PHYSICAL REVIEW D VOLUME 6, NUMBER 8 15 OCTOBER 1972

Tests of Light-Cone Commutators. II

Duane A. Dicus*
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

and

Vigdor L. Teplitz†
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 01239

(Received 31 March 1972)

Light-cone commutators are used to derive fixed-mass sum rules for structure functions evaluated away from the forward direction. Both spin-averaged and spin-dependent sum rules are considered. As in the forward direction these sum rules relate the structure functions to form factors of bilocal operators. The bilocal operators can, in turn, be related to the deep-inelastic limits of the structure functions by Fourier transformations. We derive these relations for the spin-averaged case. In addition we use the Mandelstam representation to show the consistency of the scaling hypothesis with crossed-channel unitarity.

I. INTRODUCTION

Starting from assumed forms for light-cone commutators of currents,[1-3] a variety of conclusions can be drawn regarding the properties of electron and neutrino scattering structure $W(q^2,\nu)$ and their scaling limits $F(\omega)$.[4] Since these functions are forward absorptive parts of the four-point current-particle amplitude it is natural to inquire into the properties of the off-forward absorptive parts $W(q_i^2,\nu,t)$ and $F(\omega,t)$.[5] Although these latter are not soon likely to be measured experimentally, any requirements on them are important constraints for any model dealing with the forward amplitudes.

In the present work we apply the methods of Dicus, Jackiw, and Teplitz (DJT)[4] to the nonforward vector amplitude and deduce several sum rules. That work, and this, is performed under the assumptions of the canonical commutators of a vector-gluon fermion-quark model.[1,3]

We derive fixed-q^2 sum rules for the W's [see Eq. (2.18)] and formulas giving the F's as Fourier transforms of the form factors of bilocal operators (2.27). In Sec. III we find fixed-q^2 sum rules for the nonforward spin-flip absorptive parts (3.5). Our conclusions are 16 fixed-q^2 sum rules of which six are extensions of $t=0$ sum rules; of the new sum rules, three are expected to converge in a Regge model.

Finally in Sec. IV we make a natural first step toward a model-independent dynamics by discussing the analyticity of the off-forward scalar-current scalar-target amplitude in the framework of the Mandelstam representation.[6] We show the consistency of the scaling hypothesis with the two-body t-channel unitarity approximation and the absence of anomalous thresholds in $F(\omega,t)$. Using this we evaluate the box-diagram contribution to $F(\omega,t)$. The answer is given in Eq. (4.15).

In Appendix A we relate the structure functions W used in the sum rules to the t-channel helicity amplitudes and also find the $q^2\to 0$ conditions on the structure functions. In Appendix B we provide the free-quark-model Born approximations to the amplitudes under discussion.

II. SPIN-AVERAGED SUM RULES AND TRANSFORMS

A. Fixed-Mass Sum Rules

We consider the Fourier-transformed commutator function

$$C_{ab}^{\nu\mu}(p_1,q_1;p_2,q_2)=\int d^4x\,e^{iQ\cdot x/2}\langle p_2|[V_a^\nu(\tfrac{1}{2}x),V_b^\mu(-\tfrac{1}{2}x)]|p_1\rangle\,. \tag{2.1}$$

$C_{ab}^{\nu\mu}$ satisfies the relation

$$C_{ab}^{\nu\mu}(p_1,q_1;p_2,q_2)=-C_{ba}^{\mu\nu}(p_1,-q_2;p_2,-q_1)\,. \tag{2.2}$$

Following Gross,[5] we expand $C_{ab}^{\nu\mu}$ in terms of conserved tensor covariants

$$C_{ab}^{\nu\mu}=\sum_{i=1}^{5}W_i^{ab}\left(g^{\nu\nu'}-\frac{q_2^\nu q_2^{\nu'}}{q_2^2}\right)A^i_{\nu'\mu'}\left(g^{\mu'\mu}-\frac{q_1^{\mu'}q_1^\mu}{q_1^2}\right), \tag{2.3}$$

where the $A^i_{\nu\mu}$ are taken to be

$$A^1_{\nu\mu}=-g_{\nu\mu}\,, \tag{2.4a}$$

$$A^2_{\nu\mu}=P_\nu P_\mu\,, \tag{2.4b}$$

$$A^3_{\nu\mu}=P_\nu\Delta_\mu-P_\mu\Delta_\nu\,, \tag{2.4c}$$

$$A^4_{\nu\mu}=P_\nu\Delta_\mu+P_\mu\Delta_\nu\,, \tag{2.4d}$$

$$A^5_{\nu\mu}=\Delta_\nu\Delta_\mu\,, \tag{2.4e}$$

where

$$P=\tfrac{1}{2}(p_1+p_2)\,, \tag{2.5a}$$

$$Q=\tfrac{1}{2}(q_1+q_2)\,, \tag{2.5b}$$

$$\Delta=q_2-q_1=p_1-p_2\,. \tag{2.5c}$$

The structure functions W_i depend on the Lorentz scalars ν, t, Q^2, and δ where $\nu=P\cdot Q=P\cdot q_1=P\cdot q_2$, $t=\Delta^2$, and $\delta=2Q\cdot\Delta$.

From (2.2) we see that

$$W_i^{(ab)}(\nu,\delta)=-W_i^{(ab)}(-\nu,-\delta),\quad i=1,2,4,5 \tag{2.6a}$$

$$W_3^{(ab)}(\nu,\delta)=W_3^{(ab)}(-\nu,-\delta)\,, \tag{2.6b}$$

while the opposite symmetries hold for $W_i^{(ab)}$. Note that W_3 and W_4 are *not* related to the spin-flip amplitudes called W_3 and W_4 in DJT.

To obtain fixed-mass sum rules from (2.1) we follow the method of DJT and integrate (2.3) over Q^-. On the right-hand side we interchange the Q^- and x integrations. The Q^- integral then gives a factor of $\delta(x^+)$. This operation is only valid in the absence of Class-II singularities as discussed in Appendix D of DJT; it is not, however, invalidated by Z graphs.

This procedure yields sum rules of the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}dQ^-W(\nu,Q^2,t,\delta)=\int dx^-d^2x_\perp e^{iQ^+x^-/2}e^{-iQ_\perp\cdot x_\perp/2}\langle p_2|[V_1(\tfrac{1}{2}x),V_2(-\tfrac{1}{2}x)]|p_1\rangle|_{x^+=0}\,. \tag{2.7}$$

The right-hand side is a light-cone commutator which we may then evaluate from the results of Ref. 1. On the left-hand side the initial and final virtual photon masses squared are given by

$$\begin{Bmatrix}q_1^2\\ q_2^2\end{Bmatrix}=Q^2+\tfrac{1}{4}t\mp\tfrac{1}{2}\delta\,. \tag{2.8}$$

To ensure q_1^2 and q_2^2 being constant as Q^- varies we may take $Q^+=\Delta^+=q_1^+=q_2^+=0$.

Returning to (2.1) we choose $\nu=+$, $\mu=\pm,i$ $(i=1,2)$ in order to limit our derivation of sum rules to those commutators found by Cornwall and Jackiw to be interaction-independent.[1] Integrating over Q^- gives

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}dQ^-\left\{g^{+\mu}W_1^{ab}+P^+\left(P^\mu-\frac{\nu q_1^\mu}{q_1^2}\right)W_2^{ab}+P^+\left(\Delta^\mu-\frac{q\cdot\Delta q_1^\mu}{q_1^2}\right)(W_3^{ab}+W_4^{ab})\right\}\\ =\int dx^-d^2x_\perp e^{-iQ_\perp\cdot x_\perp/2}\langle p_2|[V_a^\mu(\tfrac{1}{2}x),V_b^+(-\tfrac{1}{2}x)]|p_1\rangle|_{x^+=0}\\ =\int dx^-d^2x_\perp e^{-iq_{2\perp}\cdot x_\perp}\langle p_2|[V_a^\mu(x),V_b^+(0)]|p_1\rangle|_{x^+=0}\,.\end{aligned} \tag{2.9}$$

The current commutators which emerge in the interacting fermion theory of Ref. 1 are the following:

$$[V_a^+(x),V_b^+(y)]_{x^+=y^+}=if_{abc}V_c^+(x)\delta(x^--y^-)\delta^2(x_\perp-y_\perp)-\tfrac{1}{4}i\partial_-^x\partial_-^y[S_{ab}(x|y)\epsilon(x^--y^-)\delta^2(x_\perp-y_\perp)]\,, \tag{2.10}$$

$$\begin{aligned}[V_a^+(x),V_b^-(y)]_{x^+=y^+}=&if_{abc}V_c^-(x)\delta(x^--y^-)\delta^2(x_\perp-y_\perp)\\ &-\tfrac{1}{2}if_{abc}\{\partial_-^x[\epsilon(x^--y^-)\delta^2(x_\perp-y_\perp)\mathcal{V}_c^-(x|y)]+\tfrac{1}{2}\partial_i^x[\epsilon(x^--y^-)\delta^2(x_\perp-y_\perp)\mathcal{V}_c^i(x|y)]\\ &\qquad-\tfrac{1}{2}\partial_i^x\epsilon^{ij}[\epsilon(x^--y^-)\delta^2(x_\perp-y_\perp)\bar{\mathcal{A}}_{jc}(x|y)]\}\\ &-\tfrac{1}{2}id_{abc}\{\partial_-^x[\epsilon(x^--y^-)\delta^2(x_\perp-y_\perp)\bar{\mathcal{V}}_c^-(x|y)]+\tfrac{1}{2}\partial_i^x[\epsilon(x^--y^-)\delta^2(x_\perp-y_\perp)\bar{\mathcal{V}}_c^i(x|y)]\\ &\qquad+\tfrac{1}{2}\partial_i^x\epsilon^{ij}[\epsilon(x^--y^-)\delta^2(x_\perp-y_\perp)\mathcal{A}_{jc}(x|y)]\}\\ &+i\epsilon(x^--y^-)\delta^2(x_\perp-y_\perp)M_{ab}(x|y)-\tfrac{1}{8}iS_{ab}(x|y)\epsilon(x^--y^-)\partial_i\partial^i\delta^2(x_\perp-y_\perp)\,.\end{aligned} \tag{2.11}$$

These results were obtained in a quark model with a vector-gluon interaction, and a symmetry-breaking mass term. In (2.11) all terms, except $S_{ab}(x|y)$ emerge with canonical manipulations; however, $S_{ab}(x|y)$ can be shown to have nonzero vacuum expectation value. The inability to compute it canonically is a reflection of the fact that the ordinary Schwinger term is not evaluated canonically, and is related to the nonoccurrence of dimension-2 fermion operators in the theory. We shall assume that S_{ab} is a c number; $S_{ab}(x|y)=\delta_{ab}S$. This assumption, which is equivalent to setting $F_L(\omega)=0$, is not true in perturbation theory.[7]

The term involving $M_{ab}(x|y)$ is present in (2.11) only when the currents are not conserved, and need not concern us any further. The remaining bilocal operators are defined as follows:

$$V_a^\mu(x|y)=\tfrac{1}{2}\bar{\psi}(x)\gamma^\mu\lambda_a\psi(y)\,, \tag{2.12a}$$

$$A_a^\mu(x|y)=\tfrac{1}{2}i\,\bar{\psi}(x)\gamma^\mu\gamma^5\lambda_a\psi(y)\,. \tag{2.12b}$$

These are bilocal non-Hermitian generalizations of the vector and axial-vector currents. We now extract the Hermitian and anti-Hermitian parts, as these are the objects which occur in (2.11):

$$\mathcal{V}_a^\mu(x|y)\equiv\tfrac{1}{2}V_a^\mu(x|y)+\tfrac{1}{2}V_a^\mu(y|x)\,, \tag{2.13a}$$

$$\bar{\mathcal{V}}_a^\mu(x|y)\equiv\frac{1}{2i}\left[V_a^\mu(x|y)-V_a^\mu(y|x)\right], \tag{2.13b}$$

$$\mathcal{A}_a^\mu(x|y)\equiv\tfrac{1}{2}A_a^\mu(x|y)+\tfrac{1}{2}A_a^\mu(y|x)\,, \tag{2.13c}$$

$$\bar{\mathcal{A}}_a^\mu(x|y)\equiv\frac{1}{2i}\left[A_a^\mu(x|y)-A_a^\mu(y|x)\right]. \tag{2.13d}$$

Finally, we expand the bilocal operators $\mathcal{V}$, $\bar{\mathcal{V}}$, $\mathcal{A}$, and $\bar{\mathcal{A}}$, of (2.11) in terms of real form factors. We keep, for use in Sec. III, terms involving the nucleon spin:

$$\langle p_2|V_a^\mu(0)|p_1\rangle=P^\mu f_a(t)+\frac{i}{P^2}\epsilon^\mu(P\Delta S)f_s^a(t)\,, \tag{2.14}$$

$$\begin{aligned}\langle p_2|\mathcal{V}_c^\mu(x|0)|p_1\rangle=\;&P^\mu V_1^c(x^2,x\cdot P,x\cdot\Delta,t)+x^\mu V_2^c(x^2,x\cdot P,x\cdot\Delta,t)\\&+i\Delta^\mu V_3^c(x^2,x\cdot P,x\cdot\Delta,t)+i\epsilon^{\mu\alpha\beta\rho}P_\alpha\Delta_\beta s_\rho V_4^c(x^2,x\cdot P,x\cdot\Delta,t)\\&+\epsilon^{\mu\alpha\beta\rho}P_\alpha x_\beta s_\rho V_5^c(x^2,x\cdot P,x\cdot\Delta,t)+i\epsilon^{\mu\alpha\beta\rho}\Delta_\alpha x_\beta s_\rho V_6^c(x^2,x\cdot P,x\cdot\Delta,t)\,,\end{aligned} \tag{2.15}$$

where $s^\mu=\bar{u}(p_2)\gamma^\mu\gamma_5 u(p_1)$. Time-reversal invariance requires that, when $x\cdot\Delta=0$, V_3^c and V_5^c must be zero. As long as we are dealing with $++$, $+-$, or $+i$ commutators in (2.1) we evaluate the bilocal form factors at $x\cdot\Delta=0$.

In the same way A_c^4, A_c^5, and A_c^6 are zero at $x\cdot\Delta=0$ in the following:

$$\langle p_2|A_a^\mu(0)|p_1\rangle=s^\mu g_s^a(t)\,, \tag{2.16}$$

$$\begin{aligned}\langle p_2|\mathcal{A}_c^\mu(x|0)|p_1\rangle=\;&s^\mu A_1^c(x^2,x\cdot P,x\cdot\Delta,t)+P^\mu x\cdot sA_2^c(x^2,x\cdot P,x\cdot\Delta,t)+x^\mu x\cdot sA_3^c(x^2,x\cdot P,x\cdot\Delta,t)\\&+iP^\mu\Delta\cdot sA_4^c(x^2,x\cdot P,x\cdot\Delta,t)+ix^\mu\Delta\cdot sA_5^c(x^2,x\cdot P,x\cdot\Delta,t)+i\Delta^\mu x\cdot sA_6^c(x^2,x\cdot P,x\cdot\Delta,t)\\&+\Delta^\mu\Delta\cdot sA_7^c(x^2,x\cdot P,x\cdot\Delta,t)+i\epsilon^{\mu\alpha\beta\rho}P_\alpha\Delta_\beta x_\rho A_8^c(x^2,x\cdot P,x\cdot\Delta,t)\,.\end{aligned} \tag{2.17}$$

Similar decompositions hold for $\bar{\mathcal{V}}$ and $\bar{\mathcal{A}}$. Summing matrix elements over initial and final proton spins as required in this section puts s^μ equal to zero in (2.14) and (2.17).

With the above formulas, fixed-mass sum rules are readily derived. Setting μ equal to $+$ in (2.9) and using (2.10) gives the $t\neq 0$ generalization of the Dashen–Fubini–Gell-Mann (DFGM) sum rule. Setting μ equal to $-$ in (2.9), using (2.11) and (2.14) through (2.17) to evaluate the right-hand side, and equating coefficients of $1/P^+$, $P\cdot Q/P^+$, $P\cdot\Delta/P^+$, and P^- gives three new sum rules in addition to the DFGM rule (which comes from the coefficient of P^-). Because δ is nonzero, interchanging μ and ν and considering

$$\int dQ^- C_{ab}^{\nu=+,\,\mu=-}$$

yields two new independent sum rules. The resulting six sum rules may be written as follows [with $\alpha=P^+x^-$ and $W_L=W_1+(\nu^2/q_1\cdot q_2)W_2$]:

$$\int_0^\infty d\nu\, W_2^{[ab]}(\nu,Q^2,t,\delta)=i\pi f_{abc}f^c(t)\,, \tag{2.18a}$$

$$\int_0^\infty d\nu\,\frac{\nu}{q_1\cdot q_2}W_2^{(ab)}=-\tfrac{1}{2}\pi d_{abc}\int_0^\infty d\alpha\,\bar{V}_1^c(x^2=0,\alpha,x\cdot\Delta=0,t), \tag{2.18b}$$

$$\int_0^\infty d\nu\, W_3^{(ab)}=\tfrac{1}{2}\pi d_{abc}\int_0^\infty d\alpha\,\bar{V}_1^c(0,\alpha,0,t)\,, \tag{2.18c}$$

$$\int_0^\infty d\nu\, W_4^{(ab)}=0\,, \tag{2.18d}$$

$$2\int_0^\infty d\nu\, W_L^{[ab]} - \frac{t(q_1^2+q_2^2)-\delta^2}{2q_1^2q_2^2 q_1\cdot q_2}\int_0^\infty d\nu\,\nu^2 W_2^{[ab]} + \delta\,\frac{q_1\cdot q_2}{q_1^2q_2^2}\int_0^\infty d\nu\,\nu W_4^{[ab]} + \frac{\delta^2 - t(q_1^2+q_2^2)}{2q_1^2q_2^2}\int_0^\infty d\nu\,\nu W_3^{[ab]} = \tfrac{1}{2}i\pi t f_{abc}\int_0^\infty d\alpha\,\alpha \bar{A}_8^c(0,\alpha,0,t)\,, \quad (2.18\text{e})$$

$$\frac{\delta}{q_1^2q_2^2}\int_0^\infty d\nu\,\nu^2 W_2^{[ab]} + \delta\,\frac{q_1\cdot q_2}{q_1^2q_2^2}\int_0^\infty d\nu\,\nu W_3^{[ab]} - \frac{t(q_1^2+q_2^2)-\delta^2}{2q_1^2q_2^2}\int_0^\infty d\nu\,\nu W_4^{[ab]} = \delta\tfrac{1}{2}\pi i f_{abc}\int_0^\infty d\alpha\,\alpha\bar{A}_8^c(0,\alpha,0,t)\,. \quad (2.18\text{f})$$

In each of the six sum rules (2.18) the structure functions W_i^{ab} stand for

$$\tfrac{1}{2}[W_i^{ab}(\nu, Q^2, t, \delta) + W_i^{ab}(\nu, Q^2, t, -\delta)], \quad i = 1, 2, 3, 5$$

and

$$\tfrac{1}{2}[W_4^{ab}(\nu, Q^2, t, \delta) - W_4^{ab}(\nu, Q^2, t, -\delta)]$$

as can be seen from the crossing relations (2.6). There exist an additional six sum rules which are identical in form to (2.18) except that the W_i^{ab} stand for

$$\tfrac{1}{2}[W_i^{ab}(\nu, Q^2, t, \delta) - W_i^{ab}(\nu, Q^2, t, -\delta)], \quad i = 1, 2, 3, 5$$

$$\tfrac{1}{2}[W_4^{ab}(\nu, Q^2, t, \delta) + W_4^{ab}(\nu, Q^2, t, -\delta)],$$

the right-hand sides are all zero, and the structure functions have the opposite symmetry under interchange of a and b.

Equation (2.18b) is the $t \neq 0$ generalization of the sum rule of Cornwall, Corrigan, and Norton.[8] Note that no sum rule involving W_5 can appear from the present method in a commutator involving V^+ since both Δ^+ and Q^+ are set to zero.

No new sum rules are added to the spinless case by considering the $(+, i)$ commutator. All the sum rules in (2.18) except for (2.18a) are apparently divergent for $t \approx 0$ in a Regge model in which

$$W_L \sim \nu^\alpha, \quad W_2 \sim \nu^{\alpha-2}, \quad W_3 \sim \nu^{\alpha-1}, \quad W_4 \sim \nu^{\alpha-1}. \quad (2.19)$$

The sum rules are, in principle, convergent for t sufficiently negative that the leading j-plane singularity α has retreated the necessary distance into the left-hand plane. As pointed out by de Alwis[9] the $t \approx 0$ case may then be evaluated by analytic continuation.[10]

B. Deep-Inelastic Transforms

As in DJT the bilocal operators are measured by the deep-inelastic limits of the W_i^{ab}. We consider the Fourier transform of the time-ordered product

$$T_{ab}^{\nu\mu}(p_1, q_1; p_2, q_2) = i\int d^4x\, e^{iQ\cdot x/2}\langle p_2|(V_a^\nu(\tfrac{1}{2}x)V_b^\mu(-\tfrac{1}{2}x))_+|p_1\rangle\,. \quad (2.20)$$

This can be expanded in terms of the $A_{\nu\mu}^i$,

$$T_{ab}^{\nu\mu} = if_{abc}f^c(t)\frac{1}{q_1\cdot q_2}[g^{\nu\mu}\nu - P^\nu q_2^\mu - P^\mu q_1^\nu] + \sum_{i=1}^{5} T_i^{ab}\left(g^{\nu\nu'} - \frac{q_2^\nu q_2^{\nu'}}{q_2^2}\right)A_{\nu'\mu'}^i\left(g^{\mu'\mu} - \frac{q_1^{\mu'}q_1^\mu}{q_1^2}\right), \quad (2.21)$$

where the $A_{\nu\mu}^i$ are given in (2.4) and the W_i^{ab} in (2.3) are the discontinuities of the T_i^{ab}.

We can now relate (2.21) to the light-cone commutator by using the Bjorken-Johnson-Low theorem on the light cone,[11]

$$T_{ab}^{\nu\mu} \underset{Q^-\to\infty}{\sim} \frac{-1}{Q^-}\int dx^- d^2x_\perp e^{iQ^+x^-/2}e^{-iQ_\perp\cdot x_\perp/2}\langle p_2|[V_a^\nu(\tfrac{1}{2}x), V_b^\mu(-\tfrac{1}{2}x)]|p_1\rangle|_{x^+=0} + \text{polynomials}\,. \quad (2.22)$$

The starting point for determining the large-Q^- behavior of the T_i^{ab} is their dispersion relations. We can assume for convenience unsubtracted dispersion relations since in the end we will keep only the imaginary parts:

$$T_i^{ab}(\nu, Q^2, t, \delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu'\,\frac{W_i^{ab}(\nu', Q^2, t, \delta)}{\nu' - \nu}\,. \quad (2.23)$$

We write the W_i^{ab} in terms of functions of the scaling variable $\omega = -Q^2/2\nu$ which go into the scaling functions when Q^2 and ν go to infinity (with ω fixed),

$$W_L^{ab}(\nu, Q^2, t, \delta) = -\frac{1}{2\omega}\tilde{F}_L^{ab}(\omega, Q^2, t, \delta) \underset{\nu\to\infty;Q^2\to\infty}{\sim} -\frac{1}{2\omega}\left[F_L^{ab}(\omega, t, \delta) + \frac{1}{Q^2}G_L^{ab}(\omega, t, \delta)\right], \tag{2.24a}$$

$$\nu W_i^{ab}(\nu, Q^2, t, \delta) = \tilde{F}_i^{ab}(\omega, Q^2, t, \delta) \to F_i^{ab}(\omega, t, \delta), \quad i = 2, 3, 4, 5. \tag{2.24b}$$

Then we can rewrite (2.23) as

$$T_L^{ab}(\nu, Q^2, t, \delta) = \frac{\omega}{4\pi}\int_{-1}^{1}\frac{d\omega'}{\omega'^2}\frac{\tilde{F}_L^{ab}(\omega', Q^2, t, \delta)}{\omega' - \omega}, \tag{2.25a}$$

$$T_i^{ab}(\nu, Q^2, t, \delta) = \frac{\omega}{\pi Q^2}\int_{-1}^{1}d\omega'\frac{\tilde{F}_i^{ab}(\omega', Q^2, t, \delta)}{\omega' - \omega}, \quad i = 2, 3, 4, 5. \tag{2.25b}$$

Letting Q^- get large makes ν and Q^2 get large with ω fixed; Eq. (2.25) becomes

$$T_L^{ab}(\nu, Q^2, t, \delta) \sim \frac{\omega}{4\pi}\int_{-1}^{1}\frac{d\omega'}{\omega'^2}\frac{F_L^{ab}(\omega', t, \delta)}{\omega' - \omega} + \frac{\omega}{8\pi Q^+Q^-}\int_{-1}^{1}\frac{d\omega'}{\omega'^2}\frac{G_L^{ab}(\omega', t, \delta)}{\omega' - \omega} \tag{2.26a}$$

and

$$T_i^{ab}(\nu, Q^2, t, \delta) \sim \frac{\omega}{2\pi Q^+Q^-}\int_{-1}^{1}d\omega'\frac{F_i^{ab}(\omega', t, \delta)}{\omega' - \omega}, \quad i = 2, 3, 4, 5. \tag{2.26b}$$

Now we simply use (2.26) with (2.21) in (2.22), setting $\mu\nu$ equal to $++$ and $+-$, and using (2.10) and (2.11) in the right-hand side of (2.22). The results are five deep-inelastic transforms:

$$F_L^{ab}(\omega, t, \delta) = 0, \tag{2.27a}$$

$$F_4^{ab}(\omega, t, \delta) = 0, \tag{2.27b}$$

$$G_L^{ab}(\omega, t, \delta) = -8i\omega^3\int_{-\infty}^{\infty}d\alpha\, e^{-i\omega\alpha}\alpha[f_{abc}V_2^c(0, \alpha, 0, t) + d_{abc}\overline{V}_2^c(0, \alpha, 0, t)] -2\omega^2 t\int_{-\infty}^{\infty}d\alpha\, e^{-i\omega\alpha}\alpha[f_{abc}\overline{A}_8^c(0, \alpha, 0, t) - d_{abc}A_8^c(0, \alpha, 0, t)], \tag{2.27c}$$

$$F_2^{ab}(\omega, t, \delta) = i\omega\int_{-\infty}^{\infty}d\alpha\, e^{-i\omega\alpha}[f_{abc}V_1^c(0, \alpha, 0, t) + d_{abc}\overline{V}_1^c(0, \alpha, 0, t)], \tag{2.27d}$$

$$F_3^{ab}(\omega, t, \delta) = \frac{1}{2\omega}F_2^{ab}(\omega, t, \delta) + \omega\int_{-\infty}^{\infty}d\alpha\, e^{-i\omega\alpha}\alpha[f_{abc}\overline{A}_8^c(0, \alpha, 0, t) - d_{abc}A_8^c(0, \alpha, 0, t)]. \tag{2.27e}$$

The bilocal form factors on the right-hand side of (2.27) are evaluated at $x^2 = 0$ and $x\cdot\Delta = 0$.

The fact that $F_L^{ab}(\omega, t, \delta)$ is zero for all t and δ follows from assuming there is no q-number Schwinger term in the commutator just as it did for $t = \delta = 0$. But now we see that

$$\int\frac{d\omega}{\omega^3}G_L^{ab}(\omega, t, \delta)$$

is not zero as in the $t = 0$ case DJT but is proportional to t. There is no transform formula for F_5; its coefficient goes as $(Q^-)^{-2}$; hence it only appears in higher-order commutators.

Equation (2.27) shows that all the scaling functions are in fact completely independent of δ. This is an explicit verification of what has been argued should be a general rule.[5,12]

Comparing (2.27d) with the fixed-mass sum rule (2.18a) we find

$$f^c(t) = V_1^c(0, 0, 0, t) \tag{2.28}$$

which states that the bilocal operator goes into the local operator (the current) as $x\to 0$.[4]

Finally, if $A_8^c(0, \alpha, 0, t)$ has no pole in α then

$$\int_0^1\frac{d\omega}{\omega}F_3^{(ab)}(\omega, t, \delta) = \int_0^1\frac{d\omega}{2\omega^2}F_2^{(ab)}(\omega, t, \delta). \tag{2.29}$$

But this also follows from the fixed-mass sum rules (2.18b) and (2.18c); we conclude that A_8^c has no pole in α.

III. FIXED-MASS SUM RULES FOR NONZERO SPIN

If we do not sum over the nucleon spin in (2.3) there are many additional tensors to be added to the list (2.4); Gerstein[13] has solved for the 13 extra independent ones. We write them in the form

$$\overline{C}^{\nu\mu}_{ab}=\sum_{i=1}^{13} R_i^{ab}\left(g^{\nu\nu'}-\frac{q_2^{\nu}q_2^{\nu'}}{q_2^{\,2}}\right) B^i_{\nu'\mu'}\left(g^{\mu'\mu}-\frac{q_1^{\mu'}q_1^{\mu}}{q_1^{\,2}}\right), \tag{3.1}$$

where $\overline{C}^{\mu\nu}_{ab}$ is the spin-dependent part of $C^{\mu\nu}_{ab}$, and the $B^i_{\nu\mu}$ are taken to be

$$B^1_{\nu\mu}=g_{\nu\mu}\frac{1}{P^2}\epsilon(PQ\Delta s), \tag{3.2a}$$

$$B^2_{\nu\mu}=\frac{1}{P^2}P_\nu P_\mu\epsilon(PQ\Delta s), \tag{3.2b}$$

$$B^3_{\nu\mu}=\frac{1}{P^2}\epsilon(PQ\Delta s)[\Delta_\nu P_\mu-\Delta_\mu P_\nu], \tag{3.2c}$$

$$B^4_{\nu\mu}=-\frac{1}{P^2}\epsilon(PQ\Delta s)[\Delta_\nu P_\mu+\Delta_\mu P_\nu], \tag{3.2d}$$

$$B^5_{\nu\mu}=\epsilon_{\nu\mu}{}^{\alpha\beta}s_\alpha Q_\beta-\frac{1}{2\nu}[P_\mu\epsilon_\nu(Qs\Delta)+P_\nu\epsilon_\mu(Qs\Delta)], \tag{3.2e}$$

$$B^6_{\nu\mu}=Q\cdot s\,\epsilon_{\nu\mu\alpha\rho}P^\alpha Q^\beta, \tag{3.2f}$$

$$B^7_{\nu\mu}=\frac{1}{P^2}[P_\mu\epsilon_\nu(P\Delta s)+P_\nu\epsilon_\mu(P\Delta s)], \tag{3.2g}$$

$$B^8_{\nu\mu}=\frac{1}{P^2}[P_\mu\epsilon_\nu(P\Delta s)-P_\nu\epsilon_\mu(P\Delta s)], \tag{3.2h}$$

$$B^9_{\nu\mu}=\frac{1}{P^2}[\Delta_\mu\epsilon_\nu(P\Delta s)-\Delta_\nu\epsilon_\mu(P\Delta s)], \tag{3.2i}$$

$$B^{10}_{\nu\mu}=-\frac{1}{P^2}[\Delta_\mu\epsilon_\nu(P\Delta s)+\Delta_\nu\epsilon_\mu(P\Delta s)], \tag{3.2j}$$

$$B^{11}_{\nu\mu}=P_\mu\epsilon_\nu(PQs)+P_\nu\epsilon_\mu(PQs), \tag{3.2k}$$

$$B^{12}_{\nu\mu}=\Delta_\mu\epsilon_\nu(PQs)+\Delta_\nu\epsilon_\mu(PQs), \tag{3.2l}$$

$$B^{13}_{\nu\mu}=\Delta_\mu\epsilon_\nu(PQs)-\Delta_\nu\epsilon_\mu(PQs), \tag{3.2m}$$

where we have introduced the notation

$$\epsilon^\mu(ABC)\equiv\epsilon^{\mu\nu\alpha\beta}A_\nu B_\alpha C_\beta, \tag{3.3a}$$

$$\epsilon(ABCD)\equiv\epsilon^{\mu\nu\alpha\beta}A_\mu B_\nu C_\alpha D_\beta. \tag{3.3b}$$

The set of $B^i_{\nu\mu}$ is related to Gerstein's by using

$$i\overline{u}(p_2)\gamma^\mu\gamma^5u(p_1)\equiv s^\mu, \tag{3.4a}$$

$$\overline{u}(p_2)\gamma^\mu u(p_1)=\frac{mP^\mu}{P^2}\overline{u}(p_2)u(p_1)-\frac{1}{2P^2}\epsilon^\mu(P\Delta s), \tag{3.4b}$$

$$\overline{u}(p_2)\sigma^{\mu\nu}u(p_1)=\frac{i}{m}\epsilon^{\mu\nu}(Ps)-\frac{i}{2P^2}\overline{u}(p_2)u(p_1)[\Delta^\mu P^\nu-\Delta^\nu P^\mu]+\frac{i}{4mP^2}[\Delta^\mu\epsilon^\nu(P\Delta s)-\Delta^\nu\epsilon^\mu(P\Delta s)]. \tag{3.4c}$$

Again the invariant amplitudes are functions of ν, Q^2, t, and δ. To simplify the algebra we will set $\delta=0$ immediately. Then time-reversal invariance requires R_4, R_8, R_{10}, R_{11}, and R_{13} to be zero.

We notice that, as $\Delta\to 0$, only $B^5_{\nu\mu}$ and $B^6_{\nu\mu}$ are not zero. In this limit the amplitudes R_5^{ab} and R_6^{ab} become the amplitudes we called W_3 and W_4 in DJT. (Here we have already used the notation W_3 and W_4 for spin-independent amplitudes.)

From (2.2) we find that

$$R_i^{(ab)}(\nu,Q^2,t,0)=R_i^{(ab)}(-\nu,Q^2,t,0)$$

for $i=1$, 2, 6, 9, and 12 while

$$R_i^{(ab)}(\nu,Q^2,t,0)=-R_i^{(ab)}(-\nu,Q^2,t,0)$$

for $i=3$, 5, and 7 with opposite symmetry for the amplitudes that are antisymmetric under interchange of a and b.

It should also be noted that the projection operator $(g^{\mu\nu}-q^\mu q^\nu/q^2)$ could be replaced by $(g^{\mu\nu}-P^\mu q^\nu/\nu)$. The first choice leads to conditions among the R_i at $q^2=0$ ensuring the absence of q^2 poles in $C^{\mu\nu}$ (or alternatively the vanishing of helicity amplitudes with longitudinal photons). The second choice leads to $\nu=0$ conditions. The presence of a $1/\nu$ term in (3.2e) follows from using the second choice for the (5) amplitude and then reexpressing the result in terms of first choice amplitudes. This somewhat awkward procedure has the advantage of displaying the generalization of the Drell-Hearn[14] sum rule in a straightforward form but introduces an extra condition $R_5^{[a,b]}(\nu=0)=0$.

We now proceed as in Sec. II and DJT to derive fixed-mass sum rules. We use the $++$, $+-$, and $+i$ commutators. The only difficult problem is in determining which "reduced" tensors (that is, the tensors $B^i_{\nu\mu}$ with $\nu\mu=++$, $+-$, or $+i$) are independent. We find, for the $+$, i case, for example, the following independent tensors: $P^i\epsilon^+(P\Delta s)$, $Q^i\epsilon^+(P\Delta s)$, $\Delta^i\epsilon^+(P\Delta s)$, $Q^i\epsilon^+(PQs)$, $\Delta^i\epsilon^+(PQs)$, $P^is^+\epsilon(Q\Delta)$, $Q^is^+\epsilon(Q\Delta)$, $\Delta^is^+\epsilon(Q\Delta)$, and $\epsilon^{+i}(Ps)$, where $\epsilon(Q\Delta)=\epsilon^{jk}Q_j\Delta_k$, $j,k=1,2$. For the +, − case we find

$$\frac{1}{P^+}\epsilon^+(Q\Delta s),\quad P^-\epsilon^+(Q\Delta s),$$

$$\frac{P_iQ^i}{P^+}\epsilon^+(Q\Delta s),\quad \frac{P_i\Delta^i}{P^+}\epsilon^+(Q\Delta s),$$

$$\frac{1}{P^+}\epsilon^+(P\Delta s),\quad \frac{P_iQ^i}{P^+}\epsilon^+(P\Delta s),$$

$$\frac{P^i\Delta_i}{P^+}\epsilon^+(P\Delta s),\quad \epsilon(Qs),$$

$$\frac{P_iQ^i}{P^+}\epsilon^+(PQs),\quad P^-\epsilon^+(P\Delta s),$$

$$\frac{P_i\Delta^i}{P^+}\epsilon^+(PQs).$$

We find 12 independent sum rules:

$$\int_0^\infty d\nu\left[\frac{1}{\nu}R_5^{(ab)}(\nu,Q^2,t,\delta=0)+R_2^{(ab)}(\nu,Q^2,t,\delta=0)\right]=0, \tag{3.5a}$$

$$\int_0^\infty d\nu[P^2Q^2R_6^{(ab)}-t(\nu R_3^{(ab)}+R_9^{(ab)})]=-\tfrac{1}{2}i\pi\, d_{abc}tP^2\int_0^\infty d\alpha\overline{V}_4^c, \tag{3.5b}$$

$$\int_0^\infty d\nu R_{12}^{(ab)}=0, \tag{3.5c}$$

$$\int_0^\infty d\nu\,[\nu R_2^{[ab]}-2R_7^{[ab]}]=\pi f_{abc}f_s^c(t), \tag{3.5d}$$

$$\int_0^\infty d\nu\,[\nu R_2^{[ab]}+R_5^{[ab]}]=-\pi f_{abc}f_s(t)-\tfrac{1}{2}\pi f_{abc}\int_0^\infty d\alpha[\overline{A}_1^c+\alpha\overline{A}_2^c], \tag{3.5e}$$

$$\int_0^\infty d\nu\left[R_1^{(ab)}+\frac{\nu}{2q^2}(\nu R_6^{(ab)}-tR_3^{(ab)}-2R_7^{(ab)})\right]=\tfrac{1}{2}i\pi\, d_{abc}\int_0^\infty d\alpha\left(\frac{2P^2Q^2}{q^2}\overline{V}_4^c-\alpha\overline{V}_6^c\right), \tag{3.5f}$$

$$\int_0^\infty d\nu\,\nu[\nu R_2^{(ab)}-2R_7^{(ab)}]=\tfrac{1}{2}\pi P^2 d_{abc}\,q_1\cdot q_2\int_0^\infty d\alpha\,\overline{V}_4^c, \tag{3.5g}$$

$$\int_0^\infty d\nu\left\{\nu R_1^{[ab]}+\tfrac{1}{2}P^2(R_5^{[ab]}+2R_7^{[ab]})-\frac{\nu^2}{q^2}(\nu R_2^{[ab]}-2R_7^{[ab]})+\frac{\nu}{2q^2}[P^2Q^2R_6^{[ab]}-t(\nu R_3^{[ab]}+R_9^{[ab]})]\right\}$$
$$=-\pi f_{abc}P^2\left[f_s^c(t)+\tfrac{1}{4}\int_0^\infty d\alpha\,\overline{A}_1^c\right], \tag{3.5h}$$

$$\int_0^\infty d\nu[R_5^{[ab]}+\nu R_6^{[ab]}-tR_3^{[ab]}]=-\tfrac{1}{2}\pi f_{abc}\int_0^\infty d\alpha[\overline{A}_1^c+\alpha\overline{A}_2^c], \tag{3.5i}$$

$$\int_0^\infty d\nu\left[R_5^{[ab]}-\frac{t\nu}{2q^2}R_{12}^{[ab]}\right]=-\tfrac{1}{2}\pi f_{abc}\int_0^\infty d\alpha[\overline{A}_1^c+t\overline{A}_7^c], \tag{3.5j}$$

where we have omitted the argument $\nu, Q^2, t, \delta=0$ of the amplitudes R in most of the sum rules. Similarly on the right-hand side the bilocal operators are each a function of $x^2=0, \alpha, x\cdot\Delta=0$, and t. The form factor $f_s(t)$ is defined in (2.14). The bilocal operators are defined in (2.15) and (2.17).

The sum of (3.5a) and (3.5b) is a generalization to $q^2\neq 0$, $t\neq 0$ of the Drell-Hearn sum rule.[14] As q^2 and t go to zero the absence of q^2 poles in $C^{\mu\nu}$ requires that $2R_2=R_5/\nu$ (see Appendix A), and the difference $\sigma_{\text{parallel}}-\sigma_{\text{antiparallel}}$ is proportional to $R_5+\nu R_6$, as is well known. The sum rules (3.5b) and (3.5i) are extensions to $t\neq 0$ of sum rules derived in DJT. Equation (3.5i) is the generalization of the Bég sum rule.

The R_i in (3.5) have apparent Regge asymptotic behavior (see Appendix A) as follows:

$$R_i\sim\nu^{\alpha+1},\quad i=1,9$$
$$R_i\sim\nu^{\alpha},\quad i=3,5,7 \tag{3.6}$$
$$R_i\sim\nu^{\alpha-1},\quad i=2,6,12.$$

When the integrands in (3.5) are expressed in terms of the helicity amplitudes (from Appendix A) some cancellations of leading powers occurs. Moreover it is shown in Appendix A that if we restrict ourselves to the contributions of even-signature trajectories to the isospin-symmetric ($I_t=0$) amplitudes and odd signature to the $I_t=1$ amplitudes the convergence of some of the sum rules is still further improved. The end result is that the integrands in (3.5) behave with ν as follows:

(a) $\nu^{\alpha-3}$, (b) $\nu^{\alpha-1}$,

(c) $\nu^{\alpha-1}$, (d) $\nu^{\alpha-2}$,

(e) $\nu^{\alpha-2}$, (f) $\nu^{\alpha-1}$,

(g) $\nu^{\alpha-1}$, (h) ν^{α},

(i) $\nu^{\alpha-2}$, (j) ν^{α}.

Thus (3.5a), (3.5d), (3.5e), and (3.5i) are convergent in a simple Regge model. Still further improvement in the asymptotic behavior of the integrands may be expected to result from the proper insertion of factorized Regge poles into the t-channel parity-conserving helicity amplitudes and, at $t=0$, from use of the conspiracy conditions.[15]

As in DJT the right-hand sides of the sum rules may be replaced by the scaling limits of the left-hand sides. Again, as in Sec. II and DJT, the fixed-mass sum rules are only valid up to the neglect of contributions from Class-II diagrams.

We could derive, as in Sec. II B, Fourier-transform sum rules for the deep-inelastic limits of the R_i. Although in principle there is no difficulty in doing this, the algebra is sufficiently tedious, and the expected results, beyond those found in Sec. II B, are sufficiently unimportant, that we have not carried out the derivations.

IV. ANALYTICITY OF THE SCALING FUNCTIONS IN TWO VARIABLES

In this section we ignore the nontrivial complications introduced by the spin of the photon and study the scaling limit of the Mandelstam representation[6] for the "scalar structure function" of the kaon.

We begin with the Mandelstam representation for the scalar photon-kaon scattering amplitude $T_K(q^2, s, t)$, in the approximation of keeping only the first (s-t) double-spectral function

$$T_K(q^2,s,t)=\int \frac{dt'}{t'-t}\,\frac{ds'}{s'-s}\,\rho_K(q^2,s',t')\,. \tag{4.1}$$

In the approximation of two-body t-channel unitarity, ρ_K is given by

$$\rho_K(q^2,s,t)=\left(\frac{t-4\mu^2}{t}\right)^{1/2} \times\int \frac{dz'dz''\,W_\pi(q^2,\nu',t)\,\mathrm{Im}T^{\pi K}_{(\nu'',t)}}{[z^2+z'^2+z''^2-1-2zz'z'']^{1/2}}\,, \tag{4.2}$$

where the structure function W_π is the s-channel discontinuity of T_π and $T^{\pi K}$ is the strong (pion-kaon) amplitude. The z's are given by (with μ the pion mass)

$$\nu=\tfrac{1}{4}[(t-4M_K^2)(t-4q^2)]^{1/2}z\,, \tag{4.3a}$$

$$\nu'=\tfrac{1}{4}[(t-4\mu^2)(t-4q^2)]^{1/2}z'\,, \tag{4.3b}$$

$$\nu''=\tfrac{1}{4}[(t-4M_K^2)(t-4\mu^2)]^{1/2}z''\,. \tag{4.3c}$$

The integrals in (4.2) are from threshold up to the curve in ν' and ν'' on which the denominator

$$k_W^{1/2}(z,z',z'')=[z^2+z'^2+z''^2-1-2zz'z'']^{1/2} \tag{4.4}$$

vanishes. The Mandelstam double-spectral-function boundary $s=s(t)$ is found by setting s' and s'' equal to their threshold value and solving $k_W=0$ for $s(t)$. W can be found from dispersing (4.2) in t:

$$W_K(q^2,\nu,t)=\int\frac{dt'}{t'-t}\left(\frac{t'-4\mu^2}{t'}\right)^{1/2}\int dz'dz''\,\frac{W_\pi(q^2,\nu',t')\mathrm{Im}T^{\pi K}_{(\nu'',t')}}{k_W^{1/2}(z(\nu,t'),z'z'')}+\text{pole terms.} \tag{4.5}$$

It is not surprising that scaling is consistent with the Mandelstam representation in the sense that, letting

$$\tilde{\omega}=-2\nu/q^2\,, \tag{4.6a}$$

$$\tilde{\omega}'=-2\nu'/q^2\,, \tag{4.6b}$$

and taking the limit $q^2\to\infty$ with $\tilde{\omega}$ fixed gives

$$-\tfrac{1}{4}k_W^{-1/2}dz'dz''\to\frac{d\tilde{\omega}'d\nu''}{(t'-4\mu^2)^{1/2}}\,[\tilde{\omega}^2(t'-4\mu^2)+\tilde{\omega}'^2(t-4M_K^2)-8\tilde{\omega}\tilde{\omega}'\nu'']^{-1/2}\,. \tag{4.7}$$

Thus the kernel in the Mandelstam iteration procedure scales.

Setting ν'' and $\tilde{\omega}$ equal to their threshold values ($\nu_T=\tfrac{1}{4}t+M\mu$ and $\tilde{\omega}=1$) the zero of $k_W^{1/2}$ in the denominator of (4.7) gives the double-spectral-function boundary $\tilde{\omega}(t)$ for the scaling function

$$F(\tilde{\omega}, t) = \lim_{q^2\to\infty;\ \tilde{\omega}\ \text{fixed}} W(q^2, \nu, t).$$

The result is

$$\tilde{\omega} = 1 + \frac{4(M+\mu)}{t-4\mu^2}\left(\mu + \tfrac{1}{2}\sqrt{t}\right). \tag{4.8}$$

The curve $\tilde{\omega}(t)$ is asymptotic to the normal threshold $\tilde{\omega}=1$ and $t=4\mu^2$. It has negative definite slope; hence there are no anomalous thresholds in the scaling function. The result (4.8) must remain valid when photon, and hadron, spin are included. Any new singularities generated by spin will merely be "kinematic."

Taking the scaling limit of (4.5) gives an integral for F_K,

$$F_K(\tilde{\omega}, t) = f_K(t)\delta(\tilde{\omega}-1) - 4\int \frac{dt'}{t'-t}\frac{1}{\sqrt{t'}}\int \frac{d\nu'' d\tilde{\omega}'}{k_F^{1/2}} F_\pi(\tilde{\omega}', t')\,\mathrm{Im}T^{K\pi}(\nu'', t'), \tag{4.9}$$

with

$$k_F(\tilde{\omega}, \tilde{\omega}', \nu'', t) = \tilde{\omega}^2(t-4\mu^2) + \tilde{\omega}'^2(t-4M^2) - 8\tilde{\omega}\tilde{\omega}'\nu''. \tag{4.10}$$

Similarly, an integral equation for $F_\pi(\tilde{\omega}, t)$ can be found by considering $W_\pi(q^2, \nu, t)$ in the (t-channel) two-body unitarity approximation

$$F_\pi(\tilde{\omega}, t) = f_\pi(t)\delta(\tilde{\omega}-1) - 4\int \frac{dt'}{t'-t}\frac{1}{\sqrt{t'}}\int \frac{d\nu'' d\tilde{\omega}'}{k_{F,\pi}^{1/2}} F_\pi(\tilde{\omega}', t')\,\mathrm{Im}T^{\pi\pi}(\nu'', t'), \tag{4.11}$$

with

$$k_{F,\pi} = (\tilde{\omega}^2 + \tilde{\omega}'^2)(t-4\mu^2) - 8\tilde{\omega}\tilde{\omega}'\nu''. \tag{4.12}$$

Equations (4.9) and (4.11) can be used to find the contributions of nontree diagrams to F_K and F_π from lower-order diagrams. Consider for example the contribution of the diagram of Fig. 1 to F_π.

In this approximation, F_π under the integral in (4.11) is $\delta(\tilde{\omega}'-1)$ and $\mathrm{Im}T^{\pi\pi}(\nu'', t')$ is $\delta(s''-M_R^2)$. The contribution to F_π is

$$F_\pi^B = -\frac{4}{M_R^2(\tilde{\omega}-1)}\int_{x_1}^{\infty}\frac{dx'}{x'-x}\left[(x'-x_1)(x'+x_2)\right]^{-1/2}, \tag{4.13}$$

with

$$x = \frac{1}{M_R^2}(t-4\mu^2), \quad x_1 = \frac{4\tilde{\omega}}{(\tilde{\omega}-1)^2}, \quad x_2 = \frac{4\mu^2}{M_R^2}. \tag{4.14}$$

It is straightforward to find from (4.13)

$$F_\pi^B(\tilde{\omega}, t) = -4\frac{1}{M_R^2(\tilde{\omega}-1)}\frac{1}{\left[(x-x_1)(x+x_2)\right]^{1/2}}\ln\frac{x_1 - x - \left[(x-x_1)(x+x_2)\right]^{1/2}}{x_1 - x + \left[(x-x_1)(x+x_2)\right]^{1/2}}. \tag{4.15}$$

For $t=0$ this is, of course, the same as the result of Jackiw and Waltz[16] for the forward quark-model box-diagram contribution.

Second double-spectral functions (t-u) can be included in the above with no extra difficulty. Third double-spectral functions, however, require a model for the scaling limit of the photoproduction amplitude. Since only $t=0$ is currently accessible to experiment the first double-spectral function is more than sufficient.

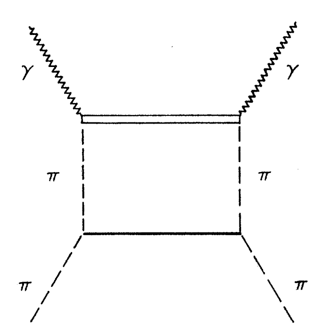

FIG. 1. Box-diagram contribution to $F_\pi(\omega, t)$. The solid line is a $\pi\pi$ scalar resonance; the double line is any single-particle-state contribution to F_π.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Roman Jackiw for many conversations and suggestions. In addition we would like to thank Dr. C. Y. Lo for help in deriving the relations in Appendix A, Dr. E. A. Paschos for a discussion on the sum rules of Sec. II, Dr. D. Palmer for assistance in deriving Eq. (3.5) correctly and for several discussions, and particularly Dr. Lo for finding a highly obscure but quite crucial error in an earlier version of Eq. (3.5).

APPENDIX A

We give here the t-channel helicity amplitudes in terms of the ($\delta=0$) scalar functions used in Secs. II and III and also the $q^2=0$ conditions which follow from the vanishing of helicity amplitudes involving longitudinal photons.

We calculate $T_{\lambda_2,\lambda_1;\sigma_2,\sigma_1}$ using

$$u_{\sigma_1}(p_1)=\begin{pmatrix}1\\ \dfrac{2\sigma_1 p}{E+m}\end{pmatrix}\chi_{\sigma_1} \quad (E^2-p^2=m^2), \tag{A1a}$$

$$v_{\sigma_2}(-p_2)=\begin{pmatrix}-\dfrac{2\sigma_2 p}{E+m}\\ 1\end{pmatrix}\chi_{\sigma_2}, \tag{A1b}$$

$$s^\mu=i(m,0,0,0), \quad \sigma_1=\sigma_2=\tfrac{1}{2} \tag{A2a}$$

$$=i(0,p,-ip,0), \quad \sigma_1=-\sigma_2=\tfrac{1}{2} \tag{A2b}$$

$$P^\mu=(0,0,0,p), \tag{A3a}$$

$$\Delta^\mu=(2E,0,0,0), \tag{A3b}$$

$$Q^\mu=(0,-q\sin\theta,0,-q\cos\theta)=\tfrac{1}{2}(q_2-q_1)^\mu, \tag{A3c}$$

$$\epsilon^\mu\begin{pmatrix}q_1\\ q_2\end{pmatrix}=\mp\frac{1}{\sqrt{2}}(0,\cos\theta,\pm i,-\sin\theta), \quad \lambda=+1 \tag{A4a}$$

$$=\pm\frac{1}{\sqrt{2}}(0,\cos\theta,\mp i,-\sin\theta), \quad \lambda=-1 \tag{A4b}$$

$$=\frac{1}{m_q}(q,\pm E\sin\theta,0,\pm E\cos\theta), \quad \lambda=0 \quad (m_q=E^2-q^2). \tag{A4c}$$

It is then straightforward, albeit tedious, to compute twelve independent t-channel parity-nonconserving helicity amplitudes in terms of the twelve scalar amplitudes W_1, W_2, W_3, W_5, R_4, R_2, R_3, R_5, R_6, R_7, R_9, and R_{12}. (W_4, R_4, R_8, R_{10}, and R_{11} are zero for $\delta=0$ by time-reversal invariance). We choose as independent helicity amplitudes:

$$T_{1,1;\frac{1}{2},\frac{1}{2}},\quad T_{1,1;-\frac{1}{2},\frac{1}{2}},\quad T_{-1,1;\frac{1}{2},\frac{1}{2}},\quad T_{-1,1;-\frac{1}{2},\frac{1}{2}},$$

$$T_{-1,-1;\frac{1}{2},\frac{1}{2}},\quad T_{1,-1;-\frac{1}{2},\frac{1}{2}},\quad T_{1,0;\frac{1}{2},\frac{1}{2}},\quad T_{1,0;-\frac{1}{2},\frac{1}{2}},$$

$$T_{-1,0;\frac{1}{2},\frac{1}{2}},\quad T_{-1,0;-\frac{1}{2},\frac{1}{2}},\quad T_{0,0;\frac{1}{2},\frac{1}{2}},\quad T_{0,0;-\frac{1}{2},\frac{1}{2}}.$$

The 12×12 matrix may then be inverted giving the (s-absorptive parts of the) scalar amplitude in terms of the (s-absorptive parts of the) t helicity amplitudes. We find[16]

$$W_L=\frac{1}{2p}(T_{1,1;\frac{1}{2},\frac{1}{2}}+2T_{-1,1;\frac{1}{2},\frac{1}{2}}+T_{-1,-1;\frac{1}{2},\frac{1}{2}})-\frac{2q^2\cos^2\theta}{p\sin^2\theta}\,\frac{1}{E^2+q^2}\,T_{-1,1;\frac{1}{2},\frac{1}{2}}, \tag{A5a}$$

$$W_2=-\frac{2}{p^3\sin^2\theta}\,T_{-1,1;\frac{1}{2},\frac{1}{2}}, \tag{A5b}$$

$$W_3=-\frac{m_q}{2\sqrt{2}\,Ep^2q\sin\theta}\left(T_{1,0;\frac{1}{2},\frac{1}{2}}-T_{-1,0;\frac{1}{2},\frac{1}{2}}+\frac{2\sqrt{2}\,E}{m_q}\,\frac{\cos\theta}{\sin\theta}\,T_{-1,1;\frac{1}{2},\frac{1}{2}}\right), \tag{A5c}$$

$$W_5=\frac{E^2+q^2}{8E^2q^2p}(T_{1,1;\frac{1}{2},\frac{1}{2}}+T_{-1,-1;\frac{1}{2},\frac{1}{2}})+\frac{1}{2q^2p\sin^2\theta}\left(1-\frac{m_q^{\,2}\sin^2\theta}{2E^2}\right)T_{-1,1;\frac{1}{2},\frac{1}{2}}$$
$$+\frac{m_q}{2\sqrt{2}\,Epq^2}\,\frac{\cos\theta}{\sin\theta}(T_{1,0;\frac{1}{2},\frac{1}{2}}-T_{-1,0;\frac{1}{2},\frac{1}{2}})-\frac{m_q^{\,2}}{4E^2q^2p}\,T_{0,0;\frac{1}{2},\frac{1}{2}}, \tag{A5d}$$

$$R_1=\frac{-1}{2Eq\sin\theta}(T_{1,1;-\frac12,\frac12}-T_{-1,1;-\frac12,\frac12}-T_{1,-1;-\frac12,\frac12})+\frac{p\cos\theta}{4m(E^2+q^2)}(T_{1,1;\frac12,\frac12}-T_{-1,-1;\frac12,\frac12}), \tag{A5e}$$

$$R_2=-\frac{1}{Ep^2q\sin^3\theta}((1-\cos\theta)T_{-1,1;-\frac12,\frac12}+(1+\cos\theta)T_{1,-1;-\frac12,\frac12})-\frac{1}{2mpq^2\cos\theta}(T_{1,1;\frac12,\frac12}-T_{-1,-1;\frac12,\frac12}), \tag{A5f}$$

$$\begin{aligned}R_3=&\frac{1}{8E^2qm}(T_{1,1;\frac12,\frac12}-T_{-1,-1;\frac12,\frac12})-\frac{E^2+q^2}{8E^3q^2p}\frac{\cos\theta}{\sin\theta}(T_{1,1;-\frac12,\frac12}-T_{-1,1;-\frac12,\frac12}-T_{1,-1;-\frac12,\frac12})\\&+\frac{1}{4Epq^2}\frac{\cos\theta}{\sin^3\theta}(1+\sin^2\theta)(T_{-1,1;-\frac12,\frac12}+T_{1,-1;-\frac12,\frac12})+\frac{1}{4Epq^2}\frac{1}{\sin^3\theta}(T_{1,-1;-\frac12,\frac12}-T_{-1,1;-\frac12,\frac12})\\&+\frac{\sqrt2 m_q}{8mE^3q}\frac{\cos\theta}{\sin\theta}(T_{1,0;\frac12,\frac12}+T_{-1,0;\frac12,\frac12})+\frac{\sqrt2 m_q}{8E^2q^2p}(T_{1,0;-\frac12,\frac12}-T_{-1,0;-\frac12,\frac12})+\frac{m_q^2}{8E^3q^2p}\frac{\cos\theta}{\sin\theta}T_{0,0;-\frac12,\frac12},\end{aligned} \tag{A5g}$$

$$R_5=\frac{1}{2qm}(T_{1,1;\frac12,\frac12}-T_{-1,-1;\frac12,\frac12}), \tag{A5h}$$

$$\begin{aligned}R_6=&\frac{E^2+q^2}{2Ep^2q^3}\frac{1}{\sin\theta}(-T_{1,1;-\frac12,\frac12}+T_{-1,1;-\frac12,\frac12}+T_{1,-1;-\frac12,\frac12})+\frac{E}{p^2q^3}\frac{\cos\theta}{\sin^3\theta}((1-\cos\theta)T_{-1,1;-\frac12,\frac12}-(1+\cos\theta)T_{1,-1;-\frac12,\frac12})\\&+\frac{m_q}{\sqrt2 mEpq^2}\frac{1}{\sin\theta}(T_{1,0;\frac12,\frac12}+T_{-1,0;\frac12,\frac12})-\frac{m_q}{\sqrt2 p^2q^3\sin^2\theta}((1+\cos\theta)T_{1,0;-\frac12,\frac12}+(1-\cos\theta)T_{-1,0;-\frac12,\frac12})\\&+\frac{m_q^2}{2Ep^2q^3\sin\theta}T_{0,0;-\frac12,\frac12},\end{aligned} \tag{A5i}$$

$$R_7=\frac{-1}{2Ep\sin\theta}(T_{-1,1;-\frac12,\frac12}-T_{1,-1;-\frac12,\frac12})-\frac{1}{4qm}(T_{1,1;\frac12,\frac12}-T_{-1,-1;\frac12,\frac12}), \tag{A5j}$$

$$\begin{aligned}R_9=&\frac{p\cos\theta}{8mE^2}(-T_{1,1;\frac12,\frac12}+T_{-1,-1;\frac12,\frac12})+\frac{E^2+q^2}{8E^3q}\sin\theta(-T_{1,1;-\frac12,\frac12}+T_{-1,1;-\frac12,\frac12}+T_{1,-1;-\frac12,\frac12})\\&-\frac{1}{4Eq}\frac{\cos^2\theta}{\sin\theta}(T_{-1,1;-\frac12,\frac12}+T_{1,-1;-\frac12,\frac12})+\frac{m_qp}{4\sqrt2 mE^3}\sin\theta(T_{1,0;\frac12,\frac12}+T_{-1,0;\frac12,\frac12})\\&+\frac{m_q}{4\sqrt2 E^2q}\cos\theta(T_{-1,0;-\frac12,\frac12}-T_{1,0;-\frac12,\frac12})+\frac{m_q^2}{8E^3q}\sin\theta\, T_{0,0;-\frac12,\frac12},\end{aligned} \tag{A5k}$$

$$R_{12}=-\frac{m_q}{2\sqrt2 Epq^2m}\frac{1}{\sin\theta}(T_{1,0;\frac12,\frac12}+T_{-1,0;\frac12,\frac12}). \tag{A5l}$$

A somewhat surprising feature of the results of (A5) is the presence of an anomalously large ν behavior in some of the amplitudes. R_9, for example, behaves as $\nu^{\alpha+1}$. Since its coefficient in the expansion of $C_{\nu\mu}$ ($B^{9}_{\nu\mu}$) is constant as $\nu\to\infty$, one would off hand expect R_9 to go as ν^α. The extra power is the result of the vanishing of the determinant of the leading ν coefficients in the inversion required to derive (A5). Alternatively defined amplitudes may be constructed without the ν anomaly; however the convergence of a given sum rule must be independent of the definition of the amplitudes in terms of which it is expressed.

There are signature and parity constraints on which trajectories can give (leading) contributions to the R's. We note the following:

(1) The leading contribution in ν to the absorptive part of a t-channel helicity amplitude is odd (or even) in ν according to whether the trajectory signature is even (or odd).

(2) For the isosymmetric amplitudes, $W_3^{(ab)}$ and $R_i^{(ab)}$ ($i=1,\ 2,\ 6,\ 9,$ and 12) are symmetric in ν while $W_i^{(ab)}$ ($i=1,\ 2,$ and 5) and $R_i^{(ab)}$ ($i=3,\ 5,$ and 7) are antisymmetric in ν.

(3) From (A5) we see that the leading contribution to $W_i^{(ab)}$ ($i=1,\ 2,$ and 5) and $R_i^{(ab)}$ ($i=3,\ 5,$ and 7) are related to the leading ν contributions to helicity absorptive parts by even powers of ν; the other R_i and W_i by odd powers. These three facts imply that only even-signature trajectories contribute to the leading behavior of the $I_t=0$ structure functions. Odd-signature contributions are reduced by (at least) one power of ν. The opposite results hold for the $I_t=1$ structure functions.

The $t=0$ results may be recaptured from (A5) by means of the derivative conspiracy conditions.[18] Alternatively, they may be found directly by setting $\Delta=0$ in (2.4) and (3.2), expressing $T_{1,1;\frac12,\frac12}$, $T_{1,-1;\frac12,\frac12}$, $T_{0,0;\frac12,\frac12}$, and $T_{1,0;-\frac12,\frac12}$ in terms of W_1, W_2, R_5, and R_6, and inverting the resulting matrix. One finds the same ν behavior and signature rules as above.

DJT are incorrect on this point: They give (in the present notation) incorrectly $R_5\sim\nu^{\alpha-1}$ and $R_6\sim\nu^{\alpha-2}$.

(The correct "anomalous ν" results are ν^{α} and $\nu^{\alpha-1}$.) From this they deduce an incorrect signature rule. The principal result that the leading power of the leading odd-signature trajectory drops out of the combination $R_5^{[ab]}+\nu R_6^{[ab]}$ at $t=0$ is, however, given correctly by DJT. This combination behaves as $\nu^{\alpha-2}$.

Results of this latter type appear in the present work: By the symmetry arguments above, combinations of amplitudes in which the leading power of a "right-signature" (even for $I_t=0$, odd for $I_t=1$) trajectory cancel must have the convergence of the contribution of that trajectory improved by ν^2 – not just ν.

The $q_\mu q^\mu = m_q{}^2 \to 0$ relations may be found by requiring the absence of $m_q{}^2$ poles in $C^{\mu\nu}$ or by requiring the vanishing of $T_{10,\frac{1}{2},\frac{1}{2}}, \ldots, T_{0,0;-\frac{1}{2}\frac{1}{2}}$ in the above list. The relations are

$$\nu W_2 + q_1\cdot q_2 W_3 \sim O(q^2)\,, \tag{A6a}$$

$$W_L + q_1\cdot q_2 W_5 \sim O(q^2)\,, \tag{A6b}$$

$$-R_L - \frac{\nu^2}{q_1\cdot q_2} R_2 + \nu R_3 - \tfrac{1}{4}P^2 R_6 + R_9 \sim O(q^2)\,, \tag{A6c}$$

$$-\nu R_2 + q_1\cdot q_2 R_3 + \tfrac{1}{2}R_5 + \tfrac{1}{2}\nu R_6 + R_7 \sim O(q^2)\,, \tag{A6d}$$

$$\tfrac{1}{2}(\nu^2 - P^2Q^2)R_6 - \nu R_7 - q_1\cdot q_2 R_9 \sim O(q^2)\,, \tag{A6e}$$

$$q_1\cdot q_2 R_{12} \sim O(q^2)\,, \tag{A6f}$$

$$q_1\cdot q_2 R_L + 2\nu^2 R_2 - 2\nu q_1\cdot q_2 R_3 - \tfrac{1}{2}\nu R_5 - \tfrac{1}{2}q^2P^2R_6 - 2\nu R_7 - 2q_1\cdot q_2 R_9 + q_1\cdot q_2 R_{12} \sim O(q^2)\,. \tag{A6g}$$

It should be noted that these relations only hold for the parts of the W's and R's which are free of δ functions, i.e., for the non-ν-pole parts of the amplitudes. A diagram which gives a ν pole for fixed q^2 may give a q^2 pole for fixed ν (see, for example, $W_2^{[a,b]}$ in Appendix B and compare with the Dashen–Fubini–Gell-Mann sum rule).

APPENDIX B

In a free-field theory with no magnetic coupling the one-nucleon contribution to the amplitudes W_i^{ab} and R_i^{ab} is (with $\delta=0$)

$$W_2^{ab} = \pi \frac{m}{P^2}\, i f_{abc}\lambda_c[\delta(Q^2+2\nu-\tfrac{1}{4}t)+\delta(Q^2-2\nu-\tfrac{1}{4}t)] + \pi\frac{m}{P^2}\, d_{abc}\lambda_c[\delta(Q^2+2\nu-\tfrac{1}{4}t)-\delta(Q^2-2\nu-\tfrac{1}{4}t)]\,,$$

$$W_3^{ab} = \tfrac{1}{2}\pi \frac{m}{P^2}\, i f_{abc}\lambda_c[\delta(Q^2+2\nu-\tfrac{1}{4}t)-\delta(Q^2-2\nu-\tfrac{1}{4}t)] + \tfrac{1}{2}\pi\frac{m}{P^2}\, d_{abc}\lambda_c[\delta(Q^2+2\nu-\tfrac{1}{4}t)+\delta(Q^2-2\nu-\tfrac{1}{4}t)]\,,$$

$$R_1 = -\frac{3}{4}\frac{P^2}{m} W_3\,,$$

$$R_2 = -\frac{1}{2}\frac{P^2}{m}\frac{1}{\nu} W_2\,,$$

$$R_5 = \frac{1}{2}\frac{P^2}{m} W_2\,,$$

$$R_7 = -\frac{1}{2}\frac{P^2}{m} W_2\,,$$

$$R_9 = -\frac{1}{2}\frac{P^2}{m} W_3\,,$$

with W_L^{ab}, W_5^{ab}, R_3^{ab}, R_6^{ab}, and R_{12}^{ab} equal to zero.

---

*Research supported by the U. S. Atomic Energy Commission under Contract No. C00-3065.

†Research supported by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-3069.

[1]J. M. Cornwall and R. Jackiw, Phys. Rev. D 4, 367 (1971).

[2]H. Fritzsch and M. Gell-Mann, in *Broken Scale Invariance and the Light Cone*, 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G. J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971), Vol. 2, p. 1.

[3]D. J. Gross and S. B. Treiman, Phys. Rev. D 4, 1059 (1971).

[4]D. Dicus, R. Jackiw, and V. Teplitz, Phys. Rev. D 4,

1733 (1971). We will refer to this paper as DJT. Also see D. Dicus, *ibid.* 5, 1367 (1972).

[5]D. J. Gross, Phys. Rev. D 4, 1130 (1971).

[6]S. Mandelstam, Phys. Rev. 112, 1344 (1958).

[7]R. Jackiw and G. Preparata, Phys. Rev. Letters 22, 975 (1969); 22, 1162(E) (1969). S. L. Adler and Wu-Ki Tung, *ibid.* 22, 978 (1969).

[8]J. M. Cornwall, J. D. Corrigan, and R. E. Norton, Phys. Rev. Letters 24, 1141 (1970); Phys. Rev. D 3, 536 (1971).

[9]S. P. de Alwis, University of Cambridge report (unpublished).

[10]A. Zee, Princeton Institute for Advanced Study report (unpublished).

[11]The proof of the light-cone BJL theorem based on the Deser-Gilbert-Sudarshan representation can easily be extended to the nonforward direction by using the Jost-Lehmann-Dyson representation. We thank Dr. David Palmer for discussions on this. See also Ref. 1 for an alternate method.

[12]J. M. Cornwall, Phys. Rev. D 2, 578 (1970).

[13]I. S. Gerstein, Phys. Rev. 161, 1631 (1967).

[14]S. D. Drell and A. C. Hearn, Phys. Rev. Letters 16, 908 (1966).

[15]C. Y. Lo, MIT report (unpublished).

[16]R. Jackiw and R. Waltz, Phys. Rev. D 6, 702 (1972).

[17]A more complete Regge analysis of the scalar amplitudes, taking into consideration factorization of the residues, has been done by C. Y. Lo (Ref. 15).

[18]E. Abers and V. L. Teplitz, Phys. Rev. 158, 1365 (1967).

PHYSICAL REVIEW D VOLUME 6, NUMBER 8 15 OCTOBER 1972

Singularity Structure of the Double-Regge Vertex in the Nonplanar Veneziano Model*

R. Roth†

Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 22 June 1972)

Previously studied models of the double-Regge vertex have exhibited only right-hand cuts in the Toller variable. We show that the nonplanar dual model of Mandelstam has left-hand cuts in the Toller variable. We also use this function to calculate all possible total (planar plus nonplanar) Reggeon-Reggeon-particle vertex functions.

I. INTRODUCTION

For planar Veneziano formulas, double-Regge limits and vertex functions have been calculated, and the contribution of double-Regge poles to the full amplitude was worked out.[1] In this paper, we will perform similar calculations for the nonplanar amplitudes of Mandelstam.[2] We will see that, whereas the planar amplitudes have only right-hand cuts in K $(=s/s_1s_2$, see Fig. 2), the nonplanar terms contribute, in addition, left-hand cuts in K. We will then introduce a vertex signature factor τ_K (Ref. 3) in order to obtain the most general possible expression for the total amplitude. Finally we will use the result thus obtained to calculate the various Reggeon-Reggeon-particle vertex functions.

II. DOUBLE-REGGE LIMIT

Mandelstam's Veneziano formula[2] corresponding to the minimal nonplanar five-point diagram, Fig. 1, is

$$A=\int du_A du_B J^{-1}\prod_p u_p{}^{-\alpha_p-1}, \tag{1}$$

where A, B are arbitrary, the product is over all channels (12, 13, 35, 45, 14, 25), and J is a Jacobian factor which transforms suitably,

$$J=\frac{u_{12}u_{45}(u_{12}+u_{45})}{(u_{25}+u_{45}+u_{35})}\ \text{ for } A=12,\quad B=45\,.$$

The u's are constrained by the conditions

$$u_{25}u_{13}+u_{12}u_{35}=1\,, \tag{2}$$

$$u_{25}u_{14}+u_{12}u_{45}=1\,, \tag{3}$$

$$u_{35}u_{14}+u_{13}u_{45}=1\,, \tag{4}$$

$$u_{25}+u_{35}+u_{45}-u_{12}-u_{13}-u_{14}=0\,. \tag{5}$$

We wish to study the double-Regge limit, $s, s_1, s_2 \to\infty$, $K=s/s_1s_2$ constant (see Fig. 2), of this non-

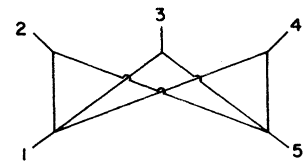

FIG. 1. Nonplanar five-point diagram.