[28] In some theories [e.g., I. S. Gerstein, R. Jackiw, B. Lee, and S. Weinberg, Phys. Rev. D 3, 2486 (1971)], it can be shown that the use of $i\mathcal{L}^{\text{int}}$ in evaluating Feynman amplitudes is equivalent to the use of $-i\mathcal{H}^{\text{int}}$ together with some appropriate noncovariant propagators. ($\mathcal{H}^{\text{int}}$ means the interaction Hamiltonian density.)

[29] The notation $\mathfrak{D}(s,u,\ldots z)$ is the set of all Feynman graphs with simple vertices (internal or external) $s,u,\ldots,z$.

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Perturbation Lagrangian Theory for Dirac Fields–Ward-Takahashi Identity and Current Algebra*

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In a class of Lagrangian field theories for Dirac spin-$\frac{1}{2}$ particles, the Bogoliubov-Parasiuk-Hepp renormalization scheme provides a proof of the operator forms of Euler-Lagrange equations of motion, Noether's theorem, and Ward-Takahashi identities. Time-ordered products for some derivatives of Dirac fields can only be defined with special care in terms of Feynman graphs. Current-algebra Ward-Takahashi identities are obtained if $\mathcal{L}^{\text{derivative}}$ is invariant under the algebra; however, Schwinger terms are absent from these identities.

I. INTRODUCTION

In the preceding paper,[1] we described perturbation theory for scalar fields in terms of objects called *vertices* and employed the Bogoliubov-Parasiuk-Hepp[2] (BPH) renormalization scheme to prove Ward-Takahashi identites,[3] which led to operator forms of Euler-Lagrange equations of motion and Noether's theorem. They also enabled us to construct the energy-momentum tensor, angular momentum current operators, and internal-symmetry currents and generators. Assuming that the derivative part, $\mathcal{L}^{\text{derivative}}$, of a Lagrangian was invariant under a symmetry algebra, we could prove current-algebra Ward-Takahashi identities. This is rather interesting, as the main bulk of current-algebra[4] results that agree with experiment comes from these identities.

The purpose of this paper is to extend these results to Dirac fields, which call for some modification. We will assume that the reader is familiar with Ref. 1, on which we will rely extensively.

The basic fields here are basis vectors of Dirac's representation[5] of the Lorentz group. They are ψ^a ($a=1,\ldots,4$) and their complex conjugate ψ^{a*} for each Dirac spin-$\frac{1}{2}$ particle. The generator of Lorentz rotation in the α-β plane is the spin matrix

$$s_{\alpha\beta}\equiv\tfrac{1}{4}[\gamma_\alpha,\gamma_\beta], \tag{1.1}$$

satisfying the commutation relation

$$[s_{\alpha\beta},s_{\mu\nu}]=g_{\alpha\nu}s_{\beta\mu}+g_{\beta\mu}s_{\alpha\nu}-g_{\alpha\mu}s_{\beta\nu}-g_{\beta\nu}s_{\alpha\mu}. \tag{1.2}$$

Repeating from Ref. 1, a *simple vertex* is the ordered pair $[f,\alpha]$, where f is a (possibly empty) sequence of fields[6] and α is an *excess-subtraction function* (see Ref. 1 for definition). A *vertex* is a finite formal linear combination of simple vertices:

$$W=a^1w^1+\cdots+a^nw^n, \tag{1.3}$$

where the coefficients a^i are polynomially bounded infinitely differentiable functions of space-time coordinates. Because of Fermi statistics, the notion of substitution needs a little modification which consists of inserting appropriate Fermi signature factors to every term in Eq. (2.4) of Ref. 1. Thus for simple vertices $w\equiv[f,\alpha]$ and $v\equiv[g,\beta]$ and any field ϕ, the *substitution* is

$$\left(\frac{\delta w}{\delta\phi}\middle| v\right)\equiv\sum_{j\in U_\phi^w}\begin{pmatrix} g,\phi,f_1,\ldots,f_{j-1}\\ \phi,f_1,\ldots,f_{j-1},g\end{pmatrix}[h^{(j)},\gamma^{(j)}]. \tag{1.4}$$

Denoting the sequences of Dirac fields in the two rows above by A and A', the signature factor[7] is defined by

$$\begin{pmatrix}A\\A'\end{pmatrix}\equiv(-)^r, \tag{1.5}$$

where r is the number of transpositions of Dirac fields required to permute the sequence A to A'.

See Ref. 1 for definitions of U_ϕ^w, $h^{(j)}$, and $\gamma^{(j)}$. For vertices $W=\sum_i a^i w^i$ and $V=\sum_j b^j v^j$, the substitution remains

$$\left(\frac{\delta W}{\delta\phi}\middle| V\right)\equiv\sum_{ij} a^i b^j\left(\frac{\delta w^i}{\delta\phi}\middle| v^j\right). \qquad (1.6)$$

The *functional derivative* and *space-time derivatives* are also unchanged from Ref. 1:

$$\frac{\delta W}{\delta\phi}\equiv\left(\frac{\delta W}{\delta\phi}\middle| 1\right), \qquad (1.7)$$

$$\partial_\mu W\equiv\sum_{\phi\in\mathfrak{F}}\left(\frac{\delta W}{\delta\phi}\middle|\partial_\mu\psi\right)+\sum_i(\partial_\mu a^i)w^i. \qquad (1.8)$$

The transformation vertex in Ref. 1 can be viewed as a mapping taking every basic field to a vertex. Thus the infinitesimal translational vertex is the mapping

$$\phi\xrightarrow{t_\mu}\partial_\mu\phi, \qquad (1.9)$$

for any basic field ϕ, and the infinitesimal Lorentz rotational vertex in the α-β plane is the mapping

$$\begin{aligned}&\psi\xrightarrow{\sigma_{\alpha\beta}}X_\alpha\partial_\beta\psi-X_\beta\partial_\alpha\psi+s_{\alpha\beta}\psi,\\&\psi^*\xrightarrow{\sigma_{\alpha\beta}}X_\alpha\partial_\beta\psi^*-X_\beta\partial_\alpha\psi^*+s^*_{\alpha\beta}\psi^*,\end{aligned} \qquad (1.10)$$

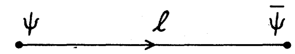

FIG. 1. Dirac propagator.

where $s_{\alpha\beta}$ is given by Eq. (1.1).

Feynman amplitudes are defined and renormalized by the BPH[2] scheme as in Ref. 1. Of course, the usual Fermi statistics is assumed. The Dirac propagator (Fig. 1) is chosen, after Zimmermann,[8] to be [9]

$$\frac{i}{(2\pi)^4}\,\frac{-\not{l}+m}{l^2-m^2+i\epsilon(\vec{l}^{\,2}+m^2)}, \qquad (1.11)$$

so that his convergence proof applies here. The Feynman amplitude for vertices $W^1,\ldots,W^n$ is denoted by

$$\mathfrak{F}(W^1,x^1;\ldots;W^n,x^n).$$

An immediate consequence is Lemma 1:

$$\frac{\partial}{\partial x^i_\mu}\mathfrak{F}(W^1,x^1;\ldots;W^n,x^n)=\mathfrak{F}(W^1,x^1;\ldots;\partial^\mu W^i,x^i;\ldots;W^n,x^n), \qquad (1.12)$$

which is proved in Appendix A of Ref. 1.

II. FUNDAMENTAL IDENTITIES AND PROPERTIES OF TIME-ORDERED PRODUCTS

Let the Lagrangian be a Lorentz-invariant vertex of the form

$$\mathfrak{L}=\mathfrak{L}^{\text{free}}+\mathfrak{L}^{\text{int}}, \qquad (2.1)$$

where[10]

$$\mathfrak{L}^{\text{free}}=\tfrac{1}{2}i[\bar\psi\overleftrightarrow{\not\partial}\psi,0]-m[\bar\psi\psi,\alpha^{\text{mass}}_{\text{Dirac}}], \qquad (2.2)$$

and the excess-subtraction function $\alpha^{\text{mass}}_{\text{Dirac}}$ of the mass term is defined by the condition[11,12]

$$\alpha^{\text{mass}}_{\text{Dirac}}(I_2)=1. \qquad (2.3)$$

In complete analogy to Appendix B of Ref. 1, we obtain, for simple vertices $w, v^1, \ldots,$ and v^n, the *first identity*,

$$\begin{aligned}\sum_{s=0}^{\infty}\sum_{r=0}^{s}(-)^r\binom{s}{r}\Bigg[&\mathfrak{F}\left(\partial_{\mu_1}\cdots\partial_{\mu_r}\left(\frac{\delta\mathfrak{L}}{\delta\partial_{\mu_1}\cdots\partial_{\mu_s}\phi}\middle|\partial_{\mu_{r+1}}\cdots\partial_{\mu_s}w\right),x;v^1,y^1;\ldots;v^n,y^n\right)\\&-i\sum_{i=1}^{n}\begin{pmatrix}v^1,\ldots,v^{i-1},v^i\\v^i,v^1,\ldots,v^{i-1}\end{pmatrix}\partial_{\mu_1}\cdots\partial_{\mu_r}\delta^4(x-y^i)\\&\times\mathfrak{F}\left(\left(\frac{\delta v^i}{\delta\partial_{\mu_1}\cdots\partial_{\mu_s}\phi}\middle|\partial_{\mu_{r+1}}\cdots\partial_{\mu_s}w\right),y^i;v^1,\ldots,y^{i-1};v^{i+1},\ldots;v^n,y^n\right)\Bigg]=0,\end{aligned} \qquad (2.4)$$

which differs from that of Ref. 1 by the presence of Fermi signature factor

$$\begin{pmatrix}v^1,\ldots,v^{i-1},v^i\\v^i,v^1,\ldots,v^{i-1}\end{pmatrix},$$

which, similarly to (1.5), is the Fermi sign involved in rearranging the sequence of fields as they appear in $v^1, \ldots, v^i$ to the sequence in $v^i, v^1, \ldots, v^{i-1}$.

If $\Lambda = (\Lambda^1, \ldots, \Lambda^M)$ is a transformation vertex, we can as before transform (2.4) into the *second identity:*

$$\frac{\partial}{\partial x_\mu} \mathfrak{F}(J_\mu^\Lambda, x^1; v^1, y^1; \cdots; v^n, y^n) = \mathfrak{F}(\delta_\Lambda \mathcal{L}, x; v^1, y^1; \cdots; v^n, y^n) - i\sum_{i=1}^{n} \begin{pmatrix} v^1, \ldots, v^{i-1}, v^i \\ v^i, v^1, \ldots, v^{i-1} \end{pmatrix} \times \Bigg[\delta(x-y^i)\mathfrak{F}(\delta_\Lambda v^i, y^i; v^1, \ldots, y^{i-1}; v^{i+1}, \ldots, y^n) + \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} (-)^r \binom{s}{r} \partial_{\mu_1} \cdots \partial_{\mu_r} \delta^4(x-y^i) \times \mathfrak{F}\left(\left(\frac{\delta v^i}{\delta \partial_{\mu_1} \cdots \partial_{\mu_s} \phi^a} \middle| \partial_{\mu_{r+1}} \cdots \partial_{\mu_s} \Lambda^a\right), y^i; v^1, \ldots, y^{i-1}; v^{i+1}, \ldots, y^n\right)\Bigg], \tag{2.5}$$

where the *transformation current vertex* is

$$J_\mu^\Lambda \equiv \sum_{s=0}^{\infty} \sum_{r=0}^{s} (-)^r \binom{s+1}{r+1} \partial_{\mu_1} \cdots \partial_{\mu_r} \left(\frac{\delta \mathcal{L}}{\delta \partial^\mu \partial_{\mu_1} \cdots \partial_{\mu_s} \phi^a} \middle| \partial_{\mu_{r+1}} \cdots \partial_{\mu_s} \Lambda^a\right) \tag{2.6}$$

and the *variation* $\delta_\Lambda W$ is defined by

$$\delta_\Lambda W \equiv \sum_{s=0}^{\infty} \left(\frac{\delta W}{\delta \partial_{\mu_1} \cdots \partial_{\mu_s} \phi^a} \middle| \partial_{\mu_1} \cdots \partial_{\mu_s} \Lambda^a\right). \tag{2.7}$$

As with Ref. 1, vacuum expectation values of time-ordered products cannot be the Feynman amplitudes, for otherwise Lemma 1 [Eq. (1.12)] and the first identity (2.4) will be in contradiction with the assumption of the uniqueness of time-ordered products. However, if we limit ourselves to nonderivative vertices, then there is no contradiction. Therefore, let us *assume* that, for any nonderivative vertex V, there exists a local operator $V_{\text{op}}(x)$ such that

$$\langle 0 | T[V_{\text{op}}^1(x^1) \cdots V_{\text{op}}^n(x^n)] | 0 \rangle = \mathfrak{F}(V^1, x^1; \cdots; V^n, x^n). \tag{2.8}$$

Thus if Λ is a nonderivative transformation vertex such that J_μ^Λ and $\delta_\Lambda \mathcal{L}$ are also nonderivative, then the second identity (2.5) is transformed into

$$\frac{\partial}{\partial x_\mu} \langle 0 | T[J_{\mu\text{op}}^\Lambda(x) v_{\text{op}}^1(y^1) \cdots v_{\text{op}}^n(y^n)] | 0 \rangle = \langle 0 | T[(\delta_\Lambda \mathcal{L})_{\text{op}}(x) v_{\text{op}}^1(y^1) \cdots v_{\text{op}}^n(y^n)] | 0 \rangle - i \sum_{i=1}^{n} \begin{pmatrix} v^1, \ldots, v^i \\ v^i, v^1, \ldots, v^{i-1} \end{pmatrix} \delta(x-y^i) \times \langle 0 | T[(\delta_\Lambda v^i)_{\text{op}}(y^i) v_{\text{op}}^1(y^1) \cdots v_{\text{op}}^{i-1}(y^{i-1}) v_{\text{op}}^{i+1}(y^{i+1}) \cdots v_{\text{op}}^n(y^n)] | 0 \rangle, \tag{2.9}$$

for nonderivative simple vertices $v^1, \ldots, v^n$. Applying the Lehmann-Symanzik-Zimmermann[13] (LSZ) reduction formula to this equation for the case where the v's are basic fields, we obtain *Noether's theorem* for this Λ:

$$\frac{\partial}{\partial x_\mu} J_{\mu\,\text{op}}^\Lambda(x) = (\delta_\Lambda \mathcal{L})_{\text{op}}(x). \tag{2.10}$$

Substituting this back into (2.9), we obtain the *Ward-Takahashi identity:*

$$\frac{\partial}{\partial x_\mu} \langle 0 | T[J_{\mu\,\text{op}}^\Lambda(x) v_{\text{op}}^1(y^1) \cdots v_{\text{op}}^n(y^n)] | 0 \rangle = \left\langle 0 \middle| T\left[\frac{\partial}{\partial x_\mu} J_{\mu\,\text{op}}^\Lambda(x) v_{\text{op}}^1(y^1) \cdots v_{\text{op}}^n(y^n)\right] \middle| 0 \right\rangle - i \sum_{i=1}^{n} \begin{pmatrix} v^1, \ldots, v^i \\ v^i, v^1, \ldots, v^{i-1} \end{pmatrix} \delta(x-y^i) \times \langle 0 | T[(\delta_\Lambda v^i)_{\text{op}}(y^i) v_{\text{op}}^1(y^1) \cdots v_{\text{op}}^{i-1}(y^{i-1}) v_{\text{op}}^{i+1}(y^{i+1}) \cdots v_{\text{op}}^n(y^n)] | 0 \rangle. \tag{2.11}$$

We will now seek a limited extension of the previous assumption about time-ordered products. Let us assume that the Lagrangian is separable into the form

$$\mathcal{L} = i\tfrac{1}{2}(1+b)[\bar{\psi}\overleftrightarrow{\partial}\psi, 0] + \mathcal{L}_1, \tag{2.12}$$

where b represents a counterterm for satisfying the condition of renormalization, and where $\mathcal{L}_1$ does not contain derivatives of Dirac fields. Consider a simple vertex with only a single derivative of Dirac field and of the form[14]

$$[(\partial_\mu \bar{\psi}^c, A), \alpha] = \pm \frac{1}{2i(1+b)}\left[\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\nu\psi^d}\,\middle|\, w\right) - \left(\frac{\delta\mathcal{L}}{\delta\partial_\nu\psi^d}\,\middle|\,\partial_\mu w\right)\right](\gamma_\nu)^{dc}, \tag{2.13}$$

where w is some nonderivative simple vertex. Then the first identity (2.4) implies

$$(\gamma_\mu)^{dc}\mathfrak{F}(\pm i\tfrac{1}{2}(1+b)[(\partial^\mu\bar{\psi}^d, A), \alpha], x; v^1, y^1; \ldots; v^n, y^n)$$
$$= \mathfrak{F}\left(\left(\frac{\delta\mathcal{L}}{\delta\psi^c}\,\middle|\, w\right), x; v^1, y^1; \ldots; v^n, y^n\right)$$
$$- i\sum_{i=1}^{n}\begin{pmatrix} v^1, \ldots, v^i \\ v^i, v^1, \ldots, v^{i-1}\end{pmatrix}\delta(x-y^i)\mathfrak{F}\left(\left(\frac{\delta v^i}{\delta\psi^c}\,\middle|\, w\right), y^i; v^1, \ldots, y^{i-1}; v^{i+1}, \ldots, y^n\right)$$

for nonderivative v's. Separating $\mathcal{L}$ as in (2.12), we find

$$\mathfrak{F}\left(\pm i(1+b)[(\partial^\mu\bar{\psi}^d, A), \alpha](\gamma_\mu)^{dc} - \left(\frac{\delta\mathcal{L}_1}{\delta\psi^c}\,\middle|\, w\right), x; v^1, y^1; \ldots; v^n, y^n\right)$$
$$= -i\sum_{i=1}^{n}\begin{pmatrix} v^1, \ldots, v^i \\ v^i, v^1, \ldots, v^{i-1}\end{pmatrix}\delta(x-y^i)\mathfrak{F}\left(\left(\frac{\delta v^i}{\delta\psi^c}\,\middle|\, w\right), y^i; v^1, \ldots, y^{i-1}; v^{i+1}, \ldots, y^n\right). \tag{2.14}$$

This obviously illustrates how the assumption that Feynman amplitudes are vacuum expectation values of time-ordered products leads to contradiction. We may bypass this contradiction by assuming that, for the vertex $[(\partial_\mu\bar{\psi}^c, A), \alpha]$ of Eq. (2.13), a local operator $[(\partial_\mu\bar{\psi}^c, A), \alpha]_{\text{op}}$ exists and its time-ordered products are given by

$$\langle 0|\, T[[(\partial_\mu\bar{\psi}^c, A), \alpha]_{\text{op}}(x)v^1_{\text{op}}(y^1)\cdots v^n_{\text{op}}(y^n)]\,|0\rangle$$
$$\equiv \mathfrak{F}([(\partial_\mu\bar{\psi}^c, A), \alpha], x; v^1, y^1; \ldots; v^n, y^n)$$
$$\pm\frac{1}{4(1+b)}(\gamma_\mu)^{dc}\sum_{i=1}^{n}\begin{pmatrix} v^1, \ldots, v^i \\ v^i, v^1, \ldots, v^{i-1}\end{pmatrix}\delta(x-y^i)\,\mathfrak{F}\left(\left(\frac{\delta v^i}{\delta\psi^d}\,\middle|\, w\right), y^i; v^1, y^1; \ldots, y^{i-1}; v^{i+1}, \ldots; v^n, y^n\right). \tag{2.15}$$

Then (2.14) simply becomes the operator equation

$$\pm i(1+b)(\gamma_\mu)^{dc}[(\partial^\mu\bar{\psi}^d, A), \alpha]_{\text{op}} = \left(\frac{\delta\mathcal{L}_1}{\delta\psi^c}\,\middle|\, w\right)_{\text{op}}. \tag{2.16}$$

Similarly for a simple vertex of the form

$$[(A, \partial_\mu\psi^c), \alpha] = \pm\frac{1}{2i(1+b)}(\gamma_\nu)^{cd}\left[\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\nu\bar{\psi}^d}\,\middle|\, w\right) - \left(\frac{\delta\mathcal{L}}{\delta\partial_\nu\bar{\psi}^d}\,\middle|\,\partial_\mu w\right)\right], \tag{2.17}$$

we define the local operator $[(A, \partial_\mu\psi^c), \alpha]_{\text{op}}$ by

$$\langle 0|\, T[[(A, \partial_\mu\psi^c), \alpha]_{\text{op}}(x)v^1_{\text{op}}(y^1)\cdots v^n_{\text{op}}(y^n)]\,|0\rangle$$
$$\equiv \mathfrak{F}([(A, \partial_\mu\psi^c), \alpha], x; v^1, y^1; \ldots; v^n, y^n)$$
$$\pm\frac{1}{4(1+b)}(\gamma_\mu)^{cd}\sum_{i=1}^{n}\begin{pmatrix} v^1, \ldots, v^i \\ v^i, v^1, \ldots, v^{i-1}\end{pmatrix}\delta(x-y^i)\mathfrak{F}\left(\left(\frac{\delta v^i}{\delta\bar{\psi}^d}\,\middle|\, w\right), y^i; v^1, \ldots, y^{i-1}; v^{i+1}, \ldots, y^n\right), \tag{2.18}$$

for nonderivative v's, and obtain the operator equation,

$$\pm i(1+b)(\gamma_\mu)^{cd}[(A, \partial^\mu\psi^d), \alpha]_{\rm op} = \left(\frac{\delta\mathcal{L}_1}{\delta\bar{\psi}^c}\bigg| w\right)_{\rm op}. \tag{2.19}$$

As a simple illustration, take $w=1$; then (2.16) and (2.19) reduce to the *Euler-Lagrange equations of motion:*

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi^c}\right)_{\rm op} - \left(\frac{\delta\mathcal{L}}{\delta\psi^c}\right)_{\rm op} = 0, \tag{2.20}$$

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\psi}^c}\right)_{\rm op} - \left(\frac{\delta\mathcal{L}}{\delta\bar{\psi}^c}\right)_{\rm op} = 0. \tag{2.21}$$

III. ENERGY-MOMENTUM TENSOR AND ANGULAR MOMENTUM CURRENT OPERATOR

The current vertex corresponding to the infinitesimal translational vertex t_ν [Eq. (1.9)] is defined by (2.6) and (2.12) to be

$$J^{t\nu}_\mu = \left(\frac{\delta\mathcal{L}}{\delta\partial^\mu\psi}\bigg|\partial_\nu\psi\right) + \left(\frac{\delta\mathcal{L}}{\delta\partial^\mu\bar{\psi}}\bigg|\partial_\nu\bar{\psi}\right). \tag{3.1}$$

Its second identity (2.5) can be stated as

$$\frac{\partial}{\partial x_\mu}\mathfrak{F}(J^{t\nu}_\mu - g_{\mu\nu}\mathcal{L}, x; v^1, y^1; \cdots; v^n, y^n) = -i\sum_{i=1}^n \delta(x-y^i)\mathfrak{F}(v^1, y^1; \cdots; \partial_\nu v^i, y^i; \cdots; v^n, y^n). \tag{3.2}$$

To transform this into an operator identity, note that all simple vertices in $J^{t\nu}_\mu$ and $\mathcal{L}$, except those of the form $[\bar{\psi}\gamma_\mu\partial_\nu\psi, 0]$ or $[\partial_\nu\bar{\psi}\gamma_\mu\psi, 0]$, are nonderivative. Now

$$[\bar{\psi}\gamma_\mu\partial_\nu\psi, 0] = -\frac{1}{2i(1+b)}\left[\partial_\nu\left(\frac{\delta\mathcal{L}}{\delta\partial^\lambda\bar{\psi}}\bigg|\bar{\psi}\gamma_\mu\gamma^\lambda\right) - \left(\frac{\delta\mathcal{L}}{\delta\partial^\lambda\bar{\psi}}\bigg|\partial_\nu\bar{\psi}\gamma_\mu\gamma^\lambda\right)\right] \tag{3.3}$$

and

$$[\partial_\nu\bar{\psi}\gamma_\mu\psi, 0] = \frac{1}{2i(1+b)}\left[\partial_\nu\left(\frac{\delta\mathcal{L}}{\delta\partial^\lambda\psi}\bigg|\gamma^\lambda\gamma_\mu\psi\right) - \left(\frac{\delta\mathcal{L}}{\delta\partial^\lambda\psi}\bigg|\gamma^\lambda\gamma_\mu\partial_\nu\psi\right)\right]. \tag{3.4}$$

Hence applying the definitions (2.15) and (2.18),

$$\begin{aligned}\langle 0|\,T[\{J^{t\nu}_{\mu\,\rm op}(x) - g_{\mu\nu}\mathcal{L}_{\rm op}(x)\}v^1_{\rm op}(y^1)\cdots v^n_{\rm op}(y^n)]\,|0\rangle &= \mathfrak{F}(J^{t\nu}_\mu - g_{\mu\nu}\mathcal{L}, x; v^1, y^1; \cdots; v^n, y^n)\\ &\quad+\frac{i}{8}\sum_{i=1}^n \delta(x-y^i)\mathfrak{F}\Bigg(v^1, y^1; \ldots, y^{i-1}; \left(\frac{\delta v^i}{\delta\bar{\psi}}\bigg|\bar{\psi}(4g_{\mu\nu} - \gamma_\mu\gamma_\nu)\right)\\ &\qquad+\left(\frac{\delta v^i}{\delta\psi}\bigg|(4g_{\nu\mu} - \gamma_\nu\gamma_\mu)\psi\right), y^i; v^{i+1}, \ldots; v^n, y^n\Bigg).\end{aligned} \tag{3.5}$$

Defining the energy-momentum tensor

$$\Theta_{\mu\nu} \equiv (J^{t\nu}_\mu)_{\rm op} - g_{\mu\nu}\mathcal{L}_{\rm op}, \tag{3.6}$$

and differentiating (3.5) by x_μ, we transform (3.2) into the Ward-Takahashi identity for $\Theta_{\mu\nu}$:

$$\begin{aligned}&\frac{\partial}{\partial x_\mu}\langle 0|\,T[\Theta_{\mu\nu}(x)v^i_{\rm op}(y^1)\cdots v^n_{\rm op}(y^n)]\,|0\rangle\\ &\quad= \frac{i}{8}\sum_{i=1}^n \partial^\mu\delta(x-y^i)\Bigg\langle 0\Bigg|\,T\Bigg[v^1_{\rm op}(y^1)\cdots v^{i-1}_{\rm op}(y^{i-1})\Bigg[\left(\frac{\delta v^i}{\delta\bar{\psi}}\bigg|\bar{\psi}(4g_{\mu\nu} - \gamma_\mu\gamma_\nu)\right)_{\rm op}(y^i)\\ &\qquad+\left(\frac{\delta v^i}{\delta\psi}\bigg|(4g_{\nu\mu} - \gamma_\nu\gamma_\mu)\psi\right)_{\rm op}(y^i)\Bigg]v^{i+1}_{\rm op}(y^{i+1})\cdots v^n_{\rm op}(y^n)\Bigg]\Bigg|0\Bigg\rangle\\ &\qquad- i\sum_{i=1}^n \delta(x-y^i)\frac{\partial}{\partial y^{i\nu}}\langle 0|\,T[v^1_{\rm op}(y^1)\cdots v^n_{\rm op}(y^n)]\,|0\rangle,\end{aligned} \tag{3.7}$$

where the v's are nonderivative. This shows[15] that the spatial integrals of $\Theta_{0\nu}$ yield[16] the momentum operators P_ν which generate the group of translations. Of course,

$$[P_\nu, v_{op}(x)] = -i\partial_\nu v_{op}(x) \tag{3.8}$$

for any nonderivative v. It also follows from (3.7) that

$$\frac{\partial}{\partial x_\mu} T[\Theta_{\mu\nu}(x)\psi(y)] = \tfrac{1}{4} i\partial^\mu \delta(x-y)(\tfrac{3}{2}g_{\mu\nu} + s_{\mu\nu})\psi(y) - i\delta(x-y)\partial_\nu\psi(y). \tag{3.8$'$}$$

Similarly, the angular momentum current operator is

$$\Sigma_{\mu\alpha\beta}(x) = x_\alpha \Theta_{\mu\beta}(x) - x_\beta\Theta_{\mu\alpha}(x) + (J_\mu^{s\,\alpha\beta})_{op}(x), \tag{3.9}$$

where $s_{\alpha\beta}$ is the transformation vertex given by (1.1). $\Sigma_{\mu\alpha\beta}$ define angular momentum generators $M_{\alpha\beta}$, which have the following commutation relations with any nonderivative vertex v:

$$[M_{\alpha\beta}, v_{op}(x)] = -i[x_\alpha\partial_\beta v_{op}(x) - x_\beta\partial_\alpha v_{op}(x) + (\delta_{s_{\alpha\beta}} v)_{op}(x)]. \tag{3.10}$$

Finally, we note that the operator corresponding to the Lagrangian vertex is[17]

$$\mathcal{L}_{op} = -\tfrac{1}{2}\left[\left(\frac{\delta\mathcal{L}_1}{\delta\bar\psi}\middle|\bar\psi\right)_{op} + \left(\frac{\delta\mathcal{L}_1}{\delta\psi}\middle|\psi\right)_{op}\right] + \mathcal{L}_{1op}, \tag{3.11}$$

so that

$$\mathcal{L}_{op} = 0, \tag{3.12}$$

if $\mathcal{L}_1$ is linear in both ψ and $\bar\psi$.

IV. INTERNAL SYMMETRY AND CURRENT ALGEBRA

Internal-symmetry transformation vertices are known to be linear in Dirac fields. Therefore, looking at (2.12), we find that, for such a transformation vertex Λ, the current vertex J_μ^Λ does not have derivatives of Dirac fields. Their Ward-Takahashi identities can easily be constructed from the second identity (2.5), and, from these identities, again we find[15,16] that the spatial integral of $J_{0\,op}^\Lambda$ is the corresponding generator of the symmetry.

To discuss current-algebra[4] results, we easily obtain Lemma 2 (similar to Lemma 2 of Ref. 1):

If Λ and Γ are two nonderivative transformation vertices with zero excess-subtraction functions and linear in Dirac fields, and if

$$\delta_\Lambda \mathcal{L}^{\text{derivative}} = 0, \tag{4.1}$$

then

$$\delta_\Lambda J_\mu^\Gamma = J_\mu^{[\Gamma,\Lambda]}. \tag{4.2}$$

Thus for a symmetry algebra of transformation vertices, with zero excess-subtraction functions, satisfying (4.1) for each Λ in the algebra, we obtain the current-algebra Ward-Takahashi identities:

$$\begin{aligned}\frac{\partial}{\partial x_\mu}\langle 0|\, T[J^\Lambda_{\mu\,op}(x)J^\Gamma_{\nu\,op}(y)v^1_{op}(z^1)\cdots v^n_{op}(z^n)]\,|0\rangle \\ = \langle 0|\, T[\partial^\mu J^\Lambda_{\mu\,op}(x)J^\Gamma_{\nu\,op}(y)v^1_{op}(z^1)\cdots v^n_{op}(z^n)]\,|0\rangle - i\delta(x-y)\langle 0|\, T[(J_\nu^{[\Gamma,\Lambda]})_{op}(y)v^1_{op}(z^1)\cdots v^n_{op}(z^n)]\,|0\rangle \\ - i\sum_{i=1}^n \delta(x-z^i)\langle 0|\, T[J^\Gamma_{\nu\,op}(y)v^1_{op}(z^1)\cdots(\delta_\Lambda v^i)_{op}(z^i)\cdots v^n_{op}(z^n)|0\rangle,\end{aligned} \tag{4.3}$$

where the v's are nonderivative simple vertices. Because the currents do not contain derivative of Dirac fields, there are *no Schwinger terms*.[18]

V. CONCLUSION

We have extended the formulation of perturbation theory of Ref. 1 for scalar fields to cover Dirac fields, and have, as before, derived Ward-Takahashi identities, Euler-Lagrange equations of motion, and Noether's theorem in operator form without recourse to equal-time canonical commutation relations. The currents under discussion include those of Poincaré invariance and internal symmetry. The content of current algebra is obtained from Ward-Takahashi identities if $\mathcal{L}^{\text{derivative}}$ is invariant under the algebra in question.

Contrary to the case of scalar fields, we are able to define local operators *not for all* vertices involving derivatives of Dirac fields, but only for those with a *single* derivative of a Dirac field and of the particular form given by (2.13) or (2.17). Furthermore this is possible only if $\mathcal{L}$ has the form (2.12). Thus in order that operators may be definable for vertices (2.13) and (2.17), $\mathcal{L}^{\text{int}}$ (apart from counterterms) does not have derivatives of Dirac fields. As it turns out, this suffices for constructing the energy-momentum tensor operators and the angular momentum current operators. Another consequence of this restriction on $\mathcal{L}^{\text{int}}$ is that Schwinger terms are absent from current-algebra Ward-Takahashi identities, if the theory has only spin-$\frac{1}{2}$ particles.

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[1]Y.-M. P. Lam, preceding paper, Phys. Rev. D 6, 2145 (1972).

[2]N. N. Bogoliubov and O. Parasiuk, Acta Math. 97, 227 (1957); K. Hepp, Commun. Math. Phys. 2, 301 (1966); W. Zimmermann, *ibid.* 15, 208 (1969). Also see N. N. Bogoliubov and D. W. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959); and W. Zimmermann, *Brandeis Lectures on Elementary Particles and Quantum Field Theory,* edited by S. Deser *et al.* (MIT Press, Cambridge, Mass., 1970), Vol. II.

[3]J. C. Ward, Phys. Rev. 78, 1824 (1950); and Y. Takahashi, Nuovo Cimento 6, 370 (1957).

[4]M. Gell-Mann, Physics 1, 63 (1964).

[5]Dirac's representation of the Lorentz group is $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$. The γ matrices we will use are defined by $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ and $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$, where $g_{\mu\mu} = (1, -1, -1, -1)$.

[6]A field is a formal space-time derivative of a basic field to arbitrary order. The set of all fields is $\mathfrak{F}$.

[7]The signature factor is so chosen that the first identity can be cast neatly in the form of Eq. (2.4).

[8]The symbol $\not{l} \equiv l^\mu\gamma_\mu$ is employed.

[9]W. Zimmermann, Commun. Math. Phys. 11, 1 (1968).

[10]Notations are $[\psi^\dagger D\psi, \alpha] \equiv D^{ab}[(\psi^{a*}, \psi^b), \alpha]$, $[\bar{\psi}D\psi, \alpha] \equiv [\psi^\dagger\gamma_0 D\psi, \alpha]$, and $\overleftrightarrow{\not{\partial}} = \gamma^\mu\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu\gamma^\mu$.

[11]According to the notation of Ref. 1, $I_2 = \{1, 2\}$.

[12]This condition (2.3) is exactly similar to $\alpha^{\text{mass}}(I_2) = 2$ of Ref. 1, and is required to prove the first identity (2.4). The reason why 1, instead of 2, appears in (2.3) is that the Dirac propagator (1.11) has dimension $(\text{mass})^{-1}$ as opposed to the dimension $(\text{mass})^{-2}$ of a scalar propagator.

[13]H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 425 (1955).

[14]Loosely speaking, for any subsequence B of A, $\alpha(\partial_\mu\bar{\psi}^c, B) = \beta(B)$ where β is the excess-subtraction function of the simple vertex w.

[15]J. Lowenstein, Phys. Rev. D 4, 2281 (1971).

[16]For the spatial integral of the zeroth component of a current to exist, the current must be conserved and the Hilbert space does not have massless particles. For a summary of these arguments, see C. A. Orzalesi, Rev. Mod. Phys. 42, 381 (1970).

[17]For the proof, note that the Lagrangian is given by (2.12), where the derivative terms are in the form of (3.3) and (3.4) so that the operator identities (2.16) and (2.19) can be applied.

[18]J. Schwinger, Phys. Rev. Letters 3, 296 (1959).