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†Permanent address.

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PHYSICAL REVIEW D VOLUME 6, NUMBER 1 1 JULY 1972

Solutions to π-π and K-π Scattering in a Model Satisfying Analyticity, Crossing Symmetry, and Approximate Unitarity*

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Phase shifts for π-π and K-π scattering are obtained from a model satisfying analyticity, exact crossing symmetry, and approximate unitarity up to $\sqrt{s} \sim 1.3$ GeV. Low-energy elastic as well as low- and high-energy total cross sections are calculated. The results are compared with available data on π-π and K-π scattering, as well as general restrictions on π-π models.

I. INTRODUCTION

We shall present, in the following, self-consistent solutions to π-π and K-π scattering, that widen the scope of applications of a model of meson-meson scattering[1,2] which satisfies

(1) Mandelstam representation,

(2) crossing symmetry,

(3) resonances in all nonexotic channels (mass spectrum),

(4) Regge behavior in all channels, and

(5) approximate unitarity.

The solutions are obtained by a partial-wave analysis of the π-π amplitude, satisfying the above conditions (1)–(4), as well as the Adler condition, and by imposing the unitarity constraints on the partial waves. The mass spectrum is predetermined in the amplitude, and is very little affected by the unitarization procedure. The exact crossing symmetry of the amplitude ensures that the partial waves satisfy the π-π crossing constraints.[2] This method yields partial-wave solutions to π-π scattering in the region from threshold to $\sqrt{s} \sim 1.2$ GeV, as well as the elastic cross sections. Total cross sections are obtained from the amplitude for all energies.

The K-π amplitude is formulated in terms of the π-π amplitude[2] almost uniquely (up to mass spectrum). This amplitude is then used to determine the phase shifts and cross sections for K-π scattering up to $\sqrt{s} \sim 1.3$ GeV. The total cross sections are determined to all energies.

In deriving these amplitudes, it was found that the diffractive part of the amplitude plays a significant role in the unitarity constraints. The model for the Pomeranchukon amplitude differs from the one developed in Ref. 2, and is, in fact, a modified version of the model described in Ref. 1.

II. DETERMINATION OF THE AMPLITUDE

We shall use the kinematical conventions of Refs. 1 and 2. The total amplitude for π-π and K-π scattering is defined by

$$A^I(s,t,u)=F^I_{\rho,K^*}(s,t,u)+P^I(s,t,u)\,, \qquad (2.1)$$

where F^I_ρ and $F^I_{K^*}$ denote the isospin amplitudes, corresponding to the purely nondiffractive (resonant) contributions to π-π and K-π scattering, respectively, while P^I denotes the Pomeranchukon amplitude. The detailed forms of F_ρ, F_{K^*}, and P^I are given in the Appendix.

Although there are quite a few parameters in the definition of the amplitude, these parameters are *not all* arbitrary. The physical solutions are required to satisfy unitarity (at least approximately in the sense described below). Thus, the search for physical, unitary solutions of the model *severely restricts the parameters in the model.*

To determine the solutions, subject to the above constraints, we project out the partial waves

$$f^I_l(s)=\frac{1}{2}\int_{-1}^{1}d\cos\theta\, f^I(s,\theta)P_l(\cos\theta)\,, \qquad (2.2)$$

where

$$f^I_l(s,\theta)=\begin{cases}\dfrac{1}{16\pi\sqrt{s}}A^I(s,t,u) & (\text{for } \pi\text{-}\pi \text{ scattering})\\[2ex] \dfrac{1}{8\pi\sqrt{s}}A^I(s,t,u) & (\text{for } K\text{-}\pi \text{ scattering})\end{cases} \qquad (2.3)$$

are the scattering amplitudes.[2] The phase shifts are determined from the relation

$$f^I_l(s)=\frac{1}{2iq}(\eta^I_l e^{2i\delta^I_l}-1)\,, \qquad (2.4)$$

where η^I_l are the inelasticity parameters, $0\le\eta^I_l\le 1$, and $\eta^I_l=1$ in the elastic region ($4m_\pi{}^2\le s\le 16m_\pi{}^2$ for π-π). Here δ^I_l is the (real) phase shift for isospin I in the lth partial wave. Inverting (2.4), we have the following relation for the inelasticity parameters:

$$\eta^I_l=|1+2iqf^I_l(s)|\,. \qquad (2.5)$$

The physical solutions to f^I_l are then restricted by the requirement that $\eta^I_l=1$ in the elastic region, and $\eta^I_l\le 1$ in the inelastic region.

This procedure was found to be very sensitive to changes in the parameters. To obtain a reasonable solution over a large range of energy, the parameters were determined almost uniquely. Thus, the amplitude determined by this method is almost completely defined by the unitarity constraints, so that the experimental properties (phase shifts, cross sections) are predicted from the model.

III. RESULTS FOR π-π SCATTERING

The inelasticity parameters for π-π scattering, determined by the above procedure for the principal partial waves, are shown in Fig. 1. It is seen that the violation of the unitarity bound in the $I=0$ S wave and the $I=1$ P wave is less than 10% below 1.2 GeV. Also shown are the estimated absorption parameters from Baton *et al.*[3] It is clear that agreement of the predicted absorption parameters with experimental results, rather than nonviolation of the unitarity bound, is the sufficient condition for unitarity to be satisfied. In that sense, violation of unitarity in these partial waves is greater than indicated above (if the experimental results are correct).

Figures 2–4 show the phase shifts for the principal partial waves. The predicted $I=0$ S wave is the "between-up" solution (cf. Morgan and Shaw[4]). The fit to the data of Baton *et al.*[3] is reasonable below the ρ mass, but diverges above this energy. The phase shift goes through 180° quite rapidly,

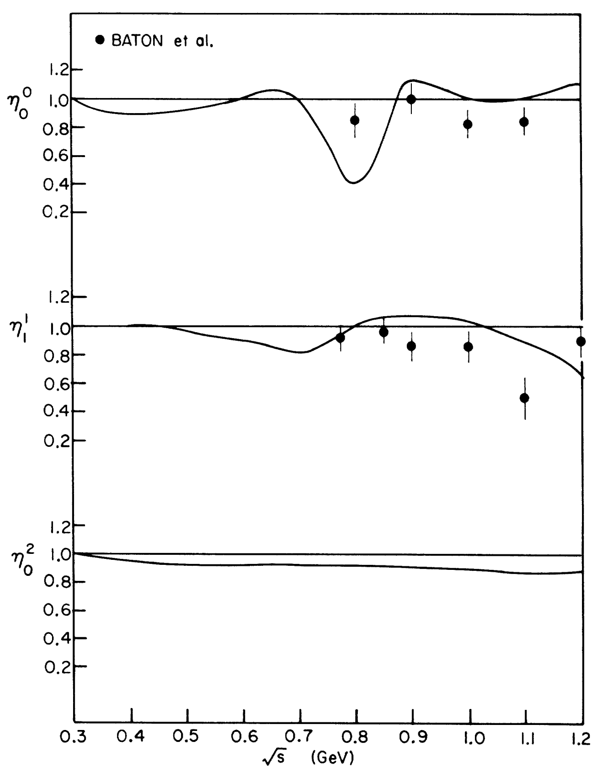

FIG. 1. Absorption parameters (η^I_l) predicted for π-π scattering. Data are from Baton *et al.* (Ref. 3).

thus giving a relatively narrow ϵ width. We find that

$$m_\epsilon = 750 \text{ MeV}, \quad \Gamma_\epsilon = 140 \text{ MeV}. \tag{3.1}$$

The data points seem to show a flattening out above the ρ mass, thus giving a wider ϵ:

$$m_\epsilon = 736 \text{ MeV}, \quad \Gamma_\epsilon = 181 \text{ MeV}. \tag{3.2}$$

The work of Baton *et al.*[3] seems to have settled the up-down ambiguity below the ρ mass, giving a unique (between) solution. However, above the ρ mass the ambiguity has not yet been resolved (down solution to S wave not shown in Fig. 2). A theoretical evaluation of the π-π phase shifts by Bennett and Johnson[5] seems to show a preference for the resonant solution to the S wave with $m_\epsilon = 760$ MeV, when the inelasticity at ~1 GeV is taken into account.

The $I=1$ P-wave phase shift fits the data of Baton *et al.*[3] and Scharenguivel *et al.*[6] Also shown in Fig. 3 is the phase shift of Baton *et al.*[3] including absorption, which agrees with the model calculation above the ρ mass. The mass and width of the ρ (Γ_ρ being the energy interval between the points where

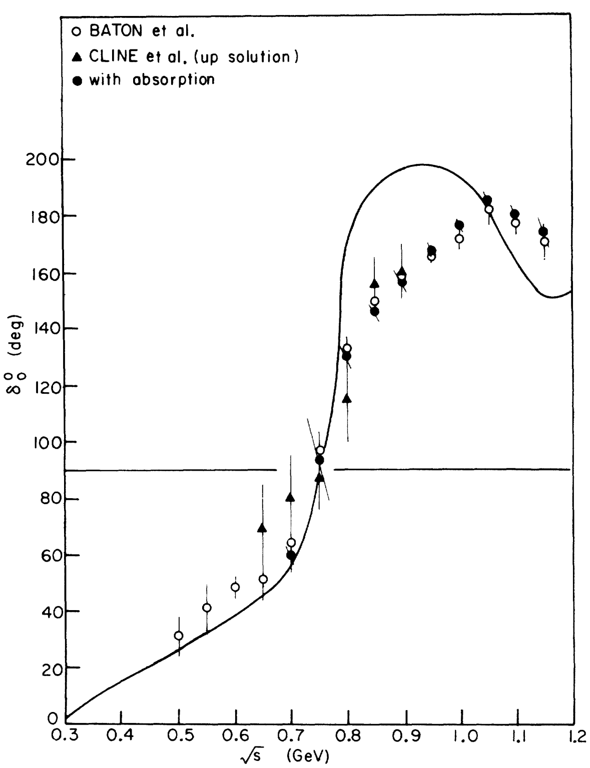

FIG. 2. Phase shift δ_0^0 (notation δ_l^I). Data of Baton *et al.* (Ref. 3) and Cline *et al.* [University of Wisconsin report, 1969 (unpublished)] (only up solution shown). Data are shown with and without absorption. The solid line is the model prediction.

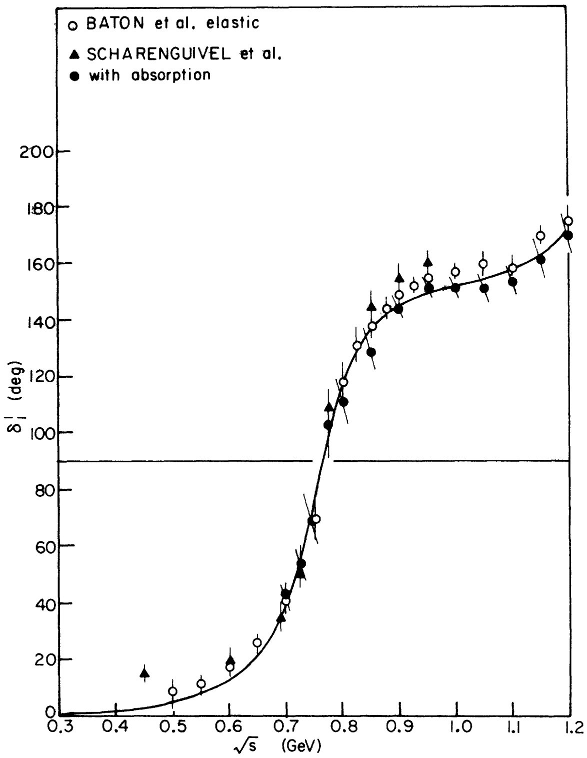

FIG. 3. Phase shift δ_1^1. Data of Baton *et al.* (Ref. 3) and Scharenguivel *et al.* (Ref. 6). Data are shown with and without absorption.

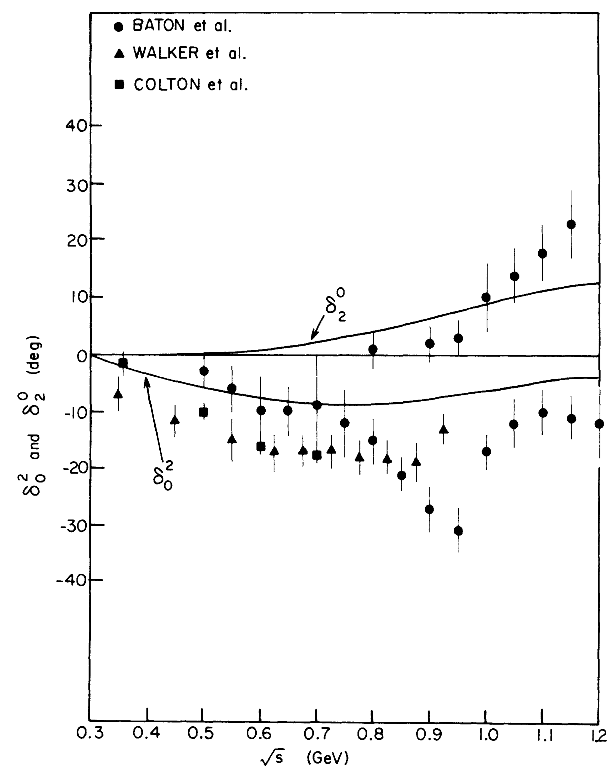

FIG. 4. Phase shifts δ_0^2 and δ_2^0. Data of Baton *et al.* (Ref. 3), Walker *et al.* and Colton *et al.* (Ref. 7).

δ_1^1 goes through 45° and 135°) are

$$m_\rho = 765 \text{ MeV}, \quad \Gamma_\rho = 125 \text{ MeV} \tag{3.3}$$

compared to those of Baton *et al.*:

$$m_\rho = 766 \text{ MeV}, \quad \Gamma_\rho = 127 \text{ MeV}. \tag{3.4}$$

The predicted $I = 2$ S wave is consistent with the data of Baton *et al.*,[3] Walker *et al.*,[7] and Colton *et al.*[7] It should be noted that the phase-shift data are extracted from the cross sections assuming $\eta_0^2 = 1$ everywhere. Since δ_0^2 is small in magnitude, this assumption can lead to considerable deviations in the estimated phase shifts when η_0^2 deviates from 1 by even a small amount. In the model, η_0^2 goes down to 0.9 at 1.2 GeV. The phase shift reaches a minimum of $-9°$ at $\sqrt{s} = 750$ MeV and decreases in magnitude for increasing s. It is to be noted that due to the small phase shift in this partial wave, unitarity violation in the elastic region is greater than the deviation of η_0^2 from 1 would seem to indicate.

The threshold parameters are determined from the partial waves[1] to be

$$a_0 = 0.19 m_\pi^{-1}, \quad a_2 = -0.032 m_\pi^{-1}, \quad a_1 = 0.028 m_\pi^{-3}. \tag{3.5}$$

It is seen that a_2 is smaller than the Weinberg[8] current-algebra value of $a_2 = -0.06 m_\pi^{-1}$, and leads to a large value of the S-wave scattering-length ratio

$$a_0/a_2 = -5.9, \tag{3.6}$$

as compared to Weinberg's value of -3.5.[8]

It is to be noted that the present model does satisfy the rigorous crossing-symmetry conditions (as it is constructed to do) as discussed in a previous paper (II) (see also Ref. 9).

A recent calculation by Carrotte and Johnson[10] of π-π partial-wave dispersion relations, using the inverse-amplitude method, yields the scattering lengths for the S waves

$$a_0 = (0.18 \pm 0.01) m_\pi^{-1}, \tag{3.7}$$

$$a_2 = -0.04 m_\pi^{-1}, \tag{3.8}$$

in reasonable agreement with (3.5).

We find that

$$L = \tfrac{1}{6}(2a_0 - 5a_2) m_\pi \tag{3.9}$$

has a value of 0.09, and the relation

$$(2a_0 - 5a_2) m_\pi = 18 m_\pi^2 a_1 \tag{3.10}$$

is satisfied to the extent that one would expect, namely, the left-hand side has the value 0.540,

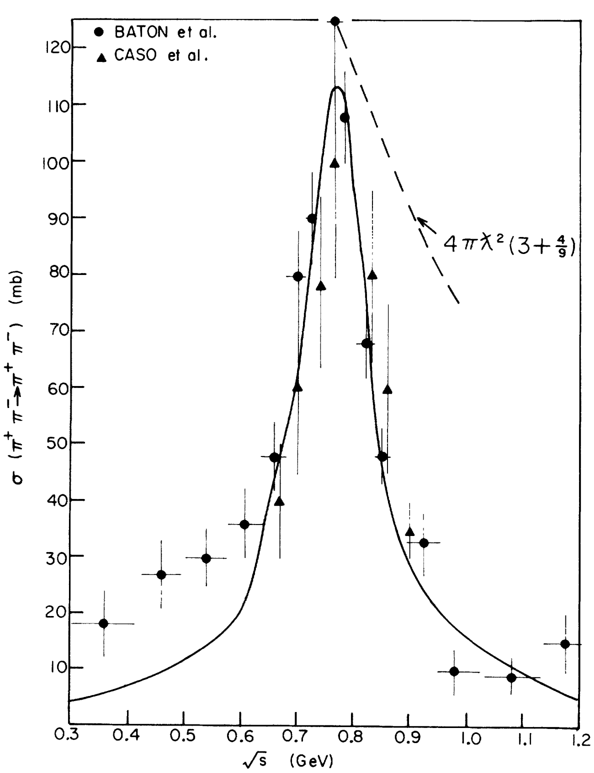

FIG. 5. Elastic cross section for $\pi^+\pi^-$ as function of c.m. energy. Data of Baton *et al.* (Ref. 3) and Caso *et al.* (Ref. 13). Unitarity limit for $S+P$ waves is also shown.

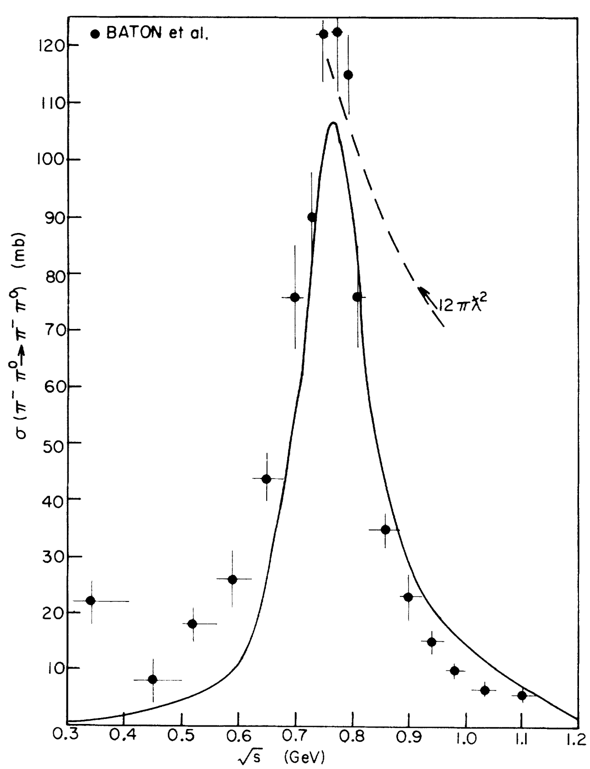

FIG. 6. Elastic cross section for $\pi^-\pi^0$. Data of Baton *et al.* (Ref. 12). Unitarity limit for P wave is also shown.

whereas the right-hand side has the value 0.525.

Another consistency check on the S-wave phase shifts at low energy is derived from K^0 decay. In the decay $K_S^0 \to 2\pi$, the branching ratio

$$\Gamma(K_S^0 \to \pi^+\pi^-)/\Gamma(K_S^0 \to \pi^0\pi^0) \tag{3.11}$$

yields the result

$$\delta_0^0(m_K) - \delta_0^2(m_K) = \pm(40^{\circ}{}^{+15^{\circ}}_{-10^{\circ}}), \tag{3.12}$$

where m_K is the kaon mass.[11]

Our phase shifts are

$$\delta_0^0(m_K) = 27^{\circ} \text{ and } \delta_0^2(m_K) = -6^{\circ}, \tag{3.13}$$

so that

$$\delta_0^0(m_K) - \delta_0^2(m_K) = 33^{\circ}, \tag{3.14}$$

consistent with the experimental results.

The elastic cross sections for $\pi^+\pi^-$ and $\pi^-\pi^0$ are calculated from the partial-wave decomposition, summing over the principal partial waves,[2] and are shown in Figs. 5 and 6. The agreement with the data of Baton *et al.*[3,12] and Caso *et al.*[13] is reasonable, except in the case of $\pi^+\pi^-$ below 600 MeV, where the deviation of η_0^0 from 1, combined with the small phase shift, gives a small cross section.

The charge-exchange cross section $\sigma(\pi^+\pi^- \to \pi^0\pi^0)$ is shown in Fig. 7, as compared with the data of Deinet *et al.*[14] and Sonderegger and Bonamy.[15] This cross section is generally small compared with the data, although it is consistent with the data of Sonderegger and Bonamy above the ρ mass. The charge-exchange cross section is typical of between-up solutions for the S wave (cf. Morgan and Shaw[4]), with a small cross section below 600 MeV. As is well known, the experimental analyses of the π-π charge-exchange cross section are plagued with large-background problems, and we must await further experimental development before we can expect this issue to be settled.

By means of the optical theorem, the total cross section for a given initial configuration is determined from the imaginary part of the scattering amplitude in the forward direction as[2]

$$\sigma_{\text{tot}}(s) = \frac{4\pi}{q} \operatorname{Im} f(s, 0). \tag{3.15}$$

The total cross section $\sigma_{\text{tot}}(\pi^+\pi^-)$ for $\pi^+\pi^-$ scattering is shown in Fig. 8. Shown for comparison are the data of Caso *et al.*[13] for the total cross section, estimated on the assumption that the amplitude is purely imaginary above 1.2 GeV. Also shown are the elastic cross section for $\pi^+\pi^-$ of Baton *et al.*,[12] as well as the elastic cross section calculated in the model. The ρ and f mesons are prominent in the cross section, whereas the g meson at $\sqrt{s} = 1.63$ GeV is not evident, due to the violation of unitarity in the $I=0$ S wave above 1.3 GeV.

The predicted total cross section for $\pi^-\pi^0$ is shown in Fig. 9, and compared to the data of Baton *et al.*[12] Since the S wave does not contribute to this process, the g meson appears as an enhancement at $\sqrt{s} = 1.63$ GeV. The mass and width of the f and g, estimated from the total cross sections, are

$$\begin{aligned} m_f &\simeq 1300 \text{ MeV}, \quad \Gamma_f \simeq 130 \text{ MeV}, \\ m_g &\simeq 1630 \text{ MeV}, \quad \Gamma_g \simeq 150 \text{ MeV}, \end{aligned} \tag{3.16}$$

which are consistent with the experimental values[16]

$$\begin{aligned} m_f &= 1269 \pm 10 \text{ MeV}, \quad \Gamma_f = 154 \pm 25 \text{ MeV}, \\ m_g &= 1660 \pm 20 \text{ MeV}, \quad \Gamma_g \lesssim 200 \text{ MeV}. \end{aligned} \tag{3.17}$$

The predicted cross sections $\sigma(\pi^0\pi^0)$ and $\sigma(\pi^+\pi^+)$ are shown in Fig. 10. The data of Alitti *et al.*,[17] Schmitz,[18] and Colton *et al.*[7] for $\sigma(\pi^+\pi^+)$ are also shown. The predicted cross section is consistent with the data, especially the data of Alitti *et al.* The cross section $\sigma(\pi^0\pi^0)$ is purely $I=0$ and $I=2$, and is therefore dominated by the ϵ and f^0. There are no reliable data on this process. The total cross sections become small at ~2 GeV, and then rise asymptotically to the value of 15.4 mb. This

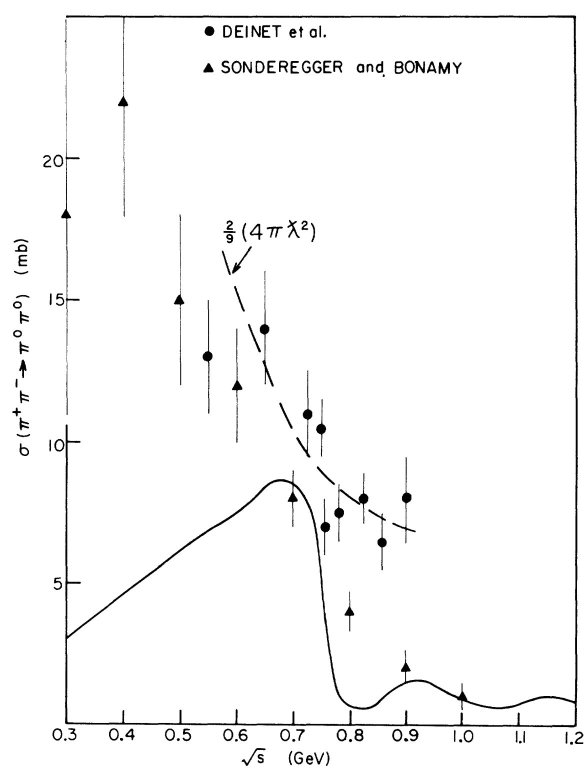

FIG. 7. Charge-exchange cross section $\sigma(\pi^+\pi^- \to \pi^0\pi^0)$. Data of Deinet *et al.* (Ref. 14) and Sonderegger and Bonamy (Ref. 15).

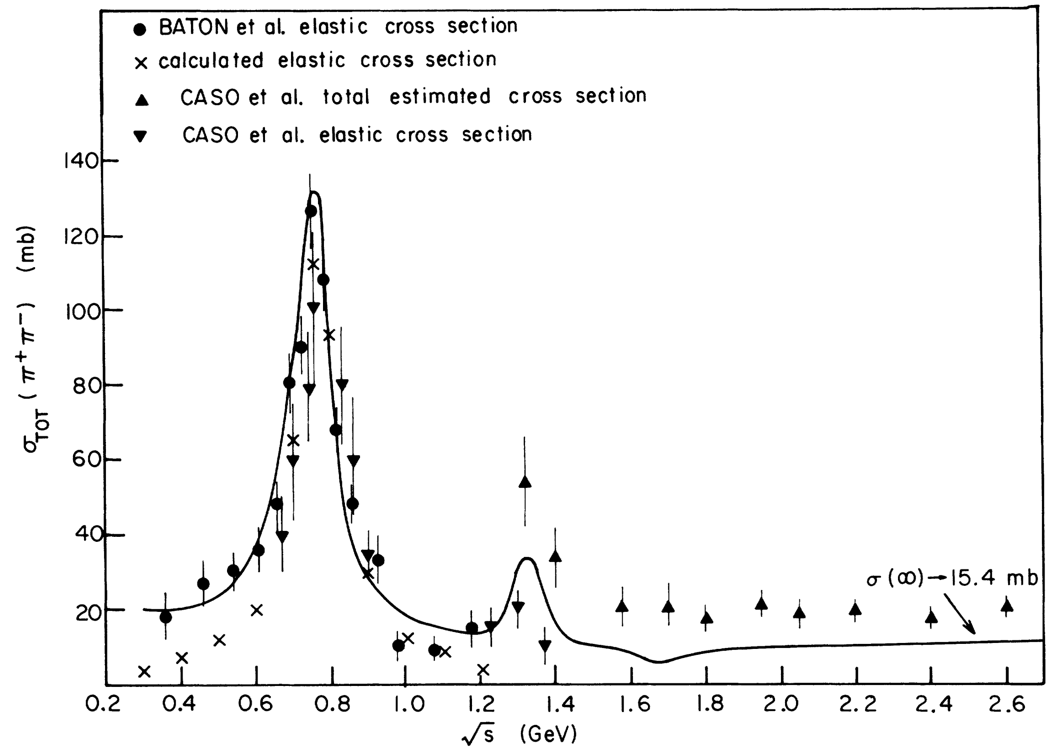

FIG. 8. Total cross section for $\pi^+\pi^-$ as a function of c.m. energy. Also shown are elastic cross sections for $\pi^+\pi^-$ from the model and from Baton *et al*. (Ref. 3) and Caso *et al*. (Ref. 13), as well as the estimated total cross section of Caso *et al*. (Ref. 13) above 1.2 GeV.

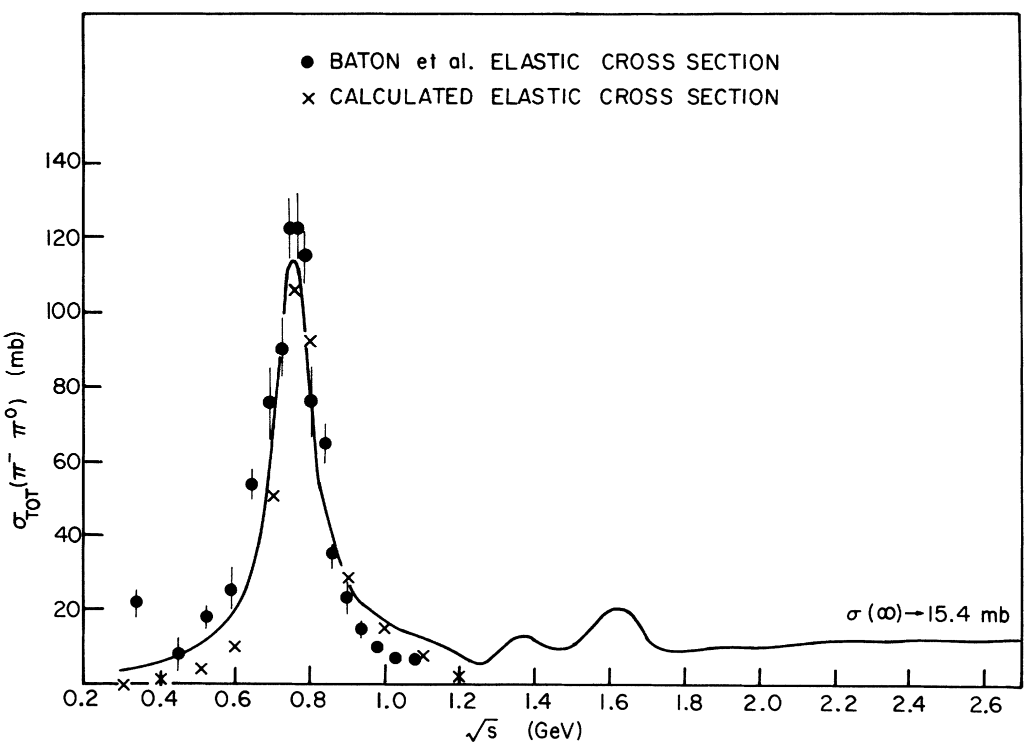

FIG. 9. Total cross section for $\pi^-\pi^0$. Also shown is the calculated elastic cross section and the data of Baton *et al*. (Ref. 12) for elastic $\pi^-\pi^0$ scattering.

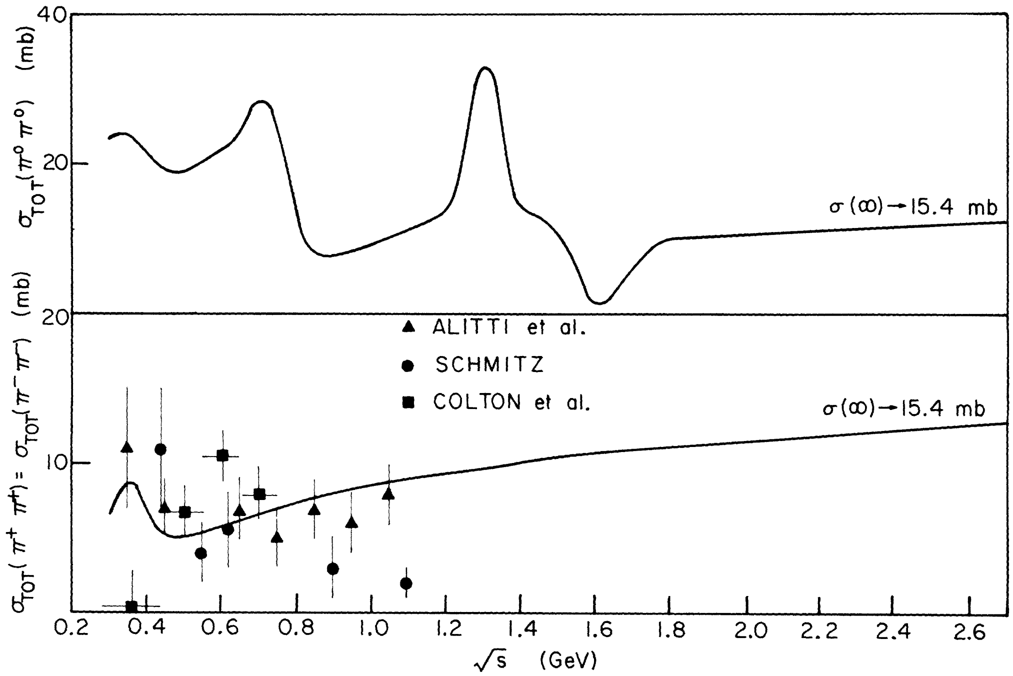

FIG. 10. Total cross sections for $\pi^0\pi^0$ and for $\pi^+\pi^+$. Data of Alitti *et al.* (Ref. 17), Schmitz (Ref. 18), and Colton *et al.* (Ref. 7).

agrees favorably with the asymptotic value of the π-π cross section obtained from factorization:

$$\sigma_{\pi\pi} = \sigma_{\pi N}{}^2/\sigma_{NN}\,, \tag{3.18}$$

which gives a value of $\sigma_{\pi\pi} \simeq 14$ mb.[19]

In Fig. 11, we show our prediction of the forward-backward asymmetry for $\pi^+\pi^-$ on the mass shell obtained from the equation[2]

$$\frac{F-B}{F+B} = \frac{3\,\mathrm{Re}[f_1(s) f_0^*(s)]}{|f_0(s)|^2 + 3|f_1(s)|^2}\,, \tag{3.19}$$

in which we neglect the D and higher waves. The data of Scharenguivel *et al.*[6] are shown for comparison.

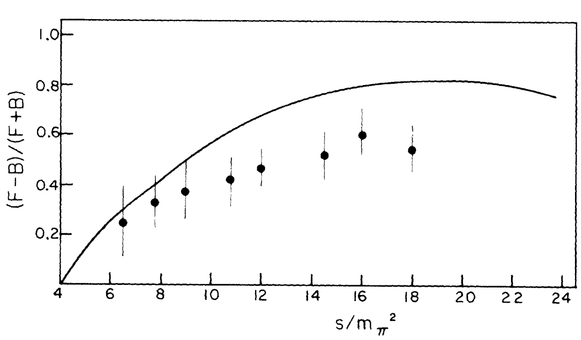

FIG. 11. Forward-backward asymmetry parameters $(F-B)/(F+B)$ for $\pi^+\pi^-$ on the mass shell. Data of Scharenguivel *et al.* (Ref. 6).

Predictions for the ratios of cross sections

$$R_1 = \frac{\sigma(\pi^+\pi^- \to \pi^0\pi^0)}{\sigma(\pi^+\pi^+ \to \pi^+\pi^+)}\,, \qquad R_2 = \frac{\sigma(\pi^+\pi^- \to \pi^0\pi^0)}{\sigma(\pi^+\pi^- \to \pi^+\pi^-)} \tag{3.20}$$

are shown in Figs. 12 and 13, together with the

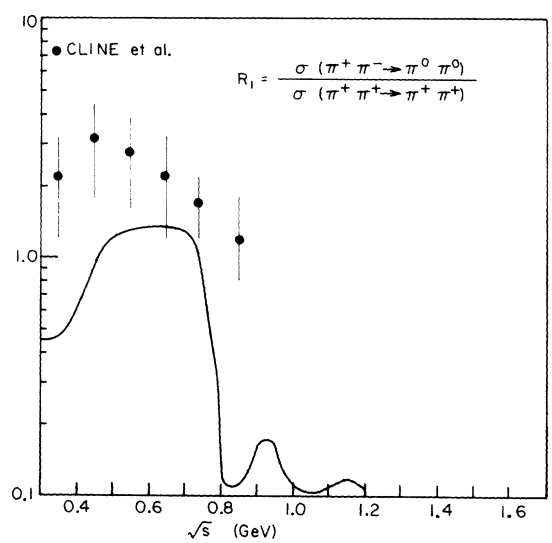

FIG. 12. $R_1 = \sigma(\pi^+\pi^- \to \pi^0\pi^0)/\sigma(\pi^+\pi^+ \to \pi^+\pi^+)$ as a function of c.m. energy. Data of Cline *et al.* (Ref. 20).

data of Cline *et al.*[20] Of course, these predictions depend sensitively on the form for the charge-exchange cross section. If, following Cline *et al.*,[20] we use R_1 and R_2 at threshold to determine the ratio of the S-wave scattering lengths, we find from R_2 that

$$a_0/a_2 = -5.8\,, \tag{3.21}$$

in agreement with the phase-shift calculation, but R_1 gives too small a value (by a factor of 2).

IV. RESULTS FOR K-π SCATTERING

The K-π scattering amplitude is formulated in terms of the π-π amplitude, already calculated, as described in the Appendix. Imposing the unitarity condition on the partial waves determines the over-all coupling constant and the Pomeranchukon coupling constant. The Pomeranchukon coupling constant required turns out to be the same as for π-π scattering, while the over-all coupling is consistent with the value derived from broken SU(3) symmetry.[21] In fact, the ρ coupling to $\pi\pi$ is determined to be

$$\gamma_{\rho\pi\pi}{}^2/4\pi = 2.28\,, \tag{4.1}$$

whereas the K^* coupling to K-π is found to be

$$\gamma_{K^*K\pi}{}^2/4\pi = 0.91\,; \tag{4.2}$$

so that their ratio is

$$\gamma_{\rho\pi\pi}{}^2/\gamma_{K^*K\pi}{}^2 = 2.50\,. \tag{4.3}$$

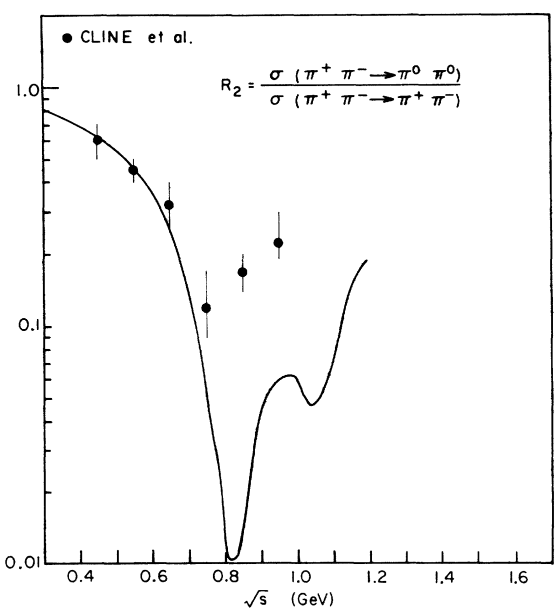

FIG. 13. $R_2 = \sigma(\pi^+\pi^- \to \pi^0\pi^0)/\sigma(\pi^+\pi^- \to \pi^+\pi^-)$. Data of Cline *et al.* (Ref. 20).

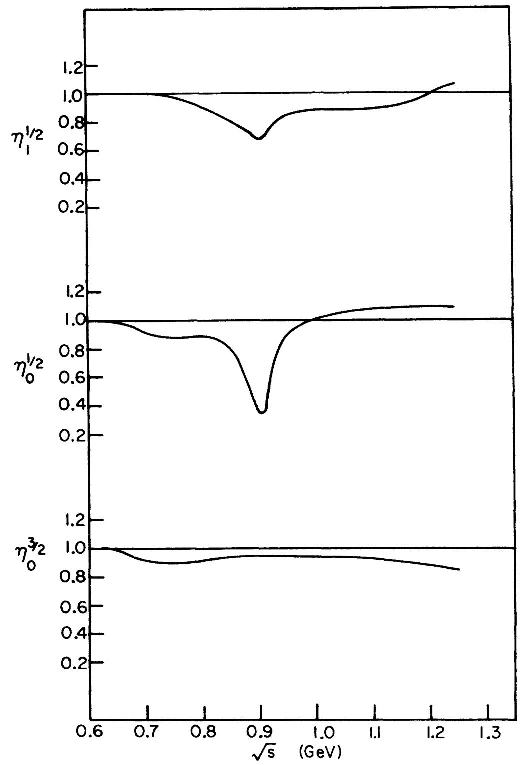

FIG. 14. Absorption parameters predicted for K-π scattering as function of c.m. energy.

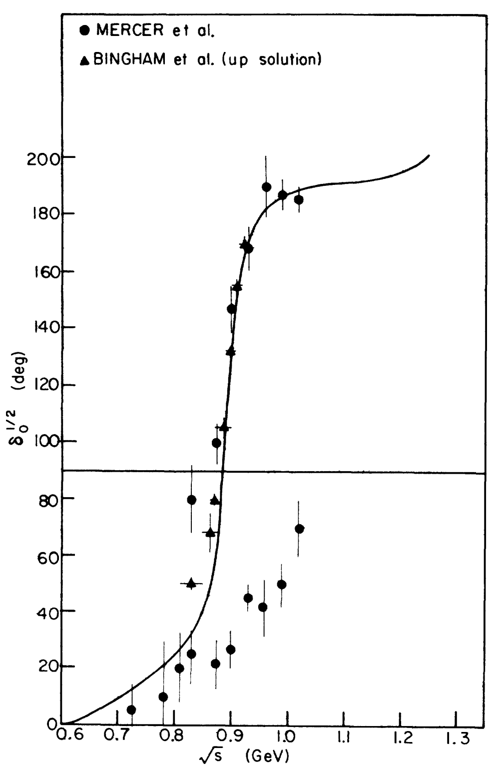

FIG. 15. Phase shift $\delta_0^{1/2}$. Data of Mercer *et al.* (Ref. 22) and the up solution of Bingham *et al.* (Ref. 23).

These are very close to the experimental values and to the calculation in SU(3):

$$\frac{\gamma_{\rho\pi\pi}{}^2}{4\pi} = 2.13 \pm 0.15 \ \ [\text{SU(3) value } 2.13],$$

$$\frac{\gamma_{K^*K\pi}{}^2}{4\pi} = 0.832 \pm 0.007 \ \ [\text{SU(3) value } 0.824], \quad (4.4)$$

$$\frac{\gamma_{\rho\pi\pi}{}^2}{\gamma_{K^*K\pi}{}^2} = 2.59 .$$

The absorption parameters in our K-π solutions are shown in Fig. 14. The violation of unitarity up to 1.3 GeV is never more than 10%.

The $I=\frac{1}{2}$ S wave is resonant (Fig. 15), predicting a mass and width for the κ meson of

$$m_\kappa = 880 \text{ MeV}, \quad \Gamma_\kappa = 45 \text{ MeV} . \quad (4.5)$$

The phase-shift data of Mercer *et al.*[22] and of Bingham *et al.*[23] (only the up solution is shown) are shown for comparison. The agreement with the data is quite good.

The $I=\frac{1}{2}$ P-wave phase shift is shown in Fig. 16 and compared to the data of Mercer *et al.*[22] The agreement with the data is excellent. The mass and width of the K^* are

$$m_{K^*} = 895 \text{ MeV}, \quad \Gamma_{K^*} = 55 \text{ MeV} \quad (4.6)$$

as compared to the experimental values[16]

$$m_{K^*} = 893 \text{ MeV}, \quad \Gamma_{K^*} = (50.1 \pm 0.8) \text{ MeV} . \quad (4.7)$$

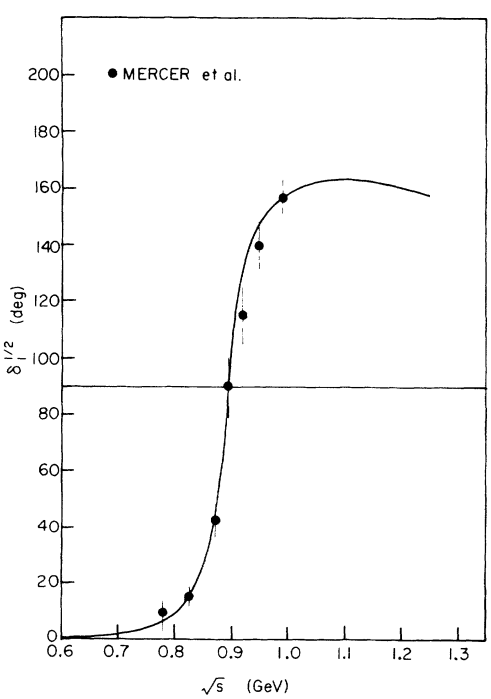

FIG. 16. Phase shift $\delta_1^{1/2}$. Data of Mercer *et al.* (Ref. 22).

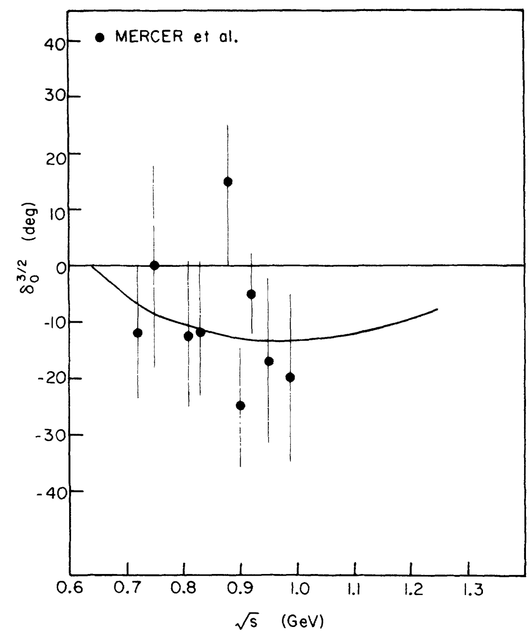

FIG. 17. Phase shift $\delta_0^{3/2}$. Data of Mercer *et al.* (Ref. 22).

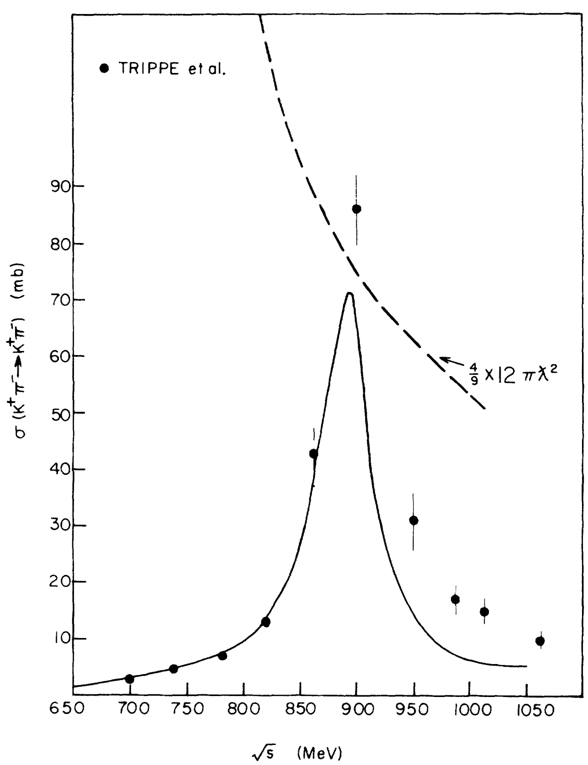

FIG. 18. Elastic cross section for $K^+\pi^-$. Data of Trippe *et al.* (Ref. 25). Also shown is the unitarity limit.

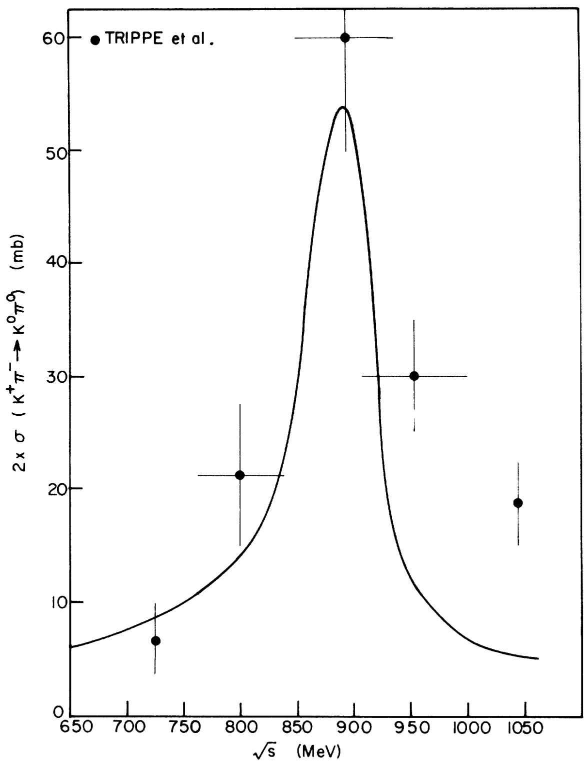

FIG. 19. Charge-exchange cross section $2\sigma(K^+\pi^- \to K^0\pi^0)$. Data of Trippe *et al.* (Ref. 25).

The $I=\frac{3}{2}$ S wave is small and negative (Fig. 17) and is consistent with the data of Mercer *et al.*[22] The higher partial waves are found to be very small below 1300 MeV, including the $I=\frac{3}{2}$ P wave.

From the partial waves, we determine the threshold parameters to be

$$a_{1/2}=0.13m_\pi^{-1},\qquad a_{3/2}=-0.078m_\pi^{-1}, \tag{4.8}$$

which are consistent with the current-algebra values

$$a_{1/2}=(0.13\pm0.02)m_\pi^{-1},\qquad a_{3/2}=-(0.07\pm0.01)m_\pi^{-1}. \tag{4.9}$$

The consistency of the π-π and K-π threshold parameters can be checked by the sum rule[24]

$$a_0^{1/2}-a_0^{3/2}=\left(\frac{3m_K}{m_K+m_\pi}\right)L, \tag{4.10}$$

which is derived in the current-algebra limit. The model yields 0.208 for the left-hand side and 0.211 for the right-hand side.

In Fig. 18, we show the predicted $K^+\pi^-$ elastic cross section and compare it to the data of Trippe *et al.*[25] The agreement with the data is good below 900 MeV, above which the inelastic effects become important. Figure 19 shows the predicted charge-exchange cross section $\sigma(K^+\pi^-\to K^0\pi^0)$. This prediction is also consistent with the data of Trippe *et al.*,[25] although the errors on the data are quite large.

The predicted total cross section for $K^+\pi^-$ is shown in Fig. 20. The K^* resonance is quite prominent, as is also the enhancement at $m_{K\pi}\approx 1400$ MeV, which corresponds to the K_N resonance at 1420 MeV with $I=\frac{1}{2}$, $J^P=2^+$. The predicted width (estimated from the total cross section) is $\Gamma_{K_N}\sim 100$ MeV, which is consistent with the experimentally quoted width $\Gamma_{K_N}=107\pm15$ MeV.[16] The cross section increases to an asymptotic value of 15 mb,

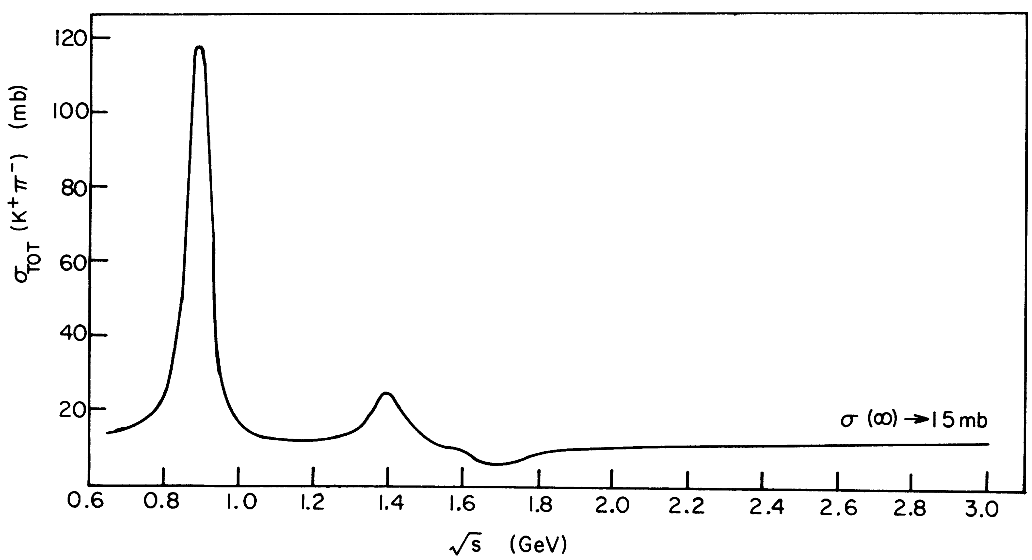

FIG. 20. Total $K^+\pi^-$ cross section as function of c.m. energy.

which is consistent with the value of ≈12 mb obtained from

$$\sigma_{K\pi} = \sigma_{KN}\sigma_{\pi N}/\sigma_{NN} , \tag{4.11}$$

based on factorization.[19]

V. CONCLUSIONS

We have succeeded in finding solutions for π-π and K-π scattering in a model satisfying Mandelstam analyticity, exact crossing symmetry, and approximate unitarity up to ~1.2–1.3 GeV. The results are in good agreement with most of the data available on the two processes, and total cross sections are reasonable to all energies. It remains to solve the phase-shift problem between 1.2 and 2 GeV, above which unitarity effects become negligible. Thus, given the mass spectrum, we can now claim that we have an approximate solution to meson-meson scattering over a fairly wide range of energy. We found that the unitarization of the scattering amplitudes placed severe restrictions on the parameters; from this it would appear that if unitarity could be imposed exactly, then, with the exception of the mass spectrum, the model would be completely determined. However, it is very unlikely that the form of the amplitude is unique; this is true of any model of this kind – including the Veneziano model.[26] But the phase-shift solutions may indeed be unique. Thus, the only solution in the model for the $I=0$ S wave in π-π and the $I=\frac{1}{2}$ S wave in K-π is the resonant solution, corresponding to fairly narrow ϵ and κ mesons. There is, as yet, no clear settlement of the experimental ambiguity between the "down-up" and "down-down" solutions for the S waves (above the ρ and K^* masses), particularly in view of the sensitivity of the data analysis to the absorption effects above the ρ and K^* masses, a fact of which we were continually reminded in our search for solutions.

We have solved a model which is unitarized in a phenomenological way, is exactly crossing symmetric and possesses satisfactory analyticity properties, but only at the price of *assuming* the mass spectrum at the outset; whether or not it is possible to obtain a model in which the mass spectrum is also predicted is unknown. Most bootstrap calculations have not been entirely successful, for one reason or another, and we are a long way from a consistent, self-contained theory of hadron dynamics. The latter circumstance is of course reflected in our semiphenomenological method of unitarizing the model. Any "honest" approach would tackle the many-body aspect "head-on." But, at the very least, we learn from model calculations of the kind presented here that there do exist solutions, consistent with the basic premises of the analytic S-matrix theory, which appear to be in fair agreement with the present data. It is difficult to say to what extent these efforts will help us find a theory of strong interactions, because it is always possible that the true theory will take an entirely different form.

ACKNOWLEDGMENT

We thank V. G. Snell for several helpful discussions.

APPENDIX

In the following, we shall describe in detail the amplitudes and parameters used to solve the model. The derivation of the physical scattering amplitudes and physical observables was done in detail in Ref. 1, and will not be repeated here.

For π-π scattering, the resonant amplitude F_ρ is given by

$$F_\rho(s,t) = -\gamma_\rho(s)\Gamma(1-\alpha_\rho(s))\big(w_\rho(t)^{\alpha_\rho(s)} + \{d_1 + d_2[1-\alpha_\rho(s)] + d_3[2-\alpha_\rho(s)][1-\alpha_\rho(s)]\}w_{1\rho}(t)^{\alpha_\rho(s)-1} + d_4[1-\alpha_\rho(s)]w_{1\rho}(t)^{\alpha_\rho(s)-2}\big) + (s \leftrightarrow t), \tag{A1}$$

including the first three satellite terms. The trajectory $\alpha_\rho(s)$ is given by the real analytic function

$$\alpha_\rho(s) = a_\rho + \frac{bs - c_\rho(4m_\pi^2 - s)^{1/2}}{\{1+[(4m_\pi^2 - s)/\Delta]^{1/2}\}^2} . \tag{A2}$$

The parameters of the ρ trajectory are determined from the spin of the ρ, $\mathrm{Re}\,\alpha_\rho(m_\rho^2)=1$, the Adler condition $\alpha_\rho(m_\pi^2)=\frac{1}{2}$, and the widths of the ρ and f^0 mesons, to be

$$a_\rho = 0.509, \quad b = 0.849,$$

$$c_\rho = 0.109, \quad \sqrt{\Delta} = 100 \tag{A3}$$

in GeV units. Then $\gamma_\rho(s)$ is given by

$$\gamma_\rho(s) = \gamma_{\pi\pi}\frac{\alpha_\rho(s)-\frac{1}{2}}{[1+x_\rho(s)]^{2b\Delta}} \times \exp\{-g_{\pi\pi}[\alpha_\rho(s)-\alpha_\rho(0)]^2\} , \tag{A4}$$

where

$$\gamma_{\pi\pi} = 159, \quad g_{\pi\pi} = 1.45 ,$$

and

$$x_\rho(s) = (4m_\pi^2 - s)^{1/2}/\sqrt{\Delta}\,. \tag{A5}$$

Moreover,

$$\begin{aligned} w_\rho(s) &= A + Bs + C(16m_\pi^2 - s)^{1/2}\,,\\ w_{1\rho}(s) &= A_1 + Bs + C(16m_\pi^2 - s)^{1/2}\,, \end{aligned} \tag{A6}$$

where

$A = 2.72$, $A_1 = 2.28$,

$B = -b$, and $C = 0.086$

in GeV units.

The satellite coefficients are

$$\begin{aligned} d_1 &= -3 - 0.45\exp\{-1.1[f(s) - f(1.4)]^2\}\,,\\ d_2 &= 2.5\exp\{-0.4[f(s) - f(1.75)]^2\}\,,\\ d_3 &= -3\exp\{-0.4[f(s) - f(2.00)]^2\}\,,\\ d_4 &= -8\exp\{-0.4[f(s) - f(1.75)]^2\}\,, \end{aligned} \tag{A7}$$

where

$$f(s) = \frac{s^2}{\{1 + [(16m_\pi^2 - s)/\Delta]^{1/2}\}^4}\,. \tag{A8}$$

The Pomeranchukon amplitude is given by

$$A_P(t,s) = \gamma_P(t)\ln\left[1 + \left(\frac{4m_\pi^2 - s}{s_0}\right)^{1/2}\right] \times [w_P(s)^{\alpha_P(t)} + d]\,, \tag{A9}$$

where

$d = -0.345$, $s_0 = 1/|B_P|$,

and

$$w_P(s) = A_P + B_P s + C_P(16m_\pi^2 - s)^{1/2}\,. \tag{A10}$$

Here,

$A_P = 0.35$, $B_P = -1$, $C_P = 0.02$.

The trajectory is given by

$$\alpha_P(s) = 1 + \frac{bs}{\left[1 + \left(\frac{16m_\pi^2 - s}{\Delta}\right)^{1/2}\right]^2}\,, \tag{A11}$$

with $b = 1$. Then,

$$\gamma_P(s) = \frac{\gamma_P[\alpha_P(s) - \alpha_P(m_\pi^2)]\exp\{-g_P[\alpha_P(s) - \alpha_P(0)]^2\}}{\left[1 + \dfrac{(16m_\pi^2 - s)^{1/2}}{2\Lambda}\right]^{4\sqrt{b}\Lambda}}\,, \tag{A12}$$

where

$\gamma_P = -24$, $g_P = 2\sqrt{\Delta}$ (in magnitude),

$\Lambda = \Delta$ (in magnitude).

The significance of the constants A, B, and C in (A6) and (A10) was described in Ref. 1. We recall that $\alpha_\rho(\pm\infty) \simeq \alpha_P(\pm\infty) \simeq -b\Delta$, so that $\gamma_\rho(s) \to 0$ and $\gamma_P(s) \to 0$ as $s \to \pm\infty$. The asymptotic properties of the Pomeranchukon were discussed in Ref. 1.

For the K-π amplitude, we have

$$\begin{aligned} F_{K^*}(s,t) = &-\gamma_{K^*}(s)\Gamma(1 - \alpha_{K^*}(s))[w_\rho(t)^{\alpha_{K^*}(s)} + d_1 w_{1\rho}(t)^{\alpha_{K^*}(s)-1}]\\ &-\gamma_\rho(t)\Gamma(1 - \alpha_\rho(t))[w_{K^*}(s)^{\alpha_\rho(t)} + d_1 w_{1K^*}(s)^{\alpha_\rho(t)-1}]\,, \end{aligned} \tag{A13}$$

where $d_1 = -3.45$. The trajectory $\alpha_{K^*}(s)$ is defined by

$$\alpha_{K^*}(s) = a_{K^*} + \frac{bs - c_{K^*}[(m_K + m_\pi)^2 - s]^{1/2}}{(1 + \{[(m_K + m_\pi)^2 - s]/\Delta\}^{1/2})^2}\,, \tag{A14}$$

where $a_{K^*} = 0.314$, $c_{K^*} = 0.061$ in GeV units. Also,

$$\gamma_{K^*}(s) = \gamma_{K\pi}\frac{[\alpha_{K^*}(s) - \frac{1}{2}]\exp\{-g_{K\pi}[\alpha_{K^*}(s) - \alpha_{K^*}(0)]^2\}}{(1 + x_{K^*})^{2b\Delta}}\,, \tag{A15}$$

where

$$\gamma_{K\pi} = 124, \quad g_{K\pi} = 1.0, \quad x_{K^*}(s) = \{[(m_K + m_\pi)^2 - s]/\Delta\}^{1/2}\,. \tag{A16}$$

The functions $w_{K^*}(s)$ and $w_{1K^*}(s)$ are

$$w_{K^*}(s) = A + Bs + C[(m_K + 2m_\pi)^2 - s]^{1/2}\,, \qquad w_{1K^*}(s) = A_1 + Bs + C[(m_K + 2m_\pi)^2 - s]^{1/2}\,, \tag{A17}$$

where A, A_1, B, and C are the same as in (A6). The Pomeranchukon amplitude for K-π scattering is

$$A_P(t,s)=\gamma_P(t)\ln\left\{1+\left[\frac{(m_K+m_\pi)^2-s}{s_0}\right]^{1/2}\right\}[w_P(s)^{\alpha_P(t)}+d]\,. \tag{A18}$$

Here $\gamma_P(t)$ is the same as in (A12), apart from the threshold factor, and the constants are the same as for π-π scattering. Moreover,

$$w_P(s)=A_P+B_Ps+C_P[(m_K+2m_\pi)^2-s]^{1/2}, \tag{A19}$$

where A_P, B_P, and C_P are the same as for π-π scattering.

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