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Massive Particles and the Spontaneous Breakdown of Dilation Invariance*

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In a recent paper an argument was given that the conservation of dilation current (vanishing of the trace of the stress-energy tensor) implied that all single-particle states have zero mass. The form of the dilation current was subsequently criticized. In this note it is shown that with a quite general form of dilation current the result is unchanged.

In a recent article (hereafter referred to as BM)[1] we have argued that for a relativistic quantum field theory for which (a) there exists a suitable energy-momentum tensor $\Theta_{\mu\nu}(x)$ such that the energy-momentum operators P_μ are related to $\Theta_{0\mu}(x)$ as $\langle A|\int d^3x\Theta_{0\mu}(x)|B\rangle=-i\langle A|P_\mu|B\rangle$, $|A\rangle$ and $|B\rangle$ belonging to a certain dense set of states; and (b) a dilation current $J_\nu(x)=x^\alpha\Theta_{\alpha\nu}(x)$ is conserved (i.e., $\Theta_\mu{}^\mu=0$); all particles must have zero mass even if the dilation invariance is broken spontaneously. There has been some criticism[2] of assumption (b) when the dilation invariance is broken spontaneously. In particular it is argued that in this case the (conserved) dilation current should be written

$$J_\nu(x)=x^\mu T_{\mu\nu}(x)+\mathcal{J}_\nu(x)\,, \tag{1}$$

where $T_{\mu\nu}$ (the Belinfante tensor in Ref. 2) satisfies (a) also but is not traceless, and $\mathcal{J}_\mu$ is a local four-current, i.e.,

$$[\mathcal{J}_\nu(x),P_\mu]=i\,\frac{\partial\mathcal{J}_\nu(x)}{\partial x^\mu}\,.$$

We show in this note that the argument of BM, slightly modified, still implies that all particles have zero mass even when the dilation current has the form given in Eq. (1).

This note is confined to answering the specific objection of Ref. 2 to BM. However, it must be made clear that the conclusion that all particles have zero mass rests on an assumption, namely that a single-particle state of finite mass behaves like a quasilocal state (see BM, p. 2968). There exist model field theories which, when solved in the tree approximation, apparently violate our conclusion. We do not know whether the tree approximation is at fault or whether our assumption is at fault.

Before proceeding with the proof, it is interesting to note that a current of the form (1) does, in fact, interpret as a dilation current. (In all that follows the notation of BM will be used.) First define

$$D_R=\int_{|\vec{x}|<R}J_0(x)\,d^3x\,. \tag{2}$$

It follows, from $\partial^\mu J_\mu(x)=0$, that

$$[[D_R,P_\mu],A]=-i[T_{0\mu}(f_R),A] \tag{3}$$

for all R greater than some large R_0 which depends upon the local Wightman polynomial, A. Equation (3) is the same as Eq. (56) of BM. It is the infinitesimal form of the property stated in Lemma I of Ref. 3 which might be taken as the defining property of dilation.

In BM the argument about the vanishing of single-particle masses was based on the relation

$$\lim_{R\to\infty}\langle A|[D_R,P_\mu]|B\rangle=i\langle A|P_\mu|B\rangle\,, \tag{4}$$

whose validity was proved from assumptions (a) and (b). In Eq. (4), $|A\rangle$ and $|B\rangle$ are arbitrary quasilocal states. In this note we shall prove that Eq. (4) continues to be valid even when assumption (b) is replaced by a dilation current given by Eq. (1). From the discussion in BM, we know that Eq. (3) will lead to Eq. (4), for local states, provided that

$$\lim_{R\to\infty}\langle 0|[D_R,P_\mu]A|0\rangle=0 \tag{5}$$

for arbitrary local Wightman polynomial A. Thus to prove Eq. (4) for local states it remains for us to prove Eq. (5) The remainder of this paper is devoted to this task.

Let us first recall[1] that assumption (a) implies:

$$\langle 0|[T_{0\nu}(x),A]|0\rangle=\int_0^\infty d\mu^2\int d^3y\,\Delta(x-y,x_0,\mu^2)\,\frac{\partial}{\partial y_i}\,\sigma^i_{1\nu}(\mu^2,y)+\int_0^\infty d\mu^2\int d^3y\,\frac{\partial}{\partial x_0}\,\Delta(x-y,x_0,\mu^2)\,\frac{\partial}{\partial y_i}\,\sigma^i_{2\nu}(\mu^2,y) \tag{6}$$

together with the following condition on the Fourier transform $\tilde{\sigma}^i_{1\nu}(\mu^2, k)$ for $\vec{k}=0$:

$$\tilde{\sigma}^i_{1\nu}(\mu^2,0)=\frac{1}{(2\pi)^3}\int d^3y\,\sigma^i_{1\nu}(\mu^2,y)=0\,. \tag{7}$$

If we assume that the current $\mathcal{J}_\mu(x)$ is local with respect to the basic fields of the field theory, then it follows quite generally[4,5] that

$$\langle 0|[\mathcal{J}_\nu(x),A]|0\rangle=\int_0^\infty d\mu^2\int d^3y\,\Delta(x-y,x_0,\mu^2)\left[\bar{\beta}_{1\nu}(\mu^2)\delta^3(y)+\frac{\partial}{\partial y_i}\,\beta^i_{1\nu}(\mu^2,y)\right]$$
$$+\int_0^\infty d\mu^2\int d^3y\,\frac{\partial}{\partial x_0}\,\Delta(x-y,x_0,\mu^2)\left[\bar{\beta}_{2\nu}(\mu^2)\delta^3(y)+\frac{\partial}{\partial y_i}\,\beta^i_{2\nu}(\mu^2,y)\right]. \tag{8}$$

In the above $\beta^i_{a\nu}$ $(a=1,2)$ are functions of compact support in y. Let us now consider the consequences of the conservation of dilation current $\partial^\mu J_\mu(x)=0$. Obviously current conservation implies

$$\frac{d}{dx_0}\langle 0|[J_0(f_R),A]|0\rangle=0,\quad R>\text{some }R_0 \tag{9}$$

which [via Eqs. (1), (2), (6), and (8)] in turn implies

$$\int_0^\infty d\mu^2[\bar{\beta}_{20}(\mu^2)+(2\pi)^3\tilde{\sigma}^i_{2i}(\mu^2,0)]\,\mu\sin(x_0\mu)=0 \tag{10}$$

and

$$\int_0^\infty d\mu^2\bar{\beta}_{10}(\mu^2)\cos(x_0\mu)=0\,, \tag{11}$$

where $\tilde{\sigma}^i_{2i}(\mu^2,0)$ denotes the Fourier transform of $\sigma^i_{2i}(\mu^2,y)$ and repeated indices imply summation. Hence

$$\bar{\beta}_{10}(\mu^2)=0 \tag{12}$$

and

$$\bar{\beta}_{20}(\mu^2)+(2\pi)^3\tilde{\sigma}^i_{2i}(\mu^2,0)=c\,\delta(\mu^2)\,. \tag{13}$$

It is easily seen from Eqs. (1), (2), (6), (8), (12), and (13) that

$$\lim_{R\to\infty}\langle 0|[D_R,A]|0\rangle=c\,. \tag{14}$$

Thus if $c\neq 0$, we have spontaneous breakdown of dilation symmetry. Equation (13) now says that massless particles are present in the spectrum and that the current $J_0(x)$ connects these states to the vacuum. This is the statement of the Goldstone theorem, for the present case. It is well known that in this case the limit $R\to\infty$ of D_R does not exist. If, on the other hand, $c=0$, then we have the conventional realization of the dilation symmetry, and the limit $R\to\infty$ of D_R does exist. It is well known that in this latter case all single-particle masses must be zero.

We are now in a position to prove Eq. (5). From Eqs. (1), (2), (6), (8), and (12) we get

$$\lim_{R\to\infty}\langle 0|[D_R,P_\mu]A|0\rangle=i\int d^3x\,x^\nu\int_0^\infty d\mu^2\int d^3y\,\frac{\partial}{\partial x_\mu}\,\Delta^{(+)}(x-y,x_0,\mu^2)\,\frac{\partial}{\partial y_i}\,\sigma^i_{1\nu}(\mu^2,y)$$
$$+i\int d^3x\,x^\nu\int_0^\infty d\mu^2\int d^3y\,\frac{\partial}{\partial x_\mu}\,\frac{\partial}{\partial x_0}\,\Delta^{(+)}(x-y,x_0,\mu^2)\,\frac{\partial}{\partial y_i}\,\sigma^i_{2\nu}(\mu^2,y)$$
$$+i\int d^3x\int_0^\infty d\mu^2\int d^3y\,\frac{\partial}{\partial x_\mu}\,\Delta^{(+)}(x-y,x_0,\mu^2)\,\frac{\partial}{\partial y_i}\,\beta^i_{10}(\mu^2,y)$$
$$+i\int d^3x\int_0^\infty d\mu^2\int d^3y\,\frac{\partial}{\partial x_\mu}\,\frac{\partial}{\partial x_0}\,\Delta^{(+)}(x-y,x_0,\mu^2)\,\frac{\partial}{\partial y_i}\,\beta^i_{20}(\mu^2,y)$$
$$+i\int d^3x\int_0^\infty d\mu^2\bar{\beta}_{20}(\mu^2)\,\frac{\partial}{\partial x_\mu}\,\frac{\partial}{\partial x_0}\,\Delta^{(+)}(x,x_0,\mu^2)\,. \tag{15}$$

By partial integration and the use of explicit representation of $\Delta^{(+)}(x, x_0, \mu^2)$ we cast the above into the form

$$\begin{aligned}\lim_{R\to\infty}\langle 0|[D_R, P_\mu]A|0\rangle = x_0(2\pi)^3\Big[\int_0^\infty d\mu^2\int d^4k\, k_\mu k_i \tilde{\sigma}_{10}^i(\mu^2,k)\, e^{ix_0k_0}\theta(k_0)\delta^3(k)\delta(k_0^2-|\vec{k}|^2-\mu^2)\\ +i\int_0^\infty d\mu^2\int d^4k\, k_\mu k_i k_0\tilde{\sigma}_{20}^i(\mu^2,k)\, e^{ix_0k_0}\theta(k_0)\delta^3(k)\delta(k_0^2-|\vec{k}|^2-\mu^2)\Big]\\ -i(2\pi)^3\Big[\int_0^\infty d\mu^2\int d^4k\, k_\mu k_i\tilde{\sigma}_{1j}^i(\mu^2,k)\, e^{ix_0k_0}\theta(k_0)\delta(k_0^2-|\vec{k}|^2-\mu^2)\frac{\partial}{\partial k^j}\delta^3(k)\\ +i\int_0^\infty d\mu^2\int d^4k\, k_\mu k_i k_0\tilde{\sigma}_{2j}^i(\mu^2,k)\, e^{ix_0k_0}\theta(k_0)\delta(k_0^2-|\vec{k}|^2-\mu^2)\frac{\partial}{\partial k^j}\delta^3(k)\Big]\\ +(2\pi)^3\Big[\int_0^\infty d\mu^2\int d^4k\, k_\mu k_i\tilde{\beta}_{10}^i(\mu^2,k)\, e^{ix_0k_0}\theta(k_0)\delta(k_0^2-|\vec{k}|^2-\mu^2)\delta^3(k)\\ +i\int_0^\infty d\mu^2\int d^4k\, k_\mu k_i k_0\tilde{\beta}_{20}^i(\mu^2,k)\, e^{ix_0k_0}\theta(k_0)\delta(k_0^2-|\vec{k}|^2-\mu^2)\delta^3(k)\Big]\\ -\int_0^\infty d\mu^2\bar{\beta}_{20}(\mu^2)\int d^4k\, k_\mu k_0 e^{ix_0k_0}\theta(k_0)\delta(k_0^2-|\vec{k}|^2-\mu^2)\delta^3(k)\,. \qquad (16)\end{aligned}$$

It is straightforward to see that the term multiplying x_0 as well as the penultimate term in Eq. (16) vanish. Similarly, it is seen that the term involving $\tilde{\sigma}_{1j}^i(\mu^2,k)$ vanishes due to Eq. (7). The remaining two terms in Eq. (16) are seen to vanish individually whenever the index μ takes up values 1, 2, or 3. Thus, finally,

$$\begin{aligned}\lim_{R\to\infty}\langle 0|[D_R, P_\mu]A|0\rangle &= \delta_{\mu 0}(2\pi)^3\int_0^\infty d\mu^2 d^4k\, k_i k_0^2\tilde{\sigma}_{2j}^i(\mu^2,k)\, e^{ix_0k_0}\theta(k_0)\delta(k_0^2-|\vec{k}|^2-\mu^2)\frac{\partial}{\partial k^j}\delta^3(k)\\ &\quad-\delta_{\mu 0}\int_0^\infty d\mu^2\bar{\beta}_{20}(\mu^2)d^4k\, k_0^2 e^{ix_0k_0}\theta(k_0)\delta(k_0^2-|\vec{k}|^2-\mu^2)\delta^3(k)\\ &= -\tfrac{1}{2}\delta_{\mu 0}\int_0^\infty d\mu^2[\bar{\beta}_{20}(\mu^2)+(2\pi)^3\tilde{\sigma}_{2i}^i(\mu^2,0)]\,\mu\, e^{ix_0\mu}\,. \qquad (17)\end{aligned}$$

From Eqs. (13) and (17) it now follows that

$$\begin{aligned}\lim_{R\to\infty}\langle 0|[D_R, P_\mu]A|0\rangle &= -\tfrac{1}{2}\delta_{\mu 0}c\int_0^\infty d\mu^2\delta(\mu^2)\mu\, e^{ix_0\mu}\\ &=0\,. \quad \text{Q.E.D.} \qquad (18)\end{aligned}$$

Thus Eq. (4) is valid for local states. It is straightforward now to extend it to quasilocal states.

One might ask, "In light of the result that all masses are zero even if the dilation is broken spontaneously, might there then not always exist a conserved and integrable current for dilation symmetry?" This question is being investigated.

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