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Partial Differential Equations with Respect to Coupling Constants: Electromagnetic Mass Difference of Hadrons*

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The partial differential equations with respect to strong and electromagnetic coupling constants are established for the S matrix, Heisenberg operators, and "out" field operators. The macrocausality relations – vanishing of derivatives with respect to coupling constants for free "in" field operators – are important in the formalism. As a consequence of macrocausality relations, observable masses do not depend on strong and electromagnetic coupling constants. This makes the observable n-p mass difference uncomputable. Thus, our approach goes along the lines of divergent (but renormalizable) field theory. We are assuming (although not discussing) the existence of nonelectromagnetic interactions which make $m_n - m_p \neq 0$ when $e = 0$. The partial differential equations with respect to strong and electromagnetic coupling constants are derived for bare n and p masses. In order to integrate these differential equations, the observable n and p must be known *a priori*. After integrating these differential equations, a formal expression for the observable n-p mass difference is obtained. This expression, compared with the usual expression from the literature, besides containing the difference of n and p mass shifts due to the electromagnetic interactions ("renormalized" by strong interactions), also contains the bare n-p mass difference and the difference of n and p mass shifts due to strong interactions. One cannot "recover" the usual expression for the n-p mass difference since our expression, as far as the observable n-p mass difference is concerned, is an identity and not the relation from which it can be computed.

I. INTRODUCTION

It has been shown on various occasions in quantum field theory that the differential equation with respect to the coupling constant for the S matrix, Heisenberg fields, and "out" fields can be quite useful.[1] They enabled one to reduce the S matrix into the closed normal form in free-field "in" operators for some models of quantum field theory.

However, if the interaction cannot be characterized by only one coupling constant and, in particular, if one is much larger than the other, for practical applications we shall need more than one set of partial differential equations for the S matrix, Heisenberg operators, and "out" field operators. This, for example, will happen when the system of particles interacts through strong and electromagnetic interactions. We shall characterize the strong interactions with just one coupling constant, g, while as usual, the electromagnetic coupling constant with e. We will vary them independently between zero and their physical values, signifying that the electromagnetic and strong interactions are different in nature.[2]

In the derivation of these partial differential equations with respect to strong and electromagnetic coupling constants, the macrocausality relations ($\partial\phi_{\text{in}}/\partial g = \partial\phi_{\text{in}}/\partial e = 0$) are important. These relations require the observable masses to be independent of coupling constants. This means that our development will be based on the divergent (renormalizable) field theory, in which the observable masses are given as input parameters. (For example, we do not know yet how to incorporate a deuteron into this formalism.)

Once having these partial differential equations, one has the means to tackle practical problems where both strong and electromagnetic interactions are important. Of all possible problems, however, we shall discuss only the question of the electromagnetic mass difference of hadrons, in particular, the n-p mass difference. As is well known, the calculated values for the n-p mass difference usually predict the proton to be heavier than the neutron, contrary to the experiments. However, according to our formalism one should not try to compute the observable n-p mass difference since by the assumption of the formalism it is not necessary to assume that the observable masses depend on coupling constants g and e.

In Sec. II, besides listing the assumptions of our formalism, we derive partial differential equations with respect to coupling constants for the S matrix, Heisenberg fields, and "out" fields.

Section III is devoted to the derivation of the decomposition of the S matrix as $S = S_s S_{\text{em}}$, where S_s describes the strong interactions only, while S_{em} is responsible for electromagnetic transitions ("renormalized" by strong interactions).

In Sec. IV we show that partial differential equations with respect to coupling constants have to be formulated with the total Hamiltonian density, if its free part contains bare masses.

Finally in Sec. V we derive partial differential equations with respect to coupling constants for neutron and proton bare masses. From them we further derive the partial differential equations for the self-masses. The Cottingham formula for the hadronic electromagnetic self-mass is also deduced. The formal expression for the observable n-p mass difference is derived. It differs from the usual expression by having some additional terms. Because of these additional terms, the expression becomes an identity as far as the observable n-p mass difference is concerned, which makes it uncomputable. Of course, we assume that there are some other (nonelectromagnetic) interactions which make $m_n \neq m_p$.

In concluding Sec. VI we make some remarks about n-p mass difference with respect to SU(2). We show that if $m_n - m_p$ is neglected and strong and electromagnetic interactions are present [SU(2) is broken] the bare n-p mass difference is calculable, and it turns out to be 0.66 MeV.

In the Appendix we discuss the SU(2) transformation properties of proton and neutron "in" field operators for cases when SU(2) is broken and exact.

II. THE DIFFERENTIAL EQUATIONS FOR THE S MATRIX AND HEISENBERG OPERATORS WITH RESPECT TO THE COUPLING CONSTANTS

We shall assume that the system of particles interacts strongly and electromagnetically only. Then, formally we can write the S matrix in the Dyson form:

$$S = T \exp\left[-i \int d^4x\, \mathcal{H}_{\text{int}}^{\text{in}}(x)\right]. \tag{1}$$

$\mathcal{H}_{\text{int}}^{\text{in}}$ denotes the interaction Hamiltonian expressed in terms of incoming field operators and their canonical conjugate operators. We denote with g and e the strengths of strong and electromagnetic couplings, allowing them to vary independently between zero and their physical values. However, before we vary the coupling constants, let us write down the most important assumptions and relations which we shall need later. The basic assumptions are macrocausality relations,

$$\frac{\partial}{\partial g}\phi_{\text{in}}(x) = \frac{\partial}{\partial e}\phi_{\text{in}}(x) = 0\,,$$
$$\frac{\partial}{\partial g}\pi_{\text{in}}(x) = \frac{\partial}{\partial e}\pi_{\text{in}}(x) = 0\,, \tag{2a}$$

where ϕ_{in} and π_{in} are common symbols for all free-field "in" operators and their canonical conjugates. As a consequence of relations (2a), no observable mass m can depend on coupling constants g and e:

$$\frac{\partial}{\partial g}m = \frac{\partial}{\partial e}m = 0\,, \tag{2b}$$

which means that the observable masses are given as input parameters. Thus our approach goes along the lines of renormalizable-divergent field theory. On the other hand, a bare mass m_0 will be generally a function of g and e and, as usual, we assume that it is connected to the mass shift Δm by the relation

$$m_0(g, e) = m - \Delta m(g, e)\,. \tag{2c}$$

Finally, we take that the S matrix in the absence of all interactions is equal to unity; i.e.,

$$S|_{g=0, e=0} = 1\,. \tag{2d}$$

Taking into account that the Heisenberg field operators and their canonical conjugate operators, which we denote with common symbols $\phi(x)$ and $\pi(x)$, are connected to $\phi_{\text{in}}(x)$ and $\pi_{\text{in}}(x)$ by the relation

$$\phi(x) = S^\dagger T(\phi_{\text{in}}(x)S)\,,$$
$$\pi(x) = S^\dagger T(\pi_{\text{in}}(x)S)\,,$$

then because of (2d), in the absence of all interactions ($e=0, g=0$), we have

$$\phi(x)|_{e=0, g=0} = \phi_{\text{in}}(x)\,,$$
$$\pi(x)|_{e=0, g=0} = \pi_{\text{in}}(x)\,. \tag{2e}$$

As a consequence of relations (2), there might be some other relations which we shall write down when needed. Let us say that macrocausality relations (2a) can be understood from the point of view of asymptotically switching-on the interaction: The system does not know anything about the interaction in the distant past.[3]

Let us now proceed with our derivation of partial differential equations with respect to coupling constants. Again using

$$\phi(x) = S^\dagger T(\phi_{\text{in}}(x)S) \quad \text{and} \quad \pi(x) = S^\dagger T(\pi_{\text{in}}(x)S)\,,$$

we get at once from (1) the following differential equations for the S matrix:

$$\frac{1}{i}\frac{\partial}{\partial g}S = -S\int d^4x\, \frac{\partial'}{\partial g}\mathcal{H}_{\text{int}}(\phi(x), \pi(x))\,,$$
$$\frac{1}{i}\frac{\partial}{\partial e}S = -S\int d^4x\, \frac{\partial'}{\partial e}\mathcal{H}_{\text{int}}(\phi(x), \pi(x))\,. \tag{3}$$

The partial derivatives $\partial'/\partial g$ and $\partial'/\partial e$ in (3) mean that while they act as ordinary derivatives $\partial/\partial g$ and $\partial/\partial e$ on the coefficients that multiply $\phi(x)$ and $\pi(x)$

in $\mathcal{H}_{int}$, they do not act on $\phi(x)$ and $\pi(x)$ at all.[4] Thus we may write formally[5]

$$\frac{\partial'}{\partial g}\phi(x)=\frac{\partial'}{\partial e}\phi(x)=0\,,\qquad \frac{\partial'}{\partial g}\pi(x)=\frac{\partial'}{\partial e}\pi(x)=0\,. \tag{4}$$

The partial differential equations (3) connect the S matrix with the Heisenberg operators $\phi(x)$ and $\pi(x)$. $\partial'\mathcal{H}_{int}/\partial g$ and $\partial'\mathcal{H}_{int}/\partial e$ pick up the parts of $\mathcal{H}_{int}$ that correspond to strong and electromagnetic interactions. However, one should note that $\partial'\mathcal{H}_{int}/\partial g$ and $\partial'\mathcal{H}_{int}/\partial e$ are dependent in general on both g and e. $\partial'\mathcal{H}_{int}/\partial g$ and $\partial'\mathcal{H}_{int}/\partial e$ are, of course, Hermitian operators in order that $S^\dagger S=SS^\dagger=1$.

To obtain the partial differential equations with respect to g and e for $\phi_{out}(x)$ and $\pi_{out}(x)$, we note that $\phi_{out}(x)=S^\dagger\phi_{in}(x)S$, $\pi_{out}(x)=S^\dagger\pi_{in}(x)S$. Then using (3) we have at once

$$\frac{1}{i}\frac{\partial}{\partial g}\phi_{out}(x)=-\left[\phi_{out}(x),\int d^4y\,\frac{\partial'}{\partial g}\mathcal{H}_{int}(y)\right],\qquad \frac{1}{i}\frac{\partial}{\partial e}\phi_{out}(x)=-\left[\phi_{out}(x),\int d^4y\,\frac{\partial'}{\partial e}\mathcal{H}_{int}(y)\right], \tag{5}$$

and the same equations for $\pi_{out}(x)$.[6]

The partial differential equations for the Heisenberg field operators $\phi(x)$ and their canonical conjugate operators $\pi(x)$ are obtained by help of the relations $\phi(x)=S^\dagger T(\phi_{in}(x)S)$, $\pi(x)=S^\dagger T(\pi_{in}(x)S)$ and the differential equations (3). We obtain

$$\frac{1}{i}\frac{\partial}{\partial g}\phi(x)=-\int\theta(x^4-y^4)d^4y\left[\phi(x),\frac{\partial'}{\partial g}\mathcal{H}_{int}(y)\right],\qquad \frac{1}{i}\frac{\partial}{\partial e}\phi(x)=-\int\theta(x^4-y^4)d^4y\left[\phi(x),\frac{\partial'}{\partial e}\mathcal{H}_{int}(y)\right], \tag{6}$$

and the same equations for $\pi(x)$.

Suppose now that we have an operator F which is a function of ϕ's and π's, and their space derivatives, for simplicity all at the same space-time point. Furthermore, let F depend also explicitly on coupling constants g and e. Then from (6) we have that F satisfies the following relations:

$$\begin{aligned}\frac{1}{i}\frac{\partial}{\partial g}F(x)&=-\int\theta(x^4-y^4)d^4y\left[F(x),\frac{\partial'}{\partial g}\mathcal{H}_{int}(y)\right]+\frac{1}{i}\frac{\partial'}{\partial g}F(x),\\ \frac{1}{i}\frac{\partial}{\partial e}F(x)&=-\int\theta(x^4-y^4)d^4y\left[F(x),\frac{\partial'}{\partial e}\mathcal{H}_{int}(y)\right]+\frac{1}{i}\frac{\partial'}{\partial e}F(x).\end{aligned} \tag{7}$$

From (7) we can derive an interesting property which the partial derivates $\partial'/\partial g$ and $\partial'/\partial e$ generally satisfy: their noncommutativity with the time derivative $\partial/\partial x^4$. Let us apply $\partial/\partial x^\mu$ to the first equation in (7),

$$\frac{1}{i}\frac{\partial}{\partial g}\frac{\partial}{\partial x^\mu}F(x)=-\int\theta(x^4-y^4)d^4y\left[\frac{\partial}{\partial x^\mu}F(x),\frac{\partial'}{\partial g}\mathcal{H}_{int}(y)\right]-g_\mu{}^4\int d^3y\left[F(\vec{x},x^4),\frac{\partial'}{\partial g}\mathcal{H}_{int}(\vec{y},x^4)\right]+\frac{\partial}{\partial x^\mu}\left(\frac{1}{i}\frac{\partial'}{\partial g}F(x)\right).$$

On the other hand, $\partial F/dx^\mu$ can be expressed in terms of ϕ's, π's, and their spatial derivatives. Thus, we can write according to (7)

$$\frac{1}{i}\frac{\partial}{\partial g}\frac{\partial}{\partial x^\mu}F(x)=-\int\theta(x^4-y^4)d^4y\left[\frac{\partial}{\partial x^\mu}F(x),\frac{\partial'}{\partial g}\mathcal{H}_{int}(y)\right]+\frac{1}{i}\frac{\partial'}{\partial g}\left(\frac{\partial}{\partial x^\mu}F(x)\right).$$

From these two equations we get that[7]

$$\frac{1}{i}\left[\frac{\partial'}{\partial g},\frac{\partial}{\partial x^\mu}\right]F(x)=-g_\mu{}^4\int d^4y\,\delta(x^4-y^4)\left[F(x),\frac{\partial'}{\partial g}\mathcal{H}_{int}(y)\right]. \tag{8}$$

The same relation we get with respect to e. In the above derivation, we assumed $[\partial/\partial g,\partial/\partial x^\mu]=0$ and $[\partial/\partial e,\partial/\partial x^\mu]=0$, since we are treating x, g, and e as independent variables. The relation (8) might be useful in current algebra in view of the fact that on the right-hand side we have an equal-time commutator. We shall not pursue this idea in this article.

III. DECOMPOSITION OF THE S MATRIX AS A PRODUCT OF TWO S MATRICES

The partial differential equations (3) when solved should give us, at least in principle, the S matrix reduced into the normal form in free-field "in" operators and as an analytical expression in the coupling constants g and e.[1,8] So far, this has been possible to achieve only for simple cases of one interaction (see, for example, Refs. 1 and 8).

Thus, presently we shall pursue a less ambitious goal: to write the S matrix as a product of two S matrices.

First, let us show how one can write $S=S_s S_{\rm em}$, S_s describing only strong interactions and $S_{\rm em}$ describing the electromagnetic interactions ("renormalized" by strong interactions).

We start with the definition

$$S=S_s S_{\rm em}, \tag{9}$$

where

$$S_s=S|_{e=0} \tag{10}$$

Then S_s, according to (3), satisfies the following differential equation:

$$\frac{1}{i}\frac{\partial}{\partial g}S_s=-S_s\int d^4x\left(\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}(x)\right)_{e=0}. \tag{11}$$

Indeed, $(\partial'\mathcal{H}_{\rm int}(x)/\partial g)_{e=0}$ will now depend on $\phi^0(x)\equiv\phi(x)|_{e=0}$ and $\pi^0(x)\equiv\pi(x)|_{e=0}$. It is easy to verify by help of (11) that S_s is unitary. Finally, from (9), (3), and (11) we get for $S_{\rm em}$

$$\frac{1}{i}\frac{\partial}{\partial e}S_{\rm em}=-S_{\rm em}\int d^4x\,\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x), \tag{12a}$$

$$\frac{1}{i}\frac{\partial}{\partial g}S_{\rm em}=-S_{\rm em}\int d^4x\,\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}(x)+\int d^4x\left(\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}(x)\right)_{e=0}S_{\rm em}. \tag{12b}$$

It is clear that $S_{\rm em}$ depends on both e and g. Thus, in order to be able to integrate (12a), we have to know $S^0_{\rm em}=S_{\rm em}|_{e=0}$ for every g. This information we should be able to get from (12b):

$$\frac{1}{i}\frac{\partial}{\partial g}S^0_{\rm em}=-\left[S^0_{\rm em},\int d^4x\left(\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}(x)\right)_{e=0}\right]. \tag{13}$$

However, before we discuss Eq. (13), let us note that the fact that we defined $S_s=S|_{e=0}$ requires $S^0_{\rm em}=1$ [see Eq. (9)]. In order to show that the same result follows from (13), we note with the help of (11) that we can write formally the solution for $S^0_{\rm em}$ satisfying (13) as

$$S^0_{\rm em}=S_s^\dagger S^{00}_{\rm em}S_s, \tag{14}$$

where $S^{00}_{\rm em}=S_{\rm em}|_{e=0,g=0}$. Since $\partial S^{00}_{\rm em}/\partial g=\partial S^{00}_{\rm em}/\partial e=0$, it does not depend on dynamics, so we can choose $S^{00}_{\rm em}=1$. Thus

$$S^0_{\rm em}=1. \tag{15}$$

Now we can write the solution of (12a) in the form

$$S_{\rm em}=\sum_n\frac{(ie)^n}{n!}\left[\left(\frac{1}{i}\frac{\partial}{\partial e}\right)^n S_{\rm em}\right]_{e=0},$$

where

$$\left[\left(\frac{1}{i}\frac{\partial}{\partial e}\right)^n S_{\rm em}\right]_{e=0}$$

are determined by the help of (15), (12a), and (16). Let us compute the first few terms. The zeroth term is unity according to (15). The first term is

$$ie\frac{1}{i}\frac{\partial}{\partial e}S_{\rm em}=-ie\int d^4x\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{e=0},$$

while for the second term we need

$$\left(\frac{1}{i}\frac{\partial}{\partial e}\right)^2 S_{\rm em}=\frac{1}{i}\frac{\partial}{\partial e}\left(\frac{1}{i}\frac{\partial}{\partial e}S_{\rm em}\right)$$

$$=-\frac{1}{i}\frac{\partial}{\partial e}S_{\rm em}\int d^4x\,\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)-S_{\rm em}\int d^4x\,\frac{1}{i}\frac{\partial}{\partial e}\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x).$$

Identifying $\partial'\mathcal{H}_{\rm int}(x)/\partial e$ with $F(x)$ in (7), we finally obtain

$$\left[\left(\frac{1}{i}\frac{\partial}{\partial e}\right)^2 S_{\rm em}\right]_{e=0}=\int d^4x d^4y\,T\left[\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{e=0}\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(y)\right)_{e=0}\right]-\int d^4x\,\frac{1}{i}\left(\frac{\partial'^2}{\partial e^2}\mathcal{H}_{\rm int}(x)\right)_{e=0}.$$

Thus, up to terms to the second order in e, we have

$$S_{\rm em}=1-i\int d^4x\left[e\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{e=0}+\frac{e^2}{2}\left(\frac{\partial'^2}{\partial e^2}\mathcal{H}_{\rm int}(x)\right)_{e=0}\right]+\tfrac{1}{2}(-i)^2e^2\int d^4xd^4y\,T\left[\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{e=0}\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(y)\right)_{e=0}\right]. \tag{16}$$

It is not difficult to find the nth term in the expansion, and the result can be cast into the Dyson form

$$S_{\rm em}=T\exp\left\{-i\int d^4x\sum_{n=1}^{\infty}\frac{e^n}{n!}\left(\frac{\partial'^n}{\partial e^n}\mathcal{H}_{\rm int}(x)\right)_{e=0}\right\}. \tag{17}$$

Let us yet note that when evaluated $(\partial'^n\mathcal{H}_{\rm int}(x)/\partial e^n)_{e=0}$ will depend on $\phi^0(x)\equiv\phi(x)|_{e=0}$ and $\pi^0(x)=\pi(x)|_{e=0}$.[9]

Secondly, we can write S equivalently as

$$S=S'_{\rm em}S'_s, \tag{18}$$

where we now define

$$S'_{\rm em}=S|_{g=0}\,. \tag{19}$$

It is clear that this second possibility is obtainable from the first by formally interchanging the roles of strong and electromagnetic interactions. Then according to (11) we write at once

$$\frac{1}{i}\frac{\partial}{\partial e}S'_{\rm em}=-S'_{\rm em}\int d^4x\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{g=0}, \tag{20}$$

where now $(\partial'\mathcal{H}_{\rm int}/\partial e)_{g=0}$ depends on $\phi'^0(x)\equiv\phi(x)|_{g=0}$ and $\pi'^0(x)\equiv\pi(x)|_{g=0}$. Similarly we get

$$\frac{1}{i}\frac{\partial}{\partial g}S'_s=-S'_s\int d^4x\,\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}(x), \tag{21a}$$

$$\frac{1}{i}\frac{\partial}{\partial e}S'_s=-S'_s\int d^4x\,\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)+\int d^4x\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{g=0}S'_s, \tag{21b}$$

where again we find that $S'^0_s\equiv S'_s|_{g=0}=1$. We now see that $S'_{\rm em}$ depends on e only, while S'_s depends on both g and e. S'_s is obtainable in principle from (21) if strong interactions are given. Of course, we expect to know $S'_{\rm em}$ as a power series in e since it only depends on electromagnetic interactions [see (20)]. To get this let us first write down some useful identities.

The S matrix must be the same no matter how we write it: i.e.,[10]

$$S=S_sS_{\rm em}=S'_{\rm em}S'_s$$

from which we get

$$S'_{\rm em}=S_{\rm em}|_{g=0}\,, \tag{22a}$$

$$S_s=S'_s|_{e=0}\,. \tag{22b}$$

Thus utilizing (17) we get $S'_{\rm em}$ at once,

$$S'_{\rm em}=T\exp\left\{-i\int d^4x\sum_{n=1}^{\infty}\frac{e^n}{n!}\left(\frac{\partial'^n}{\partial e^n}\mathcal{H}_{\rm int}(x)\right)_{e=0,g=0}\right\}. \tag{23}$$

Of course, we get the same expression directly from (20).

IV. REFORMULATION OF THE THEORY IN TERMS OF THE TOTAL HAMILTONIAN DENSITY

Everything we said so far is correct providing that the total Hamiltonian density operator is split in such a way that the free part contains observable masses of particles that interact. Since, according to our basic assumptions – macrocausality relations [see (2a) and (2b)] – the observable masses do not depend on the coupling constants g and e, $\partial'\mathcal{H}_f/\partial e=\partial'\mathcal{H}_f/\partial g=0$. In this case we can freely make the substitutions

$$\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}\to\frac{\partial'}{\partial g}\mathcal{H},\quad \frac{\partial'}{\partial e}\mathcal{H}_{\rm int}\to\frac{\partial'}{\partial e}\mathcal{H},$$

where

$$\mathcal{H}=\mathcal{H}_f+\mathcal{H}_{\rm int},$$

$\mathcal{H}_f$ being the free part of the Hamiltonian density. Then all the equations can be written with $\partial'\mathcal{H}/\partial g$ and $\partial'\mathcal{H}/\partial e$ in place of $\partial'\mathcal{H}_{\rm int}/\partial g$ and $\partial'\mathcal{H}_{\rm int}/\partial e$. On the other hand, if our decomposition of the total Hamiltonian density is such that $\mathcal{H}_f$ contains the bare masses[11] rather than the physical ones, then we can claim that in formulas such as (3), (5), (6), (7), (8), (11), (12a), (12b), (13), (16), and (17) the total Hamiltonian density must appear; i.e., in all these formulas we must have $\partial'\mathcal{H}/\partial g$ and $\partial'\mathcal{H}/\partial e$ instead of $\partial'\mathcal{H}_{\rm int}/\partial g$ and $\partial'\mathcal{H}_{\rm int}/\partial e$. The proof is quite simple. $\mathcal{H}_f$ is a sum of individual terms corresponding to each particle that participates in interaction:

$$\mathcal{H}_f(x)=\sum_i\mathcal{H}_f^{(i)}(x). \tag{24}$$

A typical expression for $\mathcal{H}_f^{(i)}$ if the ith particle is a spin-$\frac{1}{2}$ fermion is

$$\begin{aligned}\mathcal{H}_f^{(i)}&=-i\pi_{(i)}\gamma^4[-i\vec{\gamma}\nabla+m_0^i(g,e)]\psi_{(i)},\\ \pi_{(i)}&=\frac{\partial\mathcal{L}}{\partial\dot{\psi}_{(i)}}=i\psi^\dagger_{(i)}+\frac{\partial\mathcal{L}_{\rm int}}{\partial\dot{\psi}_{(i)}},\end{aligned} \tag{25}$$

where, as already mentioned earlier [see (2c)], a bare mass $m_0^i(g,e)$ is assumed to be generally a function of the coupling constants g and e. Now, since the hypothesis of the adiabatic switching of the interactions in the far past and future means considering the coupling constants weakly dependent on time in these regions $[g(t)=ge^{-\epsilon_1|t|}$, $e(t)=ee^{-\epsilon_2|t|}$, $\epsilon_{1,2}\to+0]$, then it follows that the bare masses become the observable masses m_i when both $g=0$ and $e=0$ (Ref. 12):

$$m_0^i(0,0)=m_i\,. \tag{26}$$

Equation (26) will be quite important later for the partial differential equation with respect to g and e which $m_0^i(g,e)$ will satisfy. Namely, in solving those differential equations (26) will be used as an initial condition which m_0^i will have to satisfy. Let us introduce Δm_i as

$$\begin{aligned}\Delta m_i(g,e)&=m_i-m_0^i(g,e),\\ \Delta m_i(0,0)&=0\,.\end{aligned} \tag{27}$$

Now we shall have for the case of a spin-$\frac{1}{2}$ particle

$$\mathcal{H}_f^{(i)}=-i\pi_{(i)}\gamma^4[-i\vec{\gamma}\nabla+m_i]\psi_{(i)}+i\Delta m_i(g,e)\pi_{(i)}\gamma^4\psi_{(i)}. \tag{28}$$

Of course, in usual formulation the term $i\Delta m_i(g,e)\pi_{(i)}\gamma^4\psi_{(i)}$ is simply added to $\mathcal{H}_{\rm int}$. In our formalism this is done automatically since, for example,

$$\frac{\partial'}{\partial e}\mathcal{H}_f^{(i)} = \frac{\partial'}{\partial e}[-i\pi_{(i)}\gamma^4\psi_{(i)}m_0^i(g,e)]$$
$$= -i\pi_{(i)}\gamma^4\psi_{(i)}\left(\frac{\partial}{\partial e}m_0^i(g,e)\right)$$
$$= \frac{\partial'}{\partial e}[i\pi_{(i)}\gamma^4\psi_{(i)}\Delta m_i(g,e)]$$
$$= i\pi_{(i)}\gamma^4\psi_{(i)}\left(\frac{\partial}{\partial e}\Delta m_i(g,e)\right),$$

when $\partial'\mathcal{H}/\partial e$ is written instead of $\partial'\mathcal{H}_{\text{int}}/\partial e$. Let us point out that relation (8), which now reads as

$$\frac{1}{i}\left[\frac{\partial'}{\partial g},\frac{\partial}{\partial x^\mu}\right]F(x) = -g_\mu{}^4\int d^4y\,\delta(x^4-y^4)\left[F(x),\frac{\partial'}{\partial g}\mathcal{H}(y)\right], \tag{8'}$$

is fully consistent with Heisenberg equations of motion. Thus, we see that in all our relations like (3), (5), (6), etc., we can write $\partial'\mathcal{H}/\partial e$ and $\partial'\mathcal{H}/\partial g$ regardless of whether the free part of the Hamiltonian density operator contains observable masses or bare masses. In view of this, we rewrite some relations here from Secs. II and III:

$$\frac{1}{i}\frac{\partial}{\partial g}S = -S\int d^4x\,\frac{\partial'}{\partial g}\mathcal{H}(x),$$
$$\frac{1}{i}\frac{\partial}{\partial e}S = -S\int d^4x\,\frac{\partial'}{\partial e}\mathcal{H}(x), \tag{3'}$$

$$\frac{1}{i}\frac{\partial}{\partial g}\phi(x) = -\int d^4y\,\theta(x^4-y^4)\left[\phi(x),\frac{\partial'}{\partial g}\mathcal{H}(y)\right],$$
$$\frac{1}{i}\frac{\partial}{\partial e}\phi(x) = -\int d^4y\,\theta(x^4-y^4)\left[\phi(x),\frac{\partial'}{\partial e}\mathcal{H}(y)\right], \tag{6'}$$

and the same equations for $\pi(x)$. If F depends on $\phi(x)$'s and $\pi(x)$'s and explicitly on coupling constants g and e, we have

$$\frac{1}{i}\frac{\partial}{\partial g}F(x) = -\int d^4y\,\theta(x^4-y^4)\left[F(x),\frac{\partial'}{\partial g}\mathcal{H}(y)\right] + \frac{1}{i}\frac{\partial'}{\partial g}F(x),$$
$$\frac{1}{i}\frac{\partial}{\partial e}F(x) = -\int d^4y\,\theta(x^4-y^4)\left[F(x),\frac{\partial'}{\partial e}\mathcal{H}(y)\right] + \frac{1}{i}\frac{\partial'}{\partial e}F(x). \tag{7'}$$

S_s and S_{em} defined by (9) satisfy

$$\frac{1}{i}\frac{\partial}{\partial g}S_s = -S_s\int d^4x\left(\frac{\partial'}{\partial g}\mathcal{H}(x)\right)_{e=0},$$
$$\frac{1}{i}\frac{\partial}{\partial e}S_{\text{em}} = -S_{\text{em}}\int d^4x\,\frac{\partial'}{\partial e}\mathcal{H}(x), \tag{11'}$$

$$S_{\text{em}}^0 \equiv S_{\text{em}}|_{e=0} = 1. \tag{12a'}$$

The solution (17) for S_{em} now becomes

$$S_{\text{em}} = T\exp\left\{-i\int d^4x\sum_{n=1}^{\infty}\frac{e^n}{n!}\left(\frac{\partial'^n}{\partial e^n}\mathcal{H}(x)\right)_{e=0}\right\}. \tag{17'}$$

From now on we shall assume that the free part of the Hamiltonian density has bare masses dependent on the coupling constants. With this assumption we must use the total Hamiltonian density in relations (3′), (6′), (7′), (11′), (12a′), and (17′). It will prove to be very useful to have $m_0(g,e)$ instead of $\Delta m(g,e)$ later in the discussion of mass differences of hadrons.

In order to proceed any further, we have to say something about the interaction. We will not assume any particular model for strong interactions in this paper. However, with the correspondence principle, we will be able to write down the first term in the expansion in e that occurs in (17′) and make reasonable assumptions about higher terms. To see how this comes about, let us write the solution for the Heisenberg electromagnetic field as

$$A^\mu(x) = A_0^\mu(x) + ie\left(\frac{1}{i}\frac{\partial}{\partial e}A^\mu(x)\right)_{e=0} + O(e^2),$$
$$A_0^\mu(x) \equiv A^\mu(x)|_{e=0}. \tag{29}$$

According to (6′)

$$\left(\frac{1}{i}\frac{\partial}{\partial e}A^\mu(x)\right)_{e=0} = -\int d^4y\,\theta(x^4-y^4)\left[A_0^\mu(x),\left(\frac{\partial'}{\partial e}\mathcal{H}(y)\right)_{e=0}\right]. \tag{30}$$

As we see from (29) and (30), we have to know what $A_0^\mu(x)$ is. Since, according to (29), we assumed that $A^\mu(x)$ is well behaved at $e=0$, we then can write according to (6′)

$$\left(\frac{1}{i}\frac{\partial}{\partial g}A^\mu(x)\right)_{e=0} = \frac{1}{i}\frac{\partial}{\partial g}A_0^\mu(x)$$
$$= -\int d^4y\,\theta(x^4-y^4)\times\left[A_0^\mu(x),\left(\frac{\partial'}{\partial g}\mathcal{H}(y)\right)_{e=0}\right]. \tag{31}$$

Now comes the important point: According to (11′), $(\partial'\mathcal{H}/\partial g)_{e=0}$ defines pure strong interactions, i.e., the interactions between hadrons only. Thus, when we expand $(\partial'\mathcal{H}/\partial e)_{e=0}$ in terms of a complete set of "in" fields only hadronic "in" fields come in the expansion. Then seeking the solution of (31) for $A_0^\mu(x)$ as a power series in g, we get

$$A_0^\mu(x) = A^\mu(x)|_{e=0,g=0}$$
$$= A_{\text{in}}^\mu(x), \tag{32}$$

where (2e) was taken into account. Indeed (31) is satisfied with (32) since, according to (2a),

$$\frac{\partial}{\partial g} A^{\mu}_{\text{in}}(x) = 0$$

and

$$\left[A^{\mu}_{\text{in}}(x), \left(\frac{\partial'}{\partial g}\mathcal{H}(y)\right)_{e=0}\right] = 0$$

$[(\partial'\mathcal{H}/\partial g)_{e=0}$ describes the interactions between hadrons only$]$.[13] Thus we can write

$$A^{\mu}(x) = A^{\mu}_{\text{in}}(x) - ie\int d^4y\,\theta(x^4 - y^4) \times \left[A^{\mu}_{\text{in}}(x), \left(\frac{\partial'}{\partial e}\mathcal{H}(y)\right)_{e=0}\right] + O(e^2). \tag{33}$$

In quantum electrodynamics

$$\left(\frac{\partial'}{\partial e}\mathcal{H}(y)\right)_{e=0} = -j^{\text{in}}_{\mu}(y) A^{\mu}_{\text{in}}(y)$$

(see Ref. 7), which has the property of being independent of the choice of hypersurface $\sigma(y)$.[14] $j^{\text{in}}_{\mu}(y)$ is a leptonic electromagnetic current, bilinear in leptonic "in" fields. We shall also maintain the same property in our case if we make a natural choice

$$\left(\frac{\partial'}{\partial e}\mathcal{H}(y)\right)_{e=0} = -[J_{\mu}(y) + j^{\text{in}}_{\mu}(y)]A^{\mu}_{\text{in}}(y), \tag{34}$$

where J_{μ} is a hadronic electromagnetic current; i.e., when expanded in terms of a complete set of "in" fields, only hadronic "in" field operators are needed for the expansion. Taking into account the commutation relations for A^{μ}_{in} fields, (33) can be rewritten in the familiar form

$$A^{\mu}(x) = A^{\mu}_{\text{in}}(x) - e\int d^4y\, D_R(x-y)[J^{\mu}(y) + j^{\mu}_{\text{in}}(y)] + O(e^2). \tag{33'}$$

There is still one point that we have to clarify: In view of the assumption that $\mathcal{H}_f$ contains bare masses, (34) implies for every one of them that

$$\left(\frac{\partial}{\partial e} m_0^i(g, e)\right)_{e=0} = 0, \tag{35}$$

since we also want

$$\left(\frac{\partial'}{\partial e}\mathcal{H}_{\text{int}}\right)_{e=0} = -(J^{\mu} + j^{\mu}_{\text{in}})A^{\text{in}}_{\mu}.$$

We shall see in Sec. V that (35) is indeed satisfied.

If we now continue computing $A^{\mu}(x)$ to second and higher orders in e, then besides $(\partial'\mathcal{H}/\partial e)_{e=0}$ the terms like $(\partial'^2\mathcal{H}/\partial e^2)_{e=0}$ etc. would come too. We shall assume that the terms like $(\partial'^2\mathcal{H}/\partial e^2)_{e=0}$ etc. exist and that their only role is to make the theory renormalizable.[15] This is indeed so in the case of quantum electrodynamics. However, we will not need to know them explicitly in our discussion of the mass differences of hadrons.

V. MASS DIFFERENCES OF HADRONS

Although in this section we shall discuss only the problem of the n-p mass difference, similar discussion could be extended to any other case in which the mass difference is expected to be electromagnetic in origin and where one-particle states are stable under strong and electromagnetic interactions.

In quantum electrodynamics one usually starts the discussion of the mass renormalization by observing that the electron is stable, thus demanding that $\langle q'|S-1|q\rangle = 0$. Since the one-electron states $|q\rangle$ and $|q'\rangle$ are "in" states, we can rewrite this condition as

$$\frac{1}{i}\frac{\partial}{\partial e}\langle q'|S|q\rangle = \left\langle q'\left|\frac{1}{i}\frac{\partial}{\partial e}S\right|q\right\rangle = 0.$$

Similarly, since proton and neutron are stable under strong and electromagnetic interactions, we then demand

$$\left\langle p'\left|\frac{1}{i}\frac{\partial}{\partial e}S\right|p\right\rangle = -\left\langle p'\left|S\int d^4x\,\frac{\partial'}{\partial e}\mathcal{H}(x)\right|p\right\rangle = 0, \tag{36a}$$

$$\left\langle p'\left|\frac{1}{i}\frac{\partial}{\partial g}S\right|p'\right\rangle = -\left\langle p'\left|S\int d^4x\,\frac{\partial'}{\partial g}\mathcal{H}(x)\right|p\right\rangle = 0, \tag{36b}$$

and the same equations for a neutron (making substitutions $p \to n$, $p' \to n'$). Furthermore, since both proton and neutron are stable under pure strong interactions, then according to (11′) and (12a′) we have

$$\left\langle p'\left|S_s\frac{1}{i}\frac{\partial}{\partial e}S_{\text{em}}\right|p\right\rangle = -\left\langle p'\left|S_sS_{\text{em}}\int d^4x\,\frac{\partial'}{\partial e}\mathcal{H}(x)\right|p\right\rangle = 0, \tag{37a}$$

$$\left\langle p'\left|\frac{1}{i}\frac{\partial}{\partial g}S_s\right|p\right\rangle = -\left\langle p'\left|S_s\int d^4x\left(\frac{\partial'}{\partial g}\mathcal{H}(x)\right)_{e=0}\right|p\right\rangle = 0, \tag{37b}$$

and the same equations for a neutron. From now on we shall concentrate on the case of a proton, since the neutron case can be obtained by simply replacing $p \to n$ and $p' \to n'$. Writing formally

$$\mathcal{H}(x) = \mathcal{H}_f^p(x) + \mathcal{H}(x) - \mathcal{H}_f^p(x),$$

the right-hand sides of (37a) and (37b) become

$$-i\left(\frac{\partial}{\partial e}m_0^p(g,e)\right)\int d^4x\langle p'|S_sS_{\rm em}\pi_p(x)\gamma^4\psi_p(x)|p\rangle=-\int d^4x\left\langle p'\left|S_sS_{\rm em}\frac{\partial'}{\partial e}[\mathcal{H}(x)-\mathcal{H}_f^p(x)]\right|p\right\rangle, \tag{38a}$$

$$-i\left(\frac{\partial}{\partial g}m_0^p(g,0)\right)\int d^4x\langle p'|S_s[\pi_p(x)\gamma^4\psi_p(x)]_{e=0}|p\rangle=-\int d^4x\left\langle p'\left|S_s\left(\frac{\partial'}{\partial g}[\mathcal{H}(x)-\mathcal{H}_f^p(x)]\right)_{e=0}\right|p\right\rangle, \tag{38b}$$

where we used

$$\frac{\partial'}{\partial e}\mathcal{H}_f^p(x)=-i\pi_p(x)\gamma^4\psi_p(x)\frac{\partial}{\partial e}m_0^p(g,e),$$

etc. The terms that multiply $\partial m_0^p(g,e)/\partial e$ and $\partial m_0^p(g,0)/\partial g$ are kinematic factors and can be computed exactly because of our assumption that strong interactions are already renormalized,[16]

$$\begin{aligned}-i\int d^4x\langle p'|S_sS_{\rm em}\pi_p(x)\gamma^4\psi_p(x)|p\rangle&=-i\int d^4x\langle p'|S_s[\pi_p(x)\gamma^4\psi_p(x)]_{e=0}|p\rangle\\&=\int d^4x\langle p'|:\overline{\psi}_{p_{\rm in}}(x)\psi_{p_{\rm in}}(x):|p\rangle\\&=2\pi\delta^{(4)}(p-p')\overline{u}(p')u(p).\end{aligned}$$

We also see that $\mathcal{H}(x)-\mathcal{H}_f^p(x)$ can be replaced by $\mathcal{H}_{\rm int}(x)$ in (38a) and (38b), since similarly $\partial'\mathcal{H}_f^{(i)}(x)/\partial g$ and $\partial'\mathcal{H}_f^{(i)}(x)/\partial e$ give no contribution for $i\neq p$. Thus, we have

$$2\pi\delta^{(4)}(p-p')\overline{u}(p')u(p)\frac{\partial}{\partial e}m_0^p(g,e)=-\int d^4x\left\langle p'\left|S_{\rm em}\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right|p\right\rangle, \tag{39a}$$

$$2\pi\delta^{(4)}(p-p')\overline{u}(p')u(p)\frac{\partial}{\partial g}m_0^p(g,0)=-\int d^4x\left\langle p'\left|S_s\left(\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}(x)\right)_{e=0}\right|p\right\rangle, \tag{39b}$$

where in (39a) we have put $\langle p'|S_s=\langle p'|$ which is consistent with (37b). With

$$\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}\right)_{e=0}=-(J^\mu+j_{\rm in}^\mu)A_\mu^{\rm in},$$

we see at once that $[\partial m_0^p(g,e)/\partial e]_{e=0}=0$ is satisfied. It is not difficult to see that this is true for any other mass $m_0^i(g,e)$.

The equations (39a) and (39b) are two differential equations necessary to solve in order to know $m_0^i(g,e)$. As we can see, in order to be able to integrate (39a) we have to know $m_0^p(g,0)$. This we can know in principle by solving (39b). Now, since we are interested in knowing $m_0^p(g,e)$ only to the order e^2, we need to know the right-hand side of (39a) only to the order e. Thus, we first expand $\partial'\mathcal{H}_{\rm int}/\partial e$ as

$$\begin{aligned}\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)&=\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{e=0}+ie\left[\frac{1}{i}\frac{\partial}{\partial e}\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)\right]_{e=0}\\&=\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{e=0}-ie\int d^4y\,\theta(x^4-y^4)\left[\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{e=0},\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(y)\right)_{e=0}\right]+ie\frac{1}{i}\left(\frac{\partial'^2}{\partial e^2}\mathcal{H}_{\rm int}(x)\right)_{e=0}.\end{aligned}$$

Second, we use the expression (17′) for $S_{\rm em}$, expanding it to the first order in e, and obtain

$$S_{\rm em}\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)=\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{e=0}-ie\int d^4y\,T\left[\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(x)\right)_{e=0}\left(\frac{\partial'}{\partial e}\mathcal{H}_{\rm int}(y)\right)_{e=0}\right]+ie\frac{1}{i}\left(\frac{\partial'^2}{\partial e^2}\mathcal{H}_{\rm int}(x)\right)_{e=0}. \tag{40}$$

The first term in (40) does not contribute to the mass renormalization since $(\partial'\mathcal{H}_{\rm int}/\partial e)_{e=0}$ is proportional to $A_\mu^{\rm in}$. The last term in (40), although it may be necessary for proton or neutron field-operator renormalization, need not be known explicitly since the mass renormalization can be achieved without it.[17] Thus we can continue with only the second term in (40) and have for either proton or neutron

$$\begin{aligned}2\pi\delta^{(4)}(p-p')\overline{u}(p')u(p)\frac{\partial}{\partial e}m_0(g,e)&=ie\int d^4xd^4y\langle p'|T[J_\mu(x)A_{\rm in}^\mu(x)J_\nu(y)A_{\rm in}^\nu(y)]|p\rangle\\&=e\int d^4xd^4y\,D_F(x-y)\langle p'|T[J_\mu(x)J^\mu(y)]|p\rangle.\end{aligned} \tag{41}$$

Or from here

$$\bar{u}(p)u(p)\frac{\partial}{\partial e}m_0(g,e)=\frac{e}{2\pi}\int d^4x d^4k\,\frac{e^{ikx}}{k_-^2}\langle p|T[J_\mu(x)J^\mu(0)]|p\rangle . \tag{42}$$

We integrate (42) as

$$m_0(g,e)=m_0(g,0)+\int_0^e de\,\frac{\partial m_0(g,e)}{\partial e} . \tag{43}$$

Denoting

$$\delta_{\rm em}m(g,e)=-\int_0^e de\,\frac{\partial m_0(g,e)}{\partial e} , \tag{44}$$

we have[18]

$$\bar{u}(p)u(p)\delta_{\rm em}m(g,e)=-\frac{1}{2\pi}\,\frac{e^2}{2}\int d^4x d^4k\,\frac{e^{ikx}}{k_-^2}\langle p|T[J_\mu(x)J^\mu(0)]|p\rangle . \tag{45}$$

$m_0(g, 0)$ in (43) has the meaning of the bare mass due to the strong interactions only. In principle we could find $m_0(g, 0)$ from (39b) if we knew the right-hand side of (39b) as a function of g. Taking into account that the observable mass m is equal to $m_0(0, 0)$, we write

$$m_0(g,0)=m+\int_0^g dg\,\frac{\partial m_0(g,0)}{\partial g} , \tag{46}$$

where we assume that $\partial m_0/\partial g$ is well behaved at $g=0$. If we denote

$$\delta_s m(g)=-\int_0^g dg\,\frac{\partial m_0(g)}{\partial g} , \tag{47}$$

we see that the total self-mass is

$$\begin{aligned}\Delta m(g,e)&=m-m_0(g,e)\\&=\delta_s m(g)+\delta_{\rm em}m(g,e) .\end{aligned} \tag{48}$$

$\delta_s m(g)$ is the nucleon self-mass due to the strong interactions only, and it satisfies

$$\begin{aligned}2\pi\delta^{(4)}(p-p')\bar{u}(p')u(p)\frac{\partial}{\partial g}\delta_s m(g)\\=\int d^4x\left\langle p'\left|S_s\left(\frac{\partial'}{\partial g}\mathcal{H}_{\rm int}(x)\right)_{e=0}\right|p\right\rangle ,\end{aligned} \tag{49}$$

with the initial condition $\delta_s m(0)=0$. Thus we get for the n-p mass difference two equivalent expressions[19]:

$$\begin{aligned}m_n-m_p=m_0^n(g,e)-m_0^p(g,e)+\delta_s m_n(g)-\delta_s m_p(g)\\+\delta_{\rm em}m_n(g,e)-\delta_{\rm em}m_p(g,e) ,\end{aligned} \tag{50a}$$

$$m_n-m_p=m_0^n(g,0)-m_0^p(g,0)+\delta_s m_n(g)-\delta_s m_p(g) . \tag{50b}$$

Relations (50a) and (50b) reflect the assumption that the observable masses are independent of coupling constants g and e since (50b) follows from (50a) by putting $e=0$. In view of this, both expressions, when read from left to right, should be viewed as identities rather than the expressions from which to compute m_n-m_p.

In the literature the expression from which one tries to compute m_n-m_p is

$$m_n-m_p=\delta_{\rm em}m_n(g,e)-\delta_{\rm em}m_p(g,e) . \tag{51}$$

Equation (51) definitely assumes the observable masses to depend on g and e coupling constants. We believe that this is an unnecessary assumption for most of the calculations. For example, when computing the amplitude for the Compton scattering in quantum electrodynamics, one never expands the "in" states of the electron and the photon in terms of e.

Even if (51) were numerically satisfied, that still would not mean that the observable masses must be functions of g and e coupling constants. Namely, according to our formalism this would simply mean that $m_0^n(g,e)-m_0^p(g,e)+\delta_s m_n(g)-\delta_s m_p(g)$ happens to be numerically equal to zero. Of course, we admit the existence of some other (nonelectromagnetic) interactions which cause $m_n-m_p\neq 0$. An example is quark-gluon interaction which gives the observable hadronic mass spectrum in terms of bare quark mass. Since the bare quark mass depends on the quark-gluon coupling constant, the observable hadronic mass spectrum will depend on the quark-gluon coupling constant too.

VI. REMARKS AND CONCLUSION

It is not difficult to see that in our formalism we have adopted a viewpoint which one meets in the usual formalism of divergent (renormalizable) quantum field theory: The observable masses m's are given, while the bare masses m_0's and self-masses Δm's are to be determined in such a way as to satisfy $m=m_0+\Delta m$. Since it is Δm that one usually computes, m_0 then is given as $m_0=m-\Delta m$.

Of course, what we get for Δm depends entirely on dynamical assumptions. For example, if we use the tree-diagram approximation for strong interactions (an approximation very popular for effective Lagrangians with chiral symmetry), then $\delta_s m_n(g)=0$ and $\delta_s m_p(g)=0$. This simply means that in the tree-diagram approximation strong-interaction dynamics does not change the mass which means that $m_n = m_0^n(g, 0)$ and $m_p = m_0^p(g, 0)$. However, in general (no tree-diagram approximation) $\delta_s m_n \neq 0$ and $\delta_s m_p \neq 0$.

Next we would like to bring SU(2) into the discussion. In view of assumption (2b) that the observable masses do not depend on coupling constants g and e, we shall have two independent SU(2)-symmetry-breaking parameters: e and, say, $\lambda \equiv m_n - m_p$ (see the Appendix). $\lambda \neq 0$ is due to some nonelectromagnetic interactions. In other words, the exact SU(2) symmetry is achieved only if independently $e=0$ and $\lambda=0$ (all other observable "electromagnetic" mass differences we assume to be expressible in terms of λ). In the Appendix we show that for $\lambda=0$ and $e\neq 0$ [SU(2) symmetry still is broken by electromagnetic interactions], we have $\delta_s m_n(g) = \delta_s m_p(g)$. Therefore, from relation (50a) we have

$$m_0^n(g, e) - m_0^p(g, e) = -[\delta_{\rm em} m_n(g, e) - \delta_{\rm em} m_p(g, e)] = 0.66\,{\rm MeV}, \qquad (52)$$

where the numerical value is valid to the order $O(e^2)$.[20] In other words, even if pure strong interactions are SU(2)-invariant [$\delta_s m_n = \delta_s m_p$, and therefore $m_0^n(g,0) - m_0^p(g,0) = m_n - m_p = 0$], the degenerate incoming system of neutron and proton acquires the nondegenerate bare masses under the influence of electromagnetic interactions ("renormalized" by strong interactions) and, of course, becomes again a degenerate outgoing system.

The case of $e=0$ and λ small is quite interesting since now we have only strong interactions. We expect that they break SU(2) only slightly. To show this we rewrite (50b) as

$$\lambda = m_0^n(g,0) - m_0^p(g,0) + \Delta(g,\lambda), \qquad (53)$$

where we defined

$$\Delta(g,\lambda) = \delta_s m_n(g) - \delta_s m_p(g)$$

(see the Appendix). In the Appendix we argue [see (A12)] that we should be able to write

$$\Delta(g,\lambda) = k_1(g)\lambda + k_2(g)\lambda^2 + O(\lambda^3).$$

From (53) we then have that

$$m_0^n(g,0) - m_0^p(g,0) = [1-k_1(g)]\lambda - k_2(g)\lambda^2 + O(\lambda^3). \qquad (54)$$

Equation (54) shows that when λ is small, which is true in our case,

$$m_0^n(g,0) - m_0^p(g,0) \approx \lambda.$$

Now, since the mass differences of the type $m_0^n(g,0) - m_0^p(g,0)$ determine the nature of SU(2) breaking, we see that it will indeed be small as long as λ is small.

However, (54) opens another completely unexpected possibility. Namely, let us suppose the following academic case: that λ is not too small; i.e., we really have to retain the terms up to the second order in λ. This presumably resembles the case of "medium-strong" breaking, which one meets in the case of broken SU(3). As we know, the "medium-strong" SU(3) symmetry breaking is caused by mass differences of the type $m_n - m_\Sigma$, which are considerably larger than $m_n - m_p$. However, despite the fact that now we take λ to be not too small, the symmetry breaking could still be quite small. This possibility could happen if $k_1(g)$ in (54) is numerically very close to unity, in which case

$$m_0^n(g,0) - m_0^p(g,0) \approx -k_2(g)\lambda^2,$$

which, of course, could be quite small.

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APPENDIX

In this appendix we wish to substantiate the claim in Sec. VI that when $m_n - m_p = 0$ and $(\partial \mathcal{H}_{\rm int}/\partial g)_{e=0}$ is invariant under SU(2) transformation, $\delta_s m_n(g) = \delta_s m_p(g)$.

We start as usual: In the presence of SU(2) symmetry breaking, the doublet of proton and neutron Heisenberg (interpolating) field operators is assumed to obey the linear transformation law,

$$[Q_i(t), \psi_\alpha(x)] = -\tfrac{1}{2}(\tau_i)_\alpha{}^\beta \psi_\beta(x). \qquad ({\rm A1})$$

$\psi_1(x)$ and $\psi_2(x)$ are proton and neutron Heisenberg field operators (we ignore Dirac indices). $Q_i(t)$ ($i=1,2,3$), the generators of the SU(2) transformations, obey the well-known commutation relations

$$[Q_i(t), Q_j(t)] = i\epsilon_{ijk}Q_k(t). \qquad ({\rm A2})$$

Relation (A1) usually suggests a simple expression for Q_i in terms of Heisenberg fields. However, Q_i can also be expressed in terms of asymptotic "in" fields by the fact that the Heisenberg fields can be expressed in terms of them ("in" fields) by solving the equations of motion. Since the connection between Heisenberg fields and "in" fields is generally nonlinear and nonlocal, then nobody will be surprised that the doublet of proton and neutron "in" field operators (ψ_1^{in} and ψ_2^{in}) will in general transform nonlocally and nonlinearly with Q_i ($i=1,2,3$) as generators:

$$[Q_i(t), \psi_\alpha^{\text{in}}(x)] = -\tfrac{1}{2}(\tau_i)_\alpha{}^\beta \psi_\beta^{\text{in}}(x) + O_{i,\alpha}(x), \qquad \text{(A3)}$$

where $O_{i,\alpha}(x)$, a complicated fermion field, depends nonlocally and nonlinearly on "in" field operators.[21] [Note that in (A1), (A3), and in what follows summation occurs only when contravariant and covariant isospinor indices are the same.] We can demonstrate the complicated nature of $O_{i,\alpha}(x)$ by working out the Jacobi identity between Q_i, Q_j, and ψ_α^{in}:

$$[Q_i(t), O_{j,\alpha}(x)] - [Q_j(t), O_{i,\alpha}(x)] = i\epsilon_{ijk}O_{k,\alpha}(x) - \tfrac{1}{2}(\tau_i)_\alpha{}^\beta O_{j,\beta}(x) + \tfrac{1}{2}(\tau_j)_\alpha{}^\beta O_{i,\beta}(x). \qquad \text{(A4)}$$

We see that $O_{i,\alpha}(x)$ definitely does not transform as a product of isovector and isospinor since if it were we would have to have $2i\epsilon_{ijk}$ instead of $i\epsilon_{ijk}$ in (A4).

Let us now apply the Dirac operator

$$D_{m_\alpha}(x) \equiv -i\gamma_\mu \frac{\partial}{\partial x_\mu} + m_\alpha \quad (m_1 = m_p,\ m_2 = m_n)$$

to both sides of (A3):

$$i[\dot{Q}_i(t), \gamma^4 \psi_\alpha^{\text{in}}(x)] = \tfrac{1}{2}(\tau_i)_\alpha{}^\beta (m_\alpha - m_\beta)\psi_\beta^{\text{in}}(x) - D_{m_\alpha}(x) O_{i,\alpha}(x). \qquad \text{(A5)}$$

The case of $i=3$ is quite simple. Since Q_3 is a constant of motion from (A5), we get $D_{m_\alpha}(x)O_{3,\alpha}(x) = 0$. On the other hand since $O_{3,\alpha}(x)$ is a fermion field, its vacuum expectation value is zero [no possibility of a spontaneous breakdown of SU(2) symmetry]. Furthermore, since Q_3 and the total Hamiltonian are diagonal at the same time, it then follows from (A3) that the matrix elements of $O_{3,\alpha}(x)$ between physical "in" states vanish. Therefore

$$O_{3,\alpha}(x) = 0,$$

which is not a surprising result.

The cases of $i=1$ and $i=2$, however, are not that simple. It is quite clear that Q_1 and Q_2 depend on the parameters of SU(2) symmetry breaking. Since, as we mentioned before, in our formalism the observable masses do not depend on g and e coupling constants, we shall have two independent SU(2)-symmetry-breaking parameters, e and, say, $\lambda \equiv m_n - m_p$ (all other "electromagnetic" observable mass differences we assume to be expressible in terms of λ). Furthermore, since Q_i ($i=1,2$) are not constants of motion, they generally depend on all coupling constants. Thus Q_i ($i=1,2$) will depend on g as well.

In view of the fact that we have two independent parameters of SU(2) symmetry breaking, of interest to us are the following three cases: (a) $\lambda=0$, $e\neq 0$; (b) $\lambda \neq 0$, $e=0$; and (c) $\lambda=0$, $e=0$.

(a) $\lambda=0$, $e\neq 0$. Here the SU(2) symmetry is broken only via electromagnetism. From (A5) we have

$$i[\dot{Q}_i(t), \gamma^4\psi_\alpha^{\text{in}}(x)] = -D_{m_\alpha}(x)O_{i,\alpha}(x), \quad i=1,2. \qquad \text{(A6)}$$

This relation clearly indicates that since $\dot{Q}_i \neq 0$ ($i=1,2$), $O_{i,\alpha}(x)\neq 0$ and the doublet of proton and neutron "in" field operators $\psi_\alpha^{\text{in}}(x)$ ($\alpha=1,2$) still transform nonlocally and nonlinearly. On the other hand, we know that the total Hamiltonian for the Heisenberg fields $H(t)$ equals the free-particle Hamiltonian for the incoming fields $H_f^{\text{in}}(t)$:

$$H(t) = H_f^{\text{in}}(t). \qquad \text{(A7)}$$

Thus $H_f^{\text{in}}(t)$ should reflect the breaking of SU(2) symmetry despite the fact that we have $m_n = m_p$, $m_{\Sigma^+} = m_{\Sigma^-} = m_{\Sigma^0}$, etc. $H_f^{\text{in}}(t)$ is not invariant under SU(2) transformations simply because in view of (A6) $O_{i,\alpha}(x)\neq 0$ ($i=1,2$). Namely, it is the nonlinear transformation law (A3) which makes $H_f^{\text{in}}(t)$ noninvariant. Furthermore, since $\dot{Q}_i\neq 0$ ($i=1,2$), Coleman's theorem[22] is not violated; i.e., $Q_i|0\rangle \neq 0$ ($i=1,2$), $(Q_1+iQ_2)|\text{neutron}\rangle \neq |\text{proton}\rangle$, etc.

(b) $\lambda\neq 0$, $e=0$. Here the breaking of SU(2) symmetry is due to mass differences of the type $m_n - m_p$, $m_{\Sigma^+} + m_{\Sigma^-} - 2m_{\Sigma^0}$, etc. This case is interesting in view of the fact that we have only strong interactions. According to (A5) we can still write

$$i[\dot{Q}_i(t), \gamma^4\psi_\alpha^{\text{in}}(x)] = \tfrac{1}{2}(\tau_i)_\alpha{}^\beta(m_\alpha - m_\beta)\psi_\beta^{\text{in}}(x) - D_{m_\alpha}(x)\,O_{i,\alpha}(x). \qquad \text{(A8)}$$

One would think that because $e=0$ maybe now $O_{i,\alpha}(x)$ vanishes. This, however, is not the case. Namely, since $Q_i(t)$ ($i=1,2$) are not constants of motion, they will depend on strong interactions and, again, in view of equations of motion, we conclude the nonlinear and nonlocal transformation laws for doublet of proton and neutron "in" field operators [see the general discussion after (A2)].

(c) $\lambda=0$, $e=0$. This is the case of exact SU(2) symmetry. Now we have $\dot{Q}_i=0$ for all i. From (A5) we get $D_{m_\alpha}(x)O_{i,\alpha}(x)=0$. However, as in the case of $i=3$, we again conclude that $O_{i,\alpha}(x)=0$, $i=1,2,3$. Now, of course, we shall have that $(Q_1+iQ_2)|\text{neutron}\rangle = |\text{proton}\rangle$, $Q_i|0\rangle = 0$, and similar relations.

Let us now justify relation (52) and the conclusion in Sec. VI. There we claimed that $\delta_{em}m_n(g,e) - \delta_{em}m_p(g,e) \neq 0$, $\delta_s m_n(g) = \delta_s m_p(g)$ when $\lambda = 0$, $e \neq 0$. For the sake of clarity, we shall use the symbols p and n to denote proton and neutron, respectively, while four-momenta we shall denote with q and q'. From (39a) we get for $\delta_{em}m_p(g,e)$ the following differential equation:

$$2\pi\delta^{(4)}(q-q')m\frac{\partial}{\partial e}\delta_{em}m_p(g,e) = \int d^4x \left\langle q',p \left| S_{em}\frac{\partial'}{\partial e}\mathcal{H}_{int}(x)\right| q,p\right\rangle, \tag{A9}$$

where we use the normalization $\bar{u}_p(q)u_p(q) = m$ (note that now $m_p = m_n = m$). Using the fact that Q_3 is a constant of motion and that $[Q_+, Q_-] = 2Q_3$ $(Q_\pm = Q_1 \pm iQ_2)$, we can write

$$\left\langle q',p\left|S_{em}\frac{\partial'}{\partial e}\mathcal{H}_{int}(x)\right|q,p\right\rangle = \left\langle q',p\left|S_{em}\frac{\partial'}{\partial e}\mathcal{H}_{int}(x)[Q_+,Q_-]\right|q,p\right\rangle .$$

Since for $\lambda = 0$ and $e \neq 0$, $Q_-|q,p\rangle \neq |q,n\rangle$ and $S_{em}\,\partial'\mathcal{H}_{int}(x)/\partial e$ is not invariant under SU(2) transformations, we conclude that in general

$$\left\langle q',p\left|S_{em}\frac{\partial'}{\partial e}\mathcal{H}_{int}(x)\right|q,p\right\rangle \neq \left\langle q',n\left|S_{em}\frac{\partial'}{\partial e}\mathcal{H}_{int}(x)\right|q,n\right\rangle ,$$

which in turn means that in general $\delta_{em}m_n(g,e) \neq \delta_{em}m_p(g,e)$.

Let us now justify the claim that $\delta_s m_n(g) = \delta_s m_p(g)$ when $\lambda = 0$, $e \neq 0$. From (39b) we can write

$$2\pi\delta^{(4)}(q-q')m\frac{\partial}{\partial g}\delta_s m_p(g) = \int d^4x\left\langle q',p\left|S_s\left(\frac{\partial'}{\partial g}\mathcal{H}_{int}(x)\right)_{e=0}\right|q,p\right\rangle . \tag{A10}$$

Despite the fact that we are interested in the case of $\lambda = 0$ and $e \neq 0$, we see that the right-hand side of (A10) is independent of e. Therefore, when we write

$$\left\langle q',p\left|S_s\left(\frac{\partial'}{\partial g}\mathcal{H}_{int}(x)\right)_{e=0}\right|q,p\right\rangle = \left\langle q',p\left|S_s\left(\frac{\partial'}{\partial g}\mathcal{H}_{int}(x)\right)_{e=0}[Q_+,Q_-]\right|q,p\right\rangle ,$$

we can take $Q_\pm$ to be from the case of $\lambda = 0$ and $e = 0$, which is the case of exact SU(2) symmetry. Now, since $S_s(\partial'\mathcal{H}_{int}(x)/\partial g)_{e=0}$ is SU(2)-invariant and $Q_-|q,p\rangle = |q,n\rangle$, we have

$$\left\langle q',p\left|S_s\left(\frac{\partial'}{\partial g}\mathcal{H}_{int}(x)\right)_{e=0}\right|q,p\right\rangle = \left\langle q',n\left|S_s\left(\frac{\partial'}{\partial g}\mathcal{H}_{int}(x)\right)_{e=0}\right|q,n\right\rangle$$

which in turn means $\delta_s m_n(g) = \delta_s m_p(g)$.

This result can be further used for the case of $e = 0$, $\lambda \neq 0$. Namely, if we now denote $\delta_s m_n(g) - \delta_s m_p(g) = \Delta(g,\lambda)$, then $\lambda = 0$ is definitely included in the domain of convergence when $\Delta(g,\lambda)$ is expressed as a power series in λ. On the other hand, from the physical point of view, we expect that $\lambda = m_n - m_p$ lies between $\lambda = 0$ and the radius of convergence of the power series. If this is so, then we can always write

$$\Delta(g,\lambda) = k_1(g)\lambda + k_2(g)\lambda^2 + O(\lambda^3), \tag{A11}$$

a result used in Sec. VI.

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[1]J. Šoln, Nuovo Cimento 32, 1301 (1964); 37, 122 (1965).

[2]The simplest thing is to assume that g multiplies the strong-interaction Lagrangian (or Hamiltonian) and that it varies between zero and unity, unity being its physical value. This would ensure that the strong interactions are switched off when g tends to zero. On the other hand, one may assume that the coupling constants describing the interactions of hadrons with hadrons are functions of g, and as $g \to 0$, they tend to zero, and as g tends to its physical value (which one can choose to be $g_{N\pi}$), they tend to their physical values. As a matter of fact, in view of the SU(3) symmetry, universal coupling of ρ mesons to other hadrons, the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin relation, etc., there is a strong indication that the strong interactions are characterized by only one independent coupling constant. [See, e.g., M. Gell-Mann, Phys. Rev. 125, 1067 (1962); Y. Ne'eman, Nucl. Phys. 26, 222 (1961); S. Okubo, Progr. Theoret. Phys. (Kyoto) 27, 949 (1962); M. Gell-Mann and F. Zachariasen, Phys. Rev. 124, 953 (1961); J. J. Sakurai, Phys. Rev. Letters 17, 1021 (1966); K. Kawarabayashi and M. Suzuki, *ibid.* 16, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966).] The chiral Lagrangians and field algebras suggest further the nonlinear dependence on the coupling constant. [See, e.g., S. Weinberg, Phys. Rev. Letters 18, 188 (1967); J. Schwinger, Phys. Letters 24B, 473 (1967); J. Wess and B. Zumino, Phys. Rev. 163, 1727 (1967); L. S. Brown, *ibid.* 163, 1802 (1967); J. Šoln, Phys. Rev. D 2,

2404 (1970).]

[3]To show the convenience of relations (2a) and (2b), let us take a scalar "in" field operator $\sigma_{\text{in}}(x)$ associated with a particle of a physical mass μ and assume that both are functions of a coupling constant g. Then from the differential equation $(\Box-\mu^2)\sigma_{\text{in}}(x)=0$, we get

$$(\Box-\mu^2)\left(\frac{\partial}{\partial g}\sigma_{\text{in}}(x)\right)=\left(\frac{\partial}{\partial g}\mu^2\right)\sigma_{\text{in}}(x).$$

From here we see that unless $\partial\mu^2/\partial g=0$, the matrix element $\langle 0|\partial\sigma_{\text{in}}(x)/\partial g|k\rangle=\infty$ for $k^2=-\mu^2$. Now we know that the S matrix can be expanded in terms of free-field "in" operators. While some matrix elements of $S|_g$ would not be singular when evaluated between some states, suddenly we would find that for the same states the matrix elements of $S|_{g+\delta g}$ are singular. Of course, $\partial\sigma_{\text{in}}(x)/\partial g$ still may contain an arbitrary term $[\partial\sigma_{\text{in}}(x)/\partial g]_0$ satisfying

$$(\Box-\mu^2)\left(\frac{\partial}{\partial g}\sigma_{\text{in}}(x)\right)_0=0.$$

Because of its arbitrariness, we have chosen this term to be zero.

There is no doubt that with observable masses being dependent on coupling constants, the formulation of a perturbation theory would be quite difficult.

Let us finally point out that the above example is actually a different demonstration of Haag's theorem, which states that the field operator $\sigma_{\text{in}}(x)$ of mass μ and the field operator $\sigma_{\text{in}}(x)+\delta\sigma_{\text{in}}(x)$ of mass $(\mu^2+\delta\mu^2)^{1/2}$ cannot be connected with unitary transformation [R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 29, No. 12, 3 (1955)].

[4]To clarify this point further, let us take the case where $\mathcal{H}_{\text{int}}(x)=g\phi^3(x)$. Then we have $\partial'\mathcal{H}_{\text{int}}(x)/\partial g=\phi^3(x)$, $\partial'\mathcal{H}_{\text{int}}(x)/\partial e=0$.

[5]Since, in general, $\mathcal{H}_{\text{int}}$ can also depend on space derivatives of ϕ's and π's, relation (4) will hold for them too.

[6]It may look strange that $\phi_{\text{out}}(x)$ and $\pi_{\text{out}}(x)$ should depend on g and e while $\phi_{\text{in}}(x)$ and $\pi_{\text{in}}(x)$ do not. However, the explanation is very simple. Namely, the free physical system long after the collision, besides carrying the observed masses, charges, etc., will also have to carry the information about the collision. Thus $\phi_{\text{out}}(x)$ and $\pi_{\text{out}}(x)$ will have to depend on coupling constants g and e. This dependence, of course, comes through the S matrix, since $\phi_{\text{out}}=S^\dagger\phi_{\text{in}}S$, $\pi_{\text{out}}=S^\dagger\pi_{\text{in}}S$.

[7]We can verify relation (8) on a simple model of spinor field $\psi(x)$ interacting with a scalar field $\sigma(x)$ with $\mathcal{L}_{\text{int}}=g\bar\psi\gamma_\mu\psi\partial^\mu\sigma$. From $\mathcal{L}_{\text{int}}$ we get

$$\mathcal{H}_{\text{int}}=-g\bar\psi\vec\gamma\psi\nabla\sigma-g\bar\psi\gamma^4\psi\pi_\sigma+\tfrac{1}{2}g^2(\bar\psi\gamma^4\psi)^2,$$

from which

$$\frac{\partial'}{\partial g}\mathcal{H}_{\text{int}}=-\bar\psi\vec\gamma\psi\nabla\sigma-\bar\psi\gamma^4\psi\pi_\sigma+g(\bar\psi\gamma^4\psi)^2.$$

Choosing $F=\sigma$ $(\partial'\sigma/\partial g=0)$, relation (8) gives us $(1/i)(\partial'\sigma/\partial g)=i\bar\psi\gamma^4\psi$. This we rewrite as

$$\frac{1}{i}\frac{\partial}{\partial g}(\dot\sigma+g\bar\psi\gamma^4\psi)=0.$$

But $\dot\sigma+g\bar\psi\gamma^4\psi$ we recongnize to be π_σ, a canonically conjugate operator to field σ. This example shows us that relation (8) could be useful in finding a canonically conjugate operator π_F is some operator F is chosen to be a Heisenberg field operator.

[8]J. Šoln, Nuovo Cimento 18, 914 (1960).

[9]It is not difficult to check (17) in quantum electrodynamics. There $S_s=1$ and $\phi^0(x)$ and $\pi^0(x)$ are "in" operators. Thus

$$\sum_{n=1}^{\infty}\frac{e^n}{n!}\left(\frac{\partial^n}{\partial e^n}\mathcal{H}_{\text{int}}(x)\right)_{e=0}=\sum_{n=1}^{\infty}\frac{e^n}{n!}\left(\frac{\partial^n}{\partial e^n}\mathcal{H}_{\text{int}}^{\text{in}}(x)\right)_{e=0}=\mathcal{H}_{\text{int}}^{\text{in}}(x)$$

with the condition $(\mathcal{H}_{\text{int}}^{\text{in}}(x))_{e=0}=0$.

[10]One shall feel the "difference" between $S=S_s S_{\text{em}}$ and $S=S'_{\text{em}}S'_s$ in practice, however. To see this let us discuss the transition amplitude $\langle\text{out}, r'|\text{in}, r\rangle$, where common indices r and r' specify states according to a complete set of commuting observables.

For choice $S=S_s S_{\text{em}}$, we have

$$\langle\text{in}, r'|S_s S_{\text{em}}|\text{in}, r\rangle=\langle\text{out(str)}, r'|S_{\text{em}}|\text{in}, r\rangle,$$

where $\langle\text{out(str)}, r'|$ is an out state in the presence of strong interactions only. We can expand $\langle\text{out(str)}, r'|\times S_{\text{em}}|\text{in}, r\rangle$ in a power series in e quite easily, since the whole matrix element depends on e only through S_{em}.

For choice $S=S'_{\text{em}}S'_s$, we have

$$\langle\text{in}, r'|S'_{\text{em}}S'_s|\text{in}, r'\rangle=\langle\text{out(em)}, r'|S'_s|\text{in}, r\rangle,$$

where now $\langle\text{out(em)}, r'|$ is an out state in the presence of electromagnetic interactions only. While we can, at least in principle, expand $\langle\text{out(em)}, r'|S'_s|\text{in}, r\rangle$ in a power series in g straightforwardly since the dependence on g comes only through S'_s, its expansion in a power series in e is slightly more involved since both $\langle\text{out(em)}, r'|$ and S'_s depend on e and must be expanded. Of course, regardless of whether one uses $S=S_s S_{\text{em}}$ or $S=S'_{\text{em}}S'_s$, one gets the same result. As a matter of fact, to the order $O(e)$ we get exactly the same expression as in the literature [see, e.g., J. J. Sakurai, *Currents and Mesons* (University of Chicago Press, Chicago, 1969), p. 38; K. Nishijima, *Fundamental Particles* (Benjamin, New York, 1963), pp. 192 and 193].

[11]The term "bare masses" should probably be renamed "interacting masses" for the following two reasons: It is the interaction that changes the observable mass into the coupling constant dependent mass. Secondly, while the experimentally observed masses are associated with "in" and "out" fields, the bare masses are associated with the interacting (Heisenberg) fields.

[12]A very nice discussion of the asymptotic conditions at $t\to\mp\infty$ can be found in an article by G. Källén, in *Fundamental Problems in Elementary Particle Physics – Proceedings of the Fourteenth Conference on Physics at the University of Brussels, October 1967* (Interscience, New York, 1968).

[13]The solution $A^\mu(x)|_{e=0}=A^\mu_{\text{in}}(x)$ is also clear in view of the fact that photons have only electromagnetic interactions with other particles.

[14]This is equivalent to saying that the part of the $\mathcal{L}_{\text{int}}$ responsible for the electromagnetic interactions does not contain derivative couplings. See, for example, K. Nishijima, *Fields and Particles* (Benjamin, New York, 1969), Chap. 5.

[15]The assumption is that the theory is renormalizable in

the presence of strong and electromagnetic interactions. We assume that the theory can be described by a Lagrangian formalism. Then as in the case of quantum electrodynamics, we assume that the renormalization requires a finite number of counterterms in the Lagrangian. From the Lagrangian we can obtain the Hamiltonian density, and if we are satisfied with the perturbation theory in both coupling constants g and e, then from (3′) we can get that

$$S=T\exp\left\{-i\int d^4x\sum_{n=1}^{\infty}\frac{1}{n!}\left[\left(g\frac{\partial'}{\partial g_0}+e\frac{\partial'}{\partial e_0}\right)^n\mathcal{H}(x)\right]_{g_0=0,\,e_0=0}\right\}.$$

This expression for S, of course, is equal to expression (1), since

$$\sum_{n=1}^{\infty}\frac{1}{n!}\left[\left(g\frac{\partial'}{\partial g_0}+e\frac{\partial'}{\partial e_0}\right)^n\mathcal{H}(x)\right]_{g_0=0,\,e_0=0}=\sum_{n=1}^{\infty}\frac{1}{n!}\left[\left(g\frac{\partial}{\partial g_0}+e\frac{\partial}{\partial e_0}\right)^n\mathcal{H}^{\text{in}}(x)\right]_{g_0=0,\,e_0=0}=\mathcal{H}^{\text{in}}_{\text{int}}(x).$$

$\mathcal{H}^{\text{in}}(x)$ here is obtained from $\mathcal{H}(x)$ by replacing Heisenberg operators with corresponding "in" operators. Under summation signs, $\mathcal{H}$ and $\mathcal{H}^{\text{in}}$ depend formally on g_0 and e_0. Now it is clear that terms like

$$\left(\frac{\partial'^2}{\partial g^2}\mathcal{H}(x)\right)_{g=0,\,e=0},\quad\left(\frac{\partial'^2}{\partial g\partial e}\mathcal{H}(x)\right)_{g=0,\,e=0},\quad\text{etc.},$$

are computable if we know the original $\mathcal{H}_{\text{int}}$ and all counterterms [including those with $\Delta m_i(g,e)$]. On the other hand, it is also clear that in the perturbation theory the form of terms like

$$\left(\frac{\partial'^2}{\partial g^2}\mathcal{H}(x)\right)_{g=0,\,e=0},\quad\left(\frac{\partial'^2}{\partial g\partial e}\mathcal{H}(x)\right)_{g=0,\,e=0},\quad\text{etc.},$$

is influenced by renormalization requirements like, for example, the stability of one-particle states and making matrix elements of interpolating (Heisenberg) field operators finite which for $t\to\mp\infty$ should approach the matrix elements of corresponding "in" and "out" operators, respectively.

However, if the expansion with respect to g is not possible, which is true in practice, then it is more practical to use the expressions for the S matrix in form (9). Now it is customary to assume that strong interactions are already renormalized; i.e., we assume that conditions which go with renormalization are satisfied. For example, one assumes that interpolating (Heisenberg) field operators, due to pure strong interactions, have finite matrix elements and that, for $t\to\mp\infty$, they approach to corresponding "in" and "out" operators, respectively. However, in order that these conditions still hold when the electromagnetism is switched on, terms like

$$\left(\frac{\partial'^2}{\partial e^2}\mathcal{H}(x)\right)_{e=0},\quad\left(\frac{\partial'^3}{\partial e^3}\mathcal{H}(x)\right)_{e=0},\quad\text{etc.},$$

should be properly chosen.

[16]To see this, let us compute

$$-i\int d^4x\,\langle p'|\,S_s\,S_{\text{em}}\,\pi_p(x)\gamma^4\psi_p(x)\,|p\rangle\,.$$

Since we are interested in $\partial m_0^2(g,e)/\partial e$ to the first order in e, we expand S_{em} and $\pi_p(x)\gamma^4\psi_p(x)$ to the first order in e according to (17′) and (6′). Since the term proportional to e contains A_μ^{in}, its contribution is zero. Thus, we have

$$-i\int d^4x\,\langle p'|\,S_s\,S_{\text{em}}\,\pi_p(x)\,\gamma^4\psi_p(x)|p\rangle=-i\int d^4x\,\langle p'|\,S_s(\pi_p(x)\gamma^4\psi_p(x))_{e=0}|p\rangle\,.$$

Now according to (37b), the strong interactions are such that $\langle p'|S_s=\langle p'|$. Then, with the assumption that there exists an energy-momentum operator P^μ which generates translations, we have

$$-i\int d^4x\,\langle p'|\,S_s\,S_{\text{em}}\,\pi_p(x)\gamma^4\psi_p(x)\,|p\rangle=-i\,(2\pi)^4\delta^{(4)}(p-p')\,\langle p'|\,[\pi_p(0)\gamma^4\psi_p(0)]_{e=0}\,|p\rangle\,.$$

The matrix element

$$\langle p'|[\pi_p(0)\gamma^4\psi_p(0)]_{e=0}|p\rangle$$

is to be computed at $x=0$. However, we can compute it at any x because of $\delta^{(4)}$ function, $p'=p$. Now the assumption that strong interactions are already renormalized means that the matrix elements of Heisenberg operators are finite and in the $x^4\to-\infty$ limit they approach the matrix elements of corresponding "in" operators. In our case, because of arbitrariness of x^4, the matrix elements are equal. Therefore,

$$-i\int d^4y\,\langle p'|\,S_s\,S_{\text{em}}\,\pi_p(y)\gamma^4\psi_p(y)|p\rangle=-i\,(2\pi)^4\delta^{(4)}(p-p')\,\langle p'|\,[\pi_p(x)\gamma^4\psi_p(x)]_{e=0}\,|p\rangle=-i\,(2\pi)^4\delta^{(4)}(p-p')\,\langle p'|\,:\pi_{p_{\text{in}}}(x)\gamma^4\psi_{p_{\text{in}}}(x):\,|p\rangle\,.$$

With $\pi_{p_{\text{in}}}(x)=i\psi^\dagger_{p_{\text{in}}}(x)$, we get result (39).

[17]To clarify this let us assume for the moment that states $|p\rangle$ and $|p'\rangle$ are off the mass shell, and expand the right-hand side of (39a) in terms of $(\not p+m_p)$:

$$\int d^4x\left\langle p'\left|S_{\text{em}}\frac{\partial'}{\partial e}\mathcal{H}_{\text{int}}(x)\right|p\right\rangle=2\pi\delta^{(4)}(p-p')\,\bar u(p')\times\left(\frac{\partial}{\partial e}a(g,e)+(\not p+m_p)\frac{\partial}{\partial e}b(g,e)+(\not p+m_p)^2\frac{\partial}{\partial e}c(g,e;\not p+m_p)\right)u(p).$$

The assumption that the mass renormalization can be achieved without

$$\left(\frac{\partial'^2}{\partial e^2}\mathcal{H}_{\text{int}}(x)\right)_{e=0}$$

means that this term can be chosen in such a way that it gives no contribution to $\partial a(g,e)/\partial e$. $\partial b(g,e)/\partial e$ will get contributions from

$$-ie\int d^4y\,T\left[\left(\frac{\partial'}{\partial e}\mathcal{H}_{\text{int}}(x)\right)_{e=0}\left(\frac{\partial'}{\partial e}\mathcal{H}_{\text{int}}(y)\right)_{e=0}\right]$$

and from

$$e\left(\frac{\partial'^2}{\partial e^2}\mathcal{H}_{\text{int}}(x)\right)_{e=0},$$

but since we are interested in the limit $(\not p+m_p)u(p)\to0$, it is of no concern to us. As a matter of fact, in quantum electrodynamics (formally obtained by putting $g=0$), the electron field-operator renormalization will give $b=0$.

[18]Expression (45) is known in the literature as the Cottingham formula for the hadronic electromagnetic self-mass [W. N. Cottingham, Ann. Phys. (N.Y.) 25, 424 (1963)]. See also R. P. Feynman and G. Speisman,

Phys. Rev. 94, 500 (1954); A. Petermann, Helv. Phys. Acta 27, 441 (1964); G. C. Wick, in *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics, 1957* (Interscience, New York, 1957); M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters 2, 7 (1959).

[19]One could start with the equivalent expression for the S-matrix $S = S'_{\rm em} S'_s$ [see relation (18)]. Since now the roles of strong and electromagnetic interactions formally are interchanged, instead of $\delta_s m(g)$ and $\delta_{\rm em} m(g,e)$, we shall have $\delta'_{\rm em}\, m(e)$ (the self-mass due to electromagnetic interactions only) and $\delta'_s\, m(e,g)$ (the self-mass due to strong interactions "renormalized" by electromagnetic interactions). Clearly now the "initial" conditions are $\delta'_{\rm em} m(0) = 0$ and $\delta'_s\, m(e,0) = 0$. However, since the total self-mass $\Delta m(g,e)$ must be the same as before, we shall have the equality

$$\begin{aligned}\Delta m(g,e) &= \delta'_{\rm em} m(e) + \delta'_s\, m(e,g)\\ &= \delta_s\, m(g) + \delta_{\rm em}\, m(g,e).\end{aligned}$$

Since $\Delta m(g,0) = \delta_s\, m(g)$, we have that $\delta_s\, m(g) = \delta'_s\, m(0,g)$. Substituting this above, one also gets

$$\delta_{\rm em} m(g,e) = \delta'_{\rm em} m(e) + \delta'_s\, m(e,g) - \delta'_s\, m(0,g).$$

In other words, we can again proceed with the discussion in terms of $\delta_s\, m(g)$ and $\delta_{\rm em}\, m(g,e)$. Let us point out that it would be quite difficult to invoke SU(2) into the discussion without writing Δm in terms of $\delta_s m$ and $\delta_{\rm em} m$.

[20]Most calculations give the negative values for $\delta_{\rm em} m_n(g,e) - \delta_{\rm em} m_p(g,e)$. The value we quote was calculated by M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters 2, 7 (1959).

[21]For the case of broken chiral symmetry this has been demonstrated in a simple field-theoretic model [J. Šoln, Phys. Rev. D 1, 2882 (1970)].

[22]S. Coleman, J. Math. Phys. 7, 787 (1966).