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Questions Pertaining to Charge Dependence of Radiative Corrections to Superallowed Fermi Transitions

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We consider the problem posed by the near equality of the *ft* values for the nine $0^P \rightarrow 0^P$ (P = + or −) superallowed Fermi transitions. So long as one is oblivious to radiative effects, this near equality of *ft* values may be (as, indeed, it has been) regarded as a triumph of the CVC (conserved-vector-current) hypothesis. However, if one takes note of the fact that CVC, which guarantees the equality of the "bare" *ft* values, is broken by electromagnetism and that the charge of the decaying nuclei varies from $Z = 1$ (for $\pi^{\pm}$) to $Z = 27$ (for Co^{54}), the relationship between CVC and the physical *ft* values becomes obscure; CVC can be upheld only if one can establish that the renormalization effects are small. In a previous contribution we used a combination of general theorems and model-dependent arguments to show that, in the limit of vanishing lepton momenta, Z-dependent renormalization effects arise only in order $Z^2\alpha^2$ (3.9% for Co^{54}). The present paper contains a fuller explanation of the aforementioned theorems as well as amplified and improved proofs. Also, topics not treated in our earlier work (corrections for finite lepton momenta, effects of real photon emission, etc.) are discussed and the model-dependent part of the argument is reviewed.

I. INTRODUCTION

One of the better known conundrums in the theory of radiative corrections to nuclear β decay is posed by the near equality of the *ft* values in the nine superallowed Fermi transitions listed in Table I. So long as one is oblivious to radiative effects there is no difficulty in regarding this near equality of *ft* values as a triumph of the CVC (conserved-vector-current) hypothesis.[1] However, a naive perturbation treatment of radiative corrections, to the CVC value of the matrix element, results in a power series in the parameter $Z^2\alpha$; since this parameter varies from $\sim\frac{1}{137}$ for π^+ to ~ 5 for Co^{54}, one might in fact argue that CVC – which guarantees the equality of the bare *ft* values – would lead to vastly different physical *ft* values for the transitions of Table I.

It is worth emphasizing that the roots of the problem lie in the possibility of a coherent pile-up of electromagnetic effects. Only when the nucleons act coherently can their total charge Ze interact with itself, or with a residual charge $(Z\mp 1)e$ following $\beta^{\pm}$ emission, to produce a term of order $Z^2\alpha$ in the *ft* value.

Some progress towards a resolution of this problem was made in a previous note.[2] We showed that a considerable amount of apparent Z dependence in the *ft* values could be exorcised by making *full* use of the CVC hypothesis supplemented with the postulate that the temporal components of the isospin current generate a local SU(2) algebra. We addressed ourselves to that part of the problem – or, more precisely, to an idealized version of the problem – whose resolution does not hinge on the details of the nuclear dynamics. Our contribution was spelled out in terms of two theorems:

Theorem I. If CVC is broken only by electromagnetism, then (a) to first order in α, (b) to zeroth order in the lepton momenta, and (c) with neglect of induced effects stemming from the axial-vector current, the *ft* value is independent of Z.

This theorem means that subject to (a), (b), and (c) all the radiative corrections of order $Z\alpha$ to the decay half-life are just the Coulombic final-state-interaction corrections; they cancel, therefore, in the comparative half-life or the *ft* value.

Theorem II. The Z-dependent renormalization effects, which do appear when one goes to higher orders in the electric charge, are such that any

TABLE I. ft values for $0 \to 0$ Fermi transitions.[a]

β Transition	ft
$\pi^+ \to \pi^0$	$(3.0 \pm 0.2) \times 10^3$
$^{10}C \to {}^{10}B$	2973^{+50}_{-45}
$^{14}O \to {}^{14}N$	3056 ± 12
$^{26}Al \to {}^{26}Mg$	3052 ± 9
$^{34}Cl \to {}^{34}S$	3096 ± 19
$^{42}Sc \to {}^{42}Ca$	3073 ± 9
$^{46}V \to {}^{46}Ti$	3083 ± 8
$^{50}Mn \to {}^{50}Cr$	3070 ± 9
$^{54}Co \to {}^{54}Fe$	3072 ± 18

[a] See R. Blin-Stoyle, in *Proceedings of the Topical Conference on Weak Interactions, CERN, 1969* (CERN, Geneva, 1969).

potential power of Z is matched or exceeded by a power of α. That is to say, a renormalization induced by n virtual photons is (formally) at most of order $(Z\alpha)^n$, not $(Z^2\alpha)^n$, as a naive counting might indicate.

These theorems together would imply that radiative corrections to the ft values arise only in order $Z^2\alpha^2$. However, before the theorems can be regarded as relevant, the following points must be considered:

(i) Theorem II permits us to write the ft values in the form

$$ft = \sum_{m,n=0}^{\infty} a_{mn}\,\alpha^m (Z\alpha)^n ,$$

where the numbers a_{mn} are *a priori* unknown. One does know, however, that the a_{mn} depend on the details of the nuclear dynamics and may therefore vary from nucleus to nucleus. How sharp is the Z dependence implicit in the a_{mn}?

(ii) Theorem I is equivalent to the statement that a_{01} vanishes if l, the charged-lepton momentum, and q, the total momentum transferred to the leptons, are set equal to zero in the matrix element. This raises questions such as: Is it not possible that terms of the form $Z\alpha(l/M)\ln(l/M)$ or $Z\alpha l/M$ (M some unknown mass scale) could become important for finite l?

(iii) How does the production of *real* soft photons which, due to imperfect energy resolution, necessarily confound the interpretation of the experimentally measured ft value, alter our result?

(iv) Is it legitimate to neglect induced effects stemming from the axial-vector current?

These difficulties were not discussed in Ref. 2. Also, because of space limitations, the treatment in Ref. 2 was rather abstracted and, in consequence, somewhat imprecise and obscure.

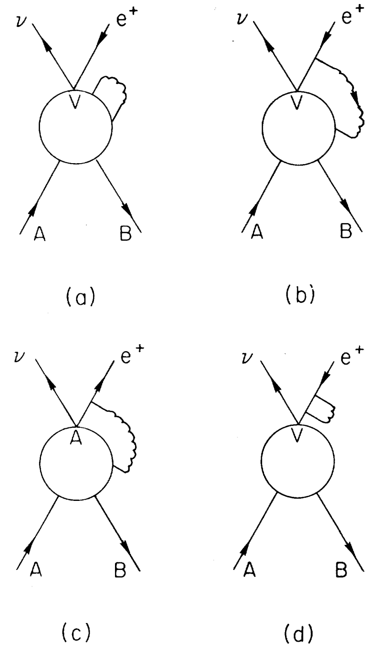

FIG. 1. Diagrams contributing to the Fermi transition in order α.

The purpose of the present paper is (a) to provide full details of the proofs of Theorems I and II and (b) to discuss the applicability of these theorems to actual decays.

II. RADIATIVE CORRECTIONS IN ORDER α

A. Derivation of Theorem I

In the local Fermi theory the contribution of virtual photons to the radiative corrections of order α are depicted in Figs. 1. Because all of the superallowed Fermi transitions (with the exception of $\pi^- \to \pi^0 + e^- + \bar{\nu}_e$) are positron emitters, we will write our expressions for this case. Of course, a completely analogous analysis can be made for negatron emitters.

It is, by now, well known that the corrections of Fig. 1(a) to the Fermi amplitude can be expressed at zero momentum transfer in the Feynman gauge as[3]

$$M_a = \frac{i\alpha G_V}{8\pi^3\sqrt{2}} L_\lambda \int \frac{d^4k}{k^2+i\epsilon} \frac{\partial}{\partial k_\lambda} V^\mu{}_\mu(k) , \qquad (2.1)$$

$$V^{\mu\rho}(k) = i\int d^4x\, e^{-ik\cdot x} \langle B|T^*(J^\mu(x) V^\rho_-(0))|A\rangle , \qquad (2.2)$$

where J^μ is the electromagnetic current, $V^\rho_- \equiv V^\rho_1 - iV^\rho_2$ is the appropriate $\Delta S = 0$ vector current,

$$L_\lambda \equiv \bar{u}_\nu \gamma_\lambda (1 - i\gamma_5) v_e$$

is the lepton current, $G_V = G_\mu \cos\theta$ is the weak-vector coupling constant, and A and B stand for the initial and final nuclear states. The contribution of Fig. 1(b) to the Fermi amplitude is given by

$$M_b^{(V)} = \frac{i\alpha G_V}{4\pi^3\sqrt{2}} \int \frac{d^4k}{k^2+i\epsilon} \bar{u}_\nu (1+i\gamma_5)\gamma_\lambda \times \frac{2l_\mu + \gamma\cdot k\gamma_\mu}{k^2+2l\cdot k+i\epsilon} v_e V^{\mu\lambda} . \qquad (2.3)$$

Using the relation

$$\gamma_\mu\gamma_\lambda\gamma_\rho = g_{\mu\lambda}\gamma_\rho - g_{\mu\rho}\gamma_\lambda + g_{\lambda\rho}\gamma_\mu + \epsilon_{\mu\lambda\rho\alpha}\gamma^\alpha\gamma_5 , \qquad (2.4)$$

Eq. (2.3) can be written as

$$M_b^{(V)} = \frac{i\alpha G_V}{4\pi^3\sqrt{2}} \int \frac{d^4k}{k^2+i\epsilon} \frac{1}{k^2+2k\cdot l+i\epsilon} \bar{u}_\nu(1+i\gamma_5)(2l_\mu\gamma_\lambda + k_\lambda\gamma_\mu + k_\mu\gamma_\lambda - g_{\mu\lambda}\gamma\cdot k + \epsilon_{\lambda\rho\mu\alpha}k^\rho\gamma^\alpha\gamma_5)v_e V^{\mu\lambda} . \qquad (2.5)$$

Using the Ward identities for $V^{\mu\lambda}$, Eq. (2.5) finally reduces to

$$M_b^{(V)} = \frac{i\alpha G_V}{4\pi^3\sqrt{2}} L_\lambda \int \frac{d^4k}{k^2+i\epsilon} \frac{1}{k^2+2l\cdot k+i\epsilon} (2l_\mu V^{\mu\lambda} - k^\lambda V^\mu{}_\mu - 2\langle B|V^\lambda|A\rangle) , \qquad (2.6)$$

where we have neglected some terms that vanish in the limit of zero momentum transfer to the leptons.

As we are only interested here in the corrections of order α and there are explicit factors of α in Eqs. (2.1) and (2.6) we can treat all the matrix elements in these expressions to zeroth order in α. In particular, the last term in Eq. (2.6) is the zeroth-order matrix element multiplied by an explicit function of α. It is clear that it will give rise to corrections of order α but not of order $Z\alpha$.

Next we turn our attention to the term proportional to $l_\mu V^{\mu\lambda}$ in Eq. (2.6). Because of the explicit factor l_μ it may appear at first glance that this contribution vanishes in the limit of zero lepton momentum. A closer examination shows, however, that the "pole terms" $V_P^{\mu\lambda}$ depicted in Fig. 2 give rise, after the k integration is performed, to terms of order $1/l$. Therefore we write

$$V^{\mu\lambda} = V^{\mu\lambda} - V_P^{\mu\lambda} + V_P^{\mu\lambda} . \qquad (2.7)$$

Note that $V^{\mu\lambda} - V_P^{\mu\lambda}$ is not singular as $k\to 0$. It is then easy to verify that the contribution of $l_\mu(V^{\mu\lambda} - V_P^{\mu\lambda})$ to Eq. (2.5) vanishes in the limit of zero lepton momentum while that of $l_\mu V_P^{\mu\lambda}$ must be retained. Keeping only terms that behave as $1/k$ as $k\to 0$, we have

$$V_P^{\mu\lambda} = -2p^\mu\langle B|V_-^\lambda|A\rangle\left(\frac{Z}{k^2-2p\cdot k+i\epsilon} + \frac{Z+1}{k^2+2p\cdot k+i\epsilon}\right), \qquad (2.8)$$

where p^μ is the nuclear four-momentum and Z is the charge of the daughter nucleus. In Eq. (2.8)

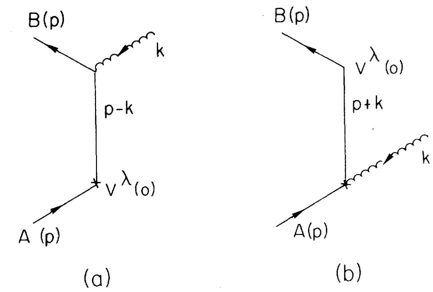

FIG. 2. The pole terms.

we have ignored form factors and terms of order $O(k)$ in the numerators as their contribution to Eq. (2.6) vanishes as $l\to 0$. Using Eq. (2.8) and carrying out the k integration one finds (as explained in detail in Appendix A) that the contribution of the term $l_\mu V_P^{\mu\lambda}$ in Eq. (2.6) to the total transition probability is $-|M^{(0)}|^2[Z\alpha\pi/\beta + O(\alpha)]$, where $M^{(0)}$ is the zeroth-order amplitude and β is the electron velocity in units of c. The first term in this result is the familiar Coulombic correction of order $Z\alpha$ which is automatically included in the calculation of the ft values, while the term $O(\alpha)$ is part of the radiative corrections of order α and is clearly independent of Z.

Next we consider the term involving $k^\lambda V^\mu{}_\mu$ in Eq. (2.6), which we rewrite as $I_1 + I_2$, where

$$I_1 = \frac{-i\alpha G_V L_\lambda}{4\pi^3\sqrt{2}} \int \frac{d^4k}{(k^2+i\epsilon)^2} k^\lambda V^\mu{}_\mu , \qquad (2.9)$$

$$I_2 = \frac{-i\alpha G_V L_\lambda}{4\pi^3\sqrt{2}} \int \frac{d^4k}{k^2+i\epsilon} k^\lambda V^\mu{}_\mu \times\left(\frac{1}{k^2+2l\cdot k+i\epsilon} - \frac{1}{k^2+i\epsilon}\right) . \qquad (2.10)$$

Using the decomposition (2.7) in Eq. (2.10), one finds that the contribution of $V^\mu{}_\mu - V^\mu_{P\mu}$ to I_2 vanishes in the limit $l\to 0$. Thus only $V^\mu_{P\mu}$ need be retained in the evaluation of I_2. Inserting the expression (2.8) into Eq. (2.10) and carrying out the k integrations, which are now explicit, one verifies (as is done explicitly in Appendix B) that all the I_2 contributions of order $Z\alpha$ to the total transition probability vanish as $l\to 0$.

The contribution I_1 is best combined with M_a given in Eq. (2.1). If it were not for the fact that these integrals are actually ultraviolet divergent, the two contributions would cancel each other as can be checked by performing a partial integration in Eq. (2.1). In fact, a finite surface term survives the cancellation. This is best treated by introducing a Feynman cutoff $\Lambda^2/(\Lambda^2-k^2)$ in the integrand of Eq. (2.1). One can then show that

$$M_a+I_1=\lim_{\Lambda\to\infty}\frac{i\alpha}{4\pi^3\sqrt{2}}G_V L_\lambda \Lambda^2\frac{\partial}{\partial\Lambda^2}\times\int\frac{d^4k}{(k^2)^2}k^\lambda\frac{\Lambda^2}{\Lambda^2-k^2}V^\mu{}_\mu\ . \qquad (2.11)$$

In order to evaluate this term all that is necessary are the terms in $V^\mu{}_\mu$ of order $1/k$ as $k\to\infty$. These, in turn, are controlled by the operators of dimensionality $\leqslant 3$ in the short-distance operator-product expansion[4] of $T(J^\mu(x)V_\mu(0))$. By abstraction from the quark models we will assume that for short distances $x_\mu\to 0$:

$$T(J_\mu(x)V^\mu_-(0))=\frac{c_1x_\mu V^\mu_-(0)}{x^4}+\text{less singular terms}, \qquad (2.12)$$

where c_1 is an undetermined constant which may be affected by the strong interactions. Note that Eq. (2.12) also follows from Wilson's enumeration of the fields of low dimensionality.[4] Under the assumption of Eq. (2.12) one can show that the hadronic part of Eq. (2.11) involves only the zeroth-order matrix element $\langle B|\,V^\lambda_-|A\rangle$ and, therefore, it cannot give rise to terms of order $Z\alpha$.

An alternative procedure is to evaluate the terms of order $1/k$ (as $k\to\infty$) in $V^\mu{}_\mu$ by appealing to the Bjorken-Johnson-Low limit.[5] One finds that such contributions are controlled by the equal-time commutator $[J_\mu(\vec{x}),V^\mu_-(0)]$. This commutator is model-dependent. It has been evaluated in the past in various models, e.g., field algebra and in the quark model with naive manipulation of operators, and in the quark gluon model taking into account the first nontrivial effects of the strong interactions.[6] Although the coefficients of this commutator differ in the three cases it is interesting to point out that the operator structure of the commutator is the same: namely, it is proportional to $V^0_-(0)$. If, by abstraction, we assume that

$$[J_\mu(\vec{x}),V^\mu_-(0)]=(\text{const})V^0_-(0)\delta^3(\vec{x})\ , \qquad (2.13)$$

one can again show that the hadronic part of Eq. (2.11) involves only the zeroth-order matrix element $\langle B|\,V^\lambda|A\rangle$ and that, therefore, it cannot give rise to corrections of order $Z\alpha$.

Finally, we consider the diagram of Fig. 1(d). Clearly this diagram is of order α and has nothing to do with the electromagnetic properties of nuclei. Theorem I is therefore established.

It is convenient at this stage to summarize the assumptions used in the derivation of this theorem. On the one hand, as we have already emphasized, a crucial role is played by the Ward identities associated with the time-time algebra. (The time-space algebra used in the derivation follows from the time-time algebra by virtue of Lorentz invariance.) On this basis we have been able to discuss in the limit of zero lepton momenta all the contributions arising from the vector current in a model-independent manner, with the exception of the surface term of Eq. (2.11). Curiously enough, this model-dependent surface term is governed by the very-high-frequency photon contributions. In order to discuss the Z dependence of this surface contribution in a manner independent of the details of nuclear structure, we have added another assumption, abstracted from models of current algebra, about the operator structure of the short-distance expansion of Eq. (2.12) or, equivalently, about the operator structure of the equal-time commutator $[J_\mu(\vec{x}),V^\mu_-(0)]$. This assumption is the simplest possible one and is embodied in Eqs. (2.12) or (2.13).

B. Emission of Real Photons

The radiatively corrected decay rate, in order α, contains contributions not only from virtual photons but also from real photons which are too soft to be detected experimentally. In the present context, the only real photons of interest are those emitted by the hadrons since they can give rise to contributions of order Ze in the amplitude. In the limit of zero momentum transfer to the leptons, however, the amplitude for infrared photon emission from the hadrons is proportional to

$$Ze\frac{p\cdot\epsilon}{p\cdot k}-(Z-1)\,e\frac{p\cdot\epsilon}{p\cdot k}\sim e\frac{p\cdot\epsilon}{p\cdot k}\ ,$$

where p is the nuclear momentum, and k and ϵ are the photon momentum and polarization vector, respectively. The amplitude is, therefore, of order e rather than Ze.

C. Corrections for Finite Lepton Momenta

The following question naturally arises: What is the order of magnitude of the terms of order $Z\alpha l$, $Z\alpha q$ which escape the domain of validity of our previous results?

These terms are model-dependent and we will limit ourselves to the following observations.

(i) The greatest apparent danger is that, because we are dealing with nuclei rather than more elementary hadrons, very small energy denominators may arise. For example, in discussing the first term in Eq. (2.6) we neglected contributions from $l_\mu(V^{\mu\lambda} - V_P^{\mu\lambda})$. Although we showed that these vanish as $l \to 0$, the possibility exists that for finite l they are actually of order $Z\alpha l/\Delta E$ where ΔE is the difference in energy between two nearby nuclear states. If $\Delta E \sim O(l)$ these terms would appear to be of order $Z\alpha$. Note, however, that schematically such a contribution is of the form

$$\frac{Z\alpha l \langle B|J^\mu(0)|I\rangle\langle I|V_-^\lambda(0)|A\rangle}{\Delta E}\Bigg|_{\vec{p}_I=\vec{p}},$$

where I is the appropriate intermediate state. If ΔE is very small, the momentum transfer between I and the initial and final states is also very small. As J^μ and V_-^λ are conserved currents (to zeroth order in e), the matrix elements $\langle B|J^\mu|I\rangle$ and $\langle I|V_-^\lambda|A\rangle$ are forbidden in such circumstances and, therefore, they are actually very small. Thus, although such small energy denominators may arise, their effect is greatly inhibited by the forbidden nature of the relevant matrix elements.

(ii) Model-dependent calculations of the terms of order $Z\alpha l$ have been carried out by Dicus and Norton[7] who have found them to be of order $Z\alpha lR$ where R is essentially a nuclear charge radius. We note that these contributions are of second order in small quantities (both $Z\alpha$ and lR are small) and may perhaps be of order of magnitude similar to other neglected effects, such as terms of order $(Z\alpha)^2$.

D. Effects Stemming from the Axial-Vector Current

Unfortunately, effects stemming from the axial-vector current can only be discussed qualitatively. With reference to Fig. 3 we note that since the transitions $A \to I$ and $I' \to B$ are induced by the axial-vector current, neither I nor I' can have the same parity and angular momentum as A or B. The electromagnetic transitions $I \to B$ and $A \to I'$ therefore vanish in the soft-photon limit.

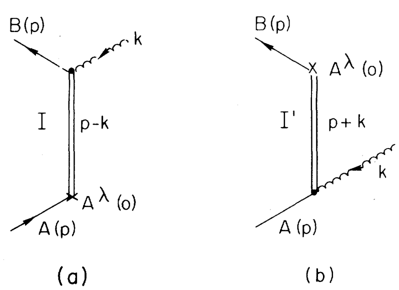

FIG. 3. Graphs involving the axial-vector current.

Now the possibility of a coherent pile-up of electromagnetic effects arises by virtue of the fact that the virtual photon can be soft enough to see the total nuclear charge; however, the photons in Fig. 3 see not the total charge but much more complicated dynamical entities, viz., the transition moments. Thus while there is no *a priori* expectation that one is dealing with an amplitude of order $Z\alpha$, one can make no statement about orders of magnitude without going into specific models.

In an independent-particle model of the nucleus, the nucleon which undergoes the Gamow-Teller transition must also be the nucleon which undergoes the electromagnetic transition; otherwise momentum will not be conserved. In this model,[8] therefore, the amplitude for the process depicted in Fig. 3 is of order α rather than $Z\alpha$.

III. MULTIPHOTON EFFECTS

A. Derivation of Theorem II

Theorem II (see Ref. 9) states that general renormalization effects, in the ft values, of order $Z^n\alpha^m$ are such that $m \geq n$. In obtaining this result we assume that it is physically meaningful to characterize the order of magnitude of nuclear matrix elements involving several currents by the product of the coupling strengths of the individual currents. Following the usual practice of quantum field theory, we further identify such coupling strengths with the zero-momentum-transfer matrix elements of the associated currents. The meaning of this assumption is discussed in Sec. II B.

We will study the derivation of this theorem for the class of diagrams depicted in Fig. 4, that is, those diagrams in which an arbitrary number of virtual photons are attached to the hadronic line. It will be immediately clear that the same result holds for those diagrams in which some or all photons are exchanged between the hadronic and leptonic lines and those diagrams involving real as well as virtual photons.

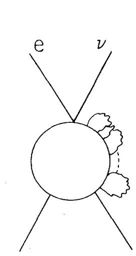

FIG. 4. Higher-order radiative correction.

The matrix element of Fig. 4 can be written as follows:

$$\langle \hat{B} | \hat{V}_\mu^+(0) | \hat{A} \rangle = -\lim_{p'^2 \to m'^2;\, p^2 \to m^2} \frac{(p'^2 - m'^2)(p^2 - m^2)}{Z'^{1/2} Z^{1/2}} \int d^4x \int d^4y\, e^{ip'x - ipy} \langle \hat{0} | T(\hat{\phi}_B(x) \hat{\phi}_A^\dagger(y) \hat{V}_\mu^+(0)) | \hat{0} \rangle\,, \tag{3.1}$$

where the ϕ's are the interpolating fields for the initial and final nuclei, the caret indicates operators which evolve in time according to the exact Hamiltonian,

$$H = H_{\text{free}} + H_{\text{st}} + H_{\text{elmg}}\,,$$

and $|\hat{0}\rangle$ is the vacuum state of H.

It will be sufficient for our purpose to study the perturbation expansion of the improper vertex function of the vector current:

$$\begin{aligned}\langle \hat{0} | T(\hat{\phi}_B(x) \hat{\phi}_A^\dagger(y) \hat{V}_\mu^+(0)) | \hat{0} \rangle &= \sum_{n,m} \int d^4x_1 \cdots d^4x_m \int d^4y_1 \cdots d^4y_n \frac{(-ie)^{n+m}}{n!\,m!} \\ &\times \langle 0 | T(\phi_B(x) \phi_A^\dagger(y) V_\mu^+(0) J_V^{\alpha_1}(x_1) A_{\alpha_1}(x_1) \cdots J_V^{\alpha_m}(x_m) A_{\alpha_m}(x_m) J_S^{\beta_1}(y_1) A_{\beta_1}(y_1) \cdots J_S^{\beta_n}(y_n) A_{\beta_n}(y_n)) | 0 \rangle\,.\end{aligned} \tag{3.2}$$

In the right-hand side of Eq. (2.2), J_V and J_S represent the isovector and isoscalar parts of the electromagnetic current, all operators evolve in time according to $H_{\text{free}} + H_{\text{st}}$ and $|0\rangle$ is the corresponding vacuum state. In particular V_μ^+ is conserved.

As explained at the beginning of this section, we characterize the coupling strength of each vertex by the zero-momentum-transfer matrix element of the associated current. Thus to the S vertices we associate a coupling strength proportional to the isoscalar charge of the system ($=\frac{1}{2}eA \sim Ze$, for all the heavier nuclei) while to the V vertices we associate a coupling strength e ($|I_3| \leq 1$ for all the nuclei under consideration). This point is important for our discussion. In fact, the theorem on the multiphoton effects must be understood in the context of this characterization.

The general term in the double summation of Eq. (3.2) is potentially of order $(Ze)^n e^m = (Z\alpha)^{(m+n)/2} Z^{(n-m)/2}$. Therefore only the terms with $n > m$ are potentially dangerous. To study these terms we fix m and consider all the terms with $n \geq m$. Next we eliminate all the A_{α_i} by pairwise contraction (with the A_{β_i} or with other A_{α_i}) in all possible ways, but we do not contract the A_{β_i} among themselves. To be specific, suppose we contract all A_{α_i} with the A_{β_i} in all possible ways. We get $n(n-1)\cdots(n-m+1)$ such terms so that this particular contribution gives

$$\begin{aligned}\sum_{n=m}^{\infty} \int d^4x_1 \cdots \int d^4x_m \int d^4y_1 \cdots \int d^4y_n \frac{(-ie)^{2m}(-i)^{n-m}}{m!\,(n-m)!} D_{\alpha_1\beta_1}(x_1 - y_1) \cdots D_{\alpha_m\beta_m}(x_m - y_m) \\ \times \langle 0 | T(\phi_B(x) \phi_A^\dagger(y) V_\mu^+(0) J_V^{\alpha_1}(x_1) \cdots J_V^{\alpha_m}(x_m) J_S^{\beta_1}(y_1) \cdots J_S^{\beta_m}(y_m) \mathcal{H}^S(y_{m+1}) \cdots \mathcal{H}^S(y_n)) | 0 \rangle\,,\end{aligned} \tag{3.3}$$

where $\mathcal{H}^S = eJ_\mu^S A^\mu$ is the "isoscalar" Hamiltonian density and the D's stand for photon propagators. Summing over n this expression becomes

$$\begin{aligned}\int d^4x_1 \cdots \int d^4x_m \int d^4y_1 \cdots \int d^4y_m \frac{(-ie)^{2m}}{m!} D_{\alpha_1\beta_1}(x_1 - y_1) \cdots D_{\alpha_m\beta_m}(x_m - y_m) \\ \times \langle 0 | T(\phi_B(x) \phi_A^\dagger(y) V_\mu^+(0) J_V^{\alpha_1}(x_1) \cdots J_V^{\alpha_m}(x_m) J_S^{\beta_1}(y_1) \cdots J_S^{\beta_m}(y_m) e^{-i\int \mathcal{H}^S(\xi) d^4\xi}) | 0 \rangle\,.\end{aligned} \tag{3.4}$$

At this stage we perform a canonical transformation so that all the operators become Heisenberg operators with respect to $H_{\text{free}} + H_{\text{st}} + H_S$, where $H_S = \int \mathcal{H}^S d^3x$. Denoting these operators by tildes, we obtain

$$\begin{aligned}\int d^4x_1 \cdots \int d^4x_m \int d^4y_1 \cdots \int d^4y_m \frac{(-ie)^{2m}}{m!} D_{\alpha_1\beta_1}(x_1 - y_1) \cdots D_{\alpha_m\beta_m}(x_m - y_m) \\ \times \langle \tilde{0} | T(\tilde{\phi}_B(x) \tilde{\phi}_A^\dagger(y) \tilde{V}_\mu^+(0) \tilde{J}_V^{\alpha_1}(x_1) \cdots \tilde{J}_V^{\alpha_m}(x_m) \tilde{J}_S^{\beta_1}(y_1) \cdots \tilde{J}_S^{\beta_m}(y_m)) | \tilde{0} \rangle\,.\end{aligned} \tag{3.5}$$

We emphasize that all the operators in this expression evolve in time according to $H_{\rm free}+H_{\rm st}+H_S$. Note that $\tilde{V}_\mu^+(0)$, $\tilde{J}_V^\mu(0)$, $\tilde{J}_S^\mu(0)$ are still conserved as H_S is invariant under isospin transformations. In particular, the Ward identities ensure that the zero-momentum-transfer matrix elements of these currents are still equal to the values in the absence of H_S. It is then clear that in Eq. (3.5) the power of Z is exactly matched by the power of α, as the $\tilde{J}_S$'s appear exactly m times in the expression.

Equations (3.3) *et seq.* have been obtained by contracting the A_{α_i} in Eq. (3.2) with the A_{β_i} in all possible ways. Had we contracted a subset of A_{α_i} among themselves while contracting the remainder with the A_{β_i} in all possible ways, an identical argument shows that the corresponding contribution is one in which the power of α exceeds the power of Z.

We conclude that in the corrections to the improper vertex function no power of Z exceeds the power of α. All the terms which in the original expansion potentially violated this result have been absorbed in the field operators and state vectors of the tilde representation, effectively becoming part of the strong interactions.

To complete the proof it is necessary to show that the same results hold for the isovector corrections to the masses and renormalization constants of the initial and final nuclei, as these appear in the reduction formula of Eq. (3.1). To show this, it is sufficient to study by an identical method the corrections to the propagators of these particles.

B. Remarks on Counting Powers of Z

Consider for example, Eq. (3.5). If this expression involved only the zero-momentum-transfer matrix elements of all the relevant currents, it is clear that our estimate of the order of magnitude of the matrix elements would be essentially exact. For example, the zero-momentum-transfer matrix element of $J_V^{\alpha_1}$ involves only the isovector charge of the nuclei which is $O(1)$ in the case of all the superallowed Fermi transitions. In such a case, the Z dependence of the matrix element would be carried completely by the matrix elements of the isoscalar currents $\tilde{J}_S^{\beta_1}$ and we would conclude, as we did, that (3.5) is indeed of order of magnitude $(Ze^2)^m$. Equation (3.5) is, however, a very complicated correlation function involving arbitrary ranges of momentum transfers. As such matrix elements seem impossible to calculate, it is clear that our characterization of their order of magnitude cannot be established with certainty. Rather, such characterization is based on the following intuitive argument: One expects that the leading dependences on Z should arise from matrix elements in which all the constituent nucleons act coherently. One naturally expects that such coherent effects will manifest themselves with greatest strength when all the relevant matrix elements are taken at zero momentum transfer. Hence, our estimate of the order of magnitude of Eq. (3.5).

We stress, however, that it is only to the extent that orders of magnitude are governed by the values of the matrix elements at zero momentum transfer (and can, therefore, be estimated in terms of a single nuclear property, viz., the total charge) that we can meaningfully talk of the Z dependence of the ft values for very different nuclear transitions.

IV. CONCLUSION

Our discussion of radiative corrections to superallowed Fermi transitions goes as far as one can go without getting involved in complex, unreliable, and unenlightening model-dependent dynamical calculations. We recognize that a complete resolution of the problem will have to wait until one has at hand a reliable calculus for handling the photon-hadron system; however, our discussion appears to have uncovered most of the interesting physics in the problem. We have shown that the near constancy of the ft values, despite the possibility of sizable radiative corrections to the CVC prediction, is a consequence of the CVC hypothesis itself supplemented with the innocuous assumption of a local commutation relation between the temporal components of the isospin current. The absence of any $Z^2\alpha$ (or, more generally, $Z^m\alpha^n$ with $m>n$) corrections in the ft value has been traced to the fact that terms of this order in the perturbation expansion arise only when the purely isoscalar part of the electromagnetic interaction comes into play; since this interaction respects CVC, such corrections cancel in the ft value. Furthermore, the $Z\alpha$ term, which can become sizable for the higher Z nuclei, has been shown to be suppressed by a combination of arguments – a model-independent current-algebra argument for transitions induced by the vector current and a model-dependent argument for the contribution of the axial-vector current to the Fermi transition. It would be interesting to see whether the axial-vector-current effects exhibit any perceptible change in going from one nuclear model to another.

Nothing in our consideration precludes the existence of radiative corrections of order $Z^2\alpha^2$ (≈4% for ^{54}Co). In the presence of model-dependent corrections of this order of magnitude, it is clear that the precise and unambiguous determination of the ft values is very difficult for the large-Z nuclei. Therefore, in the determination of $G_V/G_\mu = \cos\theta$ (which is relevant to the analysis of the universality of the weak interactions) the consideration of the ft values for low-Z Fermi transitions is much to be preferred.

APPENDIX A

In this appendix we discuss the evaluation of the contribution of $2l_\mu V_P^{\mu\lambda}$ to Eq. (2.6) and show in a simple manner how the usual Coulomb corrections of order α arise from the corresponding integral.

Inserting Eq. (2.8) into Eq. (2.6), we see that the contribution of $2l_\mu V_P^{\mu\lambda}$ to $M_b^{(v)}$ is given by

$$I_3 = I_3^{(1)} + I_3^{(2)} \;, \tag{A1}$$

$$I_3^{(1)} = -\frac{iZ\alpha}{\pi^3}(p\cdot l)M^{(0)}\int \frac{d^4k}{k^2-\lambda_{\min}{}^2+i\epsilon}\,\frac{1}{k^2+2l\cdot k+i\epsilon}\left(\frac{1}{k^2-2p\cdot k+i\epsilon}+\frac{1}{k^2+2p\cdot k+i\epsilon}\right) \;, \tag{A2}$$

$$I_3^{(2)} = -\frac{i\alpha}{\pi^3}(p\cdot l)M^{(0)}\int \frac{d^4k}{k^2-\lambda_{\min}{}^2+i\epsilon}\,\frac{1}{k^2+2l\cdot k+i\epsilon}\,\frac{1}{k^2+2p\cdot k+i\epsilon} \;, \tag{A3}$$

where

$$M^{(0)} = \frac{G_V}{\sqrt{2}}\, L_\lambda \langle B|\, V_-^\lambda(0)|A\rangle \tag{A4}$$

is the zeroth-order matrix element and we have introduced an infinitesimal photon mass $\lambda_{\min}$. Note that $I_3^{(2)}$ is explicitly of order α. It is, in fact, part of the usual radiative corrections of order α and will not be discussed here. The integral in $I_3^{(1)}$ has been evaluated exactly,[10] but the details are rather lengthy and cumbersome. For our purposes it is more illuminating to introduce the following approximation : In the terms within the large parentheses of Eq. (A2) we neglect k^2 relative to $2p\cdot k$. Thus in Eq. (A2) we replace

$$\left(\frac{1}{k^2-2p\cdot k+i\epsilon}+\frac{1}{k^2+2p\cdot k+i\epsilon}\right) \to \frac{1}{-2p\cdot k+i\epsilon}+\frac{1}{2p\cdot k+i\epsilon} = -2i\pi\delta(2p\cdot k)$$
$$= -i(\pi/M)\delta(k_0) \;, \tag{A5}$$

where the last equality holds in the rest system of the nucleus (M is the nuclear mass). The rationale behind this approximation is simple enough: The terms of order $1/l$ in the integration of Eq. (A2) arise from the "small k" region and therefore we can neglect k^2 in front of $2p\cdot k$. In fact, one can check that the error introduced by this approximation in the evaluation of $I_3^{(1)}$ is of order $O(Z\alpha\bar{l}/M)$.[11] Using the approximation indicated in Eq. (A5) and specializing to the nuclear rest frame, we obtain

$$I_3^{(1)} = -\frac{Z\alpha M^{(0)}}{\pi^2}\, l_0 \int \frac{d^3k}{[(\vec{k})^2+\lambda_{\min}{}^2][(\vec{k})^2+2\vec{l}\cdot\vec{k}-i\epsilon]} + O(Z\alpha l/M) \;. \tag{A6}$$

Calling $K=|\vec{k}|$, $L=|l|$, introducing polar coordinates, and carrying out the angular integration, we obtain

$$I_3^{(1)} = -\frac{2Z\alpha}{\pi} M^{(0)} l_0 \int_0^\infty \frac{K^2 dK}{K^2+\lambda_{\min}{}^2}\,\frac{1}{2LK}\ln\left(\frac{K^2+2LK}{K^2-2LK-i\epsilon}\right) \;. \tag{A7}$$

In Eq. (A7) as well as in the following expressions the very small errors of order $O(Z\alpha l/M)$ are not indicated explicitly. The integral in Eq. (A7) contains a real as well as an imaginary part. Because $I_3^{(1)}$ is proportional to $M^{(0)}$, to order α only the real part of the integral can contribute to the transition probability. Evaluating the integral, we obtain

$$\mathrm{Re}\int_0^\infty \frac{K^2dK}{K^2+\lambda_{\min}{}^2}\,\frac{1}{2LK}\ln\left(\frac{K^2+2LK}{K^2-2LK-i\epsilon}\right) = \frac{1}{2L}\int_0^{2L}\frac{dK}{K}\ln\left(\frac{K+2L}{2L-K}\right)+\frac{1}{2L}\int_{2L}^\infty \frac{dK}{K}\ln\left(\frac{K+2L}{K-2L}\right)$$
$$= \frac{1}{L}\int_0^1 \frac{du}{u}\ln\left(\frac{1+u}{1-u}\right)$$
$$= \frac{\pi^2}{4L} \;. \tag{A8}$$

In Eq. (A8) we have set $\lambda_{\min}=0$, as the real part of the integral is infrared convergent. Thus,

$$I_3^{(1)}=-\frac{2Z\alpha}{\pi}M^{(0)}l_0\left[\frac{\pi^2}{4L}+\frac{i\pi}{2L}\ln\left(\frac{2L}{\lambda_{\min}}\right)\right]=-\frac{Z\alpha\pi}{2\beta}M^{(0)}-i\frac{Z\alpha M^{(0)}}{\beta}\ln\left(\frac{2L}{\lambda_{\min}}\right)\quad, \tag{A9}$$

where $\beta\equiv L/l_0$ is the positron velocity in units of c and, for completeness, we have included the contribution from the imaginary part of the integral in Eq. (A7). The interference of Eq. (A9) with the zeroth-order matrix element $M^{(0)}$ gives a contribution $-|M^{(0)}|^2(Z\alpha\pi/\beta)$ to the transition probability. This is, in fact, the usual Coulomb correction of order α corresponding to a positron emitter.

APPENDIX B

In this appendix we discuss the evaluation of the quantity

$$I_{2P}=-\frac{i\alpha G_V L_\lambda}{4\pi^3\sqrt{2}}\int\frac{d^4k}{k^2+i\epsilon}k^\lambda(V_P)^\mu{}_\mu\left(\frac{1}{k^2+2l\cdot k+i\epsilon}-\frac{1}{k^2+i\epsilon}\right)\quad, \tag{B1}$$

which is the contribution of $(V_P)^\mu{}_\mu$ to I_2 [Eq. (2.10)]. Our aim is to show that in the limit $l\to 0$, $q\to 0$, I_{2P} does not give rise to contributions of order $Z\alpha$ to the total transition probability.

Inserting Eq. (2.8) into Eq. (B1), we obtain

$$I_{2P}=I_{2P}^{(1)}+I_{2P}^{(2)}\quad, \tag{B2}$$

$$I_{2P}^{(1)}=\frac{iZ\alpha G_V}{2\pi^3\sqrt{2}}p^\mu\langle B|V_{-\mu}|A\rangle L_\lambda\int\frac{d^4k}{k^2+i\epsilon}k^\lambda\left(\frac{1}{k^2-2p\cdot k+i\epsilon}+\frac{1}{k^2+2p\cdot k+i\epsilon}\right)\frac{1}{k^2+2l\cdot k+i\epsilon}\quad, \tag{B3}$$

$$I_{2P}^{(2)}=\frac{i\alpha G_V}{2\pi^3\sqrt{2}}p^\mu\langle B|V_{-\mu}|A\rangle L_\lambda\int\frac{d^4k}{k^2+i\epsilon}\frac{k^\lambda}{k^2+2p\cdot k+i\epsilon}\left(\frac{1}{k^2+2l\cdot k+i\epsilon}-\frac{1}{k^2+i\epsilon}\right)\quad. \tag{B4}$$

In the above expressions we have taken into account the fact that the integral obtained by setting $l=0$ in Eq. (B3) is identically zero, because of the oddness of the corresponding integrand.

Defining

$$J_2^\lambda=\frac{8i}{(2\pi)^2}\int\frac{d^4k}{k^2+i\epsilon}\frac{k^\lambda}{k^2-2p\cdot k+i\epsilon}\frac{1}{k^2-2l\cdot k+i\epsilon}\quad, \tag{B5}$$

$$\hat{J}_2^\lambda\equiv\frac{8i}{(2\pi)^2}\int\frac{d^4k}{k^2+i\epsilon}\frac{k^\lambda}{k^2+2p\cdot k+i\epsilon}\frac{1}{k^2-2l\cdot k+i\epsilon}\quad, \tag{B6}$$

we have

$$I_{2P}^{(1)}=-\frac{Z\alpha G_V}{4\pi\sqrt{2}}p^\mu\langle B|V_{-\mu}|A\rangle L_\lambda(J_2^\lambda+\hat{J}_2^\lambda)\ . \tag{B7}$$

Performing the integrals one verifies[12]

$$\mathrm{Re}(J_2^\lambda+\hat{J}_2^\lambda)=O(l^\lambda/M^2)\ , \tag{B8}$$

$$i\,\mathrm{Im}(J_2^\lambda+\hat{J}_2^\lambda)=i\,\mathrm{Im}\hat{J}_2^\lambda=-\frac{i2\pi}{ML}\left(p^\lambda\frac{l_0}{M}-l^\lambda\right)\quad. \tag{B9}$$

The contribution of $\mathrm{Re}(J_2^\lambda+\hat{J}_2^\lambda)$ to Eq. (B7) is explicitly of order $O(l^\lambda/M^2)$ and vanishes in the limit $l^\lambda\to 0$. On the other hand $\mathrm{Im}(J_2^\lambda+\hat{J}_2^\lambda)$ is of zeroth order in l (although not independent of l). Therefore, the contribution of Eq. (B9) must be treated explicitly. Inserting Eq. (B9) into Eq. (B7) one finds that in the limit $l\to 0$, $q\to 0$, the interference of $I_{2P}^{(1)}$ with the zeroth-order matrix element $M^{(0)}$ vanishes after the summation over the spin states of the final positron is performed. This completes our proof that $I_{2P}^{(1)}$ does not contribute terms of order $Z\alpha$ to the total transition probability in the limit l, $q\to 0$.

Regarding $I_{2P}^{(2)}$ we note that it involves an explicit factor α. Therefore, one can treat the hadronic part of the matrix element to zeroth order in α. It is then easy to verify that in the limit $q\to 0$, $I_{2P}^{(2)}$ does not contribute terms of order $Z\alpha$.

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[9]See also M. A. B. Bég, in *Brookhaven National Laboratory Lectures in Science* (Gordon and Breach, New York, 1970), Vol. 5; A. Sirlin, Ann. Phys. (N.Y.) 61, 294 (1970).

[10]A. Sirlin (unpublished). The exact result of one of the two integrals of Eq. (A2), namely,

$$J_1 \equiv \frac{8i}{(2\pi)^2}\int \frac{d^4k}{k^2-\lambda_{\min}{}^2+i\epsilon} \times \frac{1}{k^2+2l\cdot k+i\epsilon}\;\frac{1}{k^2+2p\cdot k+i\epsilon},$$

was given in R. E. Behrends, R. J. Finkelstein, and A. Sirlin, Phys. Rev. 101, 866 (1956) [see Eq. (7a) *et seq.*].

[11]More precisely, the error is $O(Z\alpha(l/M)\ln(l/M))$ or $O(Z\alpha(l/M))$, where l in these expressions stands for the positron energy, the positron momentum, or its mass. Because of the smallness of (l/M) these terms are negligible for all practical purposes.

[12]For the exact evaluation of J_2^λ, see Behrends *et al.*, Ref. 10, Eq. (7b) *et seq.* The integral $\mathcal{J}_2^\lambda$ can be obtained by similar methods.

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Bootstrap Model of Inclusive Reactions*

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We study a model in which multiparticle processes are envisaged as ocurring via the formation of a fireball and a leading particle. We impose a bootstrap condition on the fireball decay distribution by demanding that it be the same, in its rest system, as the over-all distribution in the c.m. system. This leads us to a set of integral equations for both single-particle and multiparticle inclusive distributions. We examine these equations in the scaling limit, we show that the multiplicity of produced particles grows logarithmically with energy and that the multiparticle distributions factorize for particles traveling in opposite directions, and we obtain a bound on the size of the two-particle distribution function. If Regge constraints are imposed, we show that the model predicts that the scaling limit for the single-particle distribution function, in the central region, is approached from below. Furthermore, many of the features of a Mueller analysis are clearly exhibited by the model. We illustrate various properties of the model by means of two examples, one of which is in qualitative agreement with experiment.

I. INTRODUCTION

Particle production is the dominant feature of high-energy collision processes, accounting for roughly 80% of all events, and a variety of models have been proposed to describe these processes.[1] Several of these models[2] consider high-energy processes occurring through the formation of one or several clusters of particles, often called fireballs, which then decay into the observed particles. A general assumption made in most of these models is that the decay of the fireball is isotropic in its own rest system; other details of the decay mechanism are part of the theoretical input which distinguishes among various versions of these models. In this paper we shall also envisage particle production processes as occurring via the formation of a fireball, but shall consider the alternative assumption that the fireball decay distribution, rather than being isotropic, is essentially the same as the distribution of particles in the entire event. This constitutes in effect a bootstrap hypothesis.

We shall imagine, for simplicity, a world of only