, it has been verified in the context of field theory that the amplitude for reaction (2), when continued analytically, indeed has an absorptive part which is proportional to the cross section for reaction (1).[3]

We see the analogy between the four-point function and the six-point function developing. The inclusive cross section and the six-point function satisfy a relationship similar to that between the total cross section and the four-point function. The O(2, 1) expansion in the six-point function corresponds to the Regge expansion in the four-point function. We therefore see that the machinery developed for the four-point function (forward-dispersion relations, finite-energy sum rules, etc.)