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[9]See *Tables of Integral Transforms*, edited by A. Erdélyi *et al.* (McGraw-Hill, New York, 1970), Vol. I, p. 146.

[10]Notice that in order for the S matrix to be unitary, W_n and therefore β must be real. As a result the input trajectory must be taken to be exchange-degenerate.

[11]This type of Reggeon calculus is in the spirit of that discussed by V. N. Gribov, Zh. Eksp. Teor. Fiz. 53, 654 (1967) [Sov. Phys. JETP 26, 414 (1966)]; H. D. I. Abarbanel, NAL Report No. Thy-28, 1972 (unpublished).

[12]Because of our normalization the poles in $Z_N(l, \vec{\Delta})$ are one unit to the left of the corresponding poles in the elastic scattering amplitude, $M_{22}(l, \vec{\Delta})$. We denote by $\alpha_N(\vec{\Delta})$ the position of the poles in $M_{22}(l, \vec{\Delta})$.

[13]In order to simplify our calculations we shall assume that $\alpha_N(\vec{\Delta})$ and $\beta_N(\vec{\Delta})$ are analytic in the right-half N plane and do not have branch points at $N = -\infty$. However, in order to ensure that the square-root branch points in the l plane exist it is really sufficient to have $\alpha_N(\vec{\Delta})$ increase like N^2 for large N.

[14]The interested reader can use the same technique to obtain the high-energy behavior of the solvable model discussed in Sec. III.

[15]W. B. Ford, *Studies on Divergent Series and Summability* (Chelsea Publishing Co., 1916), p. 263.

[16]This is a general feature of the eikonal model, which holds independent of the form of W_n. It was first pointed out by S.-J. Chang and T.-M. Yan, Phys. Rev. D 4, 537 (1971).

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Hadronic Eikonal Model

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A model for hadron-hadron scattering proceeding via the exchange of mesonic systems with isospin is constructed. The result can be cast in eikonal form with an effective Regge-pole potential. Extensions of the model are discussed and comparisons with other models are made.

I. INTRODUCTION

In the quest for an understanding of the phenomenon of high-energy hadron-hadron scattering, many different approaches have been used. One of the oldest, the so-called eikonal approximation,[1-3] has recently received a great deal of attention in the literature because of major theoretical advances in its employment. A gigantic program has been carried out by Cheng and Wu[2] using Feynman diagrams to calculate various combinations of electron, positron, and photon scattering and production; and other authors,[4] using a variety of different field-theoretic techniques, have studied large classes of diagrams using a variety of models. The hope of course is that these studies will provide important clues for a realistic description of the physical hadron scattering amplitudes.

All of these eikonal-model calculations are based, however, on very simple field theories, which neglect both isospin and the possibility of exchanges of particles with spin greater than one; though following the work of Chang and Weinberg, Eichten[5] has considered the possibility of an eikonal approximation for a particle acting in an external field with arbitrary spin.

What we have done is to calculate the scattering amplitude $a + b \to a' + b'$, where a, a', b, b' are arbitrary one-particle hadron states (either stable hadrons or hadron resonances in the narrow-width approximation) in the limit of infinite $s = (p_a + p_b)^2$, taking into account exchange of an arbitrary number of mesons and allowing excitation by the mesons of the hadron states a, b to arbitrary hadron resonance states. The mesons are constrained to lie on the ρ-A_2-f_0 exchange-degenerate trajectory and the hadron states to have no self-mass or vertex corrections.

The organization of the paper is as follows: In Sec. II we analyze the amplitude for the scattering $a + b \to c + d$ by absorption and emission of an arbitrary number of isospin-one, spin-one, massive ρ mesons. Using infinite-momentum techniques we sum the leading terms of each order of perturbation theory to obtain an exponentiated eikonal form for the scattering amplitude.[6] The key to the procedure is an effective commutativity of the vertices, which holds even though the exchange particles have isospin; this is a consequence of the superconvergence relations for ρ hadron $\to \rho$ hadron scattering. We emphasize this point since this is precisely what distinguishes our work from other attempts along these lines; if we had considered exchange of fictitious charged as well as neutral photons whose sources were the isospin currents, the commutativity condition would not have held and one could not obtain an exponential result for the scattering amplitude. The difference lies in the well-known fact that the current hadron $\to$ current hadron′ scattering amplitude has a fixed pole at $J = 1$ whereas the ρ hadron $\to \rho$ hadron′ does not.

In Sec. III we review the derivation of the necessary superconvergence relation in the approximation of saturation by narrow-width resonances, much along the lines of the work of Bardakci and Segrè,[7] and extend these considerations to superconvergence relations for higher-spin particles, namely the mesons lying on the ρ-A_2-f_0 exchange-degenerate trajectory.

In Sec. IV, using the general class of relations derived in Sec. III, we extend the calculations of Sec. II to include exchange of not only ρ mesons, but all the mesons on the ρ-A_2-f_0 trajectory. We then use the van Hove[8] model to sum all meson exchanges in lowest order of perturbation theory and our techniques of Sec. II to sum up all orders of perturbation theory. The final answer is a Reggeized eikonal expansion, i.e., an amplitude in which a Regge propagator appears in the eikonal function, rather than an elementary-particle propagator.

An alternate approach which bypasses many intermediate steps is also sketched. The idea is to start directly by exchanging Regge trajectories rather than elementary particles and then to obtain the analog of a superconvergence relation for Reggeon hadron $\to$ Reggeon hadron′ scattering by considering the six-point function in the triple-Regge limit region.

In Sec. V, we present an algebraic derivation of the results of Sec. II, analogous to the canonical approach to massive quantum electrodynamics (QED) developed in Ref. 3. The various restrictions we have imposed on our model are further clarified, this time within a purely algebraic framework. The algebraic approach also indicates that we have, in all probability, within our system an infinite isotopic-spin multiplet with a resultant E(3) algebra.

Section VI contains a discussion on inelastic scattering within our model. The algebraic formalism introduced in Sec. V is useful for such a discussion, and suggests ways of realizing inelasticity effects. Of course, the basic premise of eikonalization has little to say about such effects, and here is where additional assumptions have to be made. Within the context of the effective Regge eikonal introduced in Sec. IV, we construct a model based on duality to simulate inelastic scattering. Such a construction is consistent with the Regge eikonal picture, although by no means necessitated in any way by our considerations up to this point. A few remarks on the Pomeranchuk singularity are also included in this section.

II. EIKONALIZATION WITH ρ-MESON EXCHANGE

In this section we consider the scattering of an initial state consisting of two hadrons a and b with momentum p_a, p_b going to a final state of two hadrons a' and b' with momentum $p_{a'}, p_{b'}$. The kinematics is such that $s = (p_a + p_b)^2 \to \infty$ while $t = (p_a - p_{a'})^2$ remains finite with of course $p_a + p_b = p_{a'} + p_{b'}$. We do not treat the process in its full generality, but restrict ourselves in this section to the class of diagrams depicted in Fig. 1, namely exchange of an arbitrary number of ρ mesons, i.e., isospin-one, massive vector mesons. The hadrons along the a to a' line and b to b' line, which we henceforth call a and b lines, are arbitrary hadron resonances, whose excitation is allowed by the quantum numbers of the reaction. Two important restrictions are that we allow no self-mass diagrams, i.e., the hadrons are all considered in the zero-width approximation, as in dual-resonance models, and also that we take the hadron-hadron-ρ coupling to be pointlike and given by its on-mass-shell value; this means that there are also no vertex corrections taken into account. We realize these are severe restrictions, but it is all we are equipped to solve at the moment.

The approach we shall use is to employ infinite-

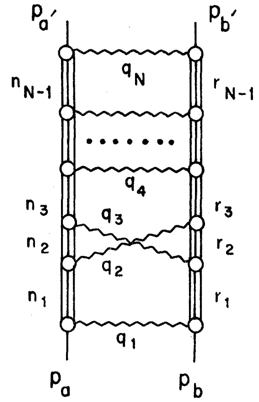

FIG. 1. Hadron-hadron scattering proceeding by meson exchange.

momentum variables $p_+ = p_0 + p_3$, $p_- = p_0 - p_3$, with $\vec{p}$ being a two-dimensional vector perpendicular to the third axis. The N exchanged ρ mesons have of course their momenta q_i restricted by the condition $\sum_{i=1}^{N} q_i = k$ where $k^2 = t$. We let $s \to \infty$ by having in the center-of-mass system $p_{a+}, p_{b-} \to \sqrt{s} \to \infty$, i.e.,

$$s = p_{a+}p_{b-} + p_{a-}p_{b+} - \vec{p}_a \cdot \vec{p}_b \sim p_{a+}p_{b-} .$$

The approximation we adopt for calculating our vertices and propagators is analogous to that of Chang and Ma[2] in that we keep only the leading term in s. Thus the vertex on line a at which the jth vector meson with isospin α_j is absorbed by the n_{j-1}th hadron is approximated by

$$\langle n_{j-1} | V_{\mu_j}^{\alpha_j}(q_j) | n_j \rangle \simeq \delta_{\mu_j,+} \sqrt{s}\, M_{n_{j-1},n_j}^{\alpha_j}(\vec{q}_j), \qquad (1)$$

and its emission by the r_{K-1}th hadron on line b by

$$\langle r_{K-1} | V_{\nu_j}^{\alpha_j}(q_j) | r_K \rangle \simeq \delta_{\nu_j,-} \sqrt{s}\, M_{r_{K-1},r_K}^{\alpha_j}(-\vec{q}_j) . \qquad (2)$$

The important kinematical feature to note about the above equations is that the matrix M depends only on $\vec{q}_j$, not on q_{j-} or q_{j+}. This is a consequence of working in the infinite-momentum frame (it is tacitly assumed throughout that q_j is a finite four-vector, i.e., we assume q_+/p_{a+}, $q_-/p_{b-} \to 0$). The key to our results will be a dynamical feature of the M matrices, namely, that they commute,

$$[M^{\alpha_i}(\vec{q}_i), M^{\alpha_j}(\vec{q}_j)]_{n,n'} = 0 , \qquad (3)$$

where the above equation is to be interpreted not as an operator equation, but as a relation between matrix elements which follows after a complete set of intermediate states, approximated by zero-width hadron resonances, is inserted. This equation, a consequence of the helicity-flip-two $\Delta I = 1$ superconvergence relation, will be discussed later on in the article. For the moment let us just assume that it holds.

The vector-meson propagator is

$$D_{\mu_j,\nu_j}(q_j) = \frac{-g_{\mu_j\nu_j} + q_{\mu_j}q_{\nu_j}/m_\rho^2}{q_j^2 - m_\rho^2}, \qquad (4)$$

but we have made the assumption that the matrix elements of Sec. V, the ρ-hadron-hadron vertices, have no off-mass-shell corrections and hence

$$q^{\mu_j} \langle | V_{\mu_j}^{\alpha_j} | \rangle = 0 , \qquad (5)$$

so the $q_\mu q_\nu$ terms may be dropped from the ρ-meson propagators. The hadron propagators have, of course, poles; the typical such term corresponding to the n_jth resonance on line a, i.e., the state a reaches after absorbing j vector mesons is

$$D_{n_j}\left(p_a + \sum_{K=1}^{j} q_K\right) = \frac{1}{\left(p_a + \sum_{K=1}^{j} q_K\right)^2 - m_{n_j}^2 + i\epsilon}$$

$$\approx \frac{1}{(p_{a+})\left(\sum_{K=1}^{j} q_{K-}\right) + i\epsilon}$$

$$\approx \frac{1}{\sqrt{s}\left(\sum_{K=1}^{j} q_{K-}\right) + i\epsilon} \qquad (6)$$

to the leading order in $\sqrt{s}$ in the denominator. We have assumed that all other terms which are at most $O(1)$ may be neglected; of course this is only true, as we stated earlier, if $q_+/p_+ \to 0$ and if the masses of the excited hadron resonances remain bounded, $m_n^2/s \to 0$. The integrals over q_j give as volume element

$$(2\pi)^4 i \delta^{(4)}\left(\sum_{j=1}^{N} q_j - k\right) \prod_{j=1}^{N} \frac{d^4 q_j}{i(2\pi)^4} . \qquad (7)$$

However, a, a' have infinite momentum along the positive z axis and b, b' along the negative z axis so that $p_{a-}, p_{a'-}, p_{b+}, p_{b'+} \to 0$ and hence $k_-, k_+ \to 0$ which reduces the above expression to

$$2(2\pi)^4 i \delta\left(\sum_{j=1}^{N} q_{j-}\right) \delta\left(\sum_{j=1}^{N} q_{j+}\right) \delta^{(2)}\left(\sum_{j=1}^{N} \vec{q}_j - \vec{k}\right) \prod_{j=1}^{N} \frac{dq_{j+}dq_{j-}d\vec{q}_j}{2i(2\pi)^4}$$

$$= 2(2\pi)^4 i \delta\left(\sum_{j=1}^{N} q_{j-}\right) \delta\left(\sum_{j=1}^{N} q_{j+}\right) \prod_{j=1}^{N} \frac{dq_{j+}dq_{j-}d\vec{q}_j}{2i(2\pi)^4} \frac{1}{(2\pi)^2} \int d\vec{b}\, e^{-i\vec{k}\cdot\vec{b}} \prod_{j=1}^{N} e^{i\vec{q}_j\cdot\vec{b}} . \qquad (8)$$

Combining all these factors we find that the sum of all N-meson-exchange diagrams is

$$T_N = 2(2\pi)^4 i \int \frac{d\vec{b}}{(2\pi)^2} e^{-i\vec{k}\cdot\vec{b}} \int \prod_{j=1}^{N} \frac{dq_{j+}dq_{j-}d\vec{q}_j e^{i\vec{q}_j\cdot\vec{b}}}{2i(2\pi)^4} \left(\frac{-g^{\mu_j\nu_j}}{q_j^2 - m_\rho^2}\right) \delta_{\mu_j,+}\delta_{\nu_j,-} \frac{s^N}{s^{N-1}} \times \frac{1}{N!}$$

$$\times \sum_{\substack{\text{permutations} \\ \text{on line } a}} \left[M^{\alpha_1}(\vec{q}_1) \frac{1}{q_{1-} + i\epsilon} M^{\alpha_2}(\vec{q}_2) \frac{1}{q_{1-} + q_2 + i\epsilon} \cdots \frac{M^{\alpha_N}(\vec{q}_N)}{\sum_K^{N-1} q_{K-} + i\epsilon} \right]_{aa'}$$

$$\times \sum_{\substack{\text{permutations} \\ \text{on line } b}} \left[M^{\alpha_1}(-\vec{q}_1) \frac{1}{q_{1+} + i\epsilon} M^{\alpha_2}(\vec{q}_2) \cdots \frac{1}{\sum_{K=1}^{N-1} q_{K+} + i\epsilon} M^{\alpha_N}(-\vec{q}_N) \right]_{bb'}, \quad (9)$$

where the permutation sums mean we must include all orderings in which the mesons are emitted at line b and absorbed at line a. Since the permuting is done separately on lines a and b, we must divide by $N!$ to compensate for the overcounting. The s^N/s^{N-1} factor arises because we have $2N$ vertices and $2(N-1)$ propagators each with a $\sqrt{s}$ factor.

The M matrices all commute with one another so that they can be arranged sequentially as $M^{\alpha_1}(\vec{q}_1)\cdots M^{\alpha_N}(\vec{q}_N)$ for any ordering. For the hadron resonance propagators we use the relation

$$\sum_{\text{permutations}} (q_{1-} + i\epsilon)^{-1}(q_{1-} + q_{2-} + i\epsilon)^{-1} \cdots \left(\sum_{k=1}^{N} q_{K-} + i\epsilon\right)^{-1} \delta\left(\sum_{j=1}^{N} q_{j-}\right) = (-2\pi i)^{N-1} \prod_{j=1}^{N} \delta(q_{j-}) \quad (10)$$

and the analogous one for the denominators on line b, with q_- replaced of course by q_+. The end result is then

$$T_N = -8i\pi^2 s \int \frac{d\vec{b} e^{-i\vec{k}\cdot\vec{b}}}{(2\pi)^2} \frac{1}{N!} [\chi(\vec{b})]^N_{aa',bb'}, \quad (11)$$

with

$$\chi(\vec{b}) = i \int \frac{dq_+ dq_- d\vec{q}\, \delta(q_-)\delta(q_+) e^{i\vec{q}\cdot\vec{b}}}{(2\pi)^2(q^2 - m_\rho^2)} M^\alpha(\vec{q}) M^\alpha(-\vec{q})$$

$$= -i \int \frac{d\vec{q}\, e^{i\vec{q}\cdot\vec{b}}}{(2\pi)^2(\vec{q}^2 + m_\rho^2)} \vec{M}(\vec{q}) \cdot \vec{M}(-\vec{q}), \quad (12)$$

where the dot product of the M's is in isospin space.

We finally obtain then an expression for the scattering amplitude by summing over N

$$T = \sum_{N=1}^{\infty} T_N = -2is \int d\vec{b}\, e^{-i\vec{k}\cdot\vec{b}} (e^{\chi(\vec{b})} - 1)_{aa',bb'}. \quad (13)$$

The total amplitude sums to an eikonal form, with the eikonal function χ expressed in terms of the ρ-hadron-hadron amplitude. It is proportional to s and, even though we exchange an arbitrary number of isospin-one mesons, the whole amplitude is an isoscalar in the s channel. From the t-channel point of view all isospin exchanges are present, which violates our intuitive notions of decreasing amplitudes for higher t-channel isospin exchanges; we will return to this point in Sec. IV. Finally we would like to comment that the extension to SU(3) is obvious.

III. SUPERCONVERGENCE RELATIONS

In this section we would like to explain why it is that the $M(\vec{q})$ matrices discussed in the previous section commute.

Consider, for arbitrary hadron states c, c' in the $s\to\infty$ limit, the difference of the scattering amplitude for $\rho^\alpha + c \to \rho^\beta + c'$ minus that for $\rho^\beta + c \to \rho^\alpha + c'$, where ρ^α and ρ^β are ρ mesons with momenta, respectively, of q and q'. We write this difference in the approximation that the scattering amplitudes are completely dominated by s- and u-channel resonances, which we label as n. We approach $s\to\infty$ by having the z components of the momenta of c and c' tend to infinity, while q and q' remain finite to leading order $p_{c+} = p_{c'+} = p_+$ and $(p_c \pm q)^2 \to \pm p_+ q_-$, $(p_{c'} \pm q)^2 \to \pm p_+ q_-$; we take as ρ-hadron-hadron vertices the form found in Sec. II, Eq. (1), and thus find for the difference $\Delta T_{\mu\nu}$ of the scattering amplitudes, $\Delta T_{\mu\nu} \simeq \delta_\mu^+ \delta_\nu^+ \Delta T_{++}$,

$$\Delta T_{++} \simeq \sum_n \left\{ \frac{p_+ M^\alpha_{cn}(\vec{q}) p_+ M^\beta_{nc'}(\vec{q}')}{p_+ q_- - m_n^2} + \frac{p_+ M^\beta_{cn}(\vec{q}') p_+ M^\alpha_{nc'}(\vec{q})}{-p_+ q_- - m_n^2} - \frac{p_+ M^\beta_{cn}(\vec{q}) p_+ M^\alpha_{nc'}(\vec{q})}{p_+ q_- - m_n^2} - \frac{p_+ M^\alpha_{cn}(\vec{q}) p_+ M^\beta_{nc'}(\vec{q}')}{-p_+ q_- - m_n^2} \right\}. \quad (14)$$

We now assume the mass spectrum of intermediate resonant states is bounded, i.e., $m_n^2 \ll p_+ q_-$ as $p_+ \to \infty$, which implies

$$\Delta T_{++} \to \frac{2p_+}{q_-}[M^\alpha(\vec{q}), M^\beta(\vec{q}\,')]_{cc'}, \tag{15}$$

where the above is meant not as an operator equation, but as a relation between matrix elements after introducing a complete set of resonant states in the commutator.

We now note, however, that the ρ trajectory is the leading trajectory contributing to ΔT_{++}, which asymptotically behaves therefore as

$$\begin{aligned}\Delta T_{++} &\sim [(p+q)^2]^{\alpha_\rho((q-q')^2)} \\ &\sim (p_+ q_-)^{\alpha_\rho((q-q')^2)} \\ &< p_+ q_- \quad \text{for} \quad (q-q')^2 < m^2{}_\rho,\end{aligned} \tag{16}$$

since the ρ trajectory $\alpha_\rho(t) < 1$ for $t = (q-q')^2 < m_\rho{}^2$. From Eq. (15), we see that ΔT_{++} behaves like p_+ as $p_+ \to \infty$ whereas from our above general argument we have seen this cannot be true; the resolution is to impose

$$[M^\alpha(\vec{q}), M^\beta(\vec{q}\,')]_{cc'} = 0, \tag{17}$$

the desired condition. This is nothing other than the well-known $\Delta I = 1$, helicity-flip-two superconvergence relation in the resonance saturation approximation. To refresh the reader's mind, what this says is that by writing

$$T_{\mu\nu}^{\Delta I=1} = p_\mu p_\nu A(s,t) + \cdots \tag{18}$$

one can show that $A(s,t) \sim s^{\alpha_\rho(t)-2}$, and hence, using a dispersion relation

$$\lim_{s\to\infty} sA(s,t) = \lim_{s\to\infty} s \int \frac{\operatorname{Im} A(s',t)}{s'-s}\, ds' = 0$$

$$\Rightarrow \int \operatorname{Im} A(s',t) = 0 \tag{19}$$

and saturating the integral over the absorptive part with zero-width resonances, one recovers Eq. (17). (For a discussion of this problem and its potential applications, see the paper by Bardakci and Segrè.[7])

A minor point is that we probably could have made our argument directly for T_{++} rather than having to consider the difference ΔT_{++}, since experience with notions of duality leads one to believe that saturation with s- and u-channel zero-width resonances is consistent with the leading Regge trajectory being the meson one, i.e., diffraction scattering is generated by effects neglected in this approximation. We shall return to this point in Sec. IV.

These arguments may be extended to higher-spin fields. Consider an isospin-one, spin-N field $\phi^\alpha_{\nu_1\nu_2\cdots\nu_N}$: Its coupling to hadrons is described by a vertex analogous to that of Eq. (1) in Sec. II for arbitrary hadron resonance states c and c',

$$\langle c|\, V^\alpha_{\nu_1\cdots\nu_N}(\vec{q})\,|c'\rangle \to \prod_{K=1}^{N} \delta_{\nu_K,+} \sqrt{s^N}\,[M_N^\alpha(\vec{q})]_{cc'}, \tag{20}$$

from which we can immediately deduce the superconvergence relation (in the usual sense of a relationship between matrix elements)

$$[M_N^\alpha(\vec{q}), M_N^\beta(\vec{q}\,')] = 0, \tag{21}$$

with M previously defined equal to M_1 (we must have $N \geq 1$). There is of course a much larger class of superconvergence relations than the simple one we have written above for high-spin particles, because of the possibility of large helicity flip Δh with a consequent asymptotic behavior of $s^{\alpha_\rho(t)-\Delta h}$, but we shall not discuss these in this article. Finally we remark that one may consider the scattering of a hadron by a spin-N_1 particle leading to hadron- plus-spin-N_2 particle, and derive then the condition

$$[M_{N_1}^\alpha(\vec{q}), M_{N_2}^\beta(\vec{q}\,')] = 0. \tag{22}$$

This will allow us to sum the contribution of higher-spin particles as well in our eikonal approximation.

The superconvergence relations of course only hold for on-mass-shell particles, which is why in our previous note[6] we only calculated the t-channel discontinuity of the amplitude. This had the effect of replacing the ρ-meson propagators by

$$D_{\mu\nu}(q) \to -g_{\mu\nu}\,\delta(q^2 - m_\rho{}^2)(-i\pi), \tag{23}$$

and placed the ρ mesons of course on the mass shell. Since the relation of Eq. (10) in Sec. II means that contributions only come from $q_- = q_+ = 0$, we then had to continue $\vec{q}$ to $\vec{q}\,' = i\vec{q}$ in order to have a nonvanishing δ function above. However all this is unnecessary in the approximation of neglecting all vertex and self-mass corrections, in which case on- and off-mass-shell propagators and vertices coincide (of course we take the physical value for the masses).

IV. REGGEIZED EIKONAL MODEL

There is an important conceptual problem which has been glossed over so far and that is the relative consistency of our expression for the scattering amplitude in Sec. II, Eq. (13), which says the scattering amplitude behaves asymptotically like $\sim s$, i.e., as if there were a fixed pole at $J=1$, with our assumption of Regge asymptotic behavior necessary for the superconvergence relation which in turn ascribed an asymptotic behavior of $s^{\alpha_\rho(t)}$ to the scattering amplitude. Furthermore, our result says that in terms of t-channel exchanges, the amplitude for $\Delta I = 1, 2, \ldots$, etc., as well as $\Delta I = 0$ (I is isospin) behaves like s asymptotically, which is inconsistent with experiment. We believe the source of our difficulty to be in the fact that ρ exchanged in the ρ-hadron scattering amplitude from which the commutativity condition Eq. (3) is derived is assumed to be a Reggeon, whereas the ρ exchanged in the eikonal sum is treated as an elementary particle. We shall show two different, but practically equivalent ways to circumvent this difficulty. The first allows for arbitrary spin exchange in the eikonal and then uses the van Hove model[8] to sum; the second starts by having the exchanged particles be Reggeons.

A. van Hove Model

In Sec. II we discussed the general superconvergence relations which led to the commutativity of $M_{N_1}(q)$ with $M_{N_2}(q')$. A simple solution to the algebraic problem of the M_N's is to assume that for all N, $M_N^\alpha = M_1^\alpha = M^\alpha$. [A relative factor with the dimensions of $(\text{mass})^{-N}$ necessary for dimensional reasons has been set equal to unity.]

In the van Hove model[8,9] one allows for the exchange of the whole set of particles lying on a Regge trajectory, with the mass relation, i.e., the functional dependence of $m^2(J)$ on J, being that of the trajectory so that for an infinitely rising trajectory we have a mass spectrum that goes to infinity. In addition we must allow M^α to be a two-by-two matrix in isospin space so that we allow isospin-zero as well as isospin-one mesons to be exchanged (e.g., the ω and f^0 in addition to the ρ and A_2) all lying on the same trajectory; i.e., exchange degeneracy holds. We then have instead of a single ρ exchange a sum over exchanges

$$\begin{aligned}\langle n | V_\mu^\alpha(\vec{q}) | n'\rangle D^{\mu\nu}(q) \langle r | V_\nu^\alpha(-\vec{q}) | r'\rangle &\simeq -\frac{s}{q^2 - m_\rho^2} \vec{M}_{nn'}(\vec{q}) \cdot \vec{M}_{rr'}(-\vec{q}) \\ &\to -\sum_{J=1}^{\infty} \frac{(s)^J}{q^2 - m^2(J)} \vec{M}_{nn'}(\vec{q}) \cdot \vec{M}_{rr'}(-\vec{q}),\end{aligned} \tag{24}$$

which can be summed for $s<1$ and then analytically continued to $s>1$ to give Regge asymptotic behavior

$$-\frac{\pi\alpha'(q^2) s^{\alpha(q^2)}}{\sin\pi\alpha(q^2)} \vec{M}_{nn'}(\vec{q}) \cdot \vec{M}_{rr'}(-\vec{q}). \tag{25}$$

This may be inserted into our calculations of Sec. II. Note that factors of $\sqrt{s} \times \sqrt{s}$ have to be removed from the above to cancel the $1/\sqrt{s}$ factors in the hadronic propagators at the sides of the generalized ladders. When this is done we obtain an expression for the scattering amplitude identical to that of Sec. II, Eq. (13) except for the fact that $\chi(\vec{b})$ is replaced now by an effective Reggeized $\chi(\vec{b})$ which we call $\chi_R(\vec{b})$:

$$\chi_R(\vec{b}) = i \int d\vec{q} \frac{e^{i\vec{q}\cdot\vec{b}}}{(2\pi)^2} \pi\alpha'(-\vec{q}^2) \times \frac{s^{\alpha(-\vec{q}^2)-1}}{\sin\pi\alpha(-\vec{q})^2} \vec{M}(\vec{q}) \cdot \vec{M}(-\vec{q}). \tag{26}$$

Note that this coincides with our previous result if, instead of a variable $\alpha(-\vec{q}^2)$, we fix α at the value one and have

$$\frac{\pi\alpha'(-\vec{q}^2)}{\sin\pi\alpha(-\vec{q}^2)} \to -\frac{1}{\vec{q}^2 + m_\rho^2}. \tag{27}$$

If, however, we use a physical value for the trajectory, we discover that $\alpha(-\vec{q}^2) < 1$; in fact, using the approximately known intercept for the ρ trajectory $\alpha_\rho(0) \sim \frac{1}{2}$, we are led to surmise $\alpha(-\vec{q}^2) \leq \frac{1}{2}$.

This has important consequences for the asymptotic behavior in s of the amplitude. First of all, we discover that the asymptotic behavior is no longer $T \sim sf(k^2)$ but rather (c_i is independent of s)

$$T \sim c_1 s^{\alpha(-\vec{k}^2)} + c_2 s^{2\alpha(-\vec{k}^2)-1} + \cdots, \tag{28}$$

so we see that the leading term has conventional Regge behavior. A further consequence is that the $\Delta I = 1, 2, 3, \ldots$ now no longer all have the same asymptotic behavior in s; to get $\Delta I = N$ one needs the Nth term in the expansion of $e^{i\chi_R}$ and this falls very rapidly to zero.

What we have derived here is the so-called Reggeized eikonal expansion first discussed by Arnold.[10] In fact in his multichannel derivation there is an interesting analog to our superconvergence relations, namely, he requires all potentials to be the same just as all our M's are, and furthermore he needs

$$[V(r), V(r')] = 0, \tag{29}$$

where $V(r)$ is the potential.

A Regge eikonal form has been derived by several other authors,[11] basically by iterating t-channel ladders which give Regge behavior. We shall discuss some of these calculations in Sec. VI; though the results appear similar to ours, they are of course very different from our calculations, relying as they do on an underlying field theory to obtain the scattering amplitude and hence neglecting isospin, hadron resonances, etc.

B. Triple-Regge Limit

The superconvergence relation (17) can also be derived directly by examining Reggeon + hadron → Reggeon + hadron scattering or alternatively a six-point function as depicted in Fig. 2. In the

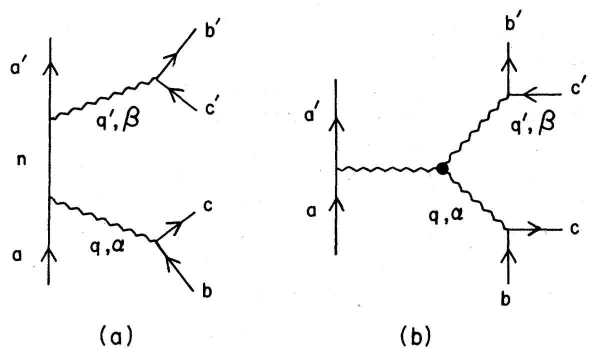

FIG. 2. (a) Amplitude for $a+b+c \to a'+b'+c'$ in Reggeon-resonance scattering approximation. (b) Triple-Reggeon description of $a+b+c \to a'+b'+c'$.

limit of $(p_b - p_c)^2 = q^2$, $(p_{b'} - p_{c'}) = q'^2$ finite with $(p_a + p_b)^2 = s$ and $(p_a + p_b - p_c)^2 = r$ both tending to infinity, we deduce the asymptotic behavior of the amplitude in Fig. 2 using the well-known triple-Regge limit as depicted in Fig. 2(b). Assume that particles b, c and b', c' are such that the leading trajectory is the ρ, i.e., the bc and $b'c'$ states have $I=1$, etc.; furthermore let us antisymmetrize in α and β so that the trajectory coupling to a, a' also has $I=1$. For simplicity let us go to the forward direction in particle a, i.e., $p_a = p_{a'}$ and then write $(p_a + p_{b'})^2 = s'$, $(p_a + p_{b'} - p_{c'})^2 = r' = r$. The asymptotic behavior of Fig. 2(b) is then given by

$$T^{\text{antisymm., triple Regge}}_{a+b+c\to a'+b'+c'} \sim \beta^{\gamma}_{aa'}(0)\beta^{\alpha}_{bc}(q^2)\beta^{\beta}_{b'c'}(q'^2)\epsilon^{\alpha\beta\gamma} \times \eta(q^2, q'^2)\left(\frac{s}{r}\right)^{\alpha(q^2)}\left(\frac{s'}{r}\right)^{\alpha(q'^2)} r^{\alpha(0)}, \tag{30}$$

where the β's are residue functions, η is the triple-Reggeon vertex, and the trajectory functions $\alpha(t)$ are all taken to be the ρ trajectory.

Alternatively, one may calculate in the resonance approximation, as shown in Fig. 2(a), and find for $r \gg$ (resonance mass)^2

$$T^{\text{antisymm., resonance}}_{a+b+c\to a'+b'+c'} \sim \beta^{\alpha}_{bc}(q^2)\beta^{\beta}_{b'c'}(q^2)\left(\frac{s}{r}\right)^{\alpha(q^2)}\left(\frac{s'}{r}\right)^{\alpha(q'^2)} \times [M^{\alpha}(\vec{q}), M^{\beta}(\vec{q}\,')]\left(\frac{r}{q_-}\right)^2 \frac{1}{r}. \tag{31}$$

The coupling of the ρ^α trajectory to the state of an incoming particle labeled by a and an outgoing resonance n is given by $[M^\alpha(q)]_{an} \times r/q_-$ with $r = (p_a + q)^2 = (p_a + q')^2$. A comparison of (30) and (31) leads once again to

$$[M^{\alpha}(\vec{q}), M^{\beta}(\vec{q}\,')] = 0, \tag{32}$$

which allows us to derive directly the eikonal expansion with the eikonal function χ_R as given in Eq. (26).

As a final comment to this section, the forms $(s/r)^{\alpha(q^2)}$, $(s'/r)^{\alpha(q'^2)}$ appeared in Eqs. (30) and (31) because the cosines of scattering angles z_{ab}, $z_{a'b'}$ for s, r, $s/r \to \infty$ and s', r, $s'/r \to \infty$ behaved, respectively, as s/r and s'/r. One might wonder then why our Reggeon propagator in (26) had the form $s^{\alpha(q^2)}$ rather than $(s/r)^{\alpha(q^2)}$; the reason is that after eikonalizing, as we saw in Eq. (10), we introduce a $\delta(q_-)$ so $r \sim p_+ q_- \to \infty$ in the considerations of this subsection, but does not in the eikonal formula where $p_+ \to \infty$ but $q_- \to 0$. One might then call into question the assumption that $r \gg m_n^2$, the resonance mass squared; such are the ambiguities with which our procedure is fraught. Our chain of limits is to first take $r \to \infty$, neglect resonance masses, sum the exchanges, which leads to the $\delta(q_-)$ factors, and then write the formal sum.

It is even more difficult to test the validity of our model than to test conventional field-theoretical eikonal models, since we have no underlying Lagrangian. The one relevant remark that we can make is that the dual-resonance model does satisfy the superconvergence relations which are the base of our calculation; however, the intermediate-state resonances in that model do have masses going all the way to infinity and it is not a correct approximation to say that the beta function, $B(s,t)$, when expressed as a sum over s-channel poles, has the large-s behavior

$$B_{if}(s,t) = \sum_n \frac{\Gamma_{i,n}\Gamma_{n,t}}{s - m_n^2} \to \frac{1}{s}\sum_n \Gamma_{i,n}\Gamma_{n,t}, \tag{33}$$

where the Γ's are vertices between initial or final states and intermediate resonances. It is possible that we are taking the correct meson propagator but our expression for meson (Reggeon) + hadron $\to$ meson (Reggeon) + hadron is incorrect.

V. OPERATOR FORMALISM

A. Dynamical Aspects

We present in this section an operator description of our previous results. Let $B^{\alpha}_{\mu}(k)$ denote the field operator for the ρ meson, in momentum space. We shall define $\mathcal{J}^{\alpha}_{\mu}(k)$ to be that operator whose matrix elements are the form factors of ρ coupling to the systems considered. We shall suppose that the dominant matrix elements of $\mathcal{J}^{\alpha}_{\mu}(k)$ in the infinite-momentum limit are between single-resonance-particle states. Under such circumstances, the effective interaction Hamiltonian for the system may be written as[12]

$$H_{\text{int}} = g : \int \frac{d^3k}{\omega_k} \mathcal{J}^{\alpha}_{\mu}(\vec{k}) B^{\mu\alpha}(\vec{k}) : + \text{H.c.} \tag{34}$$

Within the eikonal approximation, we shall be interested only in those H_{int} which involve integration over "slow" $\vec{k}$'s. We normalize the operators by

$$[B_\mu^{\dagger\alpha}(\vec{k}),\ B_\nu^\beta(\vec{k}')] = g_{\mu\nu}\,\omega_k\,\delta^{(3)}(\vec{k}-\vec{k}')\,. \tag{35}$$

When we look at fast-moving sources along the z direction, we get

$$H_{\text{int}} = g : \int \frac{d^3k}{2\omega_k}\,[\mathcal{J}_+^\alpha(\vec{k})B_-^\alpha(\vec{k}) + \mathcal{J}_-^\alpha(\vec{k})B_+^\alpha(\vec{k})] : + \text{H.c.} \tag{36}$$

to leading order in the energy of the sources.

To sum multiple meson exchanges we first perform a canonical transformation to get rid of all explicit B dependence. (We shall defer all discussions of inelastic scattering.) The relevant unitary transformation $U = e^{iF}$ we require is generated, as in Ref. 3, by

$$F = g : \int \frac{d^3k}{2\omega_k}\left[\frac{\mathcal{J}_+^\alpha(\vec{k})B_-^\alpha(\vec{k})}{\omega - k_z} + \frac{\mathcal{J}_-^\alpha(\vec{k})B_+^\alpha(\vec{k})}{\omega + k_z}\right]. \tag{37}$$

In evaluating the result of this transformation, we shall need to know $[\mathcal{J}_\pm^\alpha(k), \mathcal{J}_\pm^\beta(k)]$. It is clear that within the eikonal approximation $\mathcal{J}_+^\alpha$ and $\mathcal{J}_-^\beta$ will commute, to leading order. What is not clear is the commutator of $\mathcal{J}_+^\alpha$ and $\mathcal{J}_+^\beta$.

The key observation we are making is that this commutator is, in fact, zero, to leading order. The basis for this observation lies in the validity of the superconvergence relations discussed in a previous section. With this additional piece of information, we may go through a procedure analogous to that carried out in the analyses of massive QED,[3] and obtain for an effective potential for quasi-elastic scattering

$$V = 2g^2 \int d^3k\, \frac{\mathcal{J}_+^\alpha(\vec{k})\mathcal{J}_-^\alpha(\vec{k})}{\vec{k}_\perp^{\,2} + m^2} \tag{38}$$

to leading order. The resulting scattering operator can now be obtained by going to the interaction picture, just as before,[3] and we get

$$\begin{aligned} V_{\text{int. pict.}}(t) &= e^{i(H_R+H_L)t}\, V e^{-i(H_R+H_L)t} \\ &= 2g^2 \int d^3k\, e^{2ik_zt}\, \frac{\mathcal{J}_+^\alpha(\vec{k})\mathcal{J}_-^\alpha(-\vec{k})}{\vec{k}_\perp^{\,2} + m^2}, \end{aligned} \tag{39}$$

where H_R and H_L are the free Hamiltonian for the resonances moving with infinite momentum to the right and left. The scattering operator simplifies by virtue of the commutativity of $\mathcal{J}$'s:

$$\begin{aligned} S &= T\exp\left[-i\int V_{\text{int. pict.}}(t)\,dt\right] \\ &= e^{-ig^2\int d^2k_\perp \frac{\mathcal{J}_+^\alpha(\vec{k}_\perp)\mathcal{J}_-^\alpha(-\vec{k}_\perp)}{\vec{k}_\perp^{\,2}+m^2}}, \end{aligned} \tag{40}$$

which is our previous result, of Sec. II, written in an operator language.

The derivation of this final equation parallels that of the analogous equation for massive-neutral-vector-meson theory.[3] The resulting amplitude gives a constant cross section just as before; our resonance approximation is unable to produce any suppression for charge-exchange cross sections. Possible ways of bringing about charge-exchange suppression have been discussed in Sec. IV.

B. Algebraic Aspects

It is interesting to look at the implications of the commutativity conditions on the algebraic structure of the theory. Let us, for the sake of argument, suppose that our system possesses only isotopic symmetry. The generators of SU(2) and the effective source of the ρ field at high energies then define the algebra of E(3):

$$\begin{aligned} &[Q^\alpha, Q^\beta] = i\epsilon^{\alpha\beta\gamma}Q^\gamma\,, \\ &[Q^\alpha, \mathcal{J}^\beta] = i\epsilon^{\alpha\beta\gamma}\mathcal{J}^\gamma\,, \\ &[\mathcal{J}^\alpha, \mathcal{J}^\beta] = 0\,. \end{aligned} \tag{41}$$

Q^α is the isotopic group generator, while $\mathcal{J}^\alpha$ is the operator whose matrix elements give the couplings of the ρ quanta to the matter system. Therefore, the states used in evaluating the eikonal operator are representations of E(3) algebra.

The algebra of E(3) is noncompact, and these representations therefore contain infinite isotopic-spin multiplets. It should be noted, however, that these multiplets are all moving very fast; the E(3) structure is one that is true only to leading order in the infinite-momentum limit. The situation is then analogous to the one in strong-coupling theories.[13]

It is probably true that conventional Yang-Mills type models do not possess such a structure, for in such models the source of the Yang-Mills quanta is in fact the isotopic-spin current. The only algebraic structure is that of SU(2).

The source of the ρ field in such models is mostly generated by a gauge principle, and, while eikonalization cannot be proved, it is probably also true that the amplitudes of such models do *not* superconverge. The defining statement for the model considered is superconvergence, so that we are *not* analyzing the class of gauge models. The gauge principle, normally, will generate a source for the ρ field which is diagonal in the multiplets of hadrons. Within our resonance approximation, no excited states will be included in the intermediate states. To simulate excited-states contribution in our scheme, we must add an interaction Lagrangian which is explicitly offdiagonal in the multiplet fields. However, the precise form which this coupling takes will not be determined by the gauge principle. Therefore, the source function need not satisfy any algebraic relations characteristic of the symmetry of the effective Lagrangian. The requirements on the asymptotic behavior of the amplitudes (i.e., the superconvergence relations), however, do place constraints on our system, and it is these constraints that enable us to eikonalize.

VI. INELASTIC EFFECTS

In all of our considerations, we have implicitly assumed that the dynamics of fast-moving states is decoupled from that of slow-moving ones. The final eikonal form for scattering is valid only in the approximation that slow-moving dynamics is absent. We have, of course, not proved that such a separation is always possible, although studies of explicit field-theory models[2,3] suggest the validity of such assumptions. Here, we take it to be one of the properties defining our model considerations.

Now, the states defining the eikonals interact via exchanges of slow-moving particles, and these, once emitted, may interact among themselves. The final form of the scattering operator, suggested by model field theories,[3] is

$$S=S_0 T\exp\left[i\int_{-\infty}^{\infty} V^s(t)dt\right], \tag{42}$$

where S_0 is the eikonal scattering operator, and $H_{\rm int}\, V_s$ is the Hamiltonian controlling the time evolution of the exchanged states.

As long as we are interested in elastic or quasi-elastic amplitudes, we may suppose that S_0 is adequate to describe scattering. We have, in Sec. V, taken into account the V^s effects, within our approximations, by means of resonances of arbitrarily high spins. As we saw in Sec. V, the net effect of such exchanges is to convert the exponent of S_0 into effective Regge eikonal factors. The full scattering operator is then

$$S=\exp i\left[\frac{g^2}{(2\pi)^2}\int d^2k_\perp M^\alpha(\vec{k}_\perp)M^\beta(-\vec{k}_\perp)\Delta_{\rm Regge}^{\alpha\beta}(\vec{k}_\perp)\right],$$

$$\Delta_{\rm Regge}^{\alpha\beta}=\frac{\delta^{\alpha\beta}\pi\alpha'(-\vec{k}_\perp{}^2)}{\sin\pi\alpha(-\vec{k}_\perp{}^2)}(W^2)^{\alpha(-\vec{k}_\perp{}^2)-1}, \tag{43}$$

$$W=p_z\to\infty .$$

The fact that absence of V^s implies absence of inelasticity can be seen from the derivation of S_0 given in Sec. V: The effective potential contains no bremsstrahlung. The construction of S_0 is exactly analogous to that of Ref. 3. We are, however, neglecting the important mechanism of production by evaporation from the exchanged particles (see first paper of Ref. 5).

We write the total Hamiltonian following the notation of Ref. 3 as

$$H=H_R+H_L+H_s+H_{\rm int},\qquad H_{\rm int}\sim \mathcal{J}_\mu^\alpha B^{\mu\alpha}. \tag{44}$$

Once again, $B_\alpha(k)$ describes slow-moving particles, and $\mathcal{J}_\pm^\alpha(k)$ consists solely of the fast-moving particles. Within the eikonal context, we require that these commute. Suppose now that we wish to evaluate the cross section for scattering into a final state involving N slow-moving mesons. Denote such a state by $|b,k_1\cdots k_N\rangle_{(-)}$. Following standard reduction procedure, as, for instance, spelled out in Schweber,[14]

$$|b,k_1\cdots k_N\rangle_{(-)}=B^{\dagger\alpha_1}(k_1)B^{\dagger\alpha_2}(k_2)\cdots B^{\dagger\alpha_N}(k_N)|b\rangle + \frac{1}{E_\beta+\omega_{k_1}+\cdots+\omega_{k_N}-H-i\epsilon}\left[\mathcal{J}^{\alpha_1}(k_1)B^{\dagger\alpha_2}(k_2)\cdots B^{\dagger\alpha_N}(k_N)+\cdots+\mathcal{J}^{\alpha_N}(k_N)B^{\dagger\alpha_1}(k_1)\cdots\right]|b\rangle . \tag{45}$$

All states above are exact eigenstates of H. Let us now say that $N=2$; the relevant cross section then depends on the amplitude

$${}_{(+)}\langle a|[\mathcal{J}^{\alpha_1}(k_1)B^{\dagger\alpha_2}(k_2)+\mathcal{J}^{\alpha_2}(k_2)B^{\dagger\alpha_1}(k_1)]|b\rangle_{(-)}=R_{ab}(k_1,k_2). \tag{46}$$

We may, if we wish, rewrite this in a form independent of $B^{\dagger\alpha_i}(k_i)$:

$$R_{ab}(k_1,k_2)={}_{(+)}\left\langle a\left|\mathcal{J}^{\alpha_1}(k_1)\frac{1}{E_a-\omega_{k_1}-H-i\epsilon}\mathcal{J}^{\alpha_2}(k_2)\right|b\right\rangle_{(-)}+{}_{(+)}\left\langle a\left|\mathcal{J}^{\alpha_2}(k_2)\frac{1}{E_a+\omega_{k_2}-H-i\epsilon}\mathcal{J}^{\alpha_1}(k_1)\right|b\right\rangle_{(-)}. \tag{47}$$

This final form for $R_{ab}(k_1,k_2)$ may be used as the starting point for a perturbative evaluation of $R_{ab}(k_1,k_2)$. Within the context of our resonance approximation, this is equivalent to examining only the Born terms. Therefore,

$$R_{ab}(k_1,k_2)=\sum_n {}_{(+)}\left\langle a\left|\frac{\mathcal{J}_{an}^{\alpha_2}(k_2)\mathcal{J}_{nb}^{\alpha_1}(k_1)}{E_a-\omega_{k_2}-E_n-i\epsilon}\right|b\right\rangle_{(-)}+\sum_n {}_{(+)}\left\langle a\left|\frac{\mathcal{J}_{an}^{\alpha_1}(k_1)\mathcal{J}_{nb}^{\alpha_2}(k_2)}{E_a+\omega_{k_2}-E_n-i\epsilon}\right|b\right\rangle_{(-)}$$

$$\simeq-\left(\frac{1}{(k_2)_z+\omega_{k_2}}\right)\langle a|[\mathcal{J}^{\alpha_2}(k_2),\mathcal{J}^{\alpha_1}(k_1)]|b\rangle . \tag{48}$$

Here, we have used infinite-momentum kinematics, with $E_a, E_n \to \infty$, and we have also supposed that the k^i's came from the right-moving system.

We have gone to some lengths to arrive at this result, which is quite apparent using the method of canonical transformation of Sec. V. We have done so to bring out a formal analogy with strong-coupling models.[15] Recall that in such models one normally solves for the equation of motion within the static limit, allowing for no inelastic effects. This forces the commutativity of the coupling matrices through a set of equations analogous to Eqs. (45)-(48). In our considerations, $P_\pm = 2W$ plays the role of the nucleon mass in the static limit, while our resonance approximation parallels the Born approximation used in strong-coupling theories.

Of course, at high energies, we expect highly inelastic effects to occur. To handle this, it will be necessary to consider one or both of the following alternatives, in light of the above circumstances. First, we may relax our resonance approximation, and second, we shall have to include slow-moving-particle dynamics. With respect to the first alternative, let us remark that the kind of inelastic effects we were talking of above is the analog of emission of slow particles by bremsstrahlung in massive QED. There, such processes are forbidden if one uses the eikonal approximation, and the result is true even if we were to include fragmentation. Thus, we expect that the first alternative, by itself, is insufficient to give inelasticity. We must, therefore, consider slow-moving dynamics. In Feynman-diagram language we must sum over lower diagrams. The actual inelastic cross sections are then proportional to the various s-channel discontinuities.

The total Hamiltonian H_{tot} of the system may be written as

$$\begin{aligned} H_{\text{tot}} &= H_R + H_L + H_s + V + V_s \\ &= H + V_s \,. \end{aligned} \tag{49}$$

Notice that inclusion of V_s into H_{tot}, where V_s describes the interaction of slow-moving states, will modify the definition of the scattering states. Where previously we had built up the complete scattering states perturbatively from eigenvectors of $H_R + H_L$, assumed to be given by narrow resonances, we must now build these from $H_R + H_L + V_s$. If in a given step in the perturbation we have a state with one or more slow-moving ρ's, denoted by $|b, k_1 \cdots k_n\rangle$, $|b\rangle$ being a narrow resonant state, then, say for $n=1$,

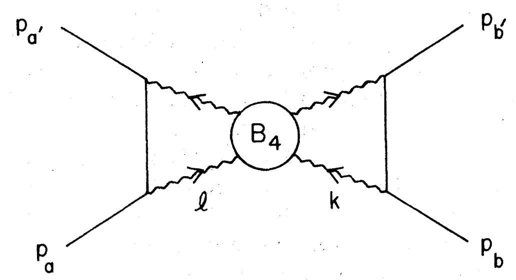

FIG. 3. Meson-meson scattering correction term in two-meson-exchange diagram.

$$|b, k_1\rangle_{\text{in}} = a^\dagger(k_1)|b\rangle_{\text{in}} + |\chi\rangle_{\text{in}} \,, \tag{50}$$

where

$$|\chi\rangle_{\text{in}} = \frac{1}{E_b - H_R - H_L - V_S + i\epsilon}\,[V_s, a_k^\dagger\,]\,|b\rangle_{\text{in}} \,. \tag{51}$$

The nonvanishing of $|\chi\rangle$ will most certainly introduce widths into the resonances. In Feynman-diagram language, the nonvanishing of χ is responsible for nontrivial vertex and mass corrections to the resonance approximations. We shall, in the following, explicitly see how this comes about. The form of the corrections will, not unexpectedly, depend on V^s.

The exact nature of V^s is, of course, at present unclear, and we must, once again, resort to models. One such model is presented in Sec. IV, where the effect of V^s is approximated by the exchange of resonances of arbitrarily high spin. The result sums up to a Regge eikonal form.

An equivalent way of understanding the result is by supposing that V^s is actually that potential that causes the scattering to take the form as prescribed by dual-resonance models. Consider then the resulting amplitude to the first nontrivial order as shown in Fig. 3. Then, temporarily ignoring isospin, the amplitude is

$$\begin{aligned} A(p_a, p_{a'}; p_b, p_{b'}) = \int d^4l\,(p_+)_a \delta(l_-) \frac{1}{(p_{a'} - p_a - l)^2 - m^2}\frac{1}{l^2 - m^2} \\ \times B_4(s,t)\,\frac{1}{(k^2 - m^2)[(p_{b'} - p_b - k)^2 - m^2]}\;\delta(k_+)(p_-)_b\, d^4k + \text{permutations}\,. \end{aligned} \tag{52}$$

The integration is over slow-moving momenta while B_4 is the standard beta function[16]:

$$
\begin{aligned}
&B_4(s,t)=\frac{\Gamma(n-\alpha(s))\Gamma(m-\alpha(t))}{\Gamma(p-\alpha(s)-\alpha(t))}\,,\\
&s=(l+k)^2\,,\quad t=(p_a-p_{a'})^2\,,\\
&(p_+)_a,(p_-)_b\sim 2W\to\infty\,;\\
&n,m,p \text{ integers}.
\end{aligned}
\tag{53}
$$

Therefore,

$$
\begin{aligned}
A(p_a,p_{a'};p_b,p_{b'})&=\int dl_+dl_-d^2l_\perp dk_+dk_-d^2k_\perp\frac{\delta(k_+)\delta(l_-)}{(k^2-m^2)(l^2-m^2)}\frac{B_4(s,t)}{[(p_{a'}-p_a-l)^2-m^2][(p_{b'}-p_b-k)^2-m^2]}\\
&=\int dl_+d^2l_\perp\frac{1}{[t+l^2-m^2-2(\vec{p}_{a'}-\vec{p}_a)_\perp\cdot\vec{l}_\perp]}\frac{1}{(l^2-m^2)}\\
&\quad\times\int\frac{dk_-d^2k_\perp}{k^2-m^2}\frac{1}{[t+k^2-m^2-2(\vec{p}_{b'}-\vec{p}_b)_\perp\cdot\vec{k}_\perp]}\sum_n\frac{\Gamma^n(s)}{t-m_n^2}\\
&=\sum_n\left[\int dl_+d^2l_\perp\frac{1}{[t+l^2-m^2-2(\vec{p}_{a'}-\vec{p}_a)_\perp\cdot\vec{l}_\perp]}\frac{1}{l^2-m^2}\tilde{\Gamma}^n_R(l,p_a,p_{a'})\right]\\
&\quad\times\frac{1}{t-m_n^2}P_n(z_t)\left[\int\frac{dk_-d^2k_\perp}{(k^2-m^2)}\frac{1}{[t+k^2-m^2-2(\vec{p}_{b'}-\vec{p}_b)_\perp\cdot\vec{k}_\perp]}\tilde{\Gamma}^n_L(k,p_b,p_{b'})\right]\\
&\equiv\sum_n\tilde{\Gamma}_R(l,p_a,p_{a'},t)\tilde{\Gamma}_L(k,p_b,p_{b'},t)\frac{P_n(z_t)}{t-m_n^2}\,.
\end{aligned}
\tag{54}
$$

We have explicitly written out the B function as a sum over its resonance poles,

$$
B_4(s,t)=\sum_n\frac{\Gamma^n(s)}{t-m_n^2}=\sum_n\frac{\tilde{\Gamma}^n_R(l,p_a,p_{a'})\tilde{\Gamma}^n_L(k,p_b,p_{b'})P_n(z_t)}{t-m_n^2}\qquad(z_t\text{ is the scattering angle}).
\tag{55}
$$

The summation is over all dual resonances; $\tilde{\Gamma}^n_R$ and $\tilde{\Gamma}^n_L$ are, in general, some Pockhammer polynomials.[17,18] $\tilde{\Gamma}_R, \tilde{\Gamma}_L$ represent the vertex-connect ed residue functions. If we assume that $\tilde{\Gamma}^n_R(t)=\tilde{\Gamma}^n_R(m_n{}^2)$, $\tilde{\Gamma}^n_L(t)=\tilde{\Gamma}^n_L(m_n{}^2)$, and only sum over the leading trajectory, we obtain the result of Sec. IV.

As explained in Sec. IV, the resulting expression yields an amplitude that goes to zero asymptotically, and therefore does not contain the Pomeranchukon trajectory. The above scheme of understanding the multi-Regge exchange, however, does give us a means of building up the Pomeranchukon singularity; it comes from nonplanar loop contributions of V^s. We make no attempt at this time to sum up such contributions, but merely point out that the model does have a means of generating such diffractive singularities.

The explicit use of the B_4 function in the above arguments is actually unnecessary. Thus, in place of B_4, we may write $B(s,t)$, where

$$
B(s,t)=\mathrm{Re}B(s,t)+i\,\mathrm{Im}B(s,t)\,.
\tag{56}
$$

For fixed t, $\mathrm{Im}B(s,t)\neq 0$ along the s and the u cuts. The basic commutativity of the eikonal vertex means that we shall be interested in the s-u crossing-symmetric part of $B(s,t)$. Then we may write

$$
A=\int d\left(\frac{l_+}{k_-}\right)J\left(\frac{l_+}{k_-}\right)d^2l_\perp d^2k_\perp\{D\}\times\left[\int_{s_0}^N ds'\,\mathrm{Re}B(s',t)+i\int_{s_0}^N ds'\,\mathrm{Im}B(s',t)\right]\,,
\tag{57}
$$

where $J(l_+/k_-)$ is the Jacobian of the transformation from k_-, l_+ to k_-l_+ and l_+/k_-, $\{D\}$ refers to the various propagators, and N is the slow-moving kinematic cutoff.

To get the results of Sec. IV, we employ the finite-energy sum rules[19] on the integral over $\mathrm{Im}B(s,t)$ to get a sum over Regge poles. As N gets large enough, we may neglect the first integral, and we recover the first term of the Regge eikonal. The integrations over the other variables give the vertex corrections referred to above.

VII. CONCLUSIONS

We have derived a formula for large-energy hadron-hadron scattering which in eikonal form sums the effect of excitation of hadron resonances and exchange of an arbitrary number of ρ mesons or ρ trajectory Reggeons; the result is the so-called Reggeized eikonal model. If nothing else, it is interesting as a complement to several calculations which derive a scattering amplitude of a similar form by using field-theoretical techniques. Our calculation does not have an underlying model skeleton, and the key ingredient is a class of strong-interaction sum rules. Unlike field-theoretical models, our calculation includes the effect of isospin and high-spin resonances; it does not give constant cross sections, however, i.e., we are unable to introduce a vacuum trajectory into our calculation.

There are several points at which our approximations may break down, and we have tried to discuss them as we went along; a key limitation is our neglect of all vertex corrections and self-mass or finite-width corrections for resonances. Just as in the dual-resonance model, these may well play a key role in obtaining constant cross sections, but we are unable to handle them systematically. We also neglect production, though some remarks on these last points are contained in Sec. VI.

One of the reasons that keeps us from examining in more detail the structure of our results is our basic ignorance of the $\rho(\rho$ trajectory$)$ + hadron $\to\rho(\rho$ trajectory$)$ + hadron scattering amplitude. From the asymptotic behavior of the graph with two Reggeons exchanged calculated [Figs. 4(a) and 4(b)] in our model, we see that our model must allow for a wrong-signature nonsense fixed pole at $J=1$ (Ref. 20) in the ρ hadron $\to\rho$ hadron scattering amplitude; otherwise our calculation of the asymptotic behavior is erroneous. At first glance, it might appear that the form for this ρ + hadron $\to\rho$ + hadron scattering amplitude employed contradicts our basic assumption of Regge-behaved amplitudes in that we are using

$$T^{\alpha\beta}(p,q)\underset{p_+\to\infty}{\sim}\frac{p_+p_+M^\alpha(q)M^\beta(q)}{p_+q_-}\,, \qquad (58)$$

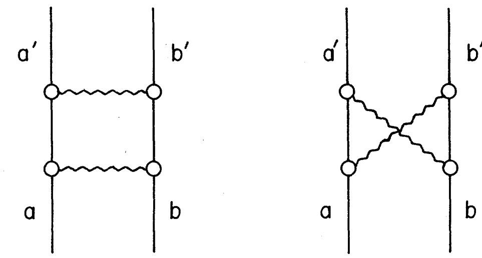

FIG. 4. Contributions to two-meson-exchange terms.

but note that once we have eikonalized, our eikonal form requires only a knowledge of the correct imaginary part of the amplitude T, not of the full amplitude, and the imaginary part of T is compatible with Regge asymptotic behavior. Note that this does not mean that we are calculating the imaginary part of the hadron-hadron scattering, as we are clearly doing more than this in our eikonal expansion; it is just that a knowledge of the imaginary part of the ρ hadron $\to\rho$ hadron amplitude is all that is needed to calculate the full hadron-hadron amplitude generated by ρ exchange in the eikonal approximation.

A final point on which we plead ignorance, unfortunately, is the connection, if any, with the Gribov Reggeon calculus.[21] Though these techniques may supersede what we have done, allowing for Pomeranchukon exchange,[22] etc., it is not clear that they will provide solutions to the questions discussed about asymptotic behavior.

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Content of the (3, 3*)+(3*, 3) Model in the Pole-Dominance Approximation

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The content of the (3,3*) + (3*,3)-symmetry-breaking model (including an isospin-violating interaction) is explored by using the spin-zero meson-dominance approximation for two- and three-point functions. By fully exploiting the pole-dominance approximation it is shown that a number of the assumptions made by Gell-Mann, Oakes, and Renner are unnecessary. The results indicate that the Hamiltonian is nearly SU(2)×SU(2)-symmetric and that a small isospin-violating interaction is present. One can conclude that (3,3*) + (3*,3)-symmetry-breaking models with Hamiltonians very different from this must contain large violations of pole dominance.

I. INTRODUCTION

Because of its simplicity the proposal[1] that the approximate SU(3)×SU(3) symmetry of the strong interactions is violated by a term in the Hamiltonian which belongs to a (3, 3*) + (3*, 3) representation of operators remains the most attractive possibility. A number of authors[2-5] have explored the physical content of the model by assuming that the matrix elements of at least some subset of the eighteen (3, 3*) + (3*, 3) operators u_i and v_i ($i=0, \ldots, 8$) are dominated by spin-zero meson poles. In the present work the pole-dominance assumption is further applied to the study of the two- and three-point functions which can be formed from the u_i and v_i.

By following the approach of Auvil and Deshpande,[5] simple expressions for the masses and decay constants of the spin-zero mesons which couple to the current divergences are found in terms of the parameters which characterize the nature of the symmetry breaking in the Hamiltonian and in the vacuum. These expressions permit a number of independent evaluations of the parameters which measure the strength of the SU(3)-symmetry violations. These evaluations are in reasonable agreement with each other and support the idea that the symmetry-breaking Hamiltonian is approximately SU(2)×SU(2)-symmetric[3] and that the vacuum state is nearly SU(3)-symmetric.

The present results depend on the assumption that the divergence of all nonconserved SU(3)×SU(3) currents are dominated by spin-zero meson poles. Although such neglect of continuum contributions represents a great simplification, it may still give a reasonable first approximation, and it provides a basis for estimating departures from this idealization of the physical world. In particular it emphasizes the fact that (3, 3*) + (3*, 3)-symmetry-breaking models which incorporate symmetry-breaking Hamiltonians which are very different from that of Gell-Mann, Oakes, and Renner (see Ref. 6, for example) must either tacitly or explicitly assume that the pole-dominance assumption is very badly violated.

The term, H', of the Hamiltonian which breaks SU(3)×SU(3) symmetry is assumed to have the form

$$H' = \epsilon_0 u_0 + \epsilon_8 u_8 + \epsilon_3 u_3 , \qquad (1)$$

where ϵ_0, ϵ_8, and ϵ_3 are constants. The u_0, u_8, and u_3 are members of a nonet of scalar operators u_i, which together with a nonet of pseudoscalars v_i transform according to the (3, 3*) + (3*, 3) rep-