Single-Particle Distributions in a Hadronic Bremsstrahlung Model*

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It is shown that the inclusive cross section for production of a single soft, neutral vector meson in pp scattering vanishes like $1/s$ at $p_{\parallel}/p_{\parallel\,\max}=0$ in the center-of-mass system for mesons radiated from the external nucleon legs. Comparison of the resulting single-pion distribution with the CERN Intersecting Storage Rings data makes it unlikely that present data in the pionization region can be understood in a conventional bremsstrahlung picture.

I. INTRODUCTION

Feynman has pointed out[1] that scaling of inclusive cross sections is suggested by the analogy to bremsstrahlung in quantum electrodynamics. In the present paper we verify this idea in a particular hadronic bremsstrahlung model of pp scattering in which the radiated vector mesons are pictured as neutral ρ or ω particles that subsequently decay into pions. We find, however, that the scaling function vanishes like X^4 as the center-of-mass ratio $p_{\parallel}/k_{\parallel\max}\equiv X$ goes to zero; that is, there is no pionization in a pure bremsstrahlung picture. We believe this vanishing of the bremsstrahlung single-particle distribution to be independent of any detailed feature of the model, although the way in which it approaches zero as $X\to 0$ may depend on specific model approximations. Comparison of the model distributions with the CERN Intersecting Storage Rings (ISR) data[2] suggests that conventional bremsstrahlung from external proton legs is not the dominant production mechanism near $X=0$.

This aspect of inelastic production is the first we have treated in which a prediction of the elementary bremsstrahlung model is qualitatively at variance with the experimental data, and clearly signals the need for inclusion of graphs of the multiperipheral type. The latter are contained within the framework of an N-ρ_0 plus ρ_0-$\pi^{\pm}$ interaction, but are not studied in this paper. In Sec. II we review the approximations necessary to obtain the soft-vector-meson (SVM) model expression for the production amplitudes in pp scattering. We then write down the resulting expressions for the total cross section and the proton and SVM inclusive cross sections and show that the single SVM distribution vanishes like $1/s$ at $X=0$, given certain reasonable (and experimentally verified) assumptions. We next estimate, in Sec. III, how fast the SVM distribution vanishes at $X=0$. In Sec. IV we take into account the decay of the ρ^0 into two pions in the manner suggested by Brink, Cottingham, and Nussinov[3] (BCN) and compare the resulting single-pion distribution with the ISR data.[2] We discuss the modifications to the BCN results due to a finite width for the ρ transverse-momentum distribution, and in the Conclusion briefly discuss ρ_0 emission in the multiperipheral manner, to produce proper pionization. An appendix is included to derive the general (and vastly more complicated) forms which appear when the narrow-width approximation is not made.

II. THE SINGLE-VECTOR-MESON DISTRIBUTION AT $X=0$

The amplitude for the emission of n soft vector mesons (SVM) each with polarization vector $\epsilon_\mu^{(i)}(\lambda_i)$ is

$$M_n=g^n M_0(p_1,p_2,p_3,p_4)\prod_{i=1}^{n} V_\mu^{(i)}\epsilon_\mu^{(i)}(\lambda_i), \qquad (1)$$

where M_0 is the full amplitude for $p_1p_2\to p_3p_4$ without emission of SVM's (but including all exchanges of hard and soft vector mesons) and

$$V_\mu^{(i)}\equiv\left(\frac{p_1}{p_1\cdot k_i}+\frac{p_2}{p_2\cdot k_i}-\frac{p_3}{p_3\cdot k_i}-\frac{p_4}{p_4\cdot k_i}\right)_\mu .$$

g is a ρNN coupling constant.

The assumptions that go into the derivation of the factorized form (1) for M_n are identical with those made in the soft-photon-emission problem for fermion-fermion scattering.[4] Their validity must, however, be reexamined in the present case in which the neutral vector meson has mass, so that the statement of "softness" is no longer frame-independent. In the numerator of any Feynman graph one neglects terms linear in k_i relative to

$\epsilon \cdot p$, where p is any one of the incoming or outgoing proton momenta. In the denominator it is necessary to neglect all $k_i \cdot k_j$ terms relative to $k \cdot p$ for all of the k's and p's. These conditions are satisfied to an accuracy $\sim m/\sqrt{s}$ when k_i is nearly at rest in the center-of-mass frame[5] ("wee" X) provided the outgoing protons retain a finite fraction of the incident proton energy. For small X ($X \ll 1$ but SVM energy $\omega \gg \mu$) terms of order X^2 are neglected relative to terms of order X. We therefore expect the results based on Eq. (1) to be a good approximation to SVM emission from external proton legs for the small-X and wee-X regions.

We emphasize that such SVM emission, which may in principle refer to ρ_0 or ω, and thence to pion production, defines the *model*, one which has been moderately successful in reproducing various aspects of different high-energy processes.[6] Apart from complicating details of the SVM-pion decay, discussed in the Appendix, such a model picture falls within the catalog of "uncorrelated jet" models.[7] Within this particular bremsstrahlung context, arithmetic complexities shall force certain *approximations*, which we shall make every attempt to justify. We make no *a priori* effort to justify the model, but merely direct the reader's attention to its previous, qualitative successes.

The cross section for the emission of n SVM's (averaged over initial and summed over final polarizations) is

$$d\sigma_n = \frac{1}{4\pi^2}\,|M_0|^2_{\text{av}}\,\frac{m^4}{E_1E_2|V_{12}|}\left(\frac{g^2}{(2\pi)^3}\right)^n \delta^4(p_1+p_2-\sum k_i - p_3 - p_4)\prod_{i=1}^{n}\left(V^{(i)}\cdot V^{(i)}\,\frac{d^3k_i}{2\omega_i}\right)\frac{d^3p_3}{E_3}\,\frac{d^3p_4}{E_4}\,, \tag{2}$$

where the subscript "av" denotes the appropriate spin averaging.

In order to construct the total cross sections and the various inclusive distributions one would like to sum $d\sigma_n$ over an infinite number of emitted SVM's. In the soft photon case this is trivial since the soft-photon-emission factors decouple from $|M_0|^2$ in momentum space in the limit (photon four-momentum) $\to 0$. We cannot here neglect the $\sum k_i$ in the δ function because at any finite energy only a finite number of massive mesons, however soft, can be emitted. We therefore write an exponential representation for the δ function and find

$$\sigma_n = \tfrac{1}{2}\int\frac{d^3p_3}{E_3}\int\frac{d^3p_4}{E_4}\,\frac{m^4}{E_1E_2|V_{12}|}\,\frac{|M_0|^2_{\text{av}}}{4\pi^2}\int\frac{d^4x}{(2\pi)^4}\,e^{ix\cdot(p_1+p_2-p_3-p_4)}\,\frac{1}{n!}\left(\frac{g^2}{(2\pi)^3}\int\frac{d^3k}{2\omega}\,e^{-ik\cdot x}\,V^2(k)\right)^n. \tag{3}$$

The factor $\frac{1}{2}1/n!$ comes from the identical particles in the final state. Thus

$$\sigma_{\text{tot}} = \sum_{n=0}^{\infty}\sigma_n$$

$$= \tfrac{1}{2}\int\frac{d^3p_3}{E_3}\int\frac{d^3p_4}{E_4}\,\frac{m^4}{E_1E_2|V_{12}|}\int\frac{d^4x}{(2\pi)^4}\,e^{ix\cdot(p_1+p_2-p_3-p_4)}\,\frac{|M_0|^2_{\text{av}}}{4\pi^2}\exp\{K(x)\}\,, \tag{4}$$

(see Ref. 8) where

$$K(x) \equiv \frac{g^2}{(2\pi)^3}\int\frac{d^3k}{2\omega}\,e^{-ik\cdot x}\,V^2(k,p_3,p_4,s)\,. \tag{5}$$

Similarly, we find for the single SVM distribution

$$2\rho(k_\perp, k_\parallel, s) \equiv 2\omega\,\frac{d\sigma}{d^3k}$$

$$= \sum_{n=1}^{\infty} 2\omega\,\frac{d\sigma_n}{d^3k}$$

$$= \frac{1}{2}\,\frac{m^4}{E_1E_2|V_{12}|}\int\frac{d^3p_3}{E_3}\int\frac{d^3p_4}{E_4}\,\frac{|M_0|^2_{\text{av}}}{4\pi^2}\int\frac{d^4x}{(2\pi)^4}\,e^{ix\cdot(p_1+p_2-p_3-p_4-k)}\,\frac{g^2}{(2\pi)^3}\,V^2(k)\exp\{K(x)\}\,. \tag{6}$$

The single-proton distribution is

$$E'\,\frac{d\sigma}{d^3p'} = \frac{m^4}{E_1E_2|V_{12}|}\int\frac{d^3p_4}{E_4}\int\frac{d^4x}{(2\pi)^4}\,e^{i(p_1+p_2-p_4-p')\cdot x}\,\frac{|M_0|^2}{4\pi^2}\exp\{K(x)\}\,. \tag{7}$$

To estimate $\rho(k_\perp, k_\parallel = 0, s)$ we note that for $p_{3\perp}$, $p_{4\perp}$, $k_\perp \sim m$, and $E_3, E_4 \gg m$

$$V^2(k)\Big|_{k_\parallel=0} \cong \frac{4k_\perp^2}{(m_\rho^2+k_\perp^2)^2}\,\frac{p_{3\perp}p_{4\perp}}{E_3E_4}\,. \tag{8}$$

Thus if the dominant contributions to the integral (6) come from regions where the transverse momenta are small and the proton elasticities large (as is experimentally the case) one can replace $k_\perp$, $p_{3\perp}$, $p_{4\perp}$, E_3, and E_4 by their average values and bring the factor $V^2(k)|_{k_\parallel=0}$ outside the integral in Eq. (6). The factor

$$|M_0|^2_{\rm av}\int d^4x\, e^{ix\cdot(p_1+p_2-p_3-p_4-k)}\exp\{K(x)\}$$

is assumed to provide the strong damping of transverse momenta and of small E_3 and E_4.[9] Then, since on the average in the center-of-mass system $E_3E_4\propto s$, we find

$$\rho(k_\perp, k_\parallel=0, s)\propto\frac{1}{s}\frac{g^2}{(2\pi)^3}\frac{m^4}{E_1E_2|V_{12}|}\int\frac{d^3p_3}{E_3}\int\frac{d^3p_4}{E_4}\int\frac{d^4x}{(2\pi)^4}e^{ix\cdot(p_1+p_2-p_3-p_4-k)}\frac{|M_0|^2_{\rm av}}{4\pi^2}\exp\{K(x)\}|_{k_\parallel=0}\,. \tag{9}$$

At $k_\parallel=0$, $e^{ix\cdot(p_1+p_2-p_3-p_4-k)}\sim e^{ix\cdot(p_1+p_2-p_3-p_4)}$, so that the remaining integral in Eq. (9) is seen to be proportional to the total cross section. Equation (9) can thus be stated as

$$\rho(k_\parallel=0, k_\perp, s)=\text{const}\,\frac{k_\perp^2}{(m_\rho^2+k_\perp^2)^2}<p_{3\perp}p_{4\perp}>g^2\,\frac{\sigma_{\rm tot}(pp)}{s}\,. \tag{10}$$

III. APPROXIMATE EVALUATION OF $\rho(X,k_\perp)$

To exhibit scaling in the model and to learn how fast the single-particle distribution vanishes as X goes to zero we must approximate the coordinate integral in Eq. (6) while retaining energy conservation. To accomplish this in a simple way we make the replacement

$$\int\frac{d^4x}{(2\pi)^4}e^{ix\cdot(p_1+p_2-p_3-p_4-k)}\exp\{K(x)\}\to\sum_\epsilon\delta^3(\vec p_3+\vec p_4+\vec k)\delta(2E-E_3-E_4-\omega_k-\epsilon)\exp\{K(0)\} \tag{11}$$

in the center-of-mass frame, where $\vec p_1+\vec p_2=0$ and $E_1=E_2=E$. ϵ is an average total center-of-mass energy carried off by the unobserved SVM's – their total three-momentum is, for simplicity, assumed to be zero in the c.m. frame – and $\sum_\epsilon$ denotes a weighted sum over ϵ. For definiteness we assume $p_{3\parallel}$ and $k_\parallel$ are in the same direction ($X\geq 0$); then for small, bounded transverse momenta and for $k_\parallel\gg m_\rho$ the δ functions on the right-hand side of (11) become

$$\tfrac{1}{2}\,\theta(1-X-\eta)\delta^2(\vec p_{3\perp}+\vec p_{4\perp}+\vec k_\perp)\delta(|p_{4\parallel}|-|p_{3\parallel}|-Xp)\delta(E_3-\eta E),$$

where η is the leading nucleon elasticity. The weighted sum over unobserved SVM energy then becomes an integral over η with weight $P(\eta)$, the probability that the leading nucleon center-of-mass energy is a fraction η of E. With the notation $K\equiv K(0)$ we then have

$$\rho(k_\perp, k_\parallel, s)\sim\frac{m^4}{E^2}\int\frac{d^3p_3}{E_3}\int\frac{d^3p_4}{E_4}\int_0^{1-X}d\eta\,\frac{|M_0|^2_{\rm av}}{32\pi^2}\frac{g^2}{(2\pi)^3}V^2(k)\exp\{K\}\delta^3(\vec p_3+\vec p_4+\vec k)\delta(E_3-\eta E)P(\eta)\,. \tag{12}$$

If we use the three-momentum δ function in Eq. (12) to do the d^3p_4 integration we can regroup the factors to get

$$\rho(k_\perp, k_\parallel, s)\sim\frac{1}{32\pi^2}\int_0^{1-X}d\eta\,P(\eta)\,\frac{\eta}{\eta+X}\int_0^\infty\delta(E_3-\eta E)dE_3\,\frac{m^4}{E^2}\int d\Omega_3\,|M_0|^2_{\rm av}\exp\{K\}\,\frac{g^2}{(2\pi)^3}V^2(k) \tag{13}$$

provided we assume[9] that large transverse momenta and small E_3 are strongly damped by the factor $|M_0|^2_{\rm av}\exp\{K\}$ in the integrand.

Since $\int d\Omega_3|M_0|^2_{\rm av}/E^2$ has the form of an elastic total cross section, which has little or no s dependence, we approximate the $d\Omega_3$ integral in (13) by neglecting the transverse-momentum dependence in $V^2(k)$ (it is a weak, $1/p_\perp^2$ dependence) and writing

$$\frac{m^4}{E^2}\frac{1}{32\pi^2}\int d\Omega_3|M_0|^2_{\rm av}\exp\{K\}\to f(k_\perp^2)\,.$$

Then

$$\rho(k_\perp, k_\parallel, s)\sim\frac{g^2}{(2\pi)^3}f(k_\perp^2)\int_0^{1-X}\frac{\eta d\eta}{\eta+X}P(\eta)V^2(k)\,, \tag{14}$$

where $V^2(k)$ is to be evaluated with

$$p_{1\mu}=(0,0,p;iE),\quad p_{2\mu}=(0,0,-p;iE),\quad p_{3\mu}=(0,0,p_3;i\eta E),$$

$$p_{4\mu}=(0,0,p_4;i(\eta+X)E),\ \text{and}\ p_4=-p_3-k_\parallel .$$

With these assumptions one finds that $V^2(k)=h(X,\eta)$, which depends on $k_\parallel$ and s only through the ratio $X=2k_\parallel/\sqrt{s}$. Explicitly,

$$h(X,\eta)=X^4\,\frac{4m^4(1-\eta^2)^2m_{\rho\perp}{}^2}{\eta^4}\,\frac{1}{(m_{\rho\perp}{}^2+X^2m^2)^2[m_{\rho\perp}{}^2+(X^2/\eta^2)m^2]^2}\,, \tag{15}$$

where $m_{\rho\perp}\equiv(m_\rho{}^2+k_\perp{}^2)^{1/2}$ is the transverse ρ mass. Insertion of the expression (15) into Eq. (14) provides an explicit form for the scaling function $\rho(X,k_\perp)$ within the model.[10]

The function $P(\eta)$ in Eq. (14) can be determined experimentally from the ISR $pp\to p+$anything data[2] by starting with Eq. (7) for the single-proton distribution and making the same sequence of approximations that led from Eq. (6) to Eq. (14). We take

$$\int\frac{d^4x}{(2\pi)^4}\,e^{i(p_1+p_2-p_3-p_4)\cdot x}\exp\{K(x)\}\to\tfrac{1}{2}\int_0^1 d\eta\,\delta^3(\vec{p}_3+\vec{p}_4)\delta(E_3-\eta E)P(\eta)\,. \tag{16}$$

In Eq. (16) $(1-\eta)$ is the fraction of the total center-of-mass energy that goes into SVM production, and the average total SVM three-momentum is assumed to be zero in the center-of-mass system. After using the momentum δ function to do the d^3p_4 integration in Eq. (7) and $\delta(E_3-\eta E)=(1/E)\times\delta(\eta-E_3/E)$ for the η integration, one has

$$E'\,\frac{d\sigma}{d^3p'}\sim\frac{1}{8\pi^2}\,\frac{m^4}{E^4}\,|M_0|^2{}_{\rm av}\exp\{K\}\,\frac{P(E'/E)}{E'/E}\,. \tag{17}$$

Equation (17) scales if $(|M_0|^2{}_{\rm av}/E^4)\exp\{K\}$ depends only on $p_\perp'^2$ and E'/E. This is a reasonable assumption, at least for small $p_\perp'^2$, since $(|M_0|^2{}_{\rm av}/E^4)$ there has the form of an elastic, forward $d\sigma/dt$, which is approximately energy-independent. We assume

$$E'\,\frac{d\sigma}{d^3p'}\propto\frac{P(E'/E)}{E'/E}$$

for fixed $k_\perp{}^2$ and determine $P(E'/E)$ from the ISR data.[2] A very crude parametrization, which is adequate for our purposes, is

$$P(\eta)=\eta+B\eta^2\ \text{ with }\ B\sim5\ \text{ for }\ k_\perp{}^2=0.16\,\text{GeV}^2.$$

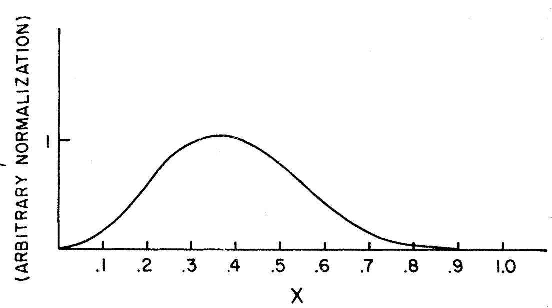

FIG. 1. Single SVM distribution function for positive values of the Feynman variable X.

Figure 1 is a plot of the single SVM distribution [Eqs. (14) and (15)] with $p(\eta)=\eta+5\eta^2$ and $R\equiv m^2/m_{\rho\perp}{}^2=\frac{3}{2}$. The shape of the single SVM spectrum is independent of B for B between 5 and 30 and its maximum varies only from 0.35 to 0.4 as R decreases from $\frac{3}{2}$ to 1.[10]

IV. SINGLE-PION SPECTRUM

Before one can compare the results of the preceding section with the ISR data it is necessary to take into account the decay of the SVM into pions in order to predict a single-pion spectrum, which is what is experimentally observed. For small transverse momentum and pion production via $\rho\to2\pi$ the analysis is very simple. It has been discussed by Brink, Cottingham, and Nussinov[3] in a calculation which shows that

$$\rho_\pi(X_\pi)\sim\rho\left(\frac{2X_\pi}{1-\gamma|\cos\theta|}\right)+\rho\left(\frac{2X_\pi}{1+\gamma|\cos\theta|}\right), \tag{18}$$

where $\gamma\equiv(1-4m_\pi{}^2/m_\rho{}^2)^{1/2}\sim0.93$ and θ is the angle of the decay pion in the ρ rest system relative to the beam direction. Thus

$$q_\perp=\tfrac{1}{2}m_\rho\,\gamma\sin\theta, \tag{19}$$

where $q_\perp$ is the transverse momentum of the pion. Equation (18) holds in the simple case where the ρ meson has no transverse momentum, and the simple relation between the pion scaling function $\rho_\pi(X_\pi)$ and the SVM scaling function $\rho(X)$ assumes $q_\parallel\gg m_\rho$. (At ISR energies $2m_\rho/\sqrt{s}\sim\frac{1}{40}$.) As written here Eq. (18) also neglects the correlation between ρ spin and pion decay distribution. The first term in Eq. (18) is associated with backward and the second with forward pions from the ρ meson in its rest system. For $q_\perp < \frac{1}{2}m_\rho \gamma \sim 0.35$ GeV it is perhaps reasonable to use the simple form, Eq. (18), since the ρ-meson transverse momenta are presumably strongly damped and most small $q_\perp$ pions come from SVM's with negligible transverse momenta. For larger $q_\perp$ for given values of $q_\perp$ and $q_\parallel$ one must integrate Eq. (18) over all possible parent SVM momenta with appropriate weighting, thus smearing out the sharp echo effect present in Eq. (18). [In visualizing the effect of Eq. (18) it is helpful to remember that the function $\rho(X)$ vanishes when its argument exceeds unity. Thus the first term in Eq. (18) shrinks the scaling function $\rho(X)$ by a scale factor >2 and produces a low X_π echo of $\rho(X)$ in the pion scaling function $\rho_\pi(X_\pi)$.] We now proceed to find the single-pion spectrum for general q, taking account of the ρ spin. If $M\cdot\epsilon(\tau)$ is the amplitude for producing a ρ of polarization τ and anything else, then

$$M\cdot\epsilon(\tau) \to \sum_\tau M\cdot\epsilon(\tau)\epsilon(\tau)\cdot(q-q')\frac{g_1}{k^2+m_\rho{}^2-i\Gamma m_\rho}$$
$$=2M\cdot q\,\frac{g_1}{k^2+m_\rho{}^2-i\Gamma m_\rho}\,, \quad (20)$$

where q and q' denote pion momenta, $k=q+q'$ is the ρ momentum, Γ is the ρ width and g_1 a $\rho\pi\pi$ coupling constant. The second step in Eq. (20) follows from the observation that $k\cdot\epsilon=0$. In the narrow-resonance approximation one neglects cross terms between amplitudes in which a given pion pair come from the same ρ meson and in which they come from different ρ mesons. (The modifications required when the narrow-resonance approximation is not made are discussed in the Appendix.) The single-pion distribution is obtained from Eq. (6) by the replacement

$$V^2(k) \to 4(V\cdot q)^2\,\frac{g_1{}^2}{(k^2+m_\rho{}^2)^2+\Gamma^2m^2}\,\frac{d^3q'}{2\omega'}$$

and integration over the unobserved pion momentum q'.[11] In the narrow-resonance approximation, one finds

$$\rho_\pi(q_\perp, q_\parallel, s)=\frac{m^4}{E_1E_2|V_{12}|}\int\frac{d^3p_3}{E_3}\int\frac{d^3p_4}{E_4}\int\frac{d^3q'}{2\omega'}\,\frac{|M_0|^2{}_{\rm av}}{4\pi^2}$$
$$\times\int\frac{d^4x}{(2\pi)^4}\,e^{ix\cdot(p_1+p_2-p_3-p_4-k)}\,\frac{g^2}{(2\pi)^3}(V\cdot q)^2\lambda\delta(k^2+m_\rho{}^2)\exp\{K(x)\}\,, \quad (21)$$

where $\lambda\equiv\lim_{\Gamma\to 0}(\pi g_1{}^2/\Gamma m_\rho)=$ const since $g_1{}^2\propto\Gamma$. One can show by a simple calculation that

$$(V\cdot q)^2|_{q_\parallel=0}=g(\vec{q}_\perp, \vec{p}_{3\perp}, \vec{p}_{4\perp})\,\frac{1}{s}$$

for $E_3, E_4 \gg m$ and $\vec{q}_\perp$, $\vec{p}_{3\perp}$, and $\vec{p}_{4\perp}=O(m)$. The basic result of the paper, Eq. (10), thus continues to hold, with, however, a different dependence on transverse momenta in the coefficient of $\sigma_{\rm tot}(pp)/s$.

Following the same series of kinematic simplifications that lead from Eq. (6) to Eq. (14) we find

$$\rho_\pi(q_\perp, q_\parallel, s)=4\lambda\,\frac{g^2}{(2\pi)^3}\int\frac{d^3q'}{2\omega'}\,\delta(k^2+m_\rho{}^2)f(k_\perp{}^2)\int_0^{1-X}d\eta\,\frac{\eta}{\eta+X}\,P(\eta)[q\cdot V(k)]^2|_{k=q+q';\,X=X_\pi+X'_\pi}\,. \quad (22)$$

The function $h(X, X_\pi, \eta)\equiv[q\cdot V(k)]^2$ scales and has the explicit form

$$h(X, X_\pi, \eta)=\left(\frac{X}{X_\pi}\,\frac{m^2(X^2\mu_\perp{}^2-m_\rho{}^2X_\pi{}^2)(1-\eta^2)}{(m_\rho{}^2+m^2X^2)(\eta^2m_\rho{}^2+m^2X^2)}\right)^2. \quad (23)$$

It is assumed that $f(k_\perp{}^2)$ provides the strong damping, and we have neglected $p_{3\perp}$ and $p_{4\perp}$ and $k_\perp$ in writing Eq. (23). $\mu_\perp$ is the transverse pion mass. The comments of Ref. 8 apply also to Eq. (22). The integral over d^3q' in Eq. (22) can be replaced by an integral over d^3k (Ref. 11) to obtain

$$\rho_\pi(X_\pi, q_\perp)\sim\frac{2\lambda g^2}{(2\pi)^3}\int\frac{d^3k}{\omega}\,\delta((k-q)^2+m_\pi{}^2)f(k_\perp{}^2)\int_0^{1-X}\frac{\eta\,d\eta}{\eta+X}\,P(\eta)h(X, X_\pi, \eta)\,. \quad (24)$$

For $f(k_\perp{}^2)=\delta(k_\perp{}^2)$ we have

$$\rho_\pi(X_\pi, q_\perp)=\frac{\lambda g^2}{2\pi^2}\int_0^1\frac{dX}{X}\,\delta((k-q)^2+m_\pi{}^2)\int_0^{1-X}\frac{\eta\,d\eta}{\eta+X}\,P(\eta)h(X, X_\pi, \eta)\,. \quad (25)$$

We can write $dk_\parallel/\omega=dX/X$ in (25) all the way to $X=0$ since $h(X, X_\pi, \eta)\propto X^6/X_\pi{}^2$ near $X=0$. For $k_\perp=0$ (and X not "wee")

$$\delta((k-q)^2+m_\pi^2)=\delta(m_\rho^2+2q_\parallel k_\parallel-2\omega_k\omega_q)$$

$$\simeq \frac{X}{m_\rho^2\gamma|\cos\theta|}\left[\delta\left(X-\frac{2X_\pi}{1-\gamma|\cos\theta|}\right)+\delta\left(X-\frac{2X_\pi}{1+\gamma|\cos\theta|}\right)\right], \tag{26}$$

and we retrieve the result of Ref. 3 [see Eq. (18) above]. The explicit form of the result in this model is obtained by inserting Eq. (26) into Eq. (25) to obtain (for $q_\perp < \gamma m_\rho/2 \sim 0.35$ GeV)

$$\rho_\pi(X_\pi, q_\perp)=\frac{\lambda g^2}{2\pi^2}\,\frac{1}{m_\rho^2\gamma|\cos\theta|}\left(\int_0^{1-X_+}\frac{\eta d\eta}{\eta+X_+}P(\eta)h(X_+, X_\pi, \eta)+\int_0^{1-X_-}\frac{\eta d\eta}{\eta+X_-}P(\eta)h(X_-, X_\pi, \eta)\right), \tag{27}$$

where $X_\pm \equiv 2X_\pi/(1\pm\gamma|\cos\theta|)$. It can be shown that

$$h(X_\pm, X_\pi, \eta)=\left(\frac{X_\pm}{X_\pi}\right)^4 R^2\,\frac{X_\pi^4\gamma^2\cos^2\theta}{(1+RX_\pm^2)^2}\,\frac{(1-\eta^2)^2}{(\eta^2+RX_\pm^2)^2}$$

so that $\rho_\pi(X_\pi, q_\perp)\propto X_\pi^4(1/X_\pi)^{3-\alpha}$ for small X_π, where $P(\eta)\sim\eta^\alpha$ for small η and $\alpha<3$. We see that unless $\alpha\leqslant -1$, a possibility clearly ruled out by the $pp\to p+$anything data, the pion scaling function vanishes at $X=0$. (Recall that $\alpha=0$ corresponds to a flat proton spectrum, and the data[2] indicate $\alpha\sim 1$.)

We have used Eq. (27) with $P(\eta)=$const and $\gamma|\cos\theta|=0.77$, corresponding to $q_\perp=0.2$ GeV, to compare this model with the $q_\perp^2=0.04$ GeV^2 $pp\to\pi^+ +$anything data of Ref. 2. The form of the results (see solid line at $q_\perp^2=0.04$ in Fig. 2) is not sensitive to the form of $P(\eta)$ for $0\leqslant\alpha<3$, nor is it sensitive to the details of the spin dependence under the same conditions.

To obtain the prediction of the model for $q_\perp \geqslant \frac{1}{2}m_\rho\gamma \sim 0.35$ GeV one must carry out the integration in Eq. (24) for $k_\perp\neq 0$. The $k_\parallel$ integral can be performed as before using the δ function. The resulting $X_\pm$ now depend on $|\vec{k}_\perp|$ and the angle ϕ between $\vec{q}_\perp$ and $\vec{k}_\perp$. The remaining $|\vec{k}_\perp|$ and ϕ integration region is determined by energy-momentum conservation and $|\cos\theta|\leqslant 1$. For $q_\perp \gg (m_\rho/2)\gamma$ one can assume that the dominant contribution to the integral comes from $k_\perp = k_{\perp\,\min}\sim q_\perp$ because of the damping factor $f(k_\perp)$. One can thus replace $|\vec{k}_\perp|$ by $|\vec{q}_\perp|$ everywhere in the integrand except in $f(k_\perp)$ and carry out the $|\vec{k}_\perp|$ integration. Furthermore, for large $q_\perp$ and $k_\perp$ the opening angle is small and we can approximate the ϕ integral by setting $\phi=0$ everywhere in the integrand. This gives (neglecting spin)

$$\rho_\pi(X_\pi, q_\perp=0.8\text{ GeV})\sim\text{const}\,[\rho(X_\pi)+\rho(1.8X_\pi)], \tag{28}$$

which illustrates the smearing out of the Brink, Cottingham, and Nussinov[3] effect at large $q_\perp$. This estimate is compared with the $q_\perp^2=0.64$ GeV^2 data in Fig. 2. The data at $q_\perp=0.4$ GeV $\sim(m_\rho/2)\gamma$ are in an intermediate range that requires a more exact evaluation of Eq. (24) than is warranted in the present case.

The results of the comparison of the model with experiment in Fig. 2 indicate that the pure bremsstrahlung picture bears little resemblance to the data even when one takes account of the kinematic bunching of backward decay pions pointed out in Ref. 2. One can also carry out a similar analysis for $\omega\to 3\pi$. In this case, even when the transverse momentum of the ω vanishes, the Brink-Cottingham-Nussinov effect is smeared out by the three final pions. Furthermore, because of the smaller phase space available to each pion, the pion distribution reproduces the SVM distribution more closely than in the $\rho\to 2\pi$ case. The inclusion of the ω among the produced SVM's therefore cannot remove the dip at $X_\pi=0$.

For comparison we have also plotted $\rho_\pi(X_\pi)$ cor-

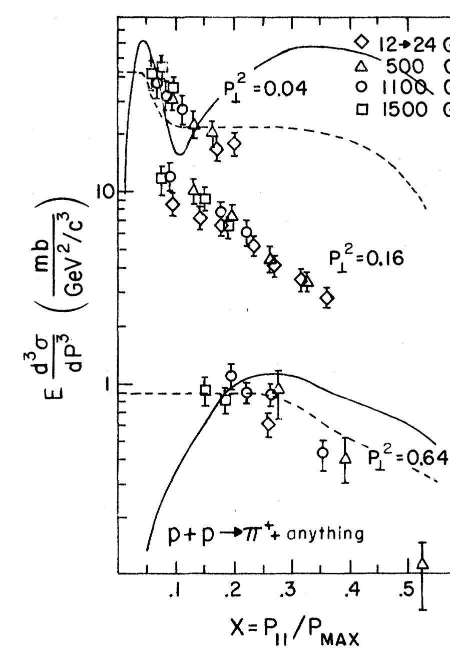

FIG. 2. Comparison of a simple ρ_0 bremsstrahlung model (solid lines) with the ISR data for $pp\to\pi^+ +$ anything (Ref. 2). The dashed lines show the shape of the single-pion distribution resulting from multiperipherally produced ρ mesons (see text).

responding to a multiperipheral-type form for the ρ-meson scaling function. These are the dashed lines in Fig. 2. The "multiperipheral" ρ-distribution, $\rho_{\text{MPM}}(X)$, was made artificially by using $\rho_{\text{MPM}}(X)=\rho(X)$ when $X>X_{\max}$ (where the behavior is presumably dominated by phase space) and setting $\rho_{\text{MPM}}(X)=\rho(X_{\max})$ for $X<X_{\max}$. $X_{\max}$ is the location of the maximum of $\rho(X)$. The results, though extremely crude, bear some resemblance to the data in that one sees a peaking for small X at small $q_\perp$ that is washed out at large $q_\perp$ as one would expect if pions are produced via ρ's. A detailed investigation of single-pion distributions using specific multiperipheral models for ρ production would be desirable.

V. CONCLUSION

The analysis presented here suggests that the simplest bremsstrahlung model of inelastic $\rho_0 \to \pi^+ + \pi^-$ production (Fig. 3) must be amended in order to produce one-particle pion distribution in agreement with the small-X ISR experiments. This conclusion is reinforced by recent single-γ experiments at ISR.[12] Assuming most photons come from π^0 decay, this experiment measures the single π^0 distribution, and the data at 90° clearly show $\rho_{\pi^0}(X=0)\neq 0$. It is reasonable to assume that the charged pion distribution is similar at $X=0$. We emphasize again that the basic prediction of the model,

$$\rho(X=0)\propto \frac{\sigma_{\text{tot}}}{s},$$

is independent of any kinematic approximations. It follows directly from the assumption that all meson emission is due to the bremsstrahlung of soft, neutral vector mesons from the external nucleon legs. Within the context of the basic interactions of the model, however, graphs involving multiperipheral ρ_0 production, such as those pictured in Fig. 4, should be considered and may be expected to yield proper pionization distributions.[13]

It is interesting to speculate on the form which experimental results will take as $k_\perp$ is measured at larger and larger values. On the basis of past experience, one has the intuitive feeling that the bremsstrahlung model should be relevant when large transverse momenta are involved, an effect which would be indicated by a dip in $\rho(k_\perp \gg m, x\sim 0)$.

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APPENDIX

We here present a functional formulation of the amplitudes, cross sections, and multiplicities of this problem in order to develop a compact way of exhibiting the complexities which are present when the narrow-width approximation is not made. These finite-width corrections have their physical origin in the necessary symmetrization of the inelastic, final-state pions. There are other obvious advantages to the functional formulation, which tend to become apparent with each sequence of simplifying approximations.

The fundamental field-theoretic interaction adopted is

$$\mathcal{L}'=ig\bar{\psi}\gamma_\mu A_\mu \psi + ig_1 A_\mu (\Pi^\dagger \cdot \partial_\mu \Pi - \partial_\mu \Pi^\dagger \cdot \Pi), \tag{A1}$$

in which we have retained only the coupling of ρ_0 field (A_μ) to nucleon fields ($\psi, \bar{\psi}$) and pion fields ($\Pi, \Pi^\dagger$), and for clarity have differentiated the corresponding coupling constants, g and g_1. The generating functional of this problem is defined by[14]

$$\mathfrak{Z}\{j_\mu, k, k^\dagger, \eta, \bar{\eta}\} = \left\langle \left(\exp i \int (j_\mu A_\mu + k^\dagger \Pi + \Pi^\dagger k + \bar{\psi}\eta + \bar{\eta}\psi) \right)_+ \right\rangle \tag{A2}$$

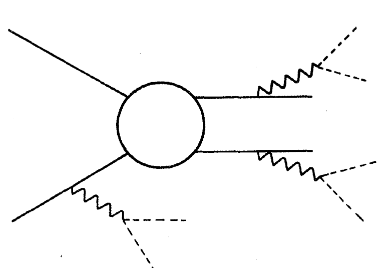

FIG. 3. A graph of the simplest bremsstrahlung type.

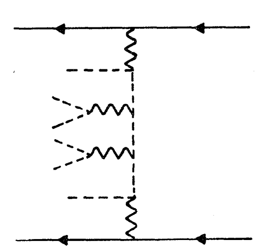

FIG. 4. A multiperipheral-type graph that is included within the general framework of the SVM model.

written in terms of the c-number sources j_μ, k, $k^\dagger$, η, $\bar{\eta}$. Because $g_1 \neq 0$, and the physical masses of ρ and $\pi^\pm$ are such that every ρ_0 produced must rapidly decay into a pion pair, we write the S matrix in the form[15]

$$\frac{S}{\langle S\rangle} =: \exp\left[Z_\pi^{-1/2}\int\left(\Pi_{\rm in}^\dagger \vec{K}\,\frac{\delta}{\delta k^\dagger} - \Pi_{\rm in}\vec{K}\,\frac{\delta}{\delta k}\right)\right] : S_F\{k, k^\dagger\}\Big|_{k=k^\dagger=0}, \tag{A3}$$

where $S_F\{k,k^\dagger\}$ denotes an effective, fermion S matrix expressed in terms of fictitious external sources $k,k^\dagger$,

$$S_F\{k,k^\dagger\} =: \exp\int Z_N^{-1/2}\left(\bar{\psi}_{\rm in}\vec{\mathfrak{D}}\,\frac{\delta}{\delta\bar{\eta}} - \frac{\delta}{\delta\eta}\,\overleftarrow{\bar{\mathfrak{D}}}\psi_{\rm in}\right) : \mathfrak{z}\{j, k, k^\dagger, \eta, \bar{\eta}\}\Big|_{j=\eta=\bar{\eta}=0}. \tag{A4}$$

We are not interested in details of the pion self-energy structure, and will subsequently make appropriate approximations such that the $Z_\pi^{-1/2}$ factor of (A3) may be replaced by unity. The operator $\Pi_{\rm in}(x)$ contains creation and destruction operators corresponding to the destruction of a π^+ and creation of a π^-, while $\Pi_{\rm in}^\dagger(x)$ may destroy a π^- and create a π^+.

The formal solution for the generating functional is given by

$$N\mathfrak{z} = \exp\left[-i\int\frac{\delta}{\delta\eta}\left(-g\gamma\cdot\frac{\delta}{\delta j}\right)\frac{\delta}{\delta\bar{\eta}} - i\int\frac{\delta}{\delta k}\left[g_1(\partial_\mu, \delta/\delta j_\mu)\right]\frac{\delta}{\delta k^\dagger}\right]\exp\left(i\int\bar{\eta}S_c\eta + \tfrac{1}{2}i\int j_\mu\Delta_c j_\mu + i\int k^\dagger D_c k\right), \tag{A5}$$

with $A(\partial_\mu, B)C \equiv AB(\partial_\mu C) - (\partial_\mu A)BC$, and where S_c, $\delta_{\mu\nu}\Delta_c$, and D_c denote (bare) massive propagators for nucleon, ρ meson, and pion, respectively; the gauge structure of the ρ propagator is quite irrelevant to the present discussion, and it is simplest to adopt the Feynman gauge. The normalization constant N is given by the vacuum-to-vacuum phase factor $\langle S\rangle$, and does not enter into any expressions for cross sections. A most convenient transformation of (A5) leads to the form[16]

$$N\mathfrak{z} = \exp\left(i\int\bar{\eta}G\left[\frac{g}{i}\frac{\delta}{\delta j}\right]\eta + L\left[\frac{g}{i}\frac{\delta}{\delta j}\right]\right)\exp\left(i\int k^\dagger\bar{D}_c\left[\frac{g_1}{i}\frac{\delta}{\delta j}\right]k + \Lambda\left[\frac{g_1}{i}\frac{\delta}{\delta j}\right]\right)\exp\left(\tfrac{1}{2}i\int j_\nu\Delta_c j_\nu\right), \tag{A6}$$

where

$$G[gA] = [m + \gamma\cdot(\partial - igA)]^{-1}, \quad \bar{D}_c[g_1A] = [m_\pi^2 - \partial^2 + ig_1(A_\nu, \partial_\nu)]^{-1},$$

$$L[gA] = \mathrm{Tr}\ln(1 - ig\gamma\cdot AS_c), \quad \Lambda[g_1A] = -\mathrm{Tr}\ln[1 - ig_1(\partial_\nu, A_\nu)D_c].$$

An alternate and frequently convenient representation of (A6) is

$$N\mathfrak{z} = \exp\left(\tfrac{1}{2}i\int j\Delta_c j\right)\exp\left(-\tfrac{1}{2}i\int\frac{\delta}{\delta A}\,\Delta_c\,\frac{\delta}{\delta A}\right)\exp\left(i\int\bar{\eta}G[gA]\eta + L[gA] + i\int k^\dagger\bar{D}_c[g_1A]k + \Lambda[g_1A]\right), \tag{A7}$$

where $A_\mu(x)$ now denotes the c-number source $\int\Delta_c(x-y)j_\mu(y)d^4y$.

We now perform a sequence of approximations on the exact expression (A7) in order to generate those graphs entering into the simplest bremsstrahlung model. The functional $L[gA]$ may be recognized as the source of all closed-fermion-loop graphs, which contribute to ρ-propagator structure and related processes; and it is dropped, $L[gA]\to 0$. The functional $\Lambda[g_1A]$, on the other hand, contains closed-pion-loop graphs contributing, among other things, that part of the ρ-propagator structure responsible for $\rho\to\pi^\pm$ decay. In the context of subsequent eikonal approximations for $G(A)$, it is instructive to expand $\Lambda[g_1A]$ to its quadratic A dependence, so that all functional operations upon the A sources may be performed exactly, with the corresponding self-energy structure everywhere inserted into Δ_c explicitly exhibited. Here, we shall simply drop $\Lambda[g_1A]$, but replace Δ_c by Δ_c', written in Breit-Wigner form, with a width Γ proportional to g_1^2. Since we are not interested in processes with ρ mesons in initial or final states, the factor

$$\exp\left(\tfrac{1}{2}i\int j\Delta_c' j\right)$$

of (A7) may be discarded; and the phase factor N shall also be omitted.

An amplitude containing two initial (p_1, p_2) and final (p_3, p_4) protons is obtained by performing functional differentiation with respect to a pair of η and $\bar{\eta}$ sources, bringing down the symmetrized factors

$$G(y_1, x_1|A)G(y_2, x_2|A) - G(y_1, x_2|A)G(y_2, x_1|A).$$

With $\eta = \bar{\eta} = 0$, one then calculates the amputated, mass shell, Fourier transforms with respect to these nucleon coordinates, using the nucleon four-momenta of the problem [as in (A12) below]. Either small- or large-momentum-transfer techniques may then be followed, with the first leading to a standard eikonal form,[17] and the second generating the appropriate wide-angle eikonal approximation.[18] The crucial step of either procedure is the extraction, from the product of $G(A)$ factors, of soft-vector-meson (SVM) dependence in the form $\exp(ig\int \mathfrak{F}_\mu A_\mu)$, with

$$\mathfrak{F}_\mu(w) = \int_0^\infty d\xi [p_1^\mu \delta(w - x_1 + \xi p_1) + p_2^\mu \delta(w - x_2 + \xi p_2) + p_3^\mu \delta(w - y_1 - \xi p_3) + p_4^\mu \delta(w - y_2 - \xi p_4)].$$

[In the small-angle $t/s \to 0$ limit of elastic scattering one passes to the limit of $p_1 - p_3 \to 0$, $p_2 - p_4 \to 0$ in $\mathfrak{F}_\mu(w)$.] With the sources k, $k^\dagger$ set equal to zero, and upon application of the functional operations of (A7), this A-dependence provides the necessary damping of the elastic nucleon scattering amplitude, and depends upon g_1 through the finite width of the Δ_c' propagator (e.g., as written by Yao[19] in a related context). Inelastic pion amplitudes are obtained by functional differentiation with respect to an appropriate number of k, $k^\dagger$ sources, before the latter are allowed to vanish; suppressing these operations, and the calculation of all Fourier transforms, etc., the general amplitude for pp scattering plus arbitrary pion production may be written as

$$M(k, k^\dagger) \sim \exp\left(-\tfrac{1}{2} i \int \frac{\delta}{\delta A} \Delta_c' \frac{\delta}{\delta A}\right) \exp\left(ig \int \mathfrak{F}_\mu A_\mu\right) \exp\left(i \int k^\dagger \bar{D}_c[g_1 A] k\right)\bigg|_{A=0}, \tag{A8}$$

where the tilde in (A8) denotes all the multiplicative, transform, and parametric integral dependence familiar from previous work.[17,18]

The functional operations of (A8) may be written in terms of self-linkages of each of the right-hand side factors, together with the cross linkages between them,

$$\begin{aligned} M(k, k^\dagger) \sim{}& \exp\left(-i \int \frac{\delta}{\delta A_1} \Delta_c' \frac{\delta}{\delta A_2}\right) \left[\exp\left(-\tfrac{1}{2} i \int \frac{\delta}{\delta A_1} \Delta_i' \frac{\delta}{\delta A_1}\right) \exp\left(ig \int \mathfrak{F} \cdot A_1\right)\right] \\ &\times \left[\exp\left(-\tfrac{1}{2} i \int \frac{\delta}{\delta A_2} \Delta_c' \frac{\delta}{\delta A_2}\right) \exp\left(i \int k^\dagger \bar{D}_c[g_1 A_2] k\right)\right]\bigg|_{A_{1,2}=0} \end{aligned} \tag{A9}$$

$$\to \exp\left(\tfrac{1}{2} i \int \mathfrak{F}_\mu \Delta_c' \mathfrak{F}_\mu\right) \exp\left(i \int k^\dagger \bar{D}_c[g_1 g \textstyle\int \Delta_c' \mathfrak{F}] k\right), \tag{A10}$$

where VM self-linkages of the $\exp(i\int k^\dagger \bar{D}_c[g_1 A] k)$ factor of (A9) have been omitted in the passage to (A10), since they correspond either to initial pions combining to form virtual ρ mesons, or to final-state pion interactions of the same form, together with pion self-energy effects. The first term on the right-hand side of (A10) contains linkages which define the elastic eikonal models, and is absorbed into the M_0 of (1). The remaining $k^\dagger$, k dependence of (A10) displays the factors needed for inelastic pion emission, and shall be written below as $\exp(i\int k^\dagger \bar{D}_c[g_1 A] k)$, where, henceforth, $A_\mu(x) \equiv g\int \Delta_c'(x-y)\mathfrak{F}_\mu(y) d^4y$. The final approximation which remains to be made is the replacement of $\bar{D}_c[g_1 A]$ by its linear A_μ dependence, thereby generating all graphs of the simplest bremsstrahlung form, as pictured in Fig. 3.

With these preliminaries, we are now in a position to discuss multiple pion emission. From (A3) and (A4), the probability amplitude for the process $p_1 + p_2 \to p_3 + p_4 + \sum_{i=1}^n (q_i + q_i')$, where q_i and q_i' denote π^+ and π^- four-momenta, respectively, is given by (all states are in-states)

$$\langle q_1 \cdots q_n, q_1' \cdots q_n', p_3, p_4 | S | p_1, p_2 \rangle = \left\langle q_1 \cdots q_n, q_1' \cdots q_n' \right| : \exp \int \left(\Pi_{\rm in}^\dagger K \frac{\delta}{\delta k^\dagger} - \Pi_{\rm in} K \frac{\delta}{\delta k}\right) : \left| 0 \right\rangle M(k, k^\dagger), \tag{A11}$$

with

$$M(k, k^\dagger) = \langle p_3, p_4 | S_F\{k, k^\dagger\} | p_1, p_2 \rangle. \tag{A12}$$

In the context of the present application, we have approximated the j, k, $k^\dagger$ dependence of (A12) by (A10). Only the negative frequency part of the $\Pi_{\rm in}$, $\Pi_{\rm in}^\dagger$ operators survives in the matrix element of (A11), connecting the pion vacuum to a state of $n\pi^+$ and $n\pi^-$ [from Eq. (A16) below it is obvious that the number of π^+

and π^- produced must be the same]; and we shall denote these operators by $(\Pi_{\rm in})^{(-)}$ and $(\Pi^\dagger_{\rm in})^{(-)}$. Thus an alternate statement of (A11) is

$$\frac{1}{n!}\left\langle q_1\cdots q_n\left|\left(\int (\Pi^\dagger_{\rm in})^{(-)}K\frac{\delta}{\delta k^\dagger}\right)^n\right|0\right\rangle\times\frac{1}{n!}\left\langle q'_1\cdots q'_n\left|\left(-\int (\Pi_{\rm in})^{(-)}K\frac{\delta}{\delta k}\right)^n\right|0\right\rangle, \tag{A13}$$

which leads to a convenient expression for the probability of emitting $n\pi^\pm$,

$$\begin{aligned}\mathcal{P}_n &= \sum_{\gamma_n}\sum_{\gamma'_n}\langle p_1p_2|S^\dagger|p_3p_4, q_1\cdots q_n, q'_1\cdots q'_n\rangle\langle q_1\cdots q_n, q'_1\cdots q'_n, p_3p_4|S|p_1p_2\rangle\\ &=\left(\frac{1}{n!}\right)^2\sum_{\gamma_n}\left\langle 0\left|\left(\int(\Pi_{\rm in})^{(+)}K\frac{\delta}{\delta k'}\right)^n\right|q_1\cdots q_n\right\rangle\left\langle q_1\cdots q_n\left|\left(\int(\Pi^\dagger_{\rm in})^{(-)}K\frac{\delta}{\delta k^\dagger}\right)^n\right|0\right\rangle\\ &\quad\times\left(\frac{1}{n!}\right)^2\sum_{\gamma'_n}\left\langle 0\left|\left(-\int(\Pi^\dagger_{\rm in})^{(+)}K\frac{\delta}{\delta k^{\dagger\prime}}\right)^n\right|q'_1\cdots q'_n\right\rangle\left\langle q'_1\cdots q'_n\left|\left(-\int(\Pi_{\rm in})^{(-)}K\frac{\delta}{\delta k}\right)^n\right|0\right\rangle\\ &\quad\times M^*(k',k^{\dagger\prime})M(k,k^\dagger)\Big|_{k=k^\dagger=k'=k^{\dagger\prime}=0},\end{aligned}\tag{A14}$$

where the $\sum_{\gamma_n}$ and $\sum_{\gamma'_n}$ denote summations over π^+ and π^- n-particle phase space; with this notation, the closure statement reads

$$\sum_{n=0}^\infty\sum_{\gamma_n}|q_1\cdots q_n\rangle\langle q_1\cdots q_n| = \sum_{n=0}^\infty\sum_{\gamma'_n}|q'_1\cdots q'_n\rangle\langle q'_1\cdots q'_n| = 1.$$

One may now invoke closure, and the simple commutation property of the in-fields, to rewrite (A14) in the form

$$\mathcal{P}_n=\left(\frac{1}{n!}\right)^2\left(i\int\frac{\delta}{\delta k'}\overleftarrow{\mathrm{K}}\cdot D_{(+)}\cdot\overrightarrow{\mathrm{K}}\frac{\delta}{\delta k^\dagger}\right)^n\left(i\int\frac{\delta}{\delta k^{\dagger\prime}}\overleftarrow{\mathrm{K}}\cdot D_{(+)}\cdot\overrightarrow{\mathrm{K}}\frac{\delta}{\delta k}\right)^n M^*(k',k^{\dagger\prime})M(k,k^\dagger)\Big|_0, \tag{A15}$$

where it is understood that the Klein-Gordon operators $K\equiv\mu^2-\partial^2$ act upon the coordinates freed by the functional differentiation. Inserting the pion source dependence of (A10), one obtains

$$\begin{aligned}\mathcal{P}_n&=\left(\frac{1}{n!}\right)^2\left(i\int\frac{\delta}{\delta k'}K\cdot D_{(+)}\cdot K\frac{\delta}{\delta k^\dagger}\right)^n\left(i\int\frac{\delta}{\delta k^{\dagger\prime}}K\cdot D_{(+)}\cdot K\frac{\delta}{\delta k}\right)^n\\ &\quad\times\exp\left(i\int k^\dagger\overline{D}_c[g_1A]k - i\int k^{\dagger\prime}\overline{D}{}^*_c[g_1A]k'\right)\Big|_0,\end{aligned}\tag{A16}$$

suppressing dependence upon all other coordinates (including the Fourier integrals over nucleon coordinates which provide four-momentum conservation in M and M^*), as in the replacement of (A12) by (A10).

From (A16) it is a straightforward matter to calculate pion cross sections and their inclusive moments. A simple representation of the quantities of (A16) yields

$$\begin{aligned}\mathcal{P}_n&=\left(\frac{1}{2\pi i}\right)^2\oint\frac{dz_1}{z_1^{n+1}}\oint\frac{dz_2}{z_2^{n+1}}\exp\left(iz_1\int\frac{\delta}{\delta k'}K\cdot D_{(+)}K\frac{\delta}{\delta k^\dagger}+iz_2\int\frac{\delta}{\delta k^{\dagger\prime}}K\cdot D_{(+)}\cdot K\frac{\delta}{\delta k}\right)\\ &\quad\times\exp\left(i\int k^\dagger\overline{D}_c[g_1A]k-i\int k^{\dagger\prime}\overline{D}{}^*_c[g_1A]k'\right)\Big|_0,\end{aligned}\tag{A17}$$

where the $z_{1,2}$ contours of (A17) circle the origin. This is a convenient form, for the functional differentiation operations may be performed exactly,[16]

$$\mathcal{P}_n=\left(\frac{1}{2\pi i}\right)^2\oint\frac{dz_1}{z_1^{n+1}}\oint\frac{dz_2}{z_2^{n+1}}\exp[-\mathrm{Tr}\ln(1-z_1z_2Q)]=\frac{1}{2\pi}\int_0^{2\pi}d\phi\, e^{-in\phi}\exp[-\mathrm{Tr}\ln(1-e^{i\phi}Q)], \tag{A18}$$

where

$$Q\equiv -D_{(+)}\cdot\overrightarrow{\mathrm{K}}\overline{D}_c[g_1A]\overleftarrow{\mathrm{K}}\cdot D^T_{(+)}\cdot\overrightarrow{\mathrm{K}}\overline{D}{}^*_c[g_1A]\overleftarrow{\mathrm{K}},$$

and the superscript T denotes transpose, $\langle x|D^T_{(+)}|y\rangle=D_{(+)}(y-x)$. A slightly more convenient representation for $\mathcal{P}_n$ is obtained by introducing fictitious sources $\varphi_{1,2}(w)$,

$$\exp[-\mathrm{Tr}\ln(1-e^{i\phi}Q)]=\exp\left(-i\int\frac{\delta}{\delta\varphi_1}Q\frac{\delta}{\delta\varphi_2}\right)\exp\left(ie^{i\phi}\int\varphi_1\varphi_2\right)\Big|_{\varphi_{1,2}=0}, \tag{A19}$$

and expanding the right-hand-side exponent of (A19), so that

$$\mathcal{P}_n = \frac{1}{n!} \exp\left(-i \int \frac{\delta}{\delta \varphi_1} Q \frac{\delta}{\delta \varphi_2}\right) \left(i \int \varphi_1 \varphi_2\right)^n \Big|_{\varphi_{1,2}=0}$$
$$= \frac{1}{n!} \left(\frac{\partial}{\partial \lambda}\right)^n \exp\left(-i \int \frac{\delta}{\delta \varphi_1} Q \frac{\delta}{\delta \varphi_2}\right) \exp\left(i\lambda \int \varphi_1 \varphi_2\right) \Big|_{\varphi_{1,2}=0,\ \lambda=0}$$

or

$$\mathcal{P}_n = \frac{1}{n!} \left(\frac{\partial}{\partial \lambda}\right)^n \exp[-\mathrm{Tr}\ \ln(1-\lambda Q)]\,|_{\lambda=0}\,. \tag{A20}$$

From (A20) all pion-pair quantities may be (formally) obtained; e.g., $P_0 = 1$, representing elastic proton scattering, while

$$\mathcal{P}_1 = \mathrm{Tr}(Q) \tag{A21}$$

and

$$\mathcal{P}_2 = \frac{1}{2!}\{(\mathrm{Tr}[Q])^2 + \mathrm{Tr}[Q^2]\}\,. \tag{A22}$$

The second term of (A22) corresponds to the cross terms obtained when calculating the $|\text{amplitude}|^2$ for two emitted pairs of π^+ and π^-, and is required by the boson symmetry of the amplitude under interchange of the two π^+, or of the two π^-. Such terms will perforce appear in all higher P_n, in addition to the "Poisson" terms, $(1/n!)(\mathrm{Tr}[Q])^n$, for the latter are statements of independent production of $n\pi^+$, and of $n\pi^-$. Important averages computed from (A20) are

$$\sum_{n=0}^{\infty} \mathcal{P}_n = \exp[-\mathrm{Tr}\ \ln(1-Q)] \tag{A23}$$

and

$$\sum_{n=1}^{\infty} n\mathcal{P}_n = \mathrm{Tr}\left[Q \frac{1}{1-Q}\right] \exp[-\mathrm{Tr}\ln(1-Q)]\,. \tag{A24}$$

It must be stressed that expressions such as (A23) and (A24) are defined for specified final proton four-momenta (and spins), and remain to be integrated over the suppressed, dummy configuration-space nucleon coordinates, which serve to express over-all conservation of four-momentum. One can trace this dependence to the nucleon x, y coordinates of $\mathfrak{F}_\mu$, which then appear in A_μ and hence in Q. In a true soft-photon description, such dependence disappears in the infrared limit $k_\mu \to 0$, but for nonzero VM mass this is not possible. All the arithmetic complications of the simple bremsstrahlung model stem from our inability to conserve four-momentum in a simple way, viz., the discussion of Sec. III, and the averaging approximation of Ref. 20. In either case, one settles upon the simplest approximation of neglecting, in the $\exp[-\mathrm{Tr}\ln(1-Q)]$ factor of (A23) and (A24), the nucleon configuration space coordinates as they appear in Q; but one then enforces four-momentum conservation in the integrals over the remaining nucleon coordinates, suppressed in (A23) and (A24), but needed when computing specific cross sections. Thus the "differential multiplicity" $\langle \nu \rangle$ defined[20] as the ratio of (A24) to (A23), and representing the number of $\pi^+\pi^-$ pairs emitted for specified final proton momenta p_3, p_4 is approximated by the same ratio omitting the $\exp[-\mathrm{Tr}\ln(1-Q)]$ factors. A further but similar approximation neglects the nucleon configuration coordinates in $\mathrm{Tr}[Q/(1-Q)]$, but "by hand" allows $p_1 + p_2 - p_3 - p_4 \neq 0$, thereby representing in a model-dependent way the missing four-momentum taken up by pion production. This approximation is too convenient to be resisted, and we shall adopt it in order to display the resulting approximation for $\langle \nu \rangle$,

$$\langle \nu \rangle \to \mathrm{Tr}\left[Q \frac{1}{1-Q}\right], \tag{A25}$$

an expression which depends only on the actual (experimentally measured) proton momenta p_1, p_2, p_3, p_4 appearing in Q. Equation (A25) shall be compared (see below) with the corresponding prediction, made in the same spirit, of Ref. 20 [Eq. (11) of Ref. 20]:

$$\langle \nu \rangle = -2\gamma \sum_{ij} \epsilon_{ij} F(t_{ij})\,. \tag{A26}$$

There is a simple, exact way of obtaining the one-pion (e.g., π^+) distribution function from (A24). When integrated over the phase space of both final nucleons (including spin summations and exact four-momentum conservation), (A24) defines a one-pion inclusive cross section σ,

$$\sigma = \sum_{n=1}^{\infty} n\sigma_n = \int \sum_n n\mathcal{P}_n\,,$$

where $\int$ denotes all final-state proton summations, with appropriate care taken in the passage from probability to cross section. Considering $D_{(+)}$ and $D^T_{(+)}$ in Q as distinct functions, one sees that

$\sum n\mathcal{P}_n$ is given by a sum over all powers of integrals over $\tilde{D}_{(+)}(k)$,

$$\sum n\mathcal{P}_n = \int d^4k\alpha(k)\tilde{D}_{(+)}(k) + \int d^4k_1\int d^4k_2\alpha(k_1, k_2)\tilde{D}_{(+)}(k_1)\tilde{D}_{(+)}(k_2) + \cdots$$
$$= \int \frac{d^3k}{2\omega}\alpha(k) + \int\frac{d^3k_1}{2\omega_1}\int\frac{d^3k_2}{2\omega_2}\alpha(k_1, k_2) + \cdots,$$

and hence

$$\frac{\delta}{\delta\tilde{D}_{(+)}(k)}\sum n\mathcal{P}_n = \alpha(k) + 2\int d^4k'\alpha(k, k')\tilde{D}_{(+)}(k') + \cdots. \tag{A27}$$

However, it is clear that

$$\int\frac{d^3k}{2\omega}\frac{\delta}{\delta\tilde{D}_{(+)}(k)}\sum n\mathcal{P}_n \neq \sum n\mathcal{P}_n,$$

and therefore $[\delta/\delta\tilde{D}_{(+)}(k)][\sum n\sigma_n]$ cannot be identified with $2\omega\, d\sigma/d^3k$. Inspection shows that the correct definition may be given by

$$2\omega\frac{d\sigma}{d^3k} \equiv \int_0^1\frac{d\lambda}{\lambda}\frac{\delta}{\delta\tilde{D}_{(+)}(k)}\sum n\sigma_n\{D_{(+)}\to\lambda D_{(+)}\}, \tag{A28}$$

for the parametric λ integral just restores the correct counting. With the aid of (A24) and (A25), one easily obtains

$$2\omega\frac{d\sigma}{d^3k} = \int\frac{\delta}{\delta\tilde{D}_{(+)}(k)}\exp[-\mathrm{Tr}\ln(1-Q)] \tag{A29}$$

or

$$2\omega\frac{d\sigma}{d^3k} = \frac{\delta}{\delta\tilde{D}_{(+)}(k)}\sigma_{\mathrm{tot}}, \tag{A30}$$

where $\sigma_{\mathrm{tot}} = \sum_{n=0}^{\infty}\sigma_n$ is the total cross section obtained by final proton summation over (A23).

A somewhat simpler derivation of (A30), which emphasizes its generality, proceeds from the observation that the exclusive $n\pi^+$ cross section σ_n is proportional to an n-fold integration over $\tilde{D}_{(+)}(k_1)\cdots\tilde{D}_{(+)}(k_n)$; hence the right-hand side of (A30), when integrated over the exhibited momentum coordinate k, must yield $\sum_n n\sigma_n$, thereby identifying $d\sigma/d^3k$. It must be noted that (A30) is correct as it stands only if the linear A approximation to $\overline{D}_c(g_1A)$ is made, so that Q depends only upon the $D_{(+)}$ (and $D^T_{(+)}$) factor exhibited in (A18). If this approximation is not made, internal $D_{(+)}$ (and $D^T_{(+)}$) dependence, contained in the virtual pion lines of $\overline{D}_c(g_1A)$ [and $\overline{D}_c^*(g_1A)$] will be present, in which case the right-hand side of (A30) must be replaced by the "partial" functional derivative with respect to the explicit $D_{(+)}$ (or $D^T_{(+)}$) dependence of Q.

The complexity of expressions of form (A29), even with a model-dependent way of handling energy-momentum conservation, is so great that one may well despair at the problem of evaluating the multiple meson integrals contained in (A24), not to mention the final proton summations involved in passing to the one-pion-inclusive cross section of (A29). Fortunately, a marked simplification occurs in the narrow-resonance limit, $\Gamma\propto g_1^2\to 0$, where all $\mathrm{Tr}[Q^n]$, $n\geq 2$, vanish. This can be understood physically by recognizing that the origin of these terms lies in the symmetrization of final-state pion amplitudes; in the limit of stable ρ_0 all such cross-term contributions must vanish, for the pions emitted (and the formalism forces pion emission) from a long-lived ρ_0 are, in principle, distinguishable. With (A18), and retaining only the linear A dependence of $\overline{D}_c[g_1A]$ (appropriate to the simple bremsstrahlung model, since every ρ_0 emitted by a nucleon can then only decay directly into a pair of pions), it is easy to show that

$$\mathrm{Tr}Q = \frac{g_1^2}{(2\pi)^6}\int d^4k\,\tilde{A}_\mu(k)\tilde{A}_\nu(k)\psi_{\mu\nu}(k), \tag{A31}$$

where

$$\psi_{\mu\nu}(k) = \int d^4q_1\tilde{D}_{(+)}(q_1)\int d^4q_2\tilde{D}_{(+)}(q_2)(q_1-q_2)_\nu(q_2-q_1)_\mu\,\delta(k-q_1-q_2) \tag{A32}$$

and

$$\tilde{A}_\mu(k) = \frac{g}{k^2+m_\rho^2-i\Gamma m_\rho}\left[e^{-ik\cdot x_1}\left(\frac{p_1}{k\cdot p_1}-\frac{p_3}{k\cdot p_3}\right)_\mu + e^{-ik\cdot x_2}\left(\frac{p_2}{k\cdot p_2}-\frac{p_4}{k\cdot p_4}\right)_\mu\right]. \tag{A33}$$

For the purpose of illustrating how momentum conservation is in principle maintained, nucleon coordinates appropriate to the simplest eikonal model ($x_1=y_1$, $x_2=y_2$) have been temporarily included in (A33). The expression for $\tilde{A}^*_\mu(k)$ is similar, except that the configuration coordinates of (A33) should be replaced by another set of dummy variables, $x'_{1,2}$, which enter into the integrals of M^*. The $x_{1,2}$ dependence of (A33) contributes both to the over-all four-momentum conserving δ function of M, and to the shift of nucleon momenta on internal lines [e.g., the replacement of $M_0(p_1,p_2,p_3,p_4)$ of (1) by $M_0(p_1,p_2,p_3+k,p_4)$, if the VM is emitted from nucleon leg p_3, etc.]; this latter effect is neglected in the simple (soft) bremsstrahlung model. For example, an eikonal amplitude containing a single VM emitted from a nucleon (and subsequently decaying into a pion pair) will require the integrals

$$\int d^4x_1 \int d^4x_2 \exp[ix_1\cdot(p_1-p_3)+ix_2\cdot(p_2-p_4)]T(x_1-x_2)\tilde{A}_\mu(k)$$

$$=(2\pi)^4\delta(p_1+p_2-p_3-p_4-k)\left[\left(\frac{p_1}{k\cdot p_1}-\frac{p_3}{k\cdot p_3}\right)_\mu \tilde{T}(p_1-p_3-k)+\left(\frac{p_2}{k\cdot p_2}-\frac{p_4}{k\cdot p_4}\right)_\mu \tilde{T}(p_1-p_3)\right]$$

$$\simeq(2\pi)^4\delta(\textstyle\sum p-k)V_\mu(k)\tilde{T}(p_1-p_3)$$

in the soft VM limit. More complicated x dependence can easily be constructed, but the essential property of these integrals will always be to provide the necessary $\delta(\sum p-\sum k)$ multiplying, in the SVM limit, products of factors $V_\mu(k)$. If one adopts the most convenient approximation of neglecting the x dependence of $\tilde{A}_\mu(k)$, and simply writing

$$\tilde{A}_\mu(k)=gV_\mu(k)(m_\rho{}^2+k^2-i\Gamma m_\rho)^{-1},$$

one must face up to the problem of providing an alternate way of generating, "by hand," the necessary momentum conservation.

In this spirit, as in (A25), the evaluation of $\psi_{\mu\nu}(k)$ is easily accomplished by an appeal to covariance; for $k_\mu\psi_{\mu\nu}(k)=0$, and hence the function $\chi(k^2)$ of the expression $\psi_{\mu\nu}=\delta_{\mu\nu}\phi(k^2)+k_\mu k_\nu\chi(k^2)$ is given by $\chi=-(1/k^2)\phi$. Note, however, that χ does not enter into the expression for $\mathrm{Tr}[Q]$. The calculation of ϕ is simplest when the pion mass vanishes, and one finds $\phi(k^2)|_{\mu=0}=-\frac{1}{6}\pi k^2\theta(-k^2)$, a positive quantity restricted to timelike k_μ. To complete the evaluation of (A31),

$$\mathrm{Tr}Q=\frac{g^2g_1{}^2}{(2\pi)^6}\int\frac{d^4k\,\phi(k^2)\sum_\mu V_\mu(k)V_\mu(-k)}{(m_\rho{}^2+k^2)^2+m_\rho{}^2\Gamma^2}, \tag{A34}$$

we refer to a previous calculation [result (11) of Ref. 20], which yielded

$$\tfrac{1}{2}g^2(2\pi)^{-4}\int\frac{d^4k\,V_\mu(k)V_\mu(-k)}{k^2+M^2-i\epsilon}=-i\gamma(M^2)\sum_{ij'}\epsilon_{ij}F(t_{ij}), \tag{A35}$$

where $\epsilon_{ij}=\pm1$ and

$$\sum\epsilon_{ij}F(t_{ij})=F(t_{13})+F(t_{24})+F(u_{14})+F(u_{23})-F(4m^2-s_{12})-F(4m^2-s_{34}).$$

Here, $\gamma(M^2)\simeq(g^2/8\pi^2)\ln(1+\mu_c{}^2/M^2)$, $F(z)\simeq-\ln(1+0.4|z|)$, and the t_{13}, t_{24}, u_{14}, u_{23}, s_{12}, s_{34} denote appropriate nucleon invariants, while μ_c represents a simple covariant cutoff limiting the magnitude of the SVM momenta. The absorptive part of (A35) yields

$$\tfrac{1}{4}g^2(2\pi)^{-3}\int d^4k\,\delta(k^2+M^2)\sum_\mu V_\mu(k)V_\mu(-k)=-\gamma(M^2)\sum_{ij}\epsilon_{ij}F(t_{ij}), \tag{A36}$$

and permits (A34) to be rewritten as

$$\mathrm{Tr}Q=-\frac{4g_1{}^2}{(2\pi)^3}\sum\epsilon_{ij}F(t_{ij})\int_0^\infty d\xi^2\phi(-\xi^2)\gamma(\xi^2)[(m_\rho{}^2-\xi^2)^2+\Gamma^2m_\rho{}^2]^{-1}. \tag{A37}$$

In the narrow-resonance approximation, $\Gamma[x^2+\Gamma^2m_\rho{}^2]^{-1}|_{\Gamma\to0}\to(\pi/m_\rho)\delta(x)$, and hence (A37) becomes

$$\mathrm{Tr}Q=\frac{\lambda m_\rho{}^2}{24\pi^2}[-2\gamma(m_\rho{}^2)\sum\epsilon_{ij}F(t_{ij})], \tag{A38}$$

where $\lambda\equiv\lim(\pi g_1{}^2/m_\rho\Gamma)|_{\Gamma\to0}$. With a redefinition of coupling constant, (A38) is just (A26).

With the continued neglect of all $\mathrm{Tr}[Q^n]$, $n\geq2$, one may apply the techniques of the previous two paragraphs to obtain, from (A23) and (A24), exactly the distributions (3), (4), (6), (7); e.g., (A23) is replaced by $\exp[\mathrm{Tr}Q]$, and σ_{tot} becomes $\int\exp[\mathrm{Tr}Q]$, where $\int$ again denotes summation over all appropriate nucleon parameters, including the configuration coordinates which produce exact four-momentum conservation. These statements are valid in the narrow-resonance limit, where it is not difficult to show that every term $\mathrm{Tr}[Q^n]$, $n\geq2$, vanishes. For example, $\mathrm{Tr}[Q^2]$ may be written in the form

$$\left(\frac{g_1^{\,2}}{(2\pi)^6}\right)^2 \int d^4k_1 \cdots \int d^4k_4\, \tilde{A}_\mu(k_1)\tilde{A}_\nu^*(k_2)\tilde{A}_\lambda(k_3)\tilde{A}_\sigma^*(k_4)\psi_{\mu\nu\lambda\sigma}\,, \tag{A39}$$

where

$$\psi_{\mu\nu\lambda\sigma}(k_1,\ldots,k_4) = \int d^4q_1\tilde{D}_{(+)}(q_1)\cdots\int d^4q_4\tilde{D}_{(+)}(q_4)(q_1-q_2)_\mu(q_2-q_3)_\nu(q_3-q_4)_\lambda(q_4-q_1)_\sigma$$
$$\times\,\delta(k_1-q_1-q_2)\delta(k_2-q_2-q_3)\delta(k_3-q_3-q_4)\delta(k_4-q_4-q_1)\,.$$

Because the simple reality structure, $\tilde{A}_\mu(k)\tilde{A}_\mu^*(k)$, of (A34) is not present here, counting powers of Γ and $g_1^{\,2}$ shows that (A39) vanishes with these quantities (one must also assume that expressions of form $\lim_{g\to 0}(1/g)[z\delta(z)]=0$); and the same property is generally true for $n>2$.

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[5]The center-of-mass frame is the appropriate coordinate system in which to define softness because $k_i \cdot k_j$ must be neglected relative to $k\cdot p$ for both incoming protons.

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[8]The factor $\exp\{K(x)\}$ contains the infrared divergence in the soft massless photon case, which is canceled in the well-known way by a similar factor from virtual soft photon exchanges appearing in $|M_0|^2$.

[9]To investigate the details of the transverse-momentum dependence provided by this factor, it is necessary to calculate virtual SVM exchanges in M_0 as has been done for pp elastic scattering [H. M. Fried and T. K. Gaisser, Phys. Rev. 179, 1491 (1969)]. In a calculation of SVM multiplicity in $pp \to ppX$ [H. M. Fried and T. K. Gaisser, Phys. Rev. D 4, 3330 (1971)] we have observed that in the case of massive SVM's there is not complete cancellation between the internal and external SVM factors; therefore, transverse-momentum damping may take place in the model. Rather than investigate the details of this damping here, however, we simply assume it as dictated by experiment.

[10]Our derivation has assumed that X is not "wee," i.e., that $k_\parallel \gg m_\rho$. But since the form we have for $\rho(X)$ vanishes like X^4 as $X\to 0$ and since we have shown in Sec. II that $\rho(X=0)$ is of order $1/s$ we can use Eqs. (14)–(15) all the way to $X=0$. Later (see Fig. 1) we also use the form of $\rho(X)$ out to $X=1$ although the basic assumption of softness is valid only for "wee" and small X regions. The result (Fig. 1) is nevertheless probably qualitatively correct even for large X since phase-space effects dominate there.

[11]The integral over d^3q' in Eq. (21) can be replaced by an integral over d^3k since

$$\int\frac{d^3q'}{2\omega'_q}\,\delta\big((q+q')^2+m_\rho^{\,2}\big)=\int\frac{d^3k}{2\omega_k}\,\delta\big((k-q)^2+m_\pi^{\,2}\big)\,.$$

Apart from spin factors, Eq. (21) is thus an integral of the single-ρ distribution function over all parent vector-meson momenta consistent with the observed pion momentum q. Such expressions are well known; for example, a similar expression was used by R. M. Sternheimer, Phys. Rev. 99, 277 (1955), in connection with $\pi^0\to 2\gamma$ decay.

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