Possible Bootstrap Origin of Thermodynamic Inclusive Spectra*

John Harte, John M. Rawls,† and H. C. Yen‡
Physics Department, Yale University, New Haven, Connecticut 06520
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A bootstrap approach is shown to reproduce a number of the results of hadronic statistical thermodynamics without the use of thermodynamical assumptions; in particular, we demonstrate a possible dynamical origin for the crucial constant-"temperature" Boltzmann factor in single-particle distributions. In our model, a fireball decays sequentially, each vertex in the decay chain being described by an asymptotic vertex function derived from a bootstrap theory of infinitely composite particles. When combined with a diffractive model for fireball production, the bootstrap description leads to good agreement with the qualitative features of inclusive reaction data.

I. INTRODUCTION

Statistical thermodynamics has proved to be a remarkably successful phenomenological tool in the study of strong interactions at high energies.[1] This approach has given reliable predictions for particle yields over a wide range of energies and masses and it seems to lead to a mass spectrum in agreement with experiment. In addition, it correctly predicts the qualitative features of single-particle momentum distributions in inclusive reactions.[2] Nevertheless, the basic assumption, that of the applicability to high-energy collisions of techniques adapted to the study of weakly coupled systems in thermal equilibrium, remains a mysterious one indeed.

From a phenomenological point of view, it is this latter assumption, as embodied in a factor of the form

$$\exp[-\beta(\vec{p}^{\,2}+\mu^2)^{1/2}], \tag{1}$$

which has been the key to the success of the model, for this factor contains the "maximum temperature of hadronic matter" $(k\beta)^{-1}$, it automatically produces the exponentials which are so characteristic of inclusive spectra,[2,3] and, when combined with a mechanism which ensures that particles are produced from objects (called fireballs) which move only forward or backward in the center-of-mass system, it leads to a transverse-momentum cutoff. It is noteworthy that, in the thermodynamic model,[1] it is simply *assumed* that fireballs move purely longitudinally.

We present here a model in which both the factors of the form (1) and the transverse-momentum cutoff follow from dynamical considerations, without thermodynamical assumptions. The basic input is the assumption of peripheral or multiperipheral production of massive objects which spew out sequentially the particles observed in the final state, each vertex in the decay chain being described by a bootstrap model.

The notion of the bootstrap has played an important role in the development of the thermodynamical model. For example, it is used in Hagedorn's derivation of the hadronic mass spectrum. Furthermore, Hagedorn has argued that the exponential falloff of elastic form factors, a behavior obtained from theories of infinitely composite particles, is also a consequence of the thermodynamic model.[1] The results of Frautschi[4] and Hamer,[5] to be discussed below, in conjunction with the present work, suggest that perhaps the bootstrap idea is central to the thermodynamic picture in the sense that many of the features of the latter are general properties of bootstrap solutions. However, we prefer not to speculate on the validity of the thermodynamic model or its assumptions. Instead, we present an alternative set of ideas which leads to similar predictions.

The combination of statistical notions with bootstrap techniques has been quite successful in reproducing thermodynamic-looking results. In particular, Frautschi[4] has derived the hadronic-mass spectrum predicted by Hagedorn[1] independently of the assumption of thermal equilibrium.[6] Exploiting further the statistical bootstrap, Hamer[5] has derived, from approximate phase-space calculations, a Boltzmann spectrum for certain inclusive reactions. In Hamer's work the dynamical assumption is that the matrix element is unity, whereas in the present work the dynamical input from the bootstrap leads to matrix elements quite different from unity. Remarkably, either assumption leads to thermodynamiclike spectra in the fireball decay; distinguishing between the two assumptions awaits further exploration of the models. One important difference between the results of the Frautschi-Hamer statistical model and ours is that the decay of very large fireball masses is more strongly suppressed in the latter. The multiplicity growth per fireball predicted by our work is therefore slower than in the statistical model. Our results are consistent with a constant multiplicity per fireball as s increases; a logarithmic or faster multiplicity growth must come from a growing multiplicity of fireballs.

Our plan of presentation is to first present a brief review of the bootstrap model (Sec. II) and then to apply these ideas to the decay of a fireball, with particular attention to the case of pion production (Sec. III). In Sec. IV, these results are combined with a phenomenological description of fireball production, thus providing a qualitative picture of inclusive cross sections. The relevant calculations are carried out in the Appendix. Because of the number of assumptions necessary to extract the inclusive cross sections and because of the uncertainties involved in the fireball-production mechanism, a *detailed* comparison with the data is not really a test of the key ideas in this work, so we attempt only to explain the qualitative features of the data. Finally, we offer a number of speculations on particle production at very high energies (Sec. V).

II. THE BOOTSTRAP

A bootstrap theory[7] of strong interactions developed by one of us provides an asymptotic description of the interactions of infinitely composite hadrons in the limit in which the Mandelstam variables characterizing a given reaction are large in absolute value. The predictions of the theory are contained in a tree theorem for the asymptotic amplitudes: Simply sum all tree graphs and describe the trilinear vertices of these graphs by a certain rapidly decreasing vertex function $\Gamma(p_1^2, p_2^2, p_3^2)$ with known asymptotic behavior. These n-particle amplitudes satisfy the exact unitarity conditions asymptotically and they describe infinitely composite particles. The self-consistent solution for Γ reads, in the limit in which at least one of the p_i^2 becomes large,

$$\Gamma(p_1^2, p_2^2, p_3^2) \sim P(p_1^2, p_2^2, p_3^2) \times \exp[-ag(p_1^2, p_2^2, p_3^2)], \qquad (2)$$

where P is power-bounded, a is a constant, and g is a function that satisfies the following conditions:

(i) $g=0$ when at least one of the p_i^2 vanishes;

(ii) if any one of the three variables, say p_1^2, becomes large, then g behaves like $(p_1^2)^{1/4}$ times a function of the remaining two variables;

(iii) if two variables, say p_1^2 and p_2^2, are large, then g behaves like $(p_1^2)^{1/4}(p_2^2)^{1/4}$ times a function of p_3^2;

(iv) if all three variables are large, g behaves like $(p_1^2)^{1/4}(p_2^2)^{1/4}(p_3^2)^{1/4}$.

According to the tree theorem, the wide-angle, high-energy elastic hadron-hadron amplitudes are related to the hadron form factors F by the expression (neglecting particle labels)

$$A(s,t,u) \approx F^2(s) + F^2(t) + F^2(u), \qquad (3a)$$

where the three terms on the right-hand side result from the three possible tree graphs in a two-body reaction. The form factor is predicted to have the asymptotic behavior

$$F(t) \sim \exp[-c(-t)^{1/4}]. \qquad (3b)$$

Both (3b) and the generalized Wu-Yang relation, Eq. (3a), give an excellent fit to the nucleon form-factor data and to the wide-angle pp elastic scattering data.[8] Comparison with data suggests that the asymptotic form in Eq. (3b) is valid for $|t|>1$ BeV^2, giving us a crude idea of where asymptopia is.

The model also makes a connection with the deep-inelastic e^-p data; in fact, the structure functions in this theory exhibit a partonlike behavior.[9] This follows from the observation that the vertex function $\Gamma(Q^2, p_1^2, p_2^2)$ is a rapidly decreasing function of the heavy photon mass $(-Q^2)^{1/2}$ if and only if $p_1^2,\ p_2^2 \neq 0$. Hence internal lines in tree graphs with zero virtual mass which couple to heavy photons play the role of partons. The observed small value for the ratio σ_L/σ_T, as well as approximate scaling, follow from these ideas; other consequences are being explored.

III. A MODEL FOR FIREBALL DECAY

Let us apply this theory to high-energy multiparticle production. We take seriously the experimental observations indicating that the produced secondary particles originate from jets or fireballs, and use the bootstrap theory to derive a momentum distribution characterizing the "decay" of a fireball. In order to confront the data, however, one must also have a description of fireball production. But such a description requires more information about the functions P and g [see Eq. (2)] than is now known. Thus we are forced to adopt a phenomenological description of fireball production; this is carried out in Sec. IV.

What is a fireball? We identify a fireball with a line in a tree graph (in the sense of the tree theorem) with a timelike momentum p_μ and a large mass $M^2=p^2$. We further make the quite natural assumption that a fireball decay proceeds in the manner illustrated in Fig. 1; arguments to support this assumption in the framework of the bootstrap model are given below. Shown is a fireball of mass M_1 which decays into n final-state hadrons with momenta $p_1, p_2, \ldots, p_n$ in the rest frame of

M_1 through a sequence or chain of decays. We emphasize that the internal lines in our tree graphs do not necessarily correspond to one-particle states, and we do not wish to imply a resonance-decay-chain mechanism.

At the first vertex in the chain, a fireball of mass M_1 decays into a fireball of mass M_2 plus a stable particle with momentum $\vec{p}_1$ and mass μ_1. The vertex function for this process is given, from the bootstrap theory, by

$$\Gamma(M_1^2, M_2^2, \mu_1^2) = P(M_1^2, M_2^2, \mu_1^2) \times \exp[-b(\mu_1^2)(M_1^2)^{1/4}(M_2^2)^{1/4}], \tag{4a}$$

where P is a power-bounded but unknown function, and $b(\mu^2)$ is a function of μ^2 satisfying

$$b(0) = 0, \tag{4b}$$

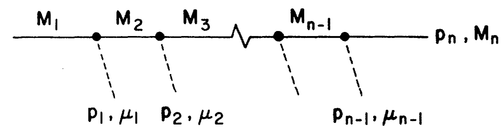

FIG. 1. A typical decay chain of a fireball of mass M_1 decaying into n final-state hadrons. $p_1, p_2, \ldots, p_n$ denote their momenta in the rest frame of M_1.

$$b(\mu^2) \approx a(\mu^2)^{1/4} \text{ as } \mu^2 \to \infty, \tag{4c}$$

and

$$\operatorname{Re} b(\mu^2) > 0. \tag{4d}$$

It has been assumed in writing Eqs. (4) that the fireball masses are sufficiently large to permit use of the asymptotic form for the vertex.

The amplitude for the particular tree graph shown in Fig. 1 is then given by

$$A(M_1; \mu_1, \vec{p}_1; \mu_2, \vec{p}_2; \ldots; M_n, \vec{p}_n) = \prod_{j=1}^{n-1} P(M_j^2, M_{j+1}^2, \mu_j^2) \exp[-b(\mu_j^2)(M_j^2)^{1/4}(M_{j+1}^2)^{1/4}]. \tag{5}$$

The total amplitude will be the sum of the amplitudes for all possible tree graphs.

To simplify the computation from here on, it is helpful to assume that all of the $n-1$ decay products indicated by dashed lines in Fig. 1 are pions. Thus we include the important case of a fireball with baryon number one which decays sequentially into $n-1$ pions and one nucleon. With this simplification, we can write the amplitude in Eq. (5) as

$$A(M_1; \vec{p}_1, \ldots, \vec{p}_{n-1}; M_n, \vec{p}_n) = B(M_1; \vec{p}_1, \ldots, \vec{p}_{n-1}; M_n, \vec{p}_n) \exp\left[-b \sum_{j=1}^{n-1} (M_j^2)^{1/4}(M_{j+1}^2)^{1/4}\right], \tag{6}$$

where $b \equiv b(\mu^2)$ is independent of j because we have chosen the emitted particles μ_j to all be pions, and the function B is just the product of all $P(M_j^2, M_{j+1}^2, \mu^2)$. The real part of b will be denoted by β, which is also a function of μ^2.

To find the total amplitude, we observe that while tree graphs topologically distinct from the one shown in Fig. 1 are allowed, they are relatively unimportant. We illustrate this point by comparing the contributions from the following two trees (see Fig. 2). Every vertex in graph 2(a) has only two heavy legs, but one of the vertices in graph 2(b) has three heavy legs. From Eq. (4), we see that the contribution from graph 2(b) is small relative to that from graph 2(a).

Thus, as a first approximation, the total amplitude for the fireball M_1 to decay into $(n-1)$ pions plus one relatively heavy particle M_n is simply the sum of all tree graphs that are topologically equivalent to the one shown in Fig. 1, that is,

$$A_T = \sum_{P(i)} A(M_1; \vec{p}_{i_1}, \vec{p}_{i_2}, \ldots, \vec{p}_{i_{n-1}}; M_n, \vec{p}_n), \tag{7}$$

where the ordered set of momenta $(\vec{p}_{i_1}, \vec{p}_{i_2}, \ldots, \vec{p}_{i_{n-1}})$ is a permutation of $(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_{n-1})$ and the summation $\sum_{P(i)}$ is over all such permutations.

In the Appendix, the summation over the tree graphs is performed and the phase-space integrals are approximated. Heavy use is made of the approximate recursion relation

$$M_j \approx M_{j+1} + (\vec{p}_j^{\,2} + \mu^2)^{1/2}, \tag{8}$$

justified in the Appendix. The key observation in our work is that *the one-fourth-power behavior in the asymptotic form for the vertex function is precisely what is needed to give a Boltzmann-like dependence on the momenta of the emitted parti-*

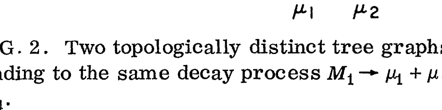

FIG. 2. Two topologically distinct tree graphs corresponding to the same decay process $M_1 \to \mu_1 + \mu_2 + \mu_3 + M_4$.

cles, which, as we have seen, is the reason for the phenomenological success of statistical thermodynamics. To see the Boltzmann factor with constant "temperature" emerge, simply note that for

$$(\vec{p}_j^{\,2}+\mu^2)\ll M_{j+1}^{\,2},$$

it follows that

$$\exp[-b(M_j^{\,2})^{1/4}(M_{j+1}^{\,2})^{1/4}] \approx \exp(-bM_{j+1})\exp[-\tfrac{1}{2}b(\vec{p}_j^{\,2}+\mu^2)^{1/2}]. \tag{9}$$

The inclusive pion spectrum for fireball decay into pions plus a nucleon is found to be [Eq. (A20)]

$$E\frac{d\sigma}{d^3p}(M_1)=\sum_{i=1}^{N-1}g_i\exp[-\beta(2i-1)(\vec{p}^{\,2}+\mu^2)^{1/2}], \tag{10}$$

where g_i is a slowly varying function of i given by Eq. (A21), and β is the real part of b. Because g_i grows slowly with i, the $i=1$ term in the sum dominates in Eq. (10) provided $|\vec{p}|$ is not too small ($\gtrsim$300 MeV); in this case, we obtain the desired Boltzmann form

$$E\frac{d\sigma}{d^3p}\approx g_1\exp\left(\frac{-(\vec{p}^{\,2}+\mu^2)^{1/2}}{kT}\right), \tag{11}$$

where we have replaced the constant β by

$$\beta^{-1}=kT. \tag{12}$$

If the g_i are independent of i, the series (10) can be summed, yielding

$$E\frac{d\sigma}{d^3p}=\frac{g\{1-\exp[-(2N-2)(\vec{p}^{\,2}+\mu^2)^{1/2}/kT]\}}{2\sinh[(\vec{p}^{\,2}+\mu^2)^{1/2}/kT]}, \tag{13}$$

an expression intermediate between Boltzmann and Bose-Einstein distributions. We have not been able to determine the value of the "temperature" within the present model, and the approximate phenomenological result $kT\approx m_\pi$ remains a mystery to us.

IV. INCLUSIVE SPECTRA IN THE C.M. FRAME

The spectrum given by Eq. (10) was derived in the rest frame of the fireball. To investigate scaling and other properties of observed inclusive spectra we must examine the effect of the Lorentz transformation from the fireball frame to the c.m. frame of the collision; this requires a model for the production of fireballs. We assume that at Brookhaven energies single-fireball production dominates and the fireball is diffractively produced (see Fig. 3). Since the production is diffractive, fireballs produced with large transverse momenta will be damped sharply by the factor e^{Bt} when the c.m. energy $s^{1/2}$ is large. Thus all fireballs will essentially be moving longitudinally. This, plus the Boltzmann-like distribution of the produced pions in the fireball frame, provides the transverse-momentum cutoff observed in production processes.

Before we discuss the detailed fireball production mechanism, let us take a simplified case and see what kind of inclusive spectra are implied by our model. Consider a fireball of mass M produced in the forward direction. If $s^{1/2}$ is the total energy in the c.m. frame of the collision, then the velocity of the fireball in the same frame is given by

$$v=\frac{1}{s+M^2-m_T^{\,2}}[s^2-2s(M^2+m_T^{\,2})+(M^2-m_T^{\,2})^2]^{1/2}, \tag{14}$$

where m_T is the mass of the target particle. In the high-energy limit, i.e.,

$$s^{1/2}\gg M, m_T, \tag{15}$$

Eq. (14) can be approximated by

$$v\approx 1-2M^2/s, \tag{16}$$

which corresponds to a Lorentz factor

$$\gamma\approx s^{1/2}/2M. \tag{17}$$

If a pion produced by a decaying fireball of mass M has momentum $p_\parallel$ and $\vec{p}_\perp$ in the c.m. frame, its energy E^f in the fireball frame will be

$$E^f=\gamma[(p_\parallel^{\,2}+\vec{p}_\perp^{\,2}+\mu^2)^{1/2}-vp_\parallel]. \tag{18}$$

Introducing the Feynman parameter

$$x=2p_\parallel/s^{1/2}, \tag{19}$$

we rewrite E^f as

$$E^f=\frac{s}{4M}\left[\left(x^2+\frac{4(\vec{p}_\perp^{\,2}+\mu^2)}{s}\right)^{1/2}-x\left(1-\frac{2M^2}{s}\right)\right], \tag{20}$$

which can be approximated by

$$E^f\approx\tfrac{1}{2}Mx+\frac{\vec{p}_\perp^{\,2}+\mu^2}{2Mx} \tag{21}$$

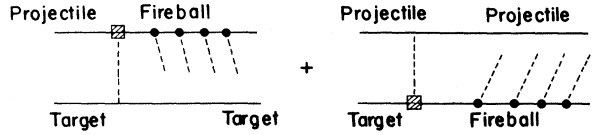

FIG. 3. Single excitation of the projectile and target, respectively. We assume that this is the predominant production mechanism at Brookhaven energies.

for

$$x \gg 2[(\vec{p}_\perp^{\,2}+\mu^2)/s]^{1/2}. \tag{22}$$

The inclusive spectrum is given by Eq. (10), or

$$f(s,x,\vec{p}_\perp) \equiv E\frac{d\sigma}{d^3p} \approx \sum_i g_i \exp\left(\frac{(2i-1)E^f}{kT}\right). \tag{23}$$

Since the right-hand side of Eq. (21) is independent of s, we see that the invariant cross section involving a given fireball, Eq. (23), scales for x satisfying (22). Scaling will thus be observed in single-particle inclusive spectra if the fireball production spectrum is s-independent. This suggests a diffractive model. The combination of a diffractive model with our description of fireball decay also ensures the presence of a $\vec{p}_\perp$ cutoff.

For smaller x, Eqs. (20) and (23) predict sizable deviations from scaling. In fact, a dip at $x=0$ develops which becomes stronger with increasing energy $s^{1/2}$. However, this behavior is restricted to pions produced from fireballs of relatively low mass. Higher-mass fireballs will produce pion distributions peaking toward smaller values of x, and they will not contribute significantly at intermediate values of x. Thus the resulting x distribution depends crucially on the fireball production mechanism.

Detailed phenomenological models of fireball production have been proposed by Jacob and Slansky[10] and by Hwa[11] which incorporate the features we require. In these models, the cross section for producing fireballs is diffractive (independent of s). Hagedorn[1] describes fireball production in terms of two arbitrary functions, the velocity weight functions. When transformed into the equivalent fireball mass distributions, the phenomenological fits to these distributions are very similar to the functional forms used by Jacob and Slansky, namely, a function rising to a low-mass maximum at about 1.5–2 GeV and falling off as M^{-2} for large M.

If we adopt any model of this type for fireball production, we find inclusive spectra very similar to those of Ref. 10, because the bulk of the data is more sensitive to the production mechanism of the fireball than to the details of the decay. We differ from the Jacob-Slansky model in our description of fireball decay. Our Bose-Einstein or Boltzmann-like form for the decay spectrum should be contrasted with their Gaussian form. This difference is reflected in the transverse-momentum distribution. For example, for intermediate values of x, we find, assuming a low fireball mass M, the $\vec{p}_\perp$ distribution to behave like

$$\exp(-\vec{p}_\perp^{\,2}/2MkTx), \tag{24}$$

while they predict

$$\exp(-\vec{p}_\perp^{\,2}/K^2), \tag{25}$$

where K depends on M but not on x. Thus when the logarithm of the inclusive spectrum is plotted against x for several values of $\vec{p}_\perp$, we predict a bunching of the curves at larger x, while they predict more nearly parallel curves.

Another aspect of the data which is sensitive to the decay mechanism is the multiplicity. We shall have more to say about this point in the next section, but we note that the apparently simple $\ln s$ behavior of the data may actually result from a combination of mechanisms, for it is found in both the Jacob-Slansky model and in this model with simple interpolating functions for $g(p_1^2, p_2^2, p_3^2)$ in Eq. (2) that two-fireball production will predominate over one-fireball production at sufficiently high energies, the transition probably occurring between Brookhaven and ISR (CERN Intersecting Storage Rings) energies. It is clear that a single-fireball picture cannot explain the existing cosmic-ray data, and yet cosmic-ray data together with accelerator data are needed to establish the $\ln s$ fit.[12] If the physics really is so different at accelerator and cosmic-ray energies, it seems unlikely that a simple theoretical picture will explain the present multiplicity data.

While single-particle distributions in the present model agree closely with those of the thermodynamic model, the models are definitely not equivalent. In the thermodynamic model a number of fireballs are produced simultaneously, even at accelerator energies. Thus correlations between emitted pions are expected to be small. We, on the other hand, predict that, in a given event involving a light fireball, the produced pions should emerge either all forward or all backward depending on whether the projectile or the target is diffractively excited. (As noted by Jacob and Slansky, heavy fireballs move sufficiently slowly in the c.m. frame that the emitted pions are not as correlated.) At higher energies, we expect a forward cone and a backward cone of produced particles, corresponding to excitation of both the target and the projectile (two-fireball production); this is observed in cosmic-ray events.[13] If these views are correct, then two-particle correlations cannot approach limiting distributions at accelerator energies. Further, the boiling of hadronic matter in the thermodynamic model does not give rise to sizable $\vec{p}_\perp$ correlations, whereas the decay-chain model predicts that pions with large $\vec{p}_\perp$ should be highly correlated both because of momentum conservation and because of the likelihood that a fireball with anomalously large t is responsible for such pions.

V. SPECULATIONS ON VERY HIGH-ENERGY PRODUCTION

The assumption of diffractive excitation is taken to mean that the excitation spectrum, denoted by $\rho(M_1)$, is independent of s, except for the shifting of the phase-space cutoff to larger values of M_1. The growth of $\langle M_1\rangle_{av}$ with s is further inhibited in our model by the rapid decrease of the production vertex, indicated by a shaded square in Fig. 3, as a function of M_1. Moreover, the multiplicity distribution $\sigma(M_1, n-1)$, given by Eq. (A16), falls off very rapidly, more rapidly than a Poisson distribution, with increasing n. The average multiplicity from a fireball of mass M_1 is given in terms of N, the maximum number of pions which is kinematically allowed, by

$$\langle n(M_1)\rangle_{av} = \sum_{n=2}^{N}(n-1)\sigma(M_1, n-1)\Big/\sum_{n=2}^{N}\sigma(M_1, n-1), \tag{26}$$

which grows at most like $\ln(M_1)$. To determine the average multiplicity, we must average over the fireball spectrum

$$\langle n\rangle_{av} = \int_{m_N}^{M_0} dM\,\rho(M)\langle n(M)\rangle_{av}\Big/\int_{m_N}^{M_0} dM\,\rho(M)\,, \tag{27}$$

where M_0 is the kinematic upper bound on the fireball mass. For large s, M_0 is proportional to $s^{1/2}$. If $\rho(M)$ behaves like M^δ for large M, one finds the following bounds on the multiplicity in the limit of large s:

$$\delta < -1, \quad \langle n\rangle \lesssim \text{constant} - O(s^{\delta+1}\ln s)$$
$$\delta \geq -1, \quad \langle n\rangle \lesssim \ln s\,. \tag{28}$$

The fastest $\rho(M)$ can grow consistent with the Froissart bound as applied to Pomeranchukon exchange is $(\ln M)/M$, which gives rise to a $\ln s$ multiplicity growth. In the work of Jacob and Slansky, and Hwa, $\rho(M)\sim M^{-2}$ and we obtain an asymptotically constant multiplicity per fireball. We note in this regard the recent prediction of Callan and Gross[14] that the multiplicity in e^+e^- annihilation via a single timelike photon is bounded.

The observed multiplicity growth, from accelerator and cosmic-ray data,[12] more nearly follows a law of the form

$$\langle n\rangle_{av} \approx \ln s \quad\text{or}\quad \langle n\rangle_{av} \approx s^{1/4}. \tag{29}$$

We are then forced to the conclusion that if $\delta < -1$, fireballs are produced with increasing multiplicities as s increases. Perhaps the simplest assumption would be to argue that *they* are produced by a multiperipheral mechanism with a Poisson distribution, with average multiplicity

$$\langle n\rangle_{av}^{\text{fireballs}} \approx c\ln s\,. \tag{30}$$

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APPENDIX

We consider the exclusive cross section for M_1 decaying into $(n-1)$ pions plus a heavy particle, say, a nucleon,

$$\begin{aligned}\sigma(M_1, n-1) &= \frac{1}{(n-1)!}\int d^3p_n\,\frac{M_n}{(\vec{p}_n^{\,2}+M_n^{\,2})^{1/2}}\left(\prod_{j=1}^{n-1}\int\frac{d^3p_j}{2(\vec{p}_j^{\,2}+\mu^2)^{1/2}}\right)|A_T|^2\\ &\qquad\times\delta^3\!\left(\sum_{j=1}^{n}\vec{p}_j\right)\delta\!\left(M_1-\sum_{j=1}^{n-1}(\vec{p}_j^{\,2}+\mu^2)^{1/2}-(\vec{p}_n^{\,2}+M_n^{\,2})^{1/2}\right)\\ &= \frac{1}{(n-1)!}\left(\prod_{j=1}^{n-1}\int\frac{d^3p_j}{2(\vec{p}_j^{\,2}+\mu^2)^{1/2}}\right)\frac{M_n}{E_n}\,|A_T|^2\,\delta\!\left(M_1-\sum_{j=1}^{n-1}(\vec{p}_j^{\,2}+\mu^2)^{1/2}-E_n\right),\end{aligned} \tag{A1}$$

where A_T is given by Eqs. (6) and (7) and

$$E_n = [M_n^{\,2}+(\vec{p}_1+\cdots+\vec{p}_{n-1})^2]^{1/2}\,.$$

The total cross section for M_1 decaying into pions plus a nucleon is then given by

$$\sigma(M_1) = \sum_{n=2}^{N}\sigma(M_1, n-1), \tag{A2}$$

where

$$N \lesssim \mu^{-1}(M_1-M_n)+1 \tag{A3}$$

is the kinematic upper bound on the number of emitted pions. The inclusive cross section for $M_1 \to \pi(\vec{p})$

+pions+1 nucleon is given by

$$2E\frac{d\sigma}{d^3p}(M_1)=\sum_{n=2}^{N}\frac{1}{(n-1)!}\sum_{i=1}^{n-1}\left[\int d^3p_n\,\frac{M_n}{(\vec{p}_n^{\,2}+M_n^{\,2})^{1/2}}\left(\prod_{j\neq i}\int\frac{d^3p_j}{2(\vec{p}_j^{\,2}+\mu^2)^{1/2}}\right)|A_T|^2\right.$$
$$\left.\times\delta^3\left(\sum_{j=1}^{n}\vec{p}_j\right)\delta\left(M_1-\sum_{j=1}^{n-1}(\vec{p}_j^{\,2}+\mu^2)^{1/2}-(\vec{p}_n^{\,2}+M_n^{\,2})^{1/2}\right)\right]_{\vec{p}_i=\vec{p}},$$

or

$$2E\frac{d\sigma}{d^3p}(M_1)=\sum_{n=2}^{N}\frac{1}{(n-1)!}\sum_{i=1}^{n-1}\left[\left(\prod_{j\neq i}\int\frac{d^3p_j}{2(\vec{p}_j^{\,2}+\mu^2)^{1/2}}\right)\frac{M_n}{E_n}|A_T|^2\,\delta\left(M_1-\sum_{j=1}^{n-1}(\vec{p}_j^{\,2}+\mu^2)^{1/2}-E_n\right)\right]_{\vec{p}_i=\vec{p}}. \tag{A4}$$

To calculate these cross sections, we must express the M_j in Eq. (6) in terms of the momenta of the emitted particles. In the rest frame of the initial fireball M_1, energy and momentum conservation provide a recurrence relation for M_j:

$$[M_j^{\,2}+(\vec{p}_1+\vec{p}_2+\cdots+\vec{p}_{j-1})^2]^{1/2}=[M_{j+1}^{\,2}+(\vec{p}_1+\cdots+\vec{p}_j)^2]^{1/2}+(\vec{p}_j^{\,2}+\mu^2)^{1/2}. \tag{A5}$$

Insertion of this into the phase-space integrals leads to intractable expressions, so we begin a series of approximations.

Since pion emission is essentially isotropic in the rest frame of fireball M_1, the randomness of the orientations of $\vec{p}_1,\vec{p}_2,\ldots,\vec{p}_j$ tends to prevent the magnitude of their vector sum from growing too large. In fact, we expect

$$\left\langle\left|\sum_{i=1}^{n}\vec{p}_i\right|\right\rangle\approx\frac{1}{n^2}\langle\vec{p}^2\rangle^{1/2}\approx n^{1/2}[\tfrac{3}{2}(0.4\text{ GeV})^2]^{1/2}=(1.5n)^{1/2}(0.4\text{ GeV}), \tag{A6}$$

where we have incorporated the fact that the observed average transverse momentum of emitted pions is $\lesssim 0.4$ GeV, independent of the energy of the collision and the identity of the colliding particles. Typically, $M_i\gtrsim 1.5$ GeV, and one finds, because of a cancellation of the corrections on the two sides of the equation, that Eq. (A5) may be approximated to a few percent accuracy by the simpler expression

$$M_j=M_{j+1}+(\vec{p}_j^{\,2}+\mu^2)^{1/2}, \tag{A7}$$

which corresponds to treating each fireball as being at rest in the rest frame of the first fireball. Such a simplified treatment, of course, must be modified if one also wishes to describe the emission of heavier particles.

Equation (A7) leads to

$$\sum_{j=1}^{n-1}(M_j^{\,2})^{1/4}(M_{j+1}^{\,2})^{1/4}\approx(n-1)M_n+\sum_{j=1}^{n-1}(j-\tfrac{1}{2})(\vec{p}_j^{\,2}+\mu^2)^{1/2}, \tag{A8}$$

which allows (6) to be written as

$$A(M_1;\vec{p}_1,\ldots,\vec{p}_{n-1};M_n,\vec{p}_n)=B\exp\left(b\sum_{j=1}^{n-1}(j-\tfrac{1}{2})(\vec{p}_j^{\,2}+\mu^2)^{1/2}\right). \tag{A9}$$

The total amplitude, given by Eq. (7), then becomes

$$A_T=\sum_{P(i)}B(i)\exp\left(-b\sum_{j=1}^{n-1}(j-\tfrac{1}{2})(\vec{p}_{i_j}^{\,2}+\mu^2)^{1/2}\right). \tag{A10}$$

To further simplify the computation, we will assume that Imb/Reb is of order unity. Then the relative phases between the amplitudes of different tree graphs oscillate very rapidly over the entire range of integration so that the phase-space integral of each of the cross terms in the expansion of $|A_T|^2$ [see Eq. (A10)] is negligible. In other words, different tree graphs can be treated as if they are incoherent. Then we can replace $|A_T|^2$ by

$$|A_T|^2\to\sum_{P(i)}|B(i)|^2\exp\left(-2\beta\sum_{j=1}^{n-1}(j-\tfrac{1}{2})(\vec{p}_{i_j}^{\,2}+\mu^2)^{1/2}\right), \tag{A11}$$

where β is the real part of b. It is assumed that the p dependence of the unknown but relatively slowly varying function $|B(i)|^2$ in front of each exponential can be neglected; we simply replace $|B(i)|^2$ by its average value $\alpha_n(M_1)$, allowing for a dependence on the mass M_1 of the first fireball.

The exclusive cross section (A1) becomes

$$\sigma(M_1,n-1)=\frac{1}{(n-1)!}\left(\prod_{j=1}^{n-1}\int\frac{d^3p_j}{2(\vec{p}_j^{\,2}+\mu^2)^{1/2}}\right)\frac{M_n}{E_n}\sum_{P(i)}\alpha_n(M_1)$$
$$\times\exp\left(-2\beta\sum_{j=1}^{n-1}(j-\tfrac{1}{2})(\vec{p}_{i_j}^{\,2}+\mu^2)^{1/2}\right)\delta\left(M_1-\sum_{j=1}^{n-1}(\vec{p}_j^{\,2}+\mu^2)^{1/2}-E_n\right), \tag{A12}$$

which can be simplified by a change of variables:

$$\sigma(M_1,n-1)=\alpha_n(M_1)\left(\prod_{j=1}^{n-1}\int\frac{d^3p_j}{2(\vec{p}_j^{\,2}+\mu^2)^{1/2}}\right)\frac{M_n}{E_n}\exp\left(-2\beta\sum_{j=1}^{n-1}(j-\tfrac{1}{2})(\vec{p}_j^{\,2}+\mu^2)^{1/2}\right)\delta\left(M_1-\sum_{j=1}^{n-1}(\vec{p}_j^{\,2}+\mu^2)^{1/2}-E_n\right). \tag{A13}$$

We will approximate E_n by M_n; this approximation, while not very accurate numerically, does serve to greatly simplify the integration and does not affect the exponential dependence of the inclusive cross section on the pion energy. Equation (A13) involves a phase-space integral which cannot be done exactly, but an approximate expression for $\sigma(M_1,n-1)$ is obtained by dropping the energy-conservation constraint.[15] In view of the rapidly decreasing integrand for large pion momenta, we expect the error made by extending to infinity the domain of integration of each pion momentum to be small. Equation (A13) then yields

$$\sigma(M_1,n-1)\approx\alpha_n(M_1)\prod_{j=1}^{n-1}\int\frac{d^3p_j}{2(\vec{p}_j^{\,2}+\mu^2)^{1/2}}\exp[-2\beta(j-\tfrac{1}{2})(\vec{p}_j^{\,2}+\mu^2)^{1/2}], \tag{A14}$$

or

$$\sigma(M_1,n-1)\approx\alpha_n(M_1)\prod_{j=1}^{n-1}\left(\frac{\mu\pi}{\beta}\right)\frac{K_1(2\beta\mu(j-\frac{1}{2}))}{(j-\frac{1}{2})}. \tag{A15}$$

The constant β, we shall see, is to be identified with $1/kT$ in the thermodynamic distribution, and hence is experimentally determined to be roughly $1/\mu$. Therefore the K_1 functions[16] can be expanded to give

$$\sigma(M_1,n-1)\approx\alpha_n(M_1)\prod_{j=1}^{n-1}\left(\frac{\mu\pi^3}{4\beta^3}\right)^{1/2}\frac{\exp[-2\beta\mu(j-\frac{1}{2})]}{(j-\frac{1}{2})^{3/2}}=\lambda_n(M_1)\frac{\exp[-\beta\mu(n-1)^2]}{[(2n-3)!!]^{3/2}}, \tag{A16}$$

where

$$\lambda_n(M_1)=\alpha_n(M_1)(2\mu\pi^3/\beta^3)^{(n-1)/2}. \tag{A17}$$

Note that the error made in using the asymptotic form of K_1 tends to underestimate σ, while the neglect of the energy conservation constraint tends to overestimate σ. In most cases, the combined correction due to these errors is small and does not alter the essential property of the solution, namely, its exponential character. However, our approximations fail if one pion carries off nearly all the available energy (in inclusive spectra, this means $x\approx 1$).

With the same approximations, the inclusive pion spectrum (A4) is reduced to

$$2E\frac{d\sigma}{d^3p}(M_1)=\sum_{n=2}^{N}\frac{\alpha_n(M_1)}{(n-1)}\sum_{i=1}^{n-1}\exp[-\beta(2i-1)(\vec{p}^{\,2}+\mu^2)^{1/2}]\prod_{j\neq i}\left(\int\frac{d^3p_j}{2(\vec{p}_j^{\,2}+\mu^2)^{1/2}}\exp[-\beta(2j-1)(\vec{p}_j^{\,2}+\mu^2)^{1/2}]\right), \tag{A18}$$

which can be expressed in terms of $\sigma(M_1,n-1)$ [see Eq. (A14)] as

$$2E\frac{d\sigma}{d^3p}(M_1)=\sum_{n=2}^{N}\left(\frac{\beta^3}{2\mu\pi^3}\right)^{1/2}\frac{\sigma(M_1,n-1)}{n-1}\sum_{i=1}^{n-1}(2i-1)^{3/2}\exp\{-\beta(2i-1)[(\vec{p}^{\,2}+\mu^2)^{1/2}-\mu]\}, \tag{A19}$$

or

$$E\frac{d\sigma}{d^3p}(M_1)=\sum_{i=1}^{N-1}g_i\exp[-\beta(2i-1)(\vec{p}^{\,2}+\mu^2)^{1/2}], \tag{A20}$$

where

$$g_i=\tfrac{1}{2}(2i-1)^{3/2}\exp[\beta(2i-1)\mu]\left(\frac{\beta^3}{2\mu\pi^3}\right)^{1/2}\sum_{n=i+1}^{N}\frac{\sigma(M_1,n-1)}{(n-1)}. \tag{A21}$$

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†Present address: Department of Physics, University of California, San Diego, California.

‡Present address: Department of Physics, Rochester University, Rochester, New York.

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Schwarz Inequality for One-Photon Annihilation, Electroproduction, and the Two-Photon Process*

Hidezumi Terazawa
The Rockefeller University, New York, New York 10021
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It is demonstrated that the Schwarz inequality gives useful relations between the one-photon annihilation process $e^+ + e^- \to$ any hadrons, the electroproduction $e + A \to e +$ anything, and the two-photon process $e^\pm + e^- \to e^\pm + e^- + A$, where A is an arbitrary hadronic state with positive charge conjugation. Particularly, the $\pi^0\gamma\gamma$ vertex function, if it decreases at all, is shown to decrease not slower than $(-q^2)^{-1/2}$ as one of the virtual-photon masses q^2 increases. Possible applications to various other processes are suggested.

Both the one-photon annihilation process $e^+ + e^- \to$ hadrons[1] and the electroproduction $e + p \to e +$ anything, especially in the Bjorken scaling region,[2] have been of great experimental and theoretical interest. In addition, the two-photon process $e^\pm + e^- \to e^\pm + e^- +$ hadrons has attracted considerable attention since its experimental feasibility was emphasized by several authors a few years ago.[3] In this paper we show that the Schwarz inequality[4] gives useful relations between the inclusive one-photon process $e^\pm + e^- \to$ any hadrons, the inclusive electroproduction $e + A \to e +$ anything, and the exclusive two-photon process $e^\pm + e^- \to e^\pm + e^- + A$, where A is an arbitrary hadron (or hadronic state) with positive charge conjugation.

To be more specific, let us take $A = \pi^0$ as the simplest example. The differential cross section for the two-photon process $e^\pm + e^- \to e^\pm + e^- + \pi^0$ at the c.m. energy E is given by

$$\frac{d\sigma}{dE_1' \, d\cos\theta_1 \, dE_2' \, d\cos\theta_2 \, d\phi} = 128\alpha^4 \frac{E^2 E_1' E_2' (E - E_1')^2 (E - E_2')^2}{(q^2 k^2)^2} \delta(P^2 - m_\pi^2) |F(q^2, k^2)|^2 \quad \text{for } m_\pi^2 \ll -q^2, \, -k^2 \ll 4E^2, \tag{1}$$

where E_i' and θ_i ($i = 1$ and 2) are the energy and angle of the scattered leptons; ϕ is the coplanarity angle; q, k, and P are the momenta of the first and second virtual photons and the pion, respectively ($P = q + k$); m_π is the pion mass; and the $\pi^0\gamma\gamma$ vertex function is defined by