Gauge-Field Theory of Particles. II. Fermions*

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We construct classes of Lagrangians which describe families of fermions containing an infinite number of particles. The Lagrangians depend on Rarita-Schwinger fields with k Lorentz indices, $k=1, 2, \ldots$, which have bilinear interactions among themselves. These Lagrangians are invariant under gauge transformations of the second kind. The physical states appear as the normal modes of these field theories, and by suitable choices of the masses of the underlying gauge fields, the physical fermions can be made to lie on linearly rising Regge trajectories. The currents have nontrivial diagonal matrix elements, and also have matrix elements between states of different spin. By considering time-ordered products of currents between single-particle states, we are able to construct on- and off-mass-shell N-point functions in the narrow-resonance approximation. Specifically, we give rules for calculating πN scattering and inelastic electron scattering in that approximation.

I. INTRODUCTION

In our previous paper on bosons[1] (hereafter denoted by I), we suggested that the physical properties of the bosons (masses and form factors) could be described by considering the particles to be normal modes of an underlying field theory possessing gauge invariance of the second kind. We showed that the prinicple of gauge invariance of the second kind generated Lagrangians containing an infinite number of fields with bilinear interactions between nearest neighbors in the Lorentz index space k. We then showed how to choose the parameters of the underlying field theory so that the physical particles lay on linearly rising Regge trajectories, and constructed vector and axial-vector currents made up of particles on the π and ρ trajectories which obeyed the SU(2) $\otimes$ SU(2) algebra. We also gave a set of rules to obtain scattering amplitudes in the narrow-resonance approximation. Explicitly we pictured scattering to take place as follows: A particle starts out as a given normal mode of the underlying field theory, and absorbs and emits external quanta with corresponding excitation and deexcitation of the underlying fields. We obtained the vertices by postulating that the external quanta coupled to the appropriate currents inherent in the Lagrangian – the ρ to the vector current, the π to the divergence of the axial-vector current.

In this paper we extend these ideas to the case of fermions. In Sec. II we show how gauge invariance of the second kind leads naturally to the study of a particular class of first-order Lagrangians, which are the analogs of the second-order boson Lagrangians previously considered. These Lagrangians have the nice property that although the field equations depend linearly on the masses of the underlying field, the Regge trajectories for the physical fermions (the normal modes) are a function only of the mass squared of the physical fermions. Thus gauge invariance seems to provide us with a special subset of possible infinite-component wave equations – those whose spectrum depends on m^2 and not on m.

The eigenvalue problem for the masses of the physical particles, which we discuss in Sec. III, is similar to the boson case, and we show how to select the masses and couplings of the underlying field theory so that the fermions lie on linearly rising Regge trajectories. We then introduce isospin and construct the vector and axial-vector currents for the nucleons. In Sec. IV we show how the matrix elements of these currents between particles on the trajectory lead to a power series for the form factors in terms of Chebychev polynomials of the second kind in $(p\cdot p')/m^2$, and their derivatives. One hopes that there exists a particular choice of the underlying parameters which will reproduce dipole form factors. By assuming the photon couples minimally to the underlying fields, we are led to the narrow-resonance approximation for on- and off-mass-shell proton Compton scattering.

Our treatment of π-N scattering is similar to that of π-ρ scattering in our previous work; the new feature here is that in addition to the couplings for external π's and ρ's described above, we also postulate that the nucleon couples to the baryonic current generated by the free Lagrangian. The procedure for constructing the scattering amplitudes is based on the general idea of using external quanta to excite and de-excite the normal modes, and is described in detail in Sec. V.

II. GAUGE INVARIANCE OF THE SECOND KIND AND CONSTRUCTION OF THE FERMION LAGRANGIAN

In I we considered the symmetric, traceless boson fields $\varphi_{\mu_1\cdots\mu_k}(x)$, and constructed the quantity

$$G_{\nu_1\cdots\nu_k}=\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}\partial_{\mu_k}\varphi_{\mu_1\cdots\mu_{k-1}}+\alpha_k\varphi_{\nu_1\cdots\nu_k} \tag{2.1}$$

(or $G_k=\delta\partial\varphi_{k-1}+\alpha_k\varphi_k$ in shorthand), where $\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}$ is the projection operator onto the space of traceless symmetric tensors. We showed that G_k was invariant under the gauge transformation of the second kind:

$$\varphi_{\mu_1\cdots\mu_k}\to\varphi_{\mu_1\cdots\mu_k}+\gamma_k\delta^{\nu_1\cdots\nu_k}_{\mu_1\cdots\mu_k}\partial_{\nu_1}\cdots\partial_{\nu_k}\Lambda(x) \tag{2.2}$$

provided $\gamma_{k-1}+\alpha_k\gamma_k=0$. This leads to the invariance of the boson Lagrangian

$$\mathfrak{L}=\sum_k\eta_k[G^k(\delta\partial\varphi_{k-1}+\alpha_k\varphi_k)-\tfrac{1}{2}G_kG^k] \tag{2.3}$$

under the transformation (2.2).

From the Lagrangian (2.3) we obtain a second-order wave equation for φ_k. For fermions we would like a first-order wave equation. To consider half-integral spin we let our fields take on a spinor index α, i.e.,

$$\varphi_k\to\psi_{k\alpha}\equiv\psi_{\mu_1\cdots\mu_k\alpha}$$

which is a Rarita-Schwinger field transforming under the $(\frac{1}{2}k,\frac{1}{2}k)\otimes[(\frac{1}{2},0)\oplus(0,\frac{1}{2})]$ representation of the homogeneous Lorentz group.

We then consider the quantity[2]

$$G_{k\mu\alpha}=i\delta\partial_\mu\psi_{k\alpha}+\overline{\alpha}_k\psi_{k\mu\alpha} \tag{2.4}$$

(where $G_{k\mu\alpha}\equiv G_{\mu_1\cdots\mu_k\mu\alpha}$), which is the fermion analog of Eq. (2.1).

Since for fermions γ^μ as well as ∂^μ is a vector, $G_{k\mu\alpha}$ is invariant not only under the transformation

$$\psi_{k\alpha}\to\psi_{k\alpha}+\lambda_k\delta\partial_{\mu_1}\cdots\partial_{\mu_k}u_\alpha(x), \tag{2.5a}$$

where $u_\alpha(x)$ is an arbitrary anticommuting c-number spinor, but also under

$$\psi_{k\alpha}\to\psi_{k\alpha}+\lambda_k\delta\gamma_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_k}u_\alpha(x) \tag{2.5b}$$

if $k\geq 1$. In choosing which of (2.5a) or (2.5b) to use, we are guided by the fact that

$$\delta\overline{\psi}^k\gamma^\mu\equiv\delta^{\mu\,\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_{k+1}}\overline{\psi}^{\nu_1\cdots\nu_k}\gamma^{\nu_{k+1}} \tag{2.6}$$

is automatically invariant under (2.5b), since for any tensor A, $\delta\gamma\gamma A=0$ (or, in other words, $\gamma^\mu\gamma^\nu$ has no symmetric traceless part). Therefore, we can construct a first-order gauge-invariant Lagrangian essentially by contracting (2.6) with (2.4) and summing over k:

$$\mathfrak{L}(x)=\sum_{k=1}^{\infty}\eta_k\overline{\psi}^k\gamma^\mu(i\delta\partial_\mu\psi_k+\overline{\alpha}_k\psi_{k\mu})+\text{H.c.} \tag{2.7}$$

This Lagrangian leads to the field equations

$$i(\gamma\cdot\delta\partial\psi_k+\partial\cdot\delta\gamma\psi_k)+\overline{\alpha}_k\gamma\cdot\psi_{k+1}+\frac{\eta_{k-1}}{\eta_k}\overline{\alpha}^*_{k-1}\delta\gamma\psi_{k-1}=0. \tag{2.8}$$

We notice that the Lagrangian (2.7) is invariant under the translation $\overline{\psi}^\mu\to\overline{\psi}^\mu+\overline{u}(x)\gamma^\mu$ if $\overline{u}(x)$ is replaced by a constant [this follows since there is no term $\overline{\alpha}_0\psi_\mu$ in (2.7)]. This leads to the conserved current

$$j^\mu_\alpha(x)=\frac{\delta\mathfrak{L}}{\delta\partial_\mu\overline{u}(x)}=\frac{-3i\eta_1}{2}[2\psi^\mu-\tfrac{1}{2}\gamma^\mu(\gamma\cdot\psi)]. \tag{2.9}$$

We notice that j^μ obeys $\gamma_\mu j^\mu=\partial_\mu j^\mu=0$.

In order to quantize this gauge-invariant theory, we must choose a particular gauge. For example, the Lorentz gauge is given by $\gamma_\mu\psi^\mu=0$. We then see from the conserved current $\partial_\mu j^\mu=0$ that $\partial_\mu\psi^\mu=0$. Thus in the Lorentz gauge, ψ^μ is a pure spin-$\frac{3}{2}$ field and obeys the equation of motion

$$i\gamma\cdot\partial\psi_\nu+\overline{\alpha}_k\gamma^\mu\psi_{\mu\nu}=0.$$

It follows that $\partial^\nu(\gamma^\mu\psi_{\mu\nu})=0$ as well. Thus, gauge invariance tells us that there are no spin-$\frac{1}{2}$ particles in the theory; ψ^μ obeys the usual equal-time commutation relations of the spin-$\frac{3}{2}$ Rarita-Schwinger field. To quantize $\psi^{\mu\nu}$, etc., it is probably best to go to the radiation gauge and follow the paper of Chang.[3] We shall not, however, pursue the question of quantization further here.

We remark that these gauge transformations of the second kind cannot be implemented by means of a unitary transformation. Each choice of gauge leads to a different quantization procedure in a different Hilbert space. As in the boson case, we can study more general Lagrangians by requiring only that the field equations are invariant under the restricted set of gauge transformations

$$\psi^k\to\psi^k+\lambda_k\delta\gamma\partial^{k-1}u(x) \tag{2.10}$$

as long as

$$(i\gamma\cdot\partial-m)u(x)=0.$$

This allows us to consider the following class of Lagrangians:

$$\mathfrak{L}=\tfrac{1}{2}\sum_{k=0}^{\infty}\eta_k\overline{\psi}^k\gamma^\mu(i\delta\partial_\mu\psi_k+\overline{\alpha}_k\psi_{k\mu}+i\overline{\beta}_k\partial^\lambda\psi_{k\lambda\mu})+\text{H.c.}, \tag{2.11}$$

from which we derive the field equations

$$i\gamma\cdot\delta\partial\psi_k + i\partial\cdot\delta\gamma\psi_k + \bar{\alpha}_k\gamma\cdot\psi_{k+1} + i\bar{\beta}_k\partial\cdot\gamma\cdot\psi_{k+2} + \left(\frac{\eta_{k-1}}{\eta_k}\right)\bar{\alpha}^*_{k-1}\delta\gamma\psi_{k-1} + \frac{i\eta_{k-2}}{\eta_k}\bar{\beta}^*_{k-2}\delta\partial\gamma\psi_{k-2} = 0. \quad (2.12)$$

The Lagrangian (2.11) and Eqs. (2.12) are invariant under the set of transformations

$$\psi_k \to \psi_k + \lambda_k^j\delta\gamma\partial^{k-j-1}u_j(x), \quad (2.13)$$

where $u_j(x)$ is an anticommuting Rarita-Schwinger spinor with j Lorentz indices satisfying

$$(i\gamma\cdot\partial - m_j)u_j = 0, \quad \gamma\cdot u_j = \partial\cdot u_j = 0. \quad (2.13a)$$

The connection between the λ_k^j in (2.13) and the parameters in (2.11) which is imposed by invariance is (see Appendix A)

$$2(k+2)(i\lambda_k^j + \bar{\alpha}_k\lambda_{k+1}^j) - im_j^2\bar{\beta}_k\lambda_{k+2}^j\frac{(k+1-j)(k+3)(k+j+3)}{(k+2)^2} = 0. \quad (2.13b)$$

Note that, by definition in (2.13), the λ_k^j are required to satisfy the boundary condition

$$\lambda_k^j = 0, \quad k \leq j. \quad (2.13c)$$

To close this section, we remark that, in the usual way, the Lagrangian (2.11) leads to the anticommutation relations:

$$\{\pi^{\mu_1\cdots\mu_k}(\vec{x}), \psi_{\nu_1\cdots\nu_k}(\vec{y})\} = -i\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}\delta^3(\vec{x}-\vec{y}), \quad (2.14)$$

where

$$\pi^{\mu_1\cdots\mu_k}(x) \equiv \frac{\delta\mathcal{L}}{\delta\partial_0\psi_{\mu_1\cdots\mu_k}} = i\eta_k\delta^{\mu_1\cdots\mu_k 0}_{\nu_1\cdots\nu_{k+1}}(\bar{\psi}^{\nu_1\cdots\nu_k}\gamma^{\nu_{k+1}}) + i\bar{\beta}_{k-2}\eta_{k-2}\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}\bar{\psi}^{\nu_1\cdots\nu_{k-2}}\gamma^{\nu_{k-1}}g^{\nu_k 0}. \quad (2.14a)$$

For $k=0$ and $k=1$, Eq. (2.14) becomes

$$\eta_0\{\psi^\dagger(\vec{x}), \psi(\vec{y})\} = -\delta^3(\vec{x}-\vec{y})$$

and

$$\tfrac{1}{2}\eta_1\{\psi^{\dagger\mu}(\vec{x}) + \bar{\psi}^0(\vec{x})\gamma^\mu - \tfrac{1}{2}g^{0\mu}\bar{\psi}^\lambda(\vec{x})\gamma_\lambda, \psi_\nu(\vec{y})\} = -\delta^\mu_\nu\delta^3(\vec{x}-\vec{y}). \quad (2.15)$$

III. MASS SPECTRUM

In order to determine the mass spectrum we assume the existence of a set of normal-mode free fields

$$\tilde{\psi}_j^{(N)}(x) \equiv \tilde{\psi}^{(N)}_{\mu_1\cdots\mu_j\alpha}(x)$$

of mass m_{Nj} and spin $j+\frac{1}{2}$. Being free fields, they satisfy

$$(i\gamma\cdot\partial - m_{Nj})\tilde{\psi}_j^{(N)} = 0, \quad \partial\cdot\tilde{\psi}_j^{(N)} = \gamma\cdot\tilde{\psi}_j^{(N)} = 0. \quad (3.1)$$

Since the underlying theory has only bilinear interactions, each field ψ_k can be written as an infinite sum over those normal modes which do not exceed its maximum spin, i.e., ψ_k can be expanded in terms of the $\tilde{\psi}_j^{(N)}$ for which $j \leq k$:

$$\psi_k = \sum_N\sum_{j\leq k}[a_k^{(Nj)}\delta\partial^{k-j}\tilde{\psi}_j^{(N)} + b_k^{(Nj)}\delta\gamma\partial^{k-j-1}\tilde{\psi}_j^{(N)}]. \quad (3.2)$$

By demanding that the wave equation (2.8) or (2.12) be satisfied for each component $\tilde{\psi}_j^{(N)}$ separately, we find that in general the coefficients $b_k^{(Nj)}$ obey an inhomogeneous difference equation in the index k (with N and j fixed), where the inhomogeneous term is a linear combination of the $a_k^{(Nj)}$. The $a_k^{(Nj)}$ themselves must satisfy a homogeneous difference equation in the index k. In the case of the fully gauge-invariant theory [Eqs. (2.8)] these difference equations are first order, while for the partially gauge-invariant case [Eqs. (2.12)] they are second order.

We assume that the physical states, represented by the $\psi_j^{(N)}$ in (3.2), are invariant under the gauge transformations. Thus the decomposition (3.2) is not invariant under the transformation (2.5b). This means that for the gauge-invariant theory, we have implicitly chosen the Hilbert space of the Lorentz gauge, and have removed all the spin-$\frac{1}{2}$ modes.

The coefficients $a_k^{(Nj)}$ and $b_k^{(Nj)}$ are related to the wave function of the particle (Nj), and the m_{Nj} are determined as the allowed eigenvalues of the difference equation that these coefficients satisfy. The input parameters $\bar{\alpha}_k$ and $\bar{\beta}_k$ are in some sense the "potential" which governs the physical properties of the system. The boundary condition is that the physical particle wave function be normalizable, or equivalently, that

$$\langle Njp|Q|Njp'\rangle = e\delta^3(\vec{p}-\vec{p}\,'), \quad (3.3)$$

where Q is the electromagnetic charge (or the third component of isospin for the neutron trajectory). This boundary condition leads to an eigenvalue condition because it limits the allowed behavior of the $a_k^{(Nj)}$ and $b_k^{(Nj)}$ for large k.

Using Eqs. (3.1) and (3.2) and various simple identities which we have recorded in Appendix A, we find the following difference equations for the a_k and b_k (note that we are now abbreviating $a_k^{(Nj)}$

by a_k, $b_k^{(Nj)}$ by b_k and m_{Nj} by m):

$$\frac{m^2(k-j)(k+j+2)(k+2)}{2(k+1)^3}a_k+i\bar{\alpha}^*_{k-1}\left(\frac{\eta_{k-1}}{\eta_k}\right)a_{k-1} -\bar{\beta}^*_{k-2}\left(\frac{\eta_{k-2}}{\eta_k}\right)a_{k-2}=0 \tag{3.4}$$

and

$$\frac{-i\bar{\beta}_k m^2(k-j+1)(k+j+3)(k+3)}{2(k+2)^3}b_{k+2}+\bar{\alpha}_k b_{k+1}+ib_k=c_k\,, \tag{3.5}$$

where

$$c_k=\frac{m^3(k-j+2)(k-j+1)(k+j+3)}{4(k+2)^3}\bar{\beta}_k a_{k+2} +\frac{im(k-j+1)\bar{\alpha}_k}{2(k+2)}a_{k+1} -\frac{m[k(k+2-j)+2]}{2(k+1)(k+2)}a_k\,. \tag{3.5a}$$

Equations (3.4) and (3.5) pertain to the partially gauge-invariant theory; we get the equations for the fully gauge-invariant case simply by setting $\bar{\beta}_k=0$ in (3.4) and (3.5).

Equations (3.4) and (3.5) with $c_k=0$ are in fact the same equation, as can be seen by setting

$$(-1)^k\eta_k a_k^*=\left(\prod_{l=0}^{k-1}\bar{\beta}_l\right)\tilde{b}_{k+1}\,. \tag{3.6}$$

Here $\tilde{b}_k$ denotes a solution to (3.5) with $c_k=0$. Furthermore, this homogeneous equation is the same as the one we found earlier, (2.13b), from gauge invariance.

We note that Eqs. (3.4) and (3.5) are the analogs of Eqs. (5.3) and (5.4) of Ref. 1 for the boson case. The form of (3.4) is very similar to (5.3) of I, and we follow basically the same procedure here as we did there. In order to be able to solve explicitly for the a_k for some particular value of j (say, $j=j_0$), we make the choice

$$\frac{\bar{\beta}_{k-2}}{(\bar{\alpha}_{k-1}\bar{\alpha}_{k-2})^*}=\frac{2\gamma k^3}{(k+1)(k+j_0+1)(k+d)(k+d+1)}\,, \tag{3.7}$$

where γ and d are arbitrary parameters. Putting $j=j_0$ in (3.4), we find the solution

$$\eta_k a_k^{(0j_0)}=\frac{(-2\gamma)^k[(k+1)!]^3}{(k-j_0)!(k+j_0+2)!(k+2)!}\frac{1}{\Gamma(k+d+2)}\times\prod_{l=0}^{k-1}(i\bar{\alpha}_l^*)\lambda(j_0), \tag{3.8}$$

where $\lambda(j_0)$ is an arbitrary normalization factor.

The corresponding mass is

$$m_{0j_0}{}^2=\frac{1}{\gamma}(d+j_0+2). \tag{3.9}$$

This solution corresponds to the lowest eigenvalue. By imposing the boundary condition that a_k shall be well behaved as $k\to\infty$, and by using the same techniques that were employed in Appendix B of I, we can show that the excited state eigenvalues are given by

$$m_{Nj_0}{}^2=\frac{1}{\gamma}[d+j_0+2+N],\quad N=1,2,\ldots \tag{3.10}$$

and that the corresponding eigenvectors a_k are Nth-degree polynomials in k times the ground-state solution (3.8). Although we cannot solve exactly for $a_k^{(j)}$ when $j\neq j_0$, we know from the work of Ref. 4 that all trajectories in this system will be asymptotically linear.

By using (3.6) we immediately obtain a solution to the homogeneous form of (3.5):

$$\tilde{b}_{k+1}^{(0j_0)}=\frac{(-2\gamma)^k[(k+1)!]^3}{(k-j_0)!(k+j_0+2)!(k+2)!}\frac{1}{\Gamma(k+d+2)}\times\prod_{l=0}^{k-1}\left(\frac{i\bar{\alpha}_l^*}{\bar{\beta}_l}\right)\tilde{\lambda}(j_0). \tag{3.11}$$

We can now solve the inhomogeneous equation by setting

$$b_k=\tilde{b}_k d_k$$

and

$$e_k=d_{k+1}-d_k\,. \tag{3.12}$$

We then find that e_k satisfies the first-order difference equation:

$$\frac{i\bar{\beta}_k m^2(k-j_0+1)(k+j_0+3)(k+3)}{2(k+2)^3}\tilde{b}_{k+2}e_{k+1}-i\tilde{b}_k e_k=c_k\,. \tag{3.13}$$

The homogeneous solution to this equation is

$$e_{k+1}=\prod_{l=j_0+1}^{k}\left[\left(\frac{\tilde{b}_l}{\tilde{b}_{l+2}}\right)\frac{2(l+2)^3}{\bar{\beta}_l m^2(l-j_0+1)(l+j_0+3)(l+3)}\right]e_{j_0+1} \equiv F_{k+1}e_{j_0+1}\,. \tag{3.14}$$

The inhomogeneous solution is then

$$e_k=F_k\left\{c+\sum_{l=j_0+1}^{k-1}f_l\right\}, \tag{3.15}$$

where

$$f_l=\frac{-ic_l}{\tilde{b}_l F_l}=\frac{-ic_l 2(l+2)^3}{\bar{\beta}_l m^2(l-j_0+1)(l+j_0+3)(l+3)F_{l+1}\tilde{b}_{l+2}}\,. \tag{3.16}$$

From (3.13) with $k=j_0$ we obtain the boundary condition

$$F_{j_0+1}c=\frac{c_{j_0}}{\bar{\alpha}^*_{j_0}\tilde{b}_{j_0+1}}\,, \tag{3.17}$$

i.e., $c=f_{j_0}$. So the solution $b_k^{(0j_0)}$ is given by the following sequence:

$$e_k = F_k \sum_{l=j_0}^{k-1} f_l\,, \tag{3.18a}$$

$$d_k = d_{j_0+1} + \sum_{l=j_0+1}^{k-1} e_l\,, \tag{3.18b}$$

$$b_k = \tilde{b}_k d_k\,. \tag{3.18c}$$

There is one constant (which we have chosen to be d_{j_0+1}) that is not determined by the boundary conditions we have so far imposed.

For the fully gauge-invariant case, we set $\bar{\beta}_k = 0$ in (3.4) and (3.5). These now become first-order difference equations, which are easily solved; however, the eigenvalue condition for the masses is no longer given by boundary conditions on one of the equations alone. Rather, we must impose a normalization condition such as (3.3) in order to fix the spectrum.

We have the difference equations

$$\frac{m^2(k-j)(k+j+2)(k+2)}{2(k+1)^3}\, a_k + i\,\bar{\alpha}^*_{k-1}\left(\frac{\eta_{k-1}}{\eta_k}\right) a_{k-1} = 0 \tag{3.19}$$

and

$$ib_k + \bar{\alpha}_k b_{k+1} = mc_k \tag{3.20}$$

with

$$c_k = \frac{i(k-j+1)}{2(k+2)}\,\bar{\alpha}_k a_{k+1} - \frac{[k(k+2-j)+2]}{2(k+1)(k+2)}\, a_k\,. \tag{3.21}$$

The solutions are

$$a_k = \left(\frac{\eta_j}{\eta_k}\right)(-i)^{k-j}\left[\frac{2(k+1)^3}{m^2(k+2)}\right]^{k-j} \frac{\lambda(j)}{(k-j)!\,(k+j+2)!} \prod_{l=j}^{k-1} \bar{\alpha}^*_l \tag{3.22}$$

and

$$b_{k+1} = (-i)^{k-j} \prod_{l=j}^{k}\left(\frac{1}{\bar{\alpha}_l}\right) m \sum_{p=j}^{k} c_p\,. \tag{3.23}$$

From the discussion of electromagnetism in Sec. IV [see especially Eq. (4.9)] we see that Eq. (3.3) implies the condition

$$\begin{aligned} 1 &= \langle Njsp=0|j^0(0)|Njsp=0\rangle \\ &= \sum_k \eta_k (-1)^j \left(\frac{m^2}{2}\right)^{k-j} \frac{j!}{(k+1)^2}(k+j+2)!\,(k-j)! \\ &\quad\times\left[(k-j+1)|a_k^{(j)}|^2 + \frac{(k+2)}{m}\,\mathrm{Im}\, a_k^{(j)*} b_k^{(j)}\right]. \end{aligned} \tag{3.24}$$

After extracting the normalization factor $|\lambda(j)|^2$, we can write this in the form

$$1 = |\lambda(j)|^2 \sum_{k=j}^{\infty} f_k^{(j)}(m^2). \tag{3.25}$$

Here $f_k^{(j)}$ depends on m^2 both through the explicit factor $(\frac{1}{2}m^2)^{k-j}$ in (3.24), and also in a more complicated way through the $a_k^{(j)}$ and the $b_k^{(j)}$. In the case of bosons, discussed in I, we found that for appropriate choices of the α_k the limit function

$$f^{(j)}(m^2) \equiv \lim_{k\to\infty} f_k^{(j)}(m^2) \tag{3.26}$$

was a simple function of m^2. Then the necessary condition for the series (3.25) to converge, namely

$$f^{(j)}(m^2) = 0\,, \tag{3.27}$$

could be solved to determine the allowed masses. Here the situation is more complicated; the condition (3.27) still in principle determines the mass spectrum for the gauge-invariant theory, although it is not easily solved.

IV. FORM FACTORS AND ELECTROMAGNETIC INTERACTIONS

We treat electromagnetic interactions by considering the photon as an external field which causes transitions from one state (normal mode) of the underlying field to another. We assume that the photon couples locally to the underlying fields, and for the case of charged fields, we obtain this coupling by the minimal prescription. For the Lagrangian (2.11) describing the proton trajectory, we let

$$\partial_\mu \psi \to (\partial_\mu - ieA_\mu)\psi \tag{4.1}$$

to obtain

$$\begin{aligned} j^\mu(x) &\equiv \frac{\delta\mathfrak{L}}{\delta e\,\partial_\mu \alpha(x)} \\ &= \tfrac{1}{2}\sum_k \eta_k[(\bar{\psi}_k\gamma)^{k+1}\delta^{k\mu}_{k+1}\psi_k + \bar{\psi}_k\delta^{k\mu}_{k+1}(\gamma\psi_k)^{k+1} + \bar{\beta}_k(\bar{\psi}_k\gamma)_{k+1}\psi^{k+1\,\mu} + \bar{\beta}^*_k\bar{\psi}^{k+1\,\mu}(\gamma\psi_k)_{k+1}]\,, \end{aligned} \tag{4.2}$$

where the notation $(A)^k B_{k\mu}$ is shorthand for

$$A^{\nu_1\cdots\nu_k}B_{\nu_1\cdots\nu_k\mu}\,.$$

We notice that the interaction with the electromagnetic field breaks the underlying gauge invariance of the Lagrangian equation (2.7), since $j_\mu A^\mu$ is not invariant under the field translation (2.5b). The original gauge transformations ensured the absence of spin-$\frac{1}{2}$ particles. However, once we introduce photons, there will be excited nucleon states of spin-$\frac{1}{2}$ coming from the spin-$\frac{3}{2}$ hadron and the spin-1 photon.

To find the matrix elements of this current between physical states, we decompose the fields ψ_k into their normal modes as in (3.2). The normal-mode fields have the following nonvanishing matrix elements between the vacuum and the proton trajectory:

$$\begin{aligned}\langle Npj\lambda|\overline{\psi}_{j'}^{(N')}(x)|0\rangle &= \frac{e^{ip\cdot x}}{(2\pi)^{3/2}}\left(\frac{m_{Nl}}{E}\right)^{1/2}\delta_{NN'}\delta_{jj'}\overline{u}^j(p,\lambda),\\ \langle 0|\psi_{j'}^{(N')}(x)|Npj\lambda\rangle &= \frac{e^{-ip\cdot x}}{(2\pi)^{3/2}}\left(\frac{m_{Nl}}{E}\right)^{1/2}\delta_{NN'}\delta_{jj'}u^j(p,\lambda).\end{aligned} \tag{4.3}$$

Here $u^j(p,\lambda)$ is a spinor for spin $j+\frac{1}{2}$ with momentum p and helicity λ, as given for example by Scadron.[5] [Note that in (4.3) the state $|Npj\lambda\rangle$ is taken to have spin $j+\frac{1}{2}$, j an integer, so that λ ranges from $-j-\frac{1}{2}$ to $j+\frac{1}{2}$.]

The most general matrix element of the current (4.2) is then

$$\begin{aligned}&\langle N'j'\Lambda'p'|j^\mu(x)|Nj\Lambda p\rangle\\ &=\left(\frac{m_N m_{N'}}{E_N E_{N'}}\right)^{1/2}\frac{e^{i(p'-p)\cdot x}}{(2\pi)^3}\sum_k \eta_k \delta^{\mu\sigma_1\cdots\sigma_k}_{\lambda_1\lambda_2\cdots\lambda_{k+1}}\\ &\quad\times\{a_k^{*\prime}a_k(i)^{k-j'}(-i)^{k-j}p'^{\lambda_{j'+1}}\cdots p'^{\lambda_k}\overline{u}^{\lambda_1\cdots\lambda_{j'}}\gamma^{\lambda_{k+1}}u_{\sigma_1\cdots\sigma_j}p_{\sigma_{j+1}}\cdots p_{\sigma_k}\\ &\quad+a_k^{*\prime}b_k(i)^{k-j'}(-i)^{k-j-1}p'^{\lambda_{j'+1}}\cdots p'^{\lambda_k}\overline{u}^{\lambda_1\cdots\lambda_{j'}}\gamma^{\lambda_{k+1}}\gamma_{\sigma_{j+1}}u_{\sigma_1\cdots\sigma_j}p_{\sigma_{j+2}}\cdots p_{\sigma_k}\\ &\quad-\beta_{k-1}[a_{k-1}^{*\prime}a_{k+1}p'_{\sigma_{j'+2}}\cdots p'_{\sigma_k}\overline{u}_{\sigma_1\cdots\sigma_{j'}}\gamma_{\sigma_{j'+1}}u^{\lambda_1\cdots\lambda_j}p^{\lambda_{j+1}}\cdots p^{\lambda_k}(i)^{k-j'-1}(-i)^{k-j+1}\\ &\quad+a_{k-1}^{*\prime}b_{k+1}(i)^{k-j'-1}(-i)^{k-j}p'_{\sigma_{j'+2}}\cdots p'_{\sigma_k}\\ &\quad\times\overline{u}_{\sigma_1\cdots\sigma_j}\gamma_{\sigma_{j''+1}}\gamma^{\lambda_{j+1}}u^{\lambda_1\cdots\lambda_j}p^{\lambda_{j+2}}\cdots p^{\lambda_{k+1}}]+\text{H.c.}\}.\end{aligned} \tag{4.4}$$

The form factor is given by

$$\begin{aligned}&\langle Nj\Lambda'p'|j^\mu(x)|Nj\Lambda p\rangle\\ &=\frac{m_N}{E}\frac{e^{i(p-p')\cdot x}}{(2\pi)^3}\sum_k\eta_k\delta^{\mu\sigma_1\cdots\sigma_k}_{\lambda_1\lambda_2\cdots\lambda_{k+1}}\{|a_k|^2[p'^{\lambda_{j+1}}\cdots p'^{\lambda_k}\overline{u}^{\lambda_1\cdots\lambda_j}\gamma^{\lambda_{k+1}}u_{\sigma_1\cdots\sigma_j}p_{\sigma_{j+1}}\cdots p_{\sigma_k}\\ &\quad+p'_{\sigma_{j+1}}\cdots p'_{\sigma_k}\overline{u}_{\sigma_1\cdots\sigma_j}\gamma^{\lambda_{k+1}}u^{\lambda_1\cdots\lambda_j}p^{\lambda_{j+1}}\cdots p^{\lambda_k}]\\ &\quad+ia_k^*b_kp'^{\lambda_{j+1}}\cdots p'^{\lambda_k}\overline{u}^{\lambda_1\cdots\lambda_j}\gamma^{\lambda_{k+1}}\gamma_{\sigma_{j+1}}u_{\sigma_1\cdots\sigma_j}p_{\sigma_{j+2}}\cdots p_{\sigma_k}\\ &\quad-ib_k^*a_kp'_{\sigma_{j+2}}\cdots p'_{\sigma_k}\overline{u}_{\sigma_1\cdots\sigma_j}\gamma_{\sigma_{j+1}}\gamma^{\lambda_{k+1}}u^{\lambda_1\cdots\lambda_j}p^{\lambda_{j+1}}\cdots p^{\lambda_k}\\ &\quad+\beta_{k-1}[a_{k-1}^*a_{k+1}p'_{\sigma_{j+2}}\cdots p'_{\sigma_k}\overline{u}_{\sigma_1\cdots\sigma_j}\gamma_{\sigma_{j+1}}u^{\lambda_1\cdots\lambda_j}p^{\lambda_{j+1}}\cdots p^{\lambda_{k+1}}\\ &\quad+a_{k+1}^*a_{k-1}p_{\sigma_{j+2}}\cdots p_{\sigma_k}\overline{u}^{\lambda_1\cdots\lambda_j}\gamma_{\sigma_{j+1}}u^{\sigma_1\cdots\sigma_j}p'^{\lambda_{j+1}}\cdots p'^{\lambda_{k+1}}\\ &\quad-ia_{k-1}^*b_{k+1}p'_{\sigma_{j+2}}\cdots p'_{\sigma_k}\overline{u}_{\sigma_1\cdots\sigma_j}\gamma_{\sigma_{j+1}}\gamma^{\lambda_{j+1}}u^{\lambda_1\cdots\lambda_j}p^{\lambda_{j+2}}\cdots p^{\lambda_{k+1}}\\ &\quad-ib_{k+1}^*a_{k-1}p_{\sigma_{j+2}}\cdots p_{\sigma_k}\overline{u}^{\lambda_1\cdots\lambda_j}\gamma^{\lambda_{j+1}}\gamma_{\sigma_{j+1}}u_{\sigma_1\cdots\sigma_j}p'^{\lambda_{j+2}}\cdots p'^{\lambda_{k+1}}]\}.\end{aligned} \tag{4.5}$$

Using Appendix C, and introducing the four-vector operators $L_\mu^{(\pm)}$ as in previous work,[6] we can write this as

$$\begin{aligned}&\frac{m_N}{E}\frac{e^{i(p'-p)\cdot x}}{(2\pi)^3}\sum_k\eta_k\left(\frac{m^2}{2}\right)^{k-j}(j!)^2(-1)^{j+1}(2j+1)^{-1}(k+1)^{-2}\langle j+\tfrac{1}{2},\Lambda'|j,\tfrac{1}{2};\lambda',\sigma'\rangle\langle j,\tfrac{1}{2};\lambda,\sigma|j+\tfrac{1}{2}\Lambda\rangle\\ &\quad\times\{\overline{u}\gamma_\lambda u[|a_k|^2(k-j)!(k+j+1)![D^{k/2\,k/2}(L^{-1}(p'))(L^{(+)\lambda}L^{(-)\mu}+L^{(+)\mu}L^{(-)\lambda})D^{k/2\,k/2}(L(p))]_{j\lambda',j\lambda}\\ &\quad-2\beta_{k-1}((k-j-1)!(k-j+1)!(k+j)!(k+j+2)!)^{1/2}\\ &\quad\times[a_{k-1}^*a_{k+1}(D^{(k-1)/2,(k-1)/2}L^{(+)\lambda}L^{(+)\mu}D^{(k+1)/2,(k+1)/2})_{j\lambda',j\lambda}\\ &\quad+a_{k+1}^*a_{k-1}(D^{(k+1)/2,(k+1)/2}L^{(-)\mu}L^{(-)\lambda}D^{(k-1)/2,(k-1)/2})_{j\lambda',j\lambda}]]\\ &\quad+i\overline{u}u(2/m)(k+2)^2(k+j-1)!(k+j)!(k+j+1)^{1/2}(k-j)^{1/2}\\ &\quad\times[(a_k^*b_k-b_{k+1}^*a_{k-1}\beta_{k-1})(D^{k/2,k/2}L^{(-)\mu}D^{(k-1)/2,(k-1)/2})_{j\lambda',j\lambda}+(\ \ )^*]\\ &\quad+\overline{u}\sigma_{\lambda_k}u(4/m)(k-j-1)!(k+j)!(k-j)^{1/2}(k+j+1)^{1/2}\\ &\quad\times[a_k^*b_k-b_{k+1}^*a_{k-1}\beta_{k-1})(D^{k/2,k/2}L^{(+)\lambda}L^{(-)k}L^{(-)\mu}D^{(k-1)/2,(k-1)/2})_{j\lambda',j\lambda}+(\ \ )^*]\}.\end{aligned} \tag{4.6}$$

To apply the normalization condition (3.3) we need the matrix element $\langle Npjj|Q|Np'jj\rangle = 2p_0\delta^3(p-p')$. Using

$$\langle kjj|L_1^{(+)}L_2^{(-)} - L_2^{(+)}L_1^{(-)}|kjj\rangle = j(k+2) \tag{4.7}$$

and

$$\begin{aligned}(L_0^{(+)}+L_0^{(-)})_{kj\sigma,k+1\,j'\sigma'} &= -\tfrac{1}{2}[(k+1-j)(k+j+2)]^{1/2}\delta_{jj'}\delta_{\sigma\sigma'},\\ (L_0^{(+)}+L_0^{(-)})_{k+1\,j\sigma,kj'\sigma'} &= \tfrac{1}{2}[(k+1-j)(k+j+2)]\delta_{jj'}\delta_{\sigma\sigma'}\end{aligned} \tag{4.8}$$

we obtain

$$1=\sum_{k=j}^{\infty}\eta_k\left(\frac{m^2}{2}\right)^{k-j}\frac{j!}{(k+1)^2}(-1)^j(k+j+2)!(k-j)!$$
$$\times\{(k-j+1)(|a_k^{(j)}|^2+\mathrm{Re}\bar\beta_{k-1}a_{k-1}^{(j)*}a_{k+1}^{(j)})+[(k+2)/m]\,\mathrm{Im}(a_k^{(j)*}b_k^{(j)}+\bar\beta_{k-1}a_{k-1}^{(j)*}b_{k+1}^{(j)})\}. \tag{4.9}$$

For discussing the inelastic electroproduction structure functions one requires the matrix element of proton → anything:

$$\langle N'j'p'\Lambda'|j^\mu(x)|N=0,\,j=0,p\Lambda\rangle=\left(\frac{mm'}{EE'}\right)^{1/2}\frac{e^{i(p'-p)\cdot x}}{(2\pi)^3}\sum_k\eta_k\delta^{\mu\sigma_1\cdots\sigma_k}_{\lambda_1\lambda_2\cdots\lambda_{k+1}}$$
$$\times[a_k^{*\prime}a_k(i)^{-j'}(p'^{\lambda_{j+1}}\cdots p'^{\lambda_k}\bar u^{\lambda_1\cdots\lambda_{j'}}\gamma^{\lambda_{k+1}}up_{\sigma_1}\cdots p_{\sigma_k}$$
$$+p'_{\sigma_{j'+1}}\cdots p'_{\sigma_k}\bar u_{\sigma_1\cdots\sigma_{j'}}\gamma^{\lambda_{k+1}}up^{\lambda_1}\cdots p^{\lambda_k})$$
$$+\beta_{k-1}(i)^{-j'}(a_{k-1}^{*\prime}a_{k+1}p^{\lambda_1}\cdots p^{\lambda_k}p'_{\sigma_{j'+2}}\cdots p'_{\sigma_k}\bar u_{\sigma_1\cdots\sigma_j}\gamma_{\sigma_{j'+1}}u$$
$$+a_{k+1}^{*\prime}a_{k-1}p'^{\lambda_{j'+1}}\cdots p'^{\lambda_{k+1}}p_{\sigma_1\cdots\sigma_{k-1}}\bar u^{\lambda_1\cdots\lambda_{j'}}\gamma_{\sigma_k}u)]. \tag{4.10}$$

We can write the proton form factors in terms of a power series in the Chebychev polynomials of the second kind $P_k(\gamma)$, by using the techniques of Appendix B. As an illustration of this, we shall compute that piece of the proton form factor [$N'=0$, $j'=0$ (i.e., spin $\frac{1}{2}$ in Eq. (4.10)] which depends only on the $a_k^{(0)}$, and which we shall call $A^{(1/2)}(p,p',\lambda,\lambda')$. In addition, there is another piece depending on the $b_k^{(0)}$ which is similar to $A^{(1/2)}$ but slightly more complicated in that it involves higher derivatives of the $P_k(\gamma)$. We have

$$A^{(1/2)}(p,p',\lambda,\lambda')=\frac{m}{(EE')^{1/2}}\frac{e^{i(p'-p)\cdot x}}{(2\pi)^3}$$
$$\times\sum_k\eta_k\delta^{\sigma_1\cdots\sigma_k\mu}_{\lambda_1\cdots\lambda_k\lambda_{k+1}}\{(p'^{\lambda_1}\cdots p'^{\lambda_k}p_{\sigma_1}\cdots p_{\sigma_k}+p'_{\sigma_1}\cdots p'_{\sigma_k}p^{\lambda_1}\cdots p^{\lambda_k})|a_k|^2\bar u(p',\lambda')\gamma^{\lambda_{k+1}}u(p,\lambda)$$
$$+\mathrm{Re}\bar\beta_k a_{k+1}^*a_{k-1}\bar u(p',\lambda')\gamma_{\sigma_k}u(p,\lambda)$$
$$\times(p'^{\lambda_1}\cdots p'^{\lambda_{k+1}}p_{\sigma_1}\cdots p_{\sigma_{k-1}}+p^{\lambda_1}\cdots p^{\lambda_{k+1}}p'_{\sigma_1}\cdots p'_{\sigma_{k-1}})\}. \tag{4.11}$$

Using Appendix B, we obtain for the sum

$$\bar u(p')\gamma^\mu u(p)F_1(q^2)+(p+p')^\mu\bar u(p')u(p)F_2(q^2),$$

where

$$F_1(q^2)=\sum_k\eta_k m^{2k}[2^k(k+1)]^{-1}\left[|a_k|^2\left(\frac{k+2}{k+1}\right)+\frac{m^2}{2}\mathrm{Re}\bar\beta_k a_{k+2}^*a_k\right]P_{k+1}'(\gamma),$$
$$F_2(q^2)=\sum_k\eta_k m^{2k-1}[2^{k+1}(k+1)]^{-1}\left\{|a_k|^2\left[\left(\frac{k+2}{k+1}\right)P_{k+1}''-(k+1)P_k'+(\gamma-2)P_k''\right]\right.$$
$$\left.-\mathrm{Re}\bar\beta_k a_{k+2}^*a_k\frac{m^2}{2}\{(k+1)P_k''+(k+3)P_{k+2}''-2(k+2)P_{k+1}''\}\right\}. \tag{4.12}$$

Realizing that a_k depends on the $\bar\alpha_k$ in the combination

$$\prod_{l=0}^{k-1}\bar\alpha_l\,, \tag{4.13}$$

we see that we have the freedom to choose the form factor independently of the choice of mass spectrum as determined by the combination of parameters (3.7). Hopefully, we will be able to find simple choices for $\bar{\alpha}_k$ so that the sum over the P_k yields a simple function, such as a pole or a dipole.

While the use of minimal coupling for the proton trajectory seems reasonable, we run into difficulty as soon as we extend this principle to the neutron trajectory. It is well known that the neutron has no charge form factor but does have a magnetic form factor. Since in our model the neutron is made up of neutral gauge fields, we automatically get a zero charge form factor, but minimal coupling would lead to zero magnetic form factor also. Thus to produce a semiphenomenological neutron magnetic form factor, we have to assume that the neutral gauge fields have magnetic moment μ and postulate a magnetic dipole interaction[7]:

$$i\gamma_\mu\partial^\mu \to i\gamma_\mu\partial^\mu - \tfrac{1}{2}\mu\sigma^{\mu\nu}F^{(\text{ext})}_{\mu\nu}\,. \tag{4.14}$$

Thus we will obtain an effective electromagnetic current:

$$j^\mu(x) = \mu\sum_k \eta_k\,\partial_\nu(\bar{\psi}^k\sigma^{\mu\nu}\psi_k). \tag{4.15}$$

This form of the current leads to the following transition matrix:

$$\langle N'p'j'\lambda'|\,j^\mu(x)|Npj\lambda\rangle = \frac{i(p'-p)_\nu}{(2\pi)^3}\left(\frac{m_N m_{N'}}{EE'}\right)^{1/2} e^{i(p'-p)\cdot x}$$
$$\times\sum_k \eta_k\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}\{[a_k'^* i^{k-j'}\bar{u}^{\nu_1\cdots\nu_{j'}}p'^{\nu_{j'+1}}\cdots p'^{\nu_k} + b_k'^* i^{k-j'-1}\bar{u}^{\nu_1\cdots\nu_{j'}}\gamma^{\nu_{j'+1}}p'^{\nu_{j'+2}}\cdots p'^{\nu_k}]$$
$$\times\sigma^{\mu\nu}[a_k(-i)^{k-j}u_{\mu_1\cdots\mu_j}p_{\mu_{j+1}}\cdots p_{\mu_k} + b_k(-i)^{k-j-1}p_{\nu_{j+2}}\cdots p_{\nu_k}\gamma_{\nu_{j+1}}u_{\nu_1\cdots\nu_j}(p)]\}. \tag{4.16}$$

The form factor is therefore

$$\langle Nj\lambda'p'|\,j^\mu(x)|Nj\lambda p\rangle = \frac{i(p'-p)^\mu}{(2\pi)^3}\left(\frac{m}{E}\right)e^{i(p'-p)\cdot x}$$
$$\times\sum_k\eta_k\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}\{|a_k|^2 p'^{\nu_{j+1}}\cdots p'^{\nu_k}p_{\mu_{j+1}}\cdots p_{\mu_k}\bar{u}^{\nu_1\cdots\nu_j}\sigma^{\mu\nu}u_{\mu_1\cdots\mu_j}$$
$$+|b_k|^2 p_{\mu_{j+2}}\cdots p_{\mu_k}p'^{\nu_{j+2}}\cdots p'^{\nu_k}\bar{u}^{\nu_1\cdots\nu_j}\gamma^{\nu_{j+1}}\sigma^{\mu\nu}u_{\mu_1\cdots\mu_j}$$
$$+ib_k^* a_k p'^{\nu_{j+2}}\cdots p'^{\nu_k}p_{\mu_{j+2}}p_{\mu_k}\bar{u}^{\nu_1\cdots\nu_j}\gamma^{\nu_{j+1}}\sigma^{\mu\nu}u_{\mu_1\cdots\mu_j}$$
$$+ia_k^* b_k p'^{\nu_{j+1}}\cdots p'^{\nu_k}p_{\mu_{j+2}}\cdots p_{\mu_k}\bar{u}^{\nu_1\cdots\nu_j}\sigma^{\mu\nu}\gamma_{\nu_{j+1}}u_{\mu_1\cdots\mu_j}\}. \tag{4.17}$$

For inelastic electron scattering the appropriate transition matrix is $j=0$ to $j=j_0$ [i.e., spin $\frac{1}{2}$ to spin $(j_0+\frac{1}{2})$]:

$$\langle N'p'j'\lambda'|\,j^\mu(x)|N=0,p,j=0,\lambda\rangle = \frac{i(p'-p)_\nu}{(2\pi)^3}\left(\frac{m\,m_N}{EE_N}\right)^{1/2}e^{i(p'-p)\cdot x}$$
$$\times\mu\sum_k\eta_k\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}[a_k'^* a_k(-i)^{j'}\bar{u}^{\nu_1\cdots\nu_{j'}}(p')\sigma^{\mu\nu}u(p)p'^{\nu_{j'+1}}\cdots p'^{\nu_k}p_{\mu_1}\cdots p_{\mu_k}$$
$$+(-i)^{j'+1}b_k'^* a_k\bar{u}^{\nu_1\cdots\nu_{j'}}\gamma^{\nu_{j'+1}}\sigma^{\mu\nu}u(p)p'^{\nu_{j+2}}\cdots p'^{\nu_k}p_{\mu_1}\cdots p_{\mu_k}]. \tag{4.18}$$

For the neutron form factor we obtain

$$\langle p'\lambda'|\,j^\mu(x)|\,p\lambda\rangle = \frac{i(p'-p)_\nu}{(2\pi)^3}\bar{u}(p')\sigma^{\mu\nu}u(p)e^{i(p'-p)\cdot x}\frac{m}{E}\,\mu\sum_k\eta_k|a_k|^2\delta^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_k}p'^{\nu_1}\cdots p'^{\nu_k}p_{\mu_1}\cdots p_{\mu_k}\,. \tag{4.19}$$

Thus the neutron magnetic form factor is

$$\tilde{F}_2(q^2) = \mu\sum_k\eta_k|a_k|^2 m^{2k}((k+1)/2^k)\,P_k(\gamma). \tag{4.20}$$

However, the fact that we need nonminimal coupling to obtain a neutron magnetic form factor seems a serious drawback to this approach.

We close this section with a brief discussion of nucleon Compton scattering and inelastic electron scattering. Assuming that the s- and u-channel poles are a good approximation to the Compton scattering amplitude to order e^2, we write the effective Hamiltonian

$$H_I = e \int j_\mu^{\text{e.m.}}(x) A^{\mu(\text{ext})} d^3x. \tag{4.21}$$

We then have, to order e^2,

$$\langle \gamma(k', \lambda') N', p', j', s' \text{ out} | \gamma(k, \lambda) N, p, j, s \text{ in} \rangle$$
$$= \delta_{fi} + \tfrac{1}{2} i^2 e^2 \epsilon^\mu(k, \lambda) \epsilon^{\nu *}(k', \lambda') \int d^4x d^4y \, e^{-i(k\cdot x - k'\cdot y)} \langle N'p'j's' | T(j_\mu(x) j_\nu(y)) | Npjs \rangle , \tag{4.22}$$

where for $j_\mu(x)$ we use either (4.2) for the proton trajectory or (4.15) for the neutron trajectory. Equation (4.22) together with the decomposition (4.3), tells us that there are an infinite number of "narrow resonances" exchanged in the s and u channels, as depicted in Fig. 1.

In our formalism, we ostensibly know $j^\mu(x)$ not only at $q^2 = 0$ but for all q^2. Since the Chebychev polynomials of the second kind are complete in a limited domain, we hope that we can invert the form factor equation [Eq. (4.12)] and find what $a_k^{(Nj)}$ (and therefore $\overline{\alpha}_k, \overline{\beta}_k$) lead to a dipole form factor. If we can do this, we can reproduce the success of Domokos *et al.*[8] in fitting W_1 and νW_2.

Explicitly,

$$W_{\mu\nu} = \tfrac{1}{2} \sum_s \sum_{N'} \langle Npjs | j_\mu(0) | N'p'j's' \rangle \langle N'p'j's' | j_\nu(0) | Npjs \rangle (2\pi)^3 \delta^4(p + q - p').$$

The matrix elements needed for this sum are given by Eq. (4.10) or (4.18).

V. STRONG-INTERACTION DYNAMICS AND π-N SCATTERING

The electromagnetic current [Eqs. (4.2) and (4.4)] leads to transitions between all the normal modes (particles on the fermion trajectory), and thus amplitudes given by the time-ordered product of two currents between physical-particle states will contain an infinite number of intermediate states. Thus, if external particles couple to these "free-field" currents, we will obtain the narrow-resonance approximation to the four-point function. We feel that choosing the infinite number of parameters in the underlying field theory to yield the correct mass spectrum and electromagnetic form factors is equivalent to putting a lot of strong-interaction information into the theory. Thus, we suggest that these free-field theory two-current (and n-current) correlation functions, sandwiched between physical single-particle normal-mode states, provide a description of strong-interaction dynamics similar to the Veneziano scheme in that it leads to the nonunitary narrow-resonance approximation.

In our scheme, however, we are able to use the currents to go off the mass shell. Thus in what follows we shall add an isospin degree of freedom to the fermions and also to the bosons discussed in I, and construct the vector, axial-vector, and baryonic currents. We can then deal with external ρ mesons, pions, and nucleons by having them couple to these currents. For example, we expect that the s- and u-channel poles in π-N scattering will be given by an expression similar to Eq. (4.22), with the vector current replaced by the axial-vector current, and the pion, coupled to $\partial_\mu A^\mu$, replacing the photon. In general, to compute a given N-point function we treat any one of the N particles as a normal mode of the underlying field and let it absorb and emit external quanta (which couple to the approximate currents) consistent with the scattering process. The underlying field is correspondingly excited and deexcited, finally returning to a normal-mode state. To ensure crossing symmetry, we sum over the various ways of choosing each external particle as the underlying field in a normal mode.

To get an idea of the structure of the currents to which the external quanta couple, we look at the trilinear interactions of ordinary field theory, treat each of the three factors as an external field, and write the interaction in the form $\mathcal{L}_{\text{int}} = \phi_\mu^{(\text{ext})} J^\mu$.

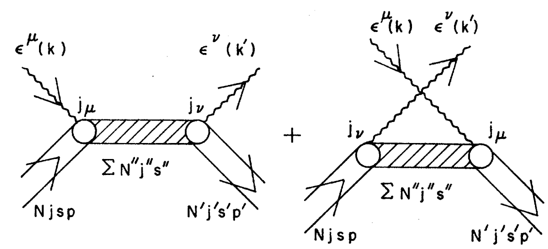

FIG. 1. Compton scattering with the exchange of a Regge trajectory.

Here $\phi_\mu^{(\mathrm{ext})}$ is a four-vector created from the external field by the appropriate use of derivatives, and J^μ is then a bilinear current which we try to generalize to the infinite component case. In particular examples, such as the vector and axial-vector currents, there may be algebraic constraints [e.g., chiral $SU(2)\otimes SU(2)$] which tell us what transformations to make on the Lagrangian to obtain the infinite-component generalizations of these bilinear currents.

Before discussing π-N scattering, let us first generalize the fermion Lagrangian to the SU(2) case, i.e., both protons and neutrons. We do this by simply attaching an isospin index $i=1, 2$ to the field $\psi_{k\mu}$. The new Lagrangian is then

$$\mathfrak{L}=\tfrac{1}{2}\sum_{k=0}^{\infty}\sum_{i=1}^{2}\eta_k\delta(\overline{\psi}_i^k\gamma^\mu)(i\partial_\mu\psi_k^i+\overline{\alpha}_k\psi_{k\mu}^i+i\overline{\beta}_k\partial^\lambda\psi_{k\lambda\mu}^i)+\mathrm{H.c.} \tag{5.1}$$

The parameters $\overline{\alpha}_k$ and $\overline{\beta}_k$ are taken to be isospin-independent, thus ensuring the isospin invariance of $\mathfrak{L}$.

We obtain the vector and axial-vector currents in the usual way by considering the transformations

$$\psi\to e^{-i\vec{\alpha}\cdot\vec{\tau}/2}\psi,\quad \overline{\psi}\to\overline{\psi}e^{i\vec{\alpha}\cdot\vec{\tau}/2} \tag{5.2a}$$

and

$$\psi\to e^{-i\gamma_5\vec{\alpha}\cdot\vec{\tau}/2}\psi,\quad \overline{\psi}\to\overline{\psi}e^{-i\gamma_5\vec{\alpha}\cdot\vec{\tau}/2}. \tag{5.2b}$$

Using the usual Gell-Mann–Lêvy equations we obtain

$$V_a^\mu=\tfrac{1}{2}\sum_k\eta_k\{\delta(\overline{\psi}^k\gamma^\mu)\tfrac{1}{2}\tau^a\psi_k+\overline{\psi}_k\tfrac{1}{2}\tau^a\delta(\gamma^\mu\psi^k)+\overline{\beta}_{k-2}[\overline{\psi}_{k-2}\gamma_\lambda\tfrac{1}{2}\tau_a\psi^{k-2\,\lambda\mu}+\overline{\psi}^{k-2\,\lambda\mu}\tfrac{1}{2}\tau^a\gamma_\lambda\psi_{k-2}]\} \tag{5.3a}$$

and

$$A_a^\mu=\tfrac{1}{2}\sum_k\eta_k\{\delta(\overline{\psi}^k\gamma^\mu)\tfrac{1}{2}\tau^a\gamma_5\psi_k+\overline{\psi}_k\tfrac{1}{2}\tau^a\delta(\gamma^\mu\gamma^5\psi^k)+\overline{\beta}_{k-2}[\overline{\psi}_{k-2}\gamma_\lambda\gamma^5\tfrac{1}{2}\tau^a\psi^{k-2\,\lambda\mu}+\overline{\psi}^{k-2\,\lambda\mu}\tfrac{1}{2}\tau^a\gamma_\lambda\gamma^5\psi_{k-2}]\}. \tag{5.3b}$$

The vector current is conserved. The matrix elements of V_μ and A_μ between physical states can be obtained using the techniques of Appendix B.

Equation (5.3) and its boson counterpart (5.9) are not covariant under the gauge transformations of the second kind. Thus, for the gauge-invariant theory the algebra of observables does not commute with these gauge transformations.[9] We therefore must verify that the S-matrix elements are gauge-invariant. This will be so because the gauge transformations change at most the spin-$\frac{1}{2}$ components of the underlying fermion fields, and the spin-0 components of the underlying boson fields. However, there are *no* physical states of spin $\frac{1}{2}$ or spin 0 in the gauge-invariant theory. Matrix elements of the current between single-particle states are therefore unaffected by these gauge transformations.

For the time components of the currents we get

$$V_a^0=\tfrac{1}{2}\sum_k\eta_k(\pi^k\tfrac{1}{2}\tau^a\psi_k+\overline{\psi}^k\tfrac{1}{2}\tau^a\overline{\pi}_k), \tag{5.4a}$$

$$A_a^0=\tfrac{1}{2}\sum_k\eta_k(\pi^k\tfrac{1}{2}\tau^a\gamma^5\psi_k+\overline{\psi}^k\tfrac{1}{2}\tau^a\gamma^5\overline{\pi}_k), \tag{5.4b}$$

where π^k is the canonical momentum [Eq. (2.14a)] and $\overline{\pi}^k=\gamma^0\pi^{k\dagger}$. Using the anticommutation relations [Eq. (2.14)] one can easily verify that V_a^0 and A_a^0 obey the chiral $SU(2)\otimes SU(2)$ algebra.

We want to describe π^-p scattering in the narrow-resonance approximation. For simplicity, we consider only the contributions from the π, ρ, and nucleon trajectories. The Lagrangian is

$$\mathfrak{L}=\sum_k\eta_k G_{a\eta}^k[\delta\partial\varphi_{k-1}^{a\eta}+\alpha_k^\eta\varphi_k^{a\eta}+\beta_k^\eta\partial\cdot\varphi_{k+1}^{a\eta}-\tfrac{1}{2}G_k^{a\eta}]+\sum_k[\eta_k\overline{\psi}_i^k\gamma^\mu(i\delta\partial_\mu\psi_k^i+\overline{\alpha}_k\psi_{k\mu}^i+i\overline{\beta}_k\partial^\lambda\psi_{k\lambda\mu}^i)+\mathrm{H.c.}]. \tag{5.5}$$

Here the first sum represents the boson Lagrangian discussed briefly in Sec. II, and treated more fully in our earlier paper. The label a is an isospin index, while η is a normality index inserted to distinguish the π from the ρ.

The ordinary field-theory poles in π^-p elastic scattering are the ρ pole in the t-channel and the neutron pole in the s channel. The t-channel pole contains the vertices

$$g_{\rho\pi\pi}\epsilon^{abc}\rho_a^\mu\varphi_b\overleftrightarrow{\partial}_\mu\varphi_c \tag{5.6a}$$

and

$$g_{\rho NN}\vec{\rho}_\mu\bar{\psi}\gamma^\mu\tfrac{1}{2}\vec{\tau}\psi. \tag{5.6b}$$

Setting $g_{\rho\pi\pi}=g_{\rho NN}=g$, this can be written, for external ρ,

$$g\vec{\rho}_\mu^{(\text{ext})}\cdot\vec{j}^\mu,$$

with

$$\vec{j}^\mu=\vec{\varphi}\times\partial^\mu\vec{\varphi}+\bar{\psi}\gamma^\mu\tfrac{1}{2}\vec{\tau}\psi, \tag{5.7}$$

i.e., the external ρ couples universally to the isospin current.

The vertex (5.6a) can also be considered (as is relevant to π^-p scattering) as an external π coupling to the divergence of the axial current, or equivalently

$$-g(\partial_\mu\vec{\varphi}^{(\text{ext})})\cdot\vec{A}^\mu$$

with

$$\vec{A}^\mu=-(\vec{\rho}^{\,\mu}\times\vec{\varphi}). \tag{5.8}$$

The natural generalization of these vector and axial-vector currents is given by Eq. (5.3) above, and for the bosons by Eqs. (6.6) and (6.10) of Ref. 1:

$$\vec{V}^\mu=-\sum_{k=0}^\infty\eta_k(\vec{G}^{k-1\,\mu}\times\vec{\varphi}_{k-1}+\beta_k\vec{G}_k\times\vec{\varphi}^{k\mu}),$$
$$\vec{A}^\mu=-\sum_{k=0}^\infty\eta_k(\vec{G}^{k-1\,\mu}\epsilon\times\vec{\varphi}_{k-1}+\beta_k^\eta\vec{G}_k^\eta\epsilon_{\eta\eta'}\times\vec{\varphi}_{\eta'}^{k\mu}). \tag{5.9}$$

Here $\epsilon_{\eta\eta'}=\delta_{\eta,\eta'+1}+\delta_{\eta+1,\eta'}$ is a matrix that interchanges the π and ρ trajectories.

We also want to interpret the vertex (5.6b) as an external nucleon coupling to the baryonic current. That is, in addition to interpreting (5.6b) as $\vec{\rho}_\mu^{(\text{ext})}\cdot\vec{V}^\mu$ with $\vec{V}$ the isospin current, we also write it as

$$\bar{\psi}\gamma^\mu\tfrac{1}{2}\vec{\tau}\psi\cdot\vec{\rho}_\mu=(\bar{\psi}_{(\text{ext})}\gamma^\mu)_\alpha j_\mu^\alpha$$

with

$$j_\mu^\alpha=\tfrac{1}{2}\vec{\tau}\psi_\alpha\cdot\vec{\rho}_\mu. \tag{5.10}$$

Similarly, the s-channel pole has the vertex

$$g_{NN\pi}\bar{\psi}\gamma_\mu\gamma_5\tfrac{1}{2}\vec{\tau}\psi\partial^\mu\vec{\varphi}\equiv\Gamma_s \tag{5.11}$$

which again we interpret in two ways, depending on which of the two particles we choose as external:

$$\Gamma_s=\vec{A}^\mu\cdot\partial_\mu\vec{\varphi}^{(\text{ext})},$$

with

$$\vec{A}^\mu=\bar{\psi}\gamma^\mu\gamma^5\tfrac{1}{2}\vec{\tau}\psi \tag{5.12}$$

or

$$\Gamma_s=(\bar{\psi}\gamma_\mu)_\alpha j^{\mu\alpha},$$

with

$$j^{\mu\alpha}=(\gamma_5\tfrac{1}{2}\vec{\tau}\psi)_\alpha\cdot\partial^\mu\vec{\varphi}. \tag{5.13}$$

The appropriate generalization of the axial-vector current is given by Eq. (5.3). It remains to find the generalization of the baryonic current as given by (5.10) and (5.13):

$$j_\alpha^\mu=g_1(\gamma_5\tfrac{1}{2}\vec{\tau}\psi)_\alpha\cdot\partial^\mu\vec{\varphi}+g_0(\tfrac{1}{2}\vec{\tau}\psi)_\alpha\cdot\vec{\rho}^{\,\mu} \tag{5.14}$$

to the infinite-component case.

We consider the following set of transformations on the fields in the Lagrangian [Eq. (5.5)] of the $\pi\rho N$ system:

$$\varphi_\eta^i(x)\to\varphi_\eta^i(x)+\bar{f}(x)\lambda_\eta^*\tfrac{1}{2}\tau^i\psi(x)+\bar{\psi}(x)\lambda_\eta\tfrac{1}{2}\tau^i f(x), \tag{5.15a}$$

$$\psi(x)\to\psi(x)+i\lambda_\eta\tfrac{1}{2}\tau^i f(x)\varphi_{i\eta}(x), \tag{5.15b}$$

$$\bar{\psi}(x)\to\bar{\psi}(x)-if(x)\lambda_\eta^*\tfrac{1}{2}\tau^i\varphi_{i\eta}(x). \tag{5.15c}$$

Here $f(x)$ is a spinor in both isospin space and space-time, and λ_η is a Dirac matrix defined to be

$g_0 I$ if $\eta=0$

$ig_1\gamma_5$ if $\eta=1$.

These transformations generate the following expression for the baryonic current:

$$j_\alpha^\mu\equiv\frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{f}_\alpha(x)}=\sum_k\eta_k\{\vec{G}_\eta^{k-1\,\mu}\cdot(\tfrac{1}{2}\vec{\tau}\lambda_\eta^*\psi_{k-1})_\alpha+\beta_k\vec{G}_k^\eta\cdot(\tfrac{1}{2}\vec{\tau}\lambda_\eta^*\psi^{k\mu})_\alpha+\vec{\varphi}_k\cdot[\tfrac{1}{2}\vec{\tau}\lambda_\eta^*\delta(\gamma^\mu\psi^k)]_\alpha+\bar{\beta}_k^*\vec{\varphi}^{k\mu\lambda}\cdot[\tfrac{1}{2}\vec{\tau}\lambda_\eta^*\delta(\gamma_\lambda\psi_k)]_\alpha\}. \tag{5.16}$$

This is the natural generalization of (5.14).

If we assume that only the π, ρ, and nucleon trajectories contribute, we are now in position to calculate π^-p scattering by coupling the external π and p to the axial-vector current and baryonic current, respectively. The diagrams are shown in Fig. 2. For example, Figs. 2(a) and 2(b) are given by the expression

$$\frac{\bar{u}(p_2)}{(2\pi)^3}\gamma^\mu\int d^4x\,d^4y\,e^{-i(k_1\cdot x-p_2\cdot y)}\left(\frac{1}{2k_{10}}\frac{m}{p_{20}}\right)^{1/2}\langle\pi^-(k_2)N=0,\,j=0,\,\eta=1|T(\partial_\lambda A^\lambda(x)j_{\mu\alpha}(y))|\,p(p_1)N=0,\,j=0,\,s\rangle. \tag{5.17}$$

To evaluate (5.17), we must use Eqs. (5.3) and (5.9) for A^λ, and Eq. (5.16) for $j_{\mu\alpha}$; we further need the decomposition (3.2) for the nucleon fields, and the corresponding decompositions (3.7) and (3.8) of I for the π and ρ fields. When we use the piece of the axial-vector current given by (5.3), we obtain the neutron trajectory in the s channel [Fig. 2(a)]; when we use (5.9), we obtain the ρ trajectory in the t channel [Fig. 2(b)].

Thus we see that in principle we can handle bosons and fermions together and calculate the tree graphs. We see that $\pi N \to \pi N^*$, etc., are known if $\pi N \to \pi N$ is. The major question is whether there is a simple choice of the $\bar{\alpha}_k$ and $\bar{\beta}_k$ leading to rising trajectories, where the matrix elements of the currents [exemplified by the power series in (4.12)] sums to some simple function (say, a dipole). If so, then it will be practical to carry out some simple calculations. As the theory stands now, going beyond the 3-point function looks like an exorbitant (and not very enlightening) amount of algebra.

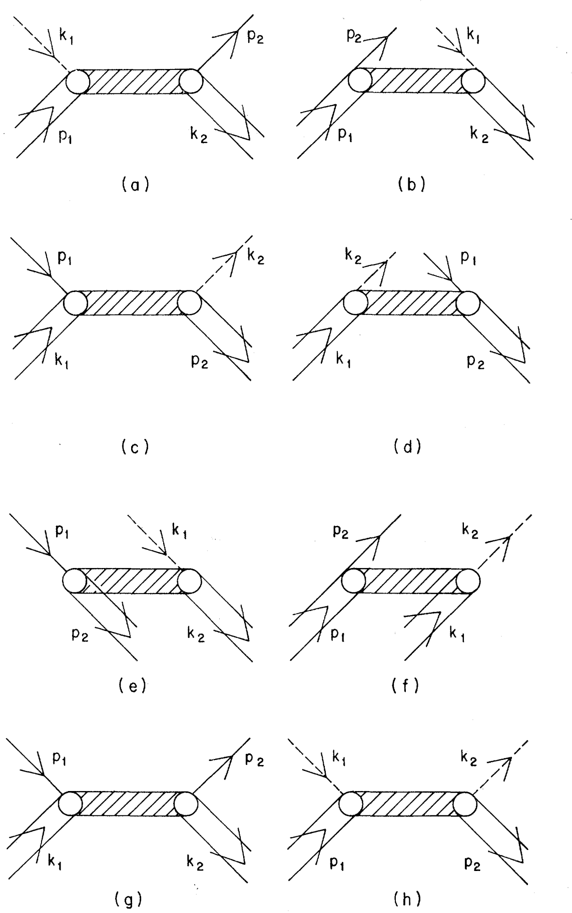

FIG. 2. Diagrams contributing to $\pi^- p$ elastic scattering. Dashed line is external π^- coupling to $\partial_\mu A^\mu$; solid line is external proton coupling to the baryonic current $j^{\mu\alpha}$. The shaded line denotes a propagating sum of normal modes.

VI. CONCLUSIONS

We have shown that there exists a free Lagrangian field theory which describes the spectrum and structure of the particles on the boson and fermion Regge trajectories. The picture we obtain is that the physical particles are the normal modes of an infinite number of gauge fields having bilinear interactions between nearest neighbors in Lorentz index space. At the free-field level the particles have nontrivial form factors.

From these free Lagrangians, we have constructed bilinear currents, such as the vector and axial-vector currents obeying the chiral SU(2) $\otimes$ SU(2) algebra. Since these currents have reasonable properties, such as causing transitions between all the physical states with the correct quantum numbers, we then assumed that these matrix elements were a close approximation to the real world, and contained strong-interaction information. We postulated that the tree-diagram N-point functions could be obtained by having the external particles couple directly to the underlying currents given by the free-field theory.

Unfortunately, the matrix elements are too complicated to allow us to sum them to see what a typical 4-point function would be. It is also not clear what role unitarity will play (in the guise of factorization). The prescription we have given for calculating scattering amplitudes does not seem *a priori* to guarantee factorization (say, of the 6-point function into two 4-point functions), and this requirement will probably limit (i.e., fix or make impossible) the choice of the infinite number of parameters of the underlying field theory. One might hope that a "group-theoretical" choice of parameters would yield simple expressions for the current matrix elements, and would allow these problems to be further explored.

Aside from this unitarity problem, there is the difficulty that the neutral particles are composites of neutral gauge fields and thus are transparent to photons unless the underlying gauge fields have intrinsic magnetic moments. We get a zero charge form factor for the neutron (which is good), but have to put the magnetic form factor in by hand (which is bad).

These problems await further investigation in the future.

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APPENDIX A

We record some useful identities relating various tensors that appear when we expand the equations of motion. Here, as in the text, $\tilde{\psi}_j$ is a spin-$(j+\frac{1}{2})$ free field with j symmetric, traceless Lorentz indices $\mu_1 \ldots \mu_j$, and a spinor index α. It satisfies

$$(i\partial_\mu \gamma^\mu - m)\tilde{\psi}_j = 0$$

and

$$\partial^{\mu_1}\tilde{\psi}_{\mu_1 \cdots \mu_j} = \gamma^{\mu_1}\tilde{\psi}_{\mu_1 \cdots \mu_j} = 0.$$

The identities are

$$\partial \cdot \delta_{k+1}\partial^{k-j+1}\tilde{\psi}_j = \frac{-m^2(k-j+1)(k+j+2)}{2(k+1)^2}\,\delta_k \partial^{k-j}\tilde{\psi}_j\,, \tag{A1}$$

$$\partial \cdot \delta_{k+1}\gamma\partial^{k-j}\tilde{\psi}_j = (-im)\frac{(j+1)}{(k+1)^2}\delta_k\partial^{k-j}\tilde{\psi}_j - \frac{m^2(k-j)(k+j+3)}{2(k+1)^2}\delta_k\gamma\partial^{k-j-1}\tilde{\psi}_j\,, \tag{A2}$$

$$\begin{aligned}\gamma \cdot \delta_{k+1}\partial^{k-j+1}\tilde{\psi}_j &= \frac{(-im)(k-j+1)}{2(k+2)}\gamma \cdot \delta_{k+1}\gamma\partial^{k-j}\tilde{\psi}_j \\ &= (-im)\frac{(k-j+1)}{k+1}\delta_k\partial^{k-j}\tilde{\psi}_j + \frac{m^2(k-j)(k-j+1)}{2(k+1)^2}\delta_k\gamma\partial^{k-j-1}\tilde{\psi}_j\,.\end{aligned} \tag{A3}$$

The symbol δ_k denotes the projection operator $\delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}$ onto tensors with k symmetric and traceless vector indices.

APPENDIX B

From I, Eq. (3.24), we have

$$\begin{aligned}G_k(p \cdot p') &= \frac{k+1}{2^k}|p|^k|p'|^k P_k(\gamma) \\ &= p'^{\nu_1}\cdots p'^{\nu_k}\delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}p_{\mu_1}\cdots p_{\mu_k}\,,\end{aligned} \tag{B1}$$

where

$$P_k(\gamma) = \frac{\sinh(k+1)\theta}{(k+1)\sinh\theta}\,, \quad \gamma = \cosh\theta = \frac{p \cdot p'}{|p||p'|}\,.$$

We want to free the covariant labels μ_i in $\delta^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_k}$ from contractions. To do this, we let

$$\begin{aligned}&p_\mu \to p_\mu + g_{\mu\alpha}\epsilon^\alpha\,, \\ &p^2 \to p^2 + 2p_\alpha\epsilon^\alpha\,, \\ &|p|^k \to |p|^k\left(1 + \frac{kp_\alpha}{|p|^2}\epsilon^\alpha\right), \\ &\gamma \equiv \frac{p \cdot p'}{|p||p'|} \to \gamma + \frac{p'_\alpha\epsilon^\alpha}{|p||p'|} - \gamma\frac{p_\alpha\epsilon^\alpha}{p^2}\,.\end{aligned} \tag{B2}$$

The following two expressions are crucial. Given

$$|p|^m|p'|^m F(\gamma) \equiv \tilde{F}_m(p,p')\,,$$

where $F(\gamma)$ is an arbitrary differentiable function, we have, under the infinitesimal translation $p_\alpha \to p_\alpha + \epsilon_\alpha$,

$$\tilde{F}_m(p,p') \to \tilde{F}_m(p,p') + \epsilon_\alpha\{|p|^{m-1}|p'|^{m-1}F'(\gamma)p'^\alpha + p^\alpha|p|^{m-2}|p'|^m[mF(\gamma) - \gamma F'(\gamma)]\}\,, \tag{B3a}$$

and similarly, under $p'_\beta \to p'_\beta + \lambda_\beta$,

$$\tilde{F}_m(p,p') \to \tilde{F}_m(p,p') + \lambda^\beta\{|p|^{m-1}|p'|^{m-1}F'(\gamma)p_\beta + p'_\beta|p'|^{m-2}|p|^m[mF(\gamma) - \gamma F'(\gamma)]\}. \tag{B3b}$$

Repeated use of (B2) and (B3) and of the identity

$$kP_k(\gamma) - \gamma P_k'(\gamma) = -P'_{k-1}(\gamma) \tag{B4}$$

leads to the following formulas:

$$G_k^\alpha \equiv p'^{\nu_1}\cdots p'^{\nu_k}\delta^{\alpha\mu_2\cdots\mu_k}_{\nu_1\cdots\nu_k}p_{\mu_2}\cdots p_{\mu_k} = \left(\frac{k+1}{k}\right)\frac{1}{2^k}|p|^{k-1}|p'|^{k-1}P_k'(\gamma)p'^\alpha - \frac{1}{2^k}|p|^{k-2}|p'|^k P'_{k-1}(\gamma)p^\alpha \tag{B5}$$

and

$$\begin{aligned} G_k^{\alpha\beta} &= p'^{\nu_1}\cdots p'^{\nu_k}\delta^{\alpha\beta\mu_3\cdots\mu_k}_{\nu_1\cdots\nu_k}p_{\mu_3}\cdots p_{\mu_k} \\ &= -g^{\alpha\beta}\frac{1}{2^k(k-1)}|p|^{k-2}|p'|^k P_{k-1}'(\gamma) - \frac{(p'^\alpha p^\beta + p'^\beta p^\alpha)}{2^k(k-1)}|p|^{k-3}|p'|^{k-1}P_{k-1}''(\gamma) \\ &\quad + \frac{(k+1)}{k(k-1)2^k}|p|^{k-2}|p'|^{k-2}P_k''(\gamma)p'^\alpha p'^\beta + \frac{1}{k2^k}|p|^{k-4}|p'|^k P_{k-2}''(\gamma)p^\alpha p^\beta. \end{aligned} \tag{B6}$$

APPENDIX C

We want to write the matrix elements of the currents in terms of a sum over two neighboring representations of the Lorentz group. We recognize that[5]

$$\begin{aligned} u_{\sigma_1\cdots\sigma_j}(p\Lambda) &= \sum_{\lambda\sigma}\langle j\tfrac{1}{2}\lambda\sigma|j+\tfrac{1}{2}\Lambda\rangle\epsilon_{\sigma_1\cdots\sigma_j}(\lambda p)u(p\sigma), \\ \bar{u}^{\lambda_1\cdots\lambda_j}(p'\Lambda') &= \sum_{\lambda'\sigma'}\epsilon^{*\lambda_1\cdots\lambda_j}(\lambda'p')\bar{u}(p'\sigma')\langle j+\tfrac{1}{2}\Lambda'|j\tfrac{1}{2}\lambda'\sigma'\rangle. \end{aligned} \tag{C1}$$

From Eq. (A6) of I we have

$$\begin{aligned} \epsilon_{\sigma_1\cdots\sigma_j}(p\lambda) &= \frac{2^{j/2}}{j!}[L_{\sigma_1}\cdots L_{\sigma_j}]_{0,jj_1\sigma_1}D^{(j/2,j/2)}_{j_1\sigma_1,j\lambda}(L(p)), \\ \epsilon^{*\lambda_1\cdots\lambda_j}(p'\lambda') &= \frac{2^{j/2}}{j!}D^{(j/2,j/2)}_{j\lambda',j_2\sigma_2}(L^{-1}(p'))[L^{\lambda_1}\cdots L^{\lambda_j}]_{jj_2\sigma_2,0}, \end{aligned} \tag{C2}$$

and

$$\delta^{\mu_1\cdots\mu_{k+1}}_{\nu_1\cdots\nu_{k+1}} = \frac{2^{k+1}(-1)^{k+1}}{[(k+1)!]^2}[L_{\nu_1}\ldots L_{\nu_{k+1}}]_{0,k+1\,j\sigma}[L^{\mu_1}\cdots L^{\mu_{k+1}}]_{k+1\,j\sigma,0}. \tag{C3}$$

Therefore

$$\begin{aligned} \delta^{\mu_1\cdots\mu_k\mu}_{\nu_1\cdots\nu_{k+1}}\epsilon^*_{\mu_1\cdots\mu_m}(p's')\epsilon^{\nu_1\cdots\nu_j}(p,s) &= \frac{2^{k+1}}{2^{(m+j)/2}}(-1)^{k+j+m+1}\frac{m!\,j!}{[(k+1)!]^2}D^{(m/2,m/2)}_{ms',j_2\sigma_2}(L^{-1}(p')) \\ &\quad\times[L^{\mu_{m+1}}\cdots L^{\mu_k}L^\mu]_{mj_2\sigma_2,k+1\,j_3\sigma_3}[L_{\nu_{j+1}}\cdots L_{\nu_{k+1}}]_{k+1\,j_3\sigma_3,jj_4\sigma_4}D^{(j/2,j/2)}_{j_4\sigma_4,js}(L(p)) \end{aligned} \tag{C4}$$

and

$$\begin{aligned} \delta^{\mu_1\cdots\mu_k\mu}_{\nu_1\cdots\nu_{k+1}}\epsilon^{*\nu_1\cdots\nu_m}(p's')\epsilon_{\mu_1\cdots\mu_j}(p,s) &= \frac{2^{k+1}}{2^{(m+j)/2}}(-1)^{k+j+m+1}\frac{m!\,j!}{[(k+1)!]^2}D^{(m/2,m/2)}_{ms',j_2\sigma_2}(L^{-1}(p')) \\ &\quad\times[L_{\nu_{m+1}}\cdots L_{\nu_{k+1}}]_{mj_2\sigma_2 jk+1j_3\sigma_3}[L^\mu L^{\mu_k}\cdots L^{\mu_{j+1}}]_{k+1\,j_3\sigma_3,jj_4\sigma_4}D^{(j/2,j/2)}_{j_4\sigma_4,js}(L(p)). \end{aligned} \tag{C5}$$

We also have from (A13)–(A15) of I

$$p_\mu L^\mu D(L(p)) = D(L(p))mL^0$$

and

$$D(L^{-1}(p'))p'_\mu L^\mu = m' L^0 D(L^{-1}(p')),$$
$$(L^0)_{kjs,k-1\,j's'} = a^j_k \delta_{jj'}\delta_{ss'},\qquad \text{(C6)}$$
$$(L^0)_{k-1\,js,kj's'} = -a^j_k \delta_{jj'}\delta_{ss'},$$

with

$$a^j_k = \tfrac{1}{2}[(k-j)(k+j+1)]^{1/2}.$$

This leads to the following identities: Using

$$A \equiv A(kjj'\Lambda\Lambda'\sigma\sigma') = (m')^{k-j'} m^{k-j} \langle j'+\tfrac{1}{2}\Lambda' | j'\tfrac{1}{2}\lambda'\sigma'\rangle \langle j\tfrac{1}{2}\Lambda\sigma | j+\tfrac{1}{2}\lambda\sigma\rangle [(2j+1)^{1/2}(2j+1)^{1/2} 2^{[(j+j')/2]} 2^{k-j-j'}(k+1)^2]^{-1}, \qquad \text{(C7)}$$

$$\delta^{\mu\sigma_1\cdots\sigma_k}_{\lambda_1\cdots\lambda_{k+1}} p'^{\lambda_{j'+1}}\cdots p'^{\lambda_k} p_{\sigma_{j+1}}\cdots p_{\sigma_k} \bar{u}^{\lambda_1\cdots\lambda_j} \gamma^{\lambda_{k+1}} \tfrac{1}{2}\tau^a \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u_{\sigma_1\cdots\sigma_j}$$
$$= \sum_{\Lambda'\sigma'\Lambda\sigma} A[(k-j)!(k+j+1)!(k-j')!(k+j'+1)!]^{1/2} \bar{u}\gamma^{\lambda_{k+1}}\tfrac{1}{2}\tau^a \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u$$
$$\times [D^{k/2,k/2}(L^{-1}(p')) L^{(+)}_{\lambda_{k+1}} L^{(-)\mu} D^{k/2,k/2}(L(p))]_{j'\lambda',j\lambda}, \qquad \text{(C8)}$$

$$\delta^{\mu\sigma_1\cdots\sigma_k}_{\lambda_1\lambda_2\cdots\lambda_{k+1}} p^{\lambda_{j+1}}\cdots p^{\lambda_k} p'_{\sigma_{j'+1}}\cdots p'_{\sigma_k} \bar{u}_{\sigma_1\cdots\sigma_j} \tfrac{1}{2}\tau^a \gamma^{\lambda_{k+1}} \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u^{\lambda_1\cdots\lambda_j}$$
$$= \sum_{\Lambda'\sigma'\Lambda\sigma} 2A[(k-j)!(k-j')!(k+j+1)!(k+j'+1)!]^{1/2}$$
$$\times \bar{u}\tfrac{1}{2}\tau^a\gamma^{\lambda_{k+1}} \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u (D^{k/2,k/2} L^{(+)\mu} L^{(-)}_{\lambda_{k+1}} D^{k/2,k/2})_{j'\Lambda',j\Lambda}, \qquad \text{(C9)}$$

$$\delta^{\mu\sigma_1\cdots\sigma_k}_{\lambda_1\lambda_2\cdots\lambda_{k+1}} p^{\lambda_{j+1}}\cdots p^{\lambda_{k+1}} p'_{\sigma_{j+1}}\cdots p'_{\sigma_{k-1}} \bar{u}_{\sigma_1\cdots\sigma_{j'}} \tfrac{1}{2}\tau^a \gamma^{\lambda_{k+1}} \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u^{\lambda_1\cdots\lambda_j}$$
$$= \sum_{\Lambda'\sigma'\Lambda\sigma} -2(m/m') A[(k-j'-1)!(k-j-1)!(k+j')!(k+j+2)!]^{1/2} \bar{u}\gamma_\lambda \tfrac{1}{2}\tau^a \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u$$
$$\times (D^{(k-1)/2,(k-1)/2} L^{(+)\lambda} L^{(+)\mu} D^{(k+1)/2,(k+1)/2})_{j'\Lambda',j\Lambda}, \qquad \text{(C10)}$$

$$\delta^{\mu\sigma_1\cdots\sigma_k}_{\lambda_1\lambda_2\cdots\lambda_{k+1}} p'^{\lambda_{j+1}}\cdots p'^{\lambda_{k+1}} p_{\sigma_{j+1}}\cdots p_{\sigma_{k-1}} \bar{u}^{\lambda_1\cdots\lambda_{j'}} \gamma_{\sigma_k} \tfrac{1}{2}\tau^a \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u_{\sigma_1\cdots\sigma_j}$$
$$= \sum -2(m'/m) A[(k+1-j')!(k+2+j')!(k-j-1)!(k+j)!]^{1/2} \bar{u}\gamma_\lambda \tfrac{1}{2}\tau^a \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u$$
$$\times (D^{(k+1)/2,(k+1)/2} L^{(-)\mu} L^{(+)\lambda} D^{(k-1)/2,(k-1)/2})_{j'\Lambda',j\Lambda}, \qquad \text{(C11)}$$

$$\delta^{\mu\sigma_1\cdots\sigma_k}_{\lambda_1\lambda_2\cdots\lambda_{k+1}} p^{\lambda_{j+1}}\cdots p^{\lambda_k} p'_{\sigma_{j'+2}}\cdots p'_{\sigma_k} \bar{u}_{\sigma_1\cdots\sigma_{j'}} \gamma_{\sigma_{j'+1}} \gamma^{\lambda_{k+1}} \begin{pmatrix}1\\ \gamma_5\end{pmatrix} \tfrac{1}{2}\tau^a u^{\lambda_1\cdots\lambda_j}$$
$$= \sum (-4/m') A[(k-j'-1)!(k+j')!(k-j)!(k+j+1)!]^{1/2}$$
$$\times \Bigg\{ \left(\frac{k+2}{2}\right)^2 \bar{u}\tau^a \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u (D^{(k-1)/2,(k-1)/2} L^{(+)\mu} D^{k/2,k/2})_{j'\Lambda',j\Lambda}$$
$$- i\bar{u}\tfrac{1}{2}\tau^a \sigma_{\lambda k} \begin{pmatrix}1\\ \gamma_5\end{pmatrix} u (D^{(k-1)/2,(k-1)/2} L^{(+)\mu} L^{(+)\lambda} L^{(-)k} D^{k/2,k/2})_{j'\Lambda',j\Lambda} \Bigg\}, \qquad \text{(C12)}$$

$$\delta^{\mu\sigma_1\cdots\sigma_k}_{\lambda_1\lambda_2\cdots\lambda_{k+1}} p_{\sigma_{j+2}}\cdots p_{\sigma_k} p'^{\lambda_{j'+1}}\cdots p'^{\lambda_k}\bar{u}^{\lambda_1\cdots\lambda_{j'}}\gamma^{\lambda_{k+1}}\tfrac{1}{2}\tau^a\begin{pmatrix}1\\ \gamma_5\end{pmatrix}\gamma_{\sigma_{j+1}}u_{\sigma_1\cdots\sigma_j}$$
$$=\sum_{\Lambda'\sigma',\Lambda\sigma}(4/m)A[(k-j')!(k+j'+1)!(k-j-1)!(k+j)!]^{1/2}$$
$$\times\left\{\frac{(k+2)^2}{2}\,\bar{u}\tfrac{1}{2}\tau^a\begin{pmatrix}1\\ -\gamma_5\end{pmatrix}u\,(D^{k/2,k/2}L^{(-)\mu}D^{(k-1)/2,(k-1)/2})_{j'\Lambda',j\Lambda}\right.$$
$$\left.-i\bar{u}\tfrac{1}{2}\tau^a\sigma_{\lambda k}\begin{pmatrix}1\\ -\gamma_5\end{pmatrix}u\,(D^{k/2,k/2}L^{(+)\lambda}L^{(-)\lambda}L^{(-)\mu}D^{(k-1)/2,(k-1)/2})_{j'\Lambda',j\Lambda}\right\}. \tag{C13}$$

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[1]F. Cooper and A. Chodos, Phys. Rev. D 4, 2374 (1971).

[2]In (2.4) we use $\bar{\alpha}_k$ rather than α_k to distinguish the fermion parameters from those characterizing the boson theory in (2.1).

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