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Perturbation Lagrangian Theory for Scalar Fields–Ward-Takahashi Identity and Current Algebra*

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Under the assumption that the time-ordered products are given by Feynman rules with Bogoliubov's renormalization scheme, it is shown that the operator forms of Euler-Lagrange equations of motion and Noether's theorem for a wide class of perturbation Lagrangian theories for scalar fields are valid. Ward-Takahashi identities for currents in Noether's theorem are also proved without recourse to equal-time commutators, and current-algebra results follow naturally in a subclass of Lagrangian field theories. Covariant Schwinger terms are present in these identities and their nature is determined.

I. INTRODUCTION

The usual text-book Lagrangian field theory[1] is formulated in terms of products of field operators at equal space-time points, which, because of the distribution nature of field operators, are not well defined. The Euler-Lagrange equations of motion and Noether's theorem are not only relations between ill-defined products, but also their derivation rests on the use of formal functional derivatives on these products. Therefore, the usual theory is not much better than a prescription for the scattering matrix. We say "not much better," because this formal theory has proved to be a useful guide[2] for constructing symmetry currents and generators of symmetry transformations. A notable example is the free-quark model from which the theory of Gell-Mann's[3] SU(3)×SU(3) current algebra is abstracted. The usual prescription is that we "quantize" the fields, the Lagrangian density, the currents, the equations of motion, and Noether's theorem of a classical field theory, but we do *not* quantize the classical derivation of the equation of motion, nor that of Noether's theorem. We will call this kind of field theory formal. From the quantized current occurring in Noether's theorem, one then shows formally that the canonical commutators imply that the spatial integral of the zeroth component of that current is a generator of the corresponding transformation. Renormalization is then nothing but a further prescription in the scheme to bring some otherwise infinite quantities finite. Hence the relationship between the scattering matrix and the Lagrangian density is, at best, formal.

Zimmermann[4] has shown that, if normal products are defined properly in the A^4 theory, then the Euler-Lagrange equation of motion is a consequence of Bogoliubov's renormalization.[5-7] The proof bears no resemblance to that in the classical theory at all. Lowenstein[8] further advanced the power of this renormalization by showing that quantization of the classical energy-momentum tensor in the same theory, when properly interpreted with regard to the subtraction scheme, yields operators with appropriate properties. In this paper, we will show that, in a wide class of Lagrangian field theories for scalar particles, Bogoliubov's renormalization scheme provides a justification for the prescriptions and consequences of the formal theory. Not only do the Lagrangian and current operators exist, but also their relationship in the classical theories (such as the Euler-Lagrange equations of motion, Noether's theorem, and how the currents are constructed from the Lagrangian) have their exact counterparts rigorously valid in operator form.

The "products" of field operators at equal space-time points will be formulated in terms of objects called vertices, which can be likened to generalizations of Zimmermann's definition of normal products.[4] Vertices will be defined purely *algebraically* for scalar particles in Sec. II, and their space-time derivatives and functional derivatives will also be introduced *algebraically*. In Sec. III, Bogoliubov's renormalization will be sketched for Feynman amplitudes of vertices, and two fundamental identities of Feynman amplitudes will be stated in Sec. IV. Then on assuming that Feynman amplitudes are time-ordered products of a quantum field theory in a limited sense (Sec. V), these identities are transformed into Ward-Takahashi identities,[9] which under suitable conditions yield generators of transformations on the physical Hilbert space. The energy-momentum tensor operators and the angular momentum current operators will also be similarly derived in Sec. VI. Section VII will show that current-algebra Ward-

Takahashi identities follow naturally under certain general conditions.

II. VERTEX: DEFINITION AND OPERATIONS

We will begin by introducing the concept of a basic field. Let there be a finite-dimensional representation of the Lorentz group decomposable only into scalar subspaces. Generalization to include Dirac's spin-$\frac{1}{2}$ subspaces will be obvious, but a slight complication necessitates its deferment to a later communication. Let $\mathfrak{B} \equiv \{\phi^1, \ldots, \phi^M\}$ be a set of basis vectors of this representation. A member of $\mathfrak{B}$ is called a *basic field*. A formal derivative[10] $\partial_{\mu_1} \cdots \partial_{\mu_n}\phi$ of a basic field ϕ of any order n is called a *field*, and let the set of all fields be $\mathfrak{F}$.

Next the concept of *excess-subtraction function* will be introduced. As its name implies, it will play an important role in specifying the number of subtractions in Bogoliubov's renormalization program.[5-7] Let I denote the set of all non-negative integers, and define $I_N \equiv \{1, 2, \ldots, N\}$ for any positive integer N. For $N=0$, define $I_0 \equiv \varnothing$, the empty set. Also define $S_N \equiv \{U : U \subset I_N\}$ for each $N \in I$. A mapping

$$\alpha : \; S_N \to I$$

is called an *excess-subtraction function* of order N if it satisfies the following two conditions:

(i) $R \subset S \subset I_N \Rightarrow \alpha(R) \leq \alpha(S)$,

(ii) $\alpha(S) = 0$ if S is either empty or has only one element.

A *simple vertex* is then defined to be the ordered pair $[f, \alpha]$, where f is a (possibly empty) sequence of N fields $f_i \in \mathfrak{F}$, and where α is an excess-subtraction function of order N. N is also called the order of the simple vertex. A simple vertex is said to be *trivial* if f is empty, and it will be denoted by 1. If f has only one element ψ, we will simply denote[11]

$$\psi \equiv [\psi, 0]. \tag{2.1}$$

The set of all simple vertices will be called $\mathfrak{V}$. A finite formal linear combination[12] of simple vertices is called a *vertex:*

$$W = a^1 w^1 + \cdots + a^n w^n ,$$

where the w's are simple vertices and where the a's are called *coefficients* and are polynomially bounded infinitely differentiable functions of x_μ ($\mu = 0, 1, 2, 3$). Note that apart from the presence of excess-subtraction functions, a vertex resembles a polynomial in fields. We will introduce operations on vertices which will further illustrate this analogy.

Let $w = [f, \alpha]$ and $v = [g, \beta]$ be two simple vertices of order N and M, respectively. Let ψ be a field and define

$$U_\psi^w \equiv \{j : j \in I_N, f_j = \psi\}.$$

For each $j \in U_\psi^w$, define a sequence $(h_1^{(j)}, \ldots, h_{N+M-1}^{(j)})$ by replacing f_j (which is ψ) in the sequence f with the entire sequence g:

$$h_i^{(j)} \equiv \begin{cases} f_i, & 1 \leq i \leq j-1 \\ g_{i-j+1}, & j \leq i \leq j+M-1 \\ f_{i-M+1}, & j+M \leq i \leq N+M-1 . \end{cases} \tag{2.2}$$

Also for $j \in U_\psi^w$, define an excess-subtraction function $\gamma^{(j)}$ of order $N+M-1$, which mainly carries the information of α and β but which also has a "memory" of the ejected f_j ($=\psi$), as follows. Let $F^{(j)}$ and $G^{(j)}$ be two mappings that map elements of f and g occurring in $h^{(j)}$ to their original positions. That is,

$$F^{(j)}(i) \equiv \begin{cases} i & \text{for } 1 \leq i \leq j-1 \\ i - M + 1 & \text{for } j+M \leq i \leq N+M-1 \end{cases}$$

and

$$G^{(j)}(i) \equiv i - j + 1 \quad \text{for } j \leq i \leq j+M-1 .$$

Let $\mathfrak{D}^{F^{(j)}}$ and $\mathfrak{D}^{G^{(j)}}$ be their domains. Then for $R \subset I_{N+M-1}$, $\gamma^{(j)}(R)$ is defined by

$$\gamma^{(j)}(R) \equiv \begin{cases} \alpha(F^{(j)}(R \cap \mathfrak{D}^{F^{(j)}})) & \text{if } R \cap \mathfrak{D}^{G^{(j)}} = \varnothing \\ \beta(G^{(j)}(R \cap \mathfrak{D}^{G^{(j)}})) & \text{if } R \cap \mathfrak{D}^{F^{(j)}} = \varnothing \\ \alpha(F^{(j)}(R \cap \mathfrak{D}^{F^{(j)}}) \cup \{j\}) + \beta(G^{(j)}(R \cap \mathfrak{D}^{G^{(j)}})) & \text{otherwise} . \end{cases} \tag{2.3}$$

The *substitution* for simple vertices w and v and field ψ is the vertex

$$\left(\frac{\delta w}{\delta \psi}\middle| v\right) \equiv \sum_{j \in U_\psi^w} [h^{(j)}, \gamma^{(j)}] . \tag{2.4}$$

For vertices $W = \sum_i a^i w^i$ and $V = \sum_j b^j v^j$, the substitution is

$$\left(\frac{\delta W}{\delta \psi}\middle| V\right) \equiv \sum_{ij} a^i b^j \left(\frac{\delta w^i}{\delta \psi}\middle| v^j\right) . \tag{2.5}$$

The *functional derivative* of W by ψ is then simply defined as

$$\frac{\delta W}{\delta \psi} \equiv \left(\frac{\delta W}{\delta \psi}\middle| 1\right) , \tag{2.6}$$

and the *space-time derivative* of W is

$$\partial_\mu W \equiv \sum_{\psi\in\mathfrak{F}} \left(\frac{\delta W}{\delta\psi} \middle| \partial_\mu \psi \right) + \sum_i (\partial_\mu a^i) w^i \,. \tag{2.7}$$

A *transformation vertex* Λ is a sequence of M vertices Λ^a ($a=1,\ldots,M$), where M is the total number of basic fields. Given any vertex W, the *variation of* W *under* Λ is

$$\delta_\Lambda W \equiv \sum_{s=0}^{\infty} \left(\frac{\delta W}{\delta \partial_{\mu_1}\cdots\partial_{\mu_s}\phi^a} \middle| \partial_{\mu_1}\cdots\partial_{\mu_s}\Lambda^a \right) . \tag{2.8}$$

An *infinitesimal translational* vertex t_μ is a transformation vertex with components

$$t_\mu^a \equiv \partial_\mu \phi^a \,, \tag{2.9}$$

and an *infinitesimal Lorentz rotational* vertex $\sigma_{\alpha\beta}$ in the $\alpha\beta$ plane is defined by

$$\sigma^a_{\alpha\beta} \equiv X_\alpha \partial_\beta \phi^a - X_\beta \partial_\alpha \phi^a \,, \tag{2.10}$$

where X_α is the function $X_\alpha(x) = x_\alpha$. These definitions are consistent with the transformation of scalar fields under the generators of the Poincaré group.

III. FEYNMAN AMPLITUDES OF VERTICES

Let $\mathcal{L}^{\text{int}}$ be a vertex with constant coefficients and let $\mathcal{L}^{\text{int}}$ transform like a Lorentz scalar:

$$\delta_{\sigma_{\alpha\beta}} \mathcal{L}^{\text{int}} = X_\alpha \partial_\beta \mathcal{L}^{\text{int}} - X_\beta \partial_\alpha \mathcal{L}^{\text{int}} \,. \tag{3.1}$$

Given n nontrivial simple vertices $w^1,\ldots,w^n$, the vertex $\mathcal{L}^{\text{int}}$ determines a set $\mathfrak{D}(w^1,\ldots,w^n)$ of Feynman graphs. Each graph consists of n external points (in one-to-one correspondence with the w's) and a number of internal points (each corresponding to a nontrivial simple vertex in $i\mathcal{L}^{\text{int}}$). Fields of different simple vertices are paired so that none is left alone. Each pair is joined by a line. Different pairings give rise to distinct graphs. Figure 1 is a typical Feynman graph.

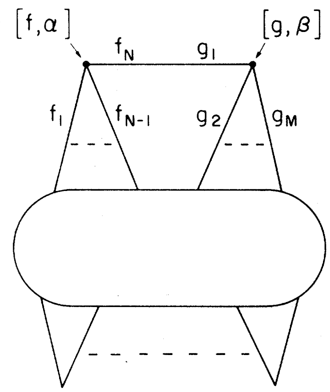

FIG. 1. A typical Feynman graph.

Given a connected Feynman graph Γ, the unrenormalized integrand $I_\Gamma(\{l\},\epsilon)$ is obtained as usual, except for a slight modification of the propagator (due to Zimmermann[13]) which is

$$i(2\pi)^{-4}[l^2 - m^2 + i\epsilon(\bar{l}^2 + m^2)]^{-1} \,. \tag{3.2}$$

Here l is the 4-momentum carried by the corresponding line, and m is the mass of the particle for this line. As usual, a derivative ∂_μ of a basic field appearing in a simple vertex introduces a factor il_μ, where l is the 4-momentum carried by the corresponding line *away* from the vertex. Every simple vertex carries a factor $(2\pi)^4$.

Bogoliubov's renormalization program[4-7] will now be sketched. Let γ be a subgraph[14] of Γ and let $v=[f^v, \alpha^v]$ be a simple vertex contained in γ. Then there is a unique set S^v of integers such that $i\in S^v \Longleftrightarrow f_i^v \in \gamma$. We define the symbol

$$\alpha^v(\gamma) \equiv \alpha^v(S^v) \,. \tag{3.3}$$

To this subgraph γ we assign a subtraction number

$$\delta(\gamma) \equiv d(\gamma) + \sum_{v\in\gamma} \alpha^v(\gamma) \,, \tag{3.4}$$

where $d(\gamma)$ is the dimension of γ (obtained by a simple power counting of momenta, including those of integration). More transparently, one determines $\delta(\gamma)$ by finding all the fields in γ and reading off their contribution to $\delta(\gamma)$ from the excess-subtraction functions. Hence the name excess-subtraction function. This marks our departure from Zimmermann's normal products,[4] where $\alpha^v(\gamma)$ is only dependent on v.[15] A *renormalization part* of Γ is a one-particle irreducible subgraph (of Γ) whose subtraction number is non-negative. A Γ *forest* U is a sequence $U_1,\ldots,U_m$ of renormalization parts of Γ such that $i<j$ implies either $U_i \subset U_j$ or $U_i \cap U_j = \emptyset$. Then the *renormalized Feynman integrand* is defined to be

$$\mathcal{R} I_\Gamma(k,\xi,\epsilon) \equiv \sum_{U\in\mathfrak{U}(\Gamma)} [-t^{\delta(U_m)}]\cdots[-t^{\delta(U_1)}] I_\Gamma \,, \tag{3.5}$$

where k are the $n-1$ independent external momenta, ξ are the integration momenta, $\mathfrak{U}(\Gamma)$ is the set of all Γ forests, and $t^{\delta(\gamma)}I$ is the sum of the first $\delta(\gamma)+1$ terms of the Taylor series of I in the external momenta variables of γ.[16] This Taylor series is expanded around the point where the external momenta of γ are zero. Then the integration of the renormalized Feynman integrand over integration momenta ξ yields (in the limit $\epsilon\to 0$) a well-tempered distribution $\bar{F}_\Gamma$ over test functions of momenta $k^1,\ldots,k^{n-1}$:

$$\bar{F}_\Gamma(w^1,\ldots,w^n; k^1,\ldots,k^{n-1}) \equiv \lim_{\epsilon\to 0^+} \int d\xi\, \mathcal{R} I_\Gamma(k,\xi,\epsilon). \tag{3.6}$$

Multiplying this by $\delta(\sum k)$ we obtain $\overline{\mathfrak{F}}_\Gamma$ as a distribution over test functions of momenta $k^1, \ldots, k^n$:

$$\overline{\mathfrak{F}}_\Gamma(w^1, k^1; \ldots; w^n, k^n) \equiv \delta^4\left(\sum_{i=1}^n k^i\right)\overline{F}_\Gamma(w^1, \ldots, w^n; k^1, \ldots, k^{n-1}). \tag{3.7}$$

Its Fourier transform is

$$\mathfrak{F}_\Gamma(w^1, x^1; \ldots; w^n, x^n) \equiv \int\left[\prod_{i=1}^n \frac{d^4k^i}{(2\pi)^4} e^{ik^i\cdot x^i}\right]\overline{\mathfrak{F}}_\Gamma(w^1, k^1; \ldots; w^n, k^n). \tag{3.8}$$

If Γ is disconnected, say $\Gamma = \cup_a \Gamma_a$ where Γ_a are connected, then $\mathfrak{F}_\Gamma$ is defined to be the product of the connected components:

$$\mathfrak{F}_\Gamma \equiv \prod_a \mathfrak{F}_{\Gamma_a}. \tag{3.9}$$

Summing over all graphs in $\mathfrak{D}(w^1, \ldots, w^n)$, the *Feynman amplitude* is

$$\mathfrak{F}(w^1, x^1; \ldots; w^n, x^n) \equiv \sum_{\Gamma\in\mathfrak{D}(w^1,\ldots,w^n)} \mathfrak{F}_\Gamma(w^1, x^1; \ldots; w^n, x^n), \tag{3.10}$$

if none of the simple vertices $w^1, \ldots, w^n$ is nontrivial. Although the summation over Γ in $\mathfrak{D}(w^1, \ldots, w^n)$ may not converge, we will assume that it does. If some of the simple vertices in $w^1, \ldots, w^n$ are trivial, $\mathfrak{F}$ is defined as before, but only on the nontrivial simple vertices in $w^1, \ldots, w^n$ and their corresponding x's.

For vertices $W^i = \sum_j a^{ij} w^{ij}$, the Feynman amplitude has a straightforward extension:

$$\mathfrak{F}(W^1, x^1; \ldots; W^n, x^n) \equiv \sum_{j_1} \cdots \sum_{j_n}\left(\prod_{i=1}^n a^{ij_i}(x^i)\right)\mathfrak{F}(w^{1j_1}, x^1; \ldots; w^{nj_n}, x^n). \tag{3.11}$$

The Feynman amplitudes are readily seen to be invariant under simultaneous permutation over $(W^1, \ldots, W^n)$ and $(x^1, \ldots, x^n)$. Another simple property is *Lemma 1:*

$$\frac{\partial}{\partial x^i_\mu}\mathfrak{F}(W^1, x^1; \ldots; W^n, x^n) = \mathfrak{F}(W^1, x^1; \ldots; \partial^\mu W^i, x^i; \ldots; W^n, x^n). \tag{3.12}$$

This is analogous to Lemma 1 of Ref. 8, and will be proved in Appendix A. Because $\mathfrak{L}^{\text{int}}$ is a Lorentz scalar, $\mathfrak{F}$ is Lorentz-invariant:

$$\sum_{i=1}^n \mathfrak{F}(W^1, x^1; \ldots; \delta_{\sigma_{\alpha\beta}} W^i, x^i; \ldots; W^n, x^n) = 0, \tag{3.13}$$

if the W's have constant coefficients.

IV. FUNDAMENTAL IDENTITIES OF FEYNMAN AMPLITUDES

Noether's theorem in classical field theory is a simple identity that follows directly from the Euler-Lagrange equation of motion. In our formulation, neither exists as yet, but we will show that our formulation is such that their analogs are valid. For this purpose we will develop two identities on Feynman amplitudes.

First of all, let us introduce the full Lagrangian vertex $\mathfrak{L}$. It is of the form

$$\mathfrak{L} = \mathfrak{L}^{\text{kin}} + \mathfrak{L}^{\text{mass}} + \mathfrak{L}^{\text{int}}, \tag{4.1}$$

where

$$\mathfrak{L}^{\text{kin}} \equiv \tfrac{1}{2}\sum_a [(\partial_\mu\phi^a, \partial^\mu\phi^a), 0], \tag{4.2}$$

$$\mathfrak{L}^{\text{mass}} \equiv -\tfrac{1}{2}\sum_a (m^a)^2[(\phi^a, \phi^a), \alpha^{\text{mass}}], \tag{4.3}$$

and the excess-subtraction function of the mass term is defined by the condition

$$\alpha^{\text{mass}}(I_2) = 2. \tag{4.4}$$

$\mathfrak{L}^{\text{int}}$ has constant coefficients, as described in Sec. III. Apart from the excess-subtraction functions, the analogy of the Lagrangian vertex to classical Lagrangian density is most obvious. It is also readily seen that $\mathfrak{L}$ is a Lorentz scalar:

$$\delta_{\sigma_{\alpha\beta}}\mathfrak{L} = X_\alpha\partial_\beta\mathfrak{L} - X_\beta\partial_\alpha\mathfrak{L}. \tag{4.5}$$

The *first identity* of Feynman amplitudes is

$$\sum_{s=0}^{\infty}\sum_{r=0}^{s}(-)^r\binom{s}{r}\left[\mathfrak{F}\left(\partial_{\mu_1}\cdots\partial_{\mu_r}\left(\frac{\delta\mathfrak{L}}{\delta\partial_{\mu_1}\cdots\partial_{\mu_s}\psi}\bigg|\partial_{\mu_{r+1}}\cdots\partial_{\mu_s}w\right),x;v^1,y^1;\ldots;v^n,y^n\right)\right.$$
$$\left.-i\sum_{i=1}^{n}\partial_{\mu_1}\cdots\partial_{\mu_r}\delta^4(x-y^i)\mathfrak{F}\left(v^1,y^1;\ldots;\left(\frac{\delta v^i}{\delta\partial_{\mu_1}\cdots\partial_{\mu_s}\psi}\bigg|\partial_{\mu_{r+1}}\cdots\partial_{\mu_s}w\right),y^i;\ldots;v^n,y^n\right)\right]=0, \tag{4.6}$$

where w and the v's are simple vertices and ψ is a basic field. The proof of this identity will be reserved for the interested reader in Appendix B. We will convert this to a more useful form. Repeating it M times, one for each Λ^a of a transformation vertex with constant coefficients, we obtain the *second identity*:

$$\frac{\partial}{\partial x_\mu}\mathfrak{F}(J_\mu^\Lambda,x;V^1,y^1;\ldots;V^n,y^n)$$
$$=\mathfrak{F}(\delta_\Lambda\mathfrak{L},x;V^1,y^1;\ldots;V^n,y^n)$$
$$-i\sum_{i=1}^{n}\left[\delta(x-y^i)\mathfrak{F}(V^1,y^1;\ldots;\delta_\Lambda V^i,y^i;\ldots;V^n,y^n)\right.$$
$$\left.+\sum_{s=1}^{\infty}\sum_{r=1}^{s}(-)^r\binom{s}{r}\partial_{\mu_1}\cdots\partial_{\mu_r}\delta^4(x-y^i)\mathfrak{F}\left(V^1,y^1;\ldots;\left(\frac{\delta V^i}{\delta\partial_{\mu_1}\cdots\partial_{\mu_s}\phi^a}\bigg|\partial_{\mu_{r+1}}\cdots\partial_{\mu_s}\Lambda^a\right),y^i;\ldots;V^n,y^n\right)\right], \tag{4.7}$$

where

$$J_\mu^\Lambda\equiv\sum_{s=0}^{\infty}\sum_{r=0}^{s}(-)^r\binom{s+1}{r+1}\partial_{\mu_1}\cdots\partial_{\mu_r}\left(\frac{\delta\mathfrak{L}}{\delta\partial^\mu\partial_{\mu_1}\cdots\partial_{\mu_s}\phi^a}\bigg|\partial_{\mu_{r+1}}\cdots\partial_{\mu_s}\Lambda^a\right) \tag{4.8}$$

is called a *transformation current vertex*, and is defined for each transformation vertex Λ.

V. TIME-ORDERED PRODUCTS AND WARD-TAKAHASHI IDENTITIES

It is usually assumed that the renormalized Feynman amplitudes are the vacuum expectation values of the time-ordered products of a local quantum field theory. By virtue of their definition, the time-ordered products are undefined at equal-time points. However, there are two compelling reasons why it is desirable for them to be well defined everywhere: (1) The power of the Ward-Takahashi identity[9] of time-ordered products in quantum electrodynamics lies on those terms defined at equal space-time points; (2) Ruelle[17] and Steinmann[18] have shown that if the time-ordered products are well defined everywhere and are Lorentz-covariant, and if the analytic continuations of their vacuum expectation values in momentum space satisfy certain generalized unitarity equations (Steinmann's relations), then they define a local quantum field theory of which they are the time-ordered products. Hence we will assume that the time-ordered products are well defined everywhere and are Lorentz-covariant.

Now the Feynman amplitudes are well defined everywhere and they are Lorentz-invariant. Whether they satisfy Steinmann's relations or not is beyond the scope of this communication. So we will assume that they are the time-ordered products of a field theory, but only in a limited sense. To be precise, let $\mathfrak{V}_m$ be the set of all simple vertices containing at most mth derivatives of basic fields, and let $\mathcal{G}_m$ be the set of those vertices which are formal linear combinations of only simple vertices in $\mathfrak{V}_m$. Vertices in $\mathcal{G}_0$ are said to be *nonderivative*. Then for any vertex V in $\mathcal{G}_1$ we assume that there is a *vertex operator* $V_{\rm op}(x)$ which is a local operator-valued distribution on the Hilbert space $\mathcal{H}$ of physical states, and that the vacuum expectation value of a time-ordered product of vertex operators $V_{\rm op}^1(x^1),\ldots,V_{\rm op}^n(x^n)$ is

$$\langle 0|\,T[\,V_{\rm op}^1(x^1)\cdots V_{\rm op}^n(x^n)]|0\rangle=\mathfrak{F}(V^1,x^1;\ldots;V^n,x^n). \tag{5.1}$$

The more general assumption that this equation is valid for V's belonging to $\mathcal{G}_m$ with $m>1$ will be shown to be inconsistent with the previous assumption that the time-ordered products are well defined everywhere.

Letting all vertices but one, say V^1, in Eq. (5.1) be basic field simple vertices, the usual Lehmann-

Symanzik-Zimmermann (LSZ) reduction formula[19] then determines the matrix elements of $V^1_{\rm op}(x^1)$ between in- and out-states. And so we find

$$\frac{\partial}{\partial x^\mu}\, V_{\rm op}(x) = (\partial_\mu V)_{\rm op}(x) \quad \text{for } V \in \mathcal{G}_0 \tag{5.2}$$

and

$$(X_\mu V)_{\rm op}(x) = x_\mu V_{\rm op}(x) \quad \text{for } V \in \mathcal{G}_1 . \tag{5.3}$$

Let us make the further restriction that the Lagrangian $\mathcal{L}$ belongs to $\mathcal{G}_1$. Then every term in the second identity (4.7) can be translated into the language of time-ordered products:

$$\begin{aligned}
&\frac{\partial}{\partial x_\mu}\langle 0|T[J^\Lambda_{\mu\rm op}(x)V^1_{\rm op}(y^1)\cdots V^n_{\rm op}(y^n)]|0\rangle \\
&\qquad = \langle 0|T[(\delta_\Lambda\mathcal{L})_{\rm op}(x)V^1_{\rm op}(y^1)\cdots V^n_{\rm op}(y^n)]|0\rangle - i\sum_{i=1}^n \delta^4(x-y^i)\langle 0|T[V^1_{\rm op}(y^1)\cdots(\delta_\Lambda V^i)_{\rm op}(y^i)\cdots V^n_{\rm op}(y^n)]|0\rangle \\
&\qquad\quad + i\sum_{i=1}^n \partial_\mu\delta^4(x-y^i)\left\langle 0\left|T\left[V^1_{\rm op}(y^1)\cdots\left(\frac{\delta V^i}{\delta\partial_\mu\phi^a}\middle|\Lambda^a\right)_{\rm op}(y^i)\cdots V^n_{\rm op}(y^n)\right]\right|0\right\rangle
\end{aligned} \tag{5.4}$$

for a nonderivative transformation vertex Λ with constant coefficients and for vertices $V^1, \ldots, V^n \in \mathcal{G}_1$. [Note that, under our restrictions, J^Λ_μ, $\delta_\Lambda\mathcal{L}$, and $\delta_\Lambda V^i$ belong to $\mathcal{G}_1$, so that $J^\Lambda_{\mu\rm op}$, $(\delta_\Lambda\mathcal{L})_{\rm op}$, and $(\delta_\Lambda V^i)_{\rm op}$ are well-defined operators.] Since $\mathcal{L} \in \mathcal{G}_1$,

$$J^\Lambda_\mu = \left(\frac{\delta\mathcal{L}}{\delta\partial^\mu\phi^a}\middle|\Lambda^a\right), \tag{5.5}$$

in complete analogy with Noether's current of classical field theory. $\delta_\Lambda\mathcal{L}$ is also analogous to the classical variation of $\mathcal{L}$ under an "infinitesimal" transformation Λ. To bring this equation to a more familiar form, let all V^i be basic fields and apply the LSZ reduction formula.[19] We then find that in (5.4) those terms involving δ functions do not survive; therefore

$$\frac{\partial}{\partial x_\mu}\langle\Psi|J^\Lambda_{\mu\rm op}(x)|\Phi\rangle = \langle\Psi|(\delta_\Lambda\mathcal{L})_{\rm op}(x)|\Phi\rangle$$

for any in-state Φ and any out-state Ψ. Hence we have the operator identity:

$$\partial^\mu J^\Lambda_{\mu\rm op}(x) = (\delta_\Lambda\mathcal{L})_{\rm op}(x) . \tag{5.6}$$

If we recall that $J^\Lambda_{\mu\rm op}$ and $(\delta_\Lambda\mathcal{L})_{\rm op}$ are the analogs of the classical Noether's current and the classical variation of $\mathcal{L}$ under Λ, it is evident that this identity is the operator analog of the classical *Noether's theorem*. The analogy to classical field theory does not stop here. If the transformation vertex Λ has the particular form $t^s = 1$ and $t^a = 0$ for $a \neq s$, then (5.6) reduces to the operator analog of the *Euler-Lagrange equation of motion*:

$$\frac{\partial}{\partial x_\mu}\left(\frac{\delta\mathcal{L}}{\delta\partial^\mu\phi^s}\right)_{\rm op}(x) = \left(\frac{\delta\mathcal{L}}{\delta\phi^s}\right)_{\rm op}(x) . \tag{5.7}$$

Returning to (5.4), let us substitute $\partial^\mu J^\Lambda_{\mu\rm op}$ for $(\delta_\Lambda\mathcal{L})_{\rm op}$, and obtain a generalization of the *Ward-Takahashi identity*[9]:

$$\begin{aligned}
&\frac{\partial}{\partial x_\mu}\langle 0|T[J^\Lambda_{\mu\rm op}(x)V^1_{\rm op}(y^1)\cdots V^n_{\rm op}(y^n)]|0\rangle \\
&\qquad = \left\langle 0\left|T\left[\frac{\partial}{\partial x_\mu}J^\Lambda_{\mu\rm op}(x)V^1_{\rm op}(y^1)\cdots V^n_{\rm op}(y^n)\right]\right|0\right\rangle - i\sum_{i=1}^n \delta(x-y^i)\langle 0|T[V^1_{\rm op}(y^1)\cdots(\delta_\Lambda V^i)_{\rm op}(y^i)\cdots V^n_{\rm op}(y^n)]|0\rangle \\
&\qquad\quad + i\sum_{i=1}^n \partial_\mu\delta(x-y^i)\left\langle 0\left|T\left[V^1_{\rm op}(y^1)\cdots\left(\frac{\delta V^i}{\delta\partial_\mu\phi^a}\middle|\Lambda^a\right)_{\rm op}(y^i)\cdots V^n_{\rm op}(y^n)\right]\right|0\right\rangle .
\end{aligned} \tag{5.8}$$

We will also call this the Ward-Takahashi identity.

It is well known that the formal theory based on equal-time commutators[20] also yields (very simply) similar identities, with the important difference that there the Schwinger terms[21] (terms proportional to spatial derivatives of δ functions) are noncovariant. This is to be expected, because the time-ordered products in the formal theory are defined with the aid of the step function of time, and hence the time-ordered products are noncovariant. Although the existence of Schwinger terms can easily be demonstrated[21] in the formal theory, their nature is very much undetermined.[22] So for practical purposes, they are usually assumed to be absent from *covariant* time-ordered products (cancellation with seagull terms which are also noncovariant). However, in the identity (5.8), terms proportional to derivatives of δ functions are both covariant and completely specified. We will call them *covariant Schwinger terms.* In operator form, they are

$$i\partial_\mu \delta(x-y)\left(\frac{\delta V}{\delta \partial_\mu \phi^a}\,\middle|\, \Lambda^a\right)_{\mathrm{op}}(y)\,,$$

and the corresponding vertex

$$\left(\frac{\delta V}{\delta \partial_\mu \phi^a}\,\middle|\, \Lambda^a\right)$$

depends only on V and Λ.

For $\mathcal{L}$ *invariant* under Λ, that is,

$$\delta_\Lambda \mathcal{L} = 0\,, \tag{5.9}$$

then from (5.6)

$$\partial^\mu J^\Lambda_{\mu\mathrm{op}}(x) = 0\,, \tag{5.10}$$

and Ref. 23 shows that an operator $G(J^\Lambda_{\mathrm{op}})$ can be rigorously defined on the physical Hilbert space if there are no massless particles. $G(J^\Lambda_{\mathrm{op}})$ corresponds to a refined definition of the usual integral $\int d^3x J^\Lambda_0(x)$ in the formal theory. After Lowenstein,[8] we can further show that under the condition (5.10) the Ward-Takahashi identity (5.8) requires $G(J^\Lambda_{\mathrm{op}})$ to be the generator corresponding to the transformation vertex Λ:

$$[G(J^\Lambda_{\mathrm{op}}),\, V_{\mathrm{op}}(x)] = -i(\delta_\Lambda V)_{\mathrm{op}}(x)\,, \tag{5.11}$$

for any vertex $V \in \mathcal{G}_1$.

Now, the description of symmetry is clear. Let there be a Lie group of linear transformations on the set $\mathcal{B}$ of basic fields. This group determines a Lie algebra of linear transformations on $\mathcal{B}$. For each element of this algebra, one constructs a transformation vertex, which in turn determines a transformation current operator. The Ward-Takahashi identity for this transformation current operator then describes how vertex operators transform under this element of the algebra. If the Lagrangian vertex is invariant under this element and if there are no massless particles, then the corresponding current operator determines a generator. If furthermore the Lagrangian vertex is invariant under the entire Lie algebra, then these generators generate a representation of the Lie group on the physical Hilbert space.

We will now end this section with a discussion on why (5.1) cannot be assumed to be valid over a wider class of vertices. Suppose that Feynman amplitudes are time-ordered products of vertices in $\mathcal{G}_2$. Then Lemma 1 and the Ward-Takahashi identity (5.8) say that

$$\left\langle 0 \left| T\left[\frac{\partial}{\partial x_\mu} J^\Lambda_{\mu\mathrm{op}}(x) V^1_{\mathrm{op}}(y^1)\cdots V^n_{\mathrm{op}}(y^n)\right] \right| 0\right\rangle = \left\langle 0 \left| T\left[\frac{\partial}{\partial x_\mu} J^\Lambda_{\mu\mathrm{op}}(x) V^1_{\mathrm{op}}(y^1)\cdots V^n_{\mathrm{op}}(y^n)\right] \right| 0\right\rangle + \text{covariant Schwinger terms.}$$

However, the time-ordered products are assumed to be well defined everywhere; hence a contradiction.

VI. ENERGY-MOMENTUM AND ANGULAR MOMENTUM TENSOR OPERATORS

To study the energy-momentum tensor, we obviously consider the infinitesimal translational vertices t_ν [Eq. (2.9)]. Now $t^a_\nu \notin \mathcal{G}_0$; hence we cannot make use of the Ward-Takahashi identity of Sec. V. So we return to the second identity (4.7) and obtain

$$\frac{\partial}{\partial x_\mu}\mathcal{F}(J^{t_\nu}_\mu - g_{\mu\nu}\mathcal{L}, x; V^1, y^1; \ldots; V^n, y^n) = -i\sum_{i=1}^n \delta(x-y^i)\mathcal{F}(V^1, y^1; \ldots; \partial_\nu V^i, y^i; \ldots; V^n, y^n)\,.$$

Note that $J^{t_\nu}_\mu$ and $\mathcal{L}$ belong to $\mathcal{G}_1$, and so $(J^{t_\nu}_\mu)_{\mathrm{op}}$ and $\mathcal{L}_{\mathrm{op}}$ are defined. Let $V^i \in \mathcal{G}_0$; then V^i_{op} and $(\partial_\nu V^i)_{\mathrm{op}}$ are also defined. Hence

$$\frac{\partial}{\partial x_\mu}\langle 0|T[\Theta_{\mu\nu}(x)V^1_{\text{op}}(y^1)\cdots V^n_{\text{op}}(y^n)]|0\rangle = -i\sum_{i=1}^n \delta(x-y^i)\langle 0|T[V^1_{\text{op}}(y^1)\cdots\partial_\nu V^i_{\text{op}}(y^i)\cdots V^n_{\text{op}}(y^n)]|0\rangle\ , \tag{6.1}$$

where

$$\Theta_{\mu\nu}(x)\equiv(J^{t\nu}_\mu)_{\text{op}}(x) - g_{\mu\nu}\mathcal{L}_{\text{op}}(x) \tag{6.2}$$

is the *energy-momentum tensor operator*. Equation (5.11) then shows that the operators

$$P_\nu \equiv G(\Theta_{\mu\nu}) \tag{6.3}$$

satisfy

$$[P_\nu, V_{\text{op}}(x)] = -i\partial_\nu V_{\text{op}}(x) \quad \forall\, V\in \mathcal{G}_0\,.$$

Hence P_ν are the *energy-momentum operators* and generate the group of translations. This implies a wider range of validity; that is,

$$[P_\nu, O(x)] = -i\partial_\nu O(x) \tag{6.4}$$

for any local operator $O(x)$.

Let us next study the infinitesimal Lorentz rotational vertices [Eq. (2.10)], and see how they will lead to the angular momentum operators. Recall that $\mathcal{L}$ is a Lorentz scalar [Eq. (4.5)]. Hence

$$J^{t\beta}_\alpha = J^{t\alpha}_\beta\ , \tag{6.5}$$

which also implies that the energy-momentum tensor is symmetric,

$$\Theta_{\alpha\beta}(x) = \Theta_{\beta\alpha}(x)\,. \tag{6.6}$$

By definition (4.8) the current vertex for $\sigma_{\alpha\beta}$ is

$$J^{\sigma\alpha\beta}_\mu = X_\alpha J^{t\beta}_\mu - X_\beta J^{t\alpha}_\mu\ , \tag{6.7}$$

and so for nonderivative vertices $V^1,\ldots,V^n$, the second identity (4.7) yields

$$\frac{\partial}{\partial x_\mu}\mathfrak{F}(J^{\sigma\alpha\beta}_\mu, x;\, V^1, y^1;\ldots;\, V^n, y^n) = \mathfrak{F}(J^{t\beta}_\alpha - J^{t\alpha}_\beta + X_\alpha\partial_\beta\mathcal{L} - X_\beta\partial_\alpha\mathcal{L}, x;\, V^1, y^1;\ldots)$$
$$-i\sum_{i=1}^n\delta(x-y^i)\mathfrak{F}(V^1,y^1;\ldots;\delta_{\sigma_{\alpha\beta}}V^i, y^i;\ldots;V^n,y^n)\,.$$

Then making use of (6.5),

$$\frac{\partial}{\partial x_\mu}\langle 0|T[\Sigma_{\mu\alpha\beta}(x)V^1_{\text{op}}(y^1)\cdots V^n_{\text{op}}(y^n)]|0\rangle = -i\sum_{i=1}^n\delta(x-y^i)\langle 0|T[V^1_{\text{op}}(y^1)\cdots(\delta_{\sigma_{\alpha\beta}}V^i)_{\text{op}}(y^i)\cdots V^n_{\text{op}}(y^n)]|0\rangle\,, \tag{6.8}$$

where

$$\Sigma_{\mu\alpha\beta}(x)\equiv x_\alpha\Theta_{\mu\beta}(x) - x_\beta\Theta_{\mu\alpha}(x) \tag{6.9}$$

is the *angular momentum current operator*. The *angular momentum operators*

$$M_{\alpha\beta}\equiv G(\Sigma_{\mu\alpha\beta}) \tag{6.10}$$

are therefore generators of Lorentz transformations on the Hilbert space, and for any nonderivative vertex operator V_{op},

$$[M_{\alpha\beta}, V_{\text{op}}(x)] = -i(\delta_{\sigma_{\alpha\beta}}V)_{\text{op}}(x)\,. \tag{6.11}$$

We wish to stress here that the energy-momentum tensor operators and the angular momentum current operators are specified by the Lagrangian vertex. Attempts have been made to alter the energy-momentum tensor in the formal theory for the purpose of symmetry,[24] or to improve[25] it so that its trace is soft. To achieve this, terms are added that contain second derivatives of fields. The corresponding procedure here is therefore to add to the Lagrangian vertex extra vertices that contain second derivatives of basic fields; but then $\mathcal{L}$ will no longer belong to $\mathcal{G}_1$. That also takes the transformation currents J^Λ_μ out of $\mathcal{G}_1$.

Therefore we can no longer define $\mathcal{L}_{\text{op}}$ and $J^{\Lambda}_{\mu\text{op}}$, and the nice connection between Lagrangian, currents, energy-momentum tensor, etc., will be lost. In other words, our Lagrangian vertex cannot simply be altered.

VII. CURRENT-ALGEBRA WARD-TAKAHASHI IDENTITIES

The current algebra of Gell-Mann[3] is usually formulated in terms of equal-time commutators of pairs of currents which correspond to a basis of the Lie algebra. Since equal-time commutators may not exist, we have to look for an alternative description of current algebra. It is well known that nearly all results of current algebra follow from the intermediate step: the current-algebra Ward-Takahashi identities. We have seen that Ward-Takahashi identities follow naturally from the Lagrangian and the transformation vertices in our formulation. Therefore we will investigate when our formulation yields those identities for current algebra.

Let us introduce some notations and a lemma. The sum of those simple vertices in $\mathcal{L}$ that contain derivatives of basic fields is a subvertex, which we shall call $\mathcal{L}^{\text{derivative}}$. Let Λ and Π be two nonderivative transformation vertices. We define their commutator[26] $[\Pi,\Lambda]$ (which is also a transformation vertex) by

$$[\Pi,\Lambda]^a=\left(\frac{\delta\Pi^a}{\delta\phi^b}\bigg|\,\Lambda^b\right)-\left(\frac{\delta\Lambda^a}{\delta\phi^b}\bigg|\,\Pi^b\right). \tag{7.1}$$

Lemma 2: If

(1) all basic fields are Bose fields,

(2) the Lagrangian $\mathcal{L}$ has at most first derivatives of basic fields,

(3) $\mathcal{L}^{\text{derivative}}$ has only zero excess-subtraction functions,

(4) Λ and Γ are two nonderivative transformation vertices with constant coefficients and zero excess-subtraction functions, and

(5) $\delta_\Lambda\mathcal{L}^{\text{derivative}}=0$, (7.2)

then

$$\delta_\Lambda J^{\Gamma}_{\mu}=J^{[\Gamma,\Lambda]}_{\mu}. \tag{7.3}$$

This lemma will be proved in Appendix C.

Lemma 2 precisely specifies the condition for current algebra. An algebra $\mathfrak{A}$ of transformation vertices is defined to be a set of nonderivative transformation vertices with constant coefficients and zero excess-subtraction functions such that it is closed under the commutator (7.1). For each $\Lambda\in\mathfrak{A}$, there is a current operator $J^{\Lambda}_{\mu\text{op}}$. If $\mathcal{L}^{\text{derivative}}$ has zero excess-subtraction functions and if it is invariant under the algebra $\mathfrak{A}$, i.e.,

$$\delta_\Lambda\mathcal{L}^{\text{derivative}}=0\quad\forall\,\Lambda\in\mathfrak{A}, \tag{7.4}$$

then Lemma 2 says that the Ward-Takahashi identities for these currents are

$$\begin{aligned}
&\frac{\partial}{\partial x_\mu}\langle 0|T[J^{\Lambda}_{\mu\text{op}}(x)J^{\Pi}_{\nu\text{op}}(y)V^1_{\text{op}}(z^1)\cdots V^n_{\text{op}}(z^n)]\,|0\rangle\\
&\qquad=\left\langle 0\left|T\left[\frac{\partial}{\partial x_\mu}J^{\Lambda}_{\mu\text{op}}(x)J^{\Pi}_{\nu\text{op}}(y)V^1_{\text{op}}(z^1)\cdots V^n_{\text{op}}(z^n)\right]\right|0\right\rangle-i\delta(x-y)\langle 0|T[(J^{[\Pi,\Lambda]}_{\nu})_{\text{op}}(y)V^1_{\text{op}}(z^1)\cdots V^n_{\text{op}}(z^n)]\,|0\rangle\\
&\qquad-i\sum_{i=1}^{n}\delta(x-z^i)\langle 0|T[J^{\Pi}_{\nu\text{op}}(y)V^1_{\text{op}}(z^1)\cdots(\delta_\Lambda V^i)_{\text{op}}(z)\cdots V^n_{\text{op}}(z^n)]\,|0\rangle\\
&\qquad+i\partial_\mu\delta(x-y)\left\langle 0\left|T\left[\left(\frac{\delta}{\delta\partial_\mu\phi^a}J^{\Pi}_{\nu}\,\bigg|\,\Lambda^a\right)_{\text{op}}(y)V^1_{\text{op}}(z^1)\cdots V^n_{\text{op}}(z^n)\right]\right|0\right\rangle\\
&\qquad+i\sum_{i=1}^{n}\partial_\mu\delta(x-z^i)\left\langle 0\left|T\left[J^{\Pi}_{\nu\text{op}}(y)V^1_{\text{op}}(z^1)\cdots\left(\frac{\delta V^i}{\delta\partial_\mu\phi^a}\,\bigg|\,\Lambda^a\right)_{\text{op}}(z^i)\cdots V^n_{\text{op}}(z^n)\right]\right|0\right\rangle.
\end{aligned} \tag{7.5}$$

The only difference between these identities and those of the formal theory is that here the Schwinger terms are again both present and covariant. In order that current-algebra Ward-Takahashi identities for a particular *Lie algebra*, say SU(3)×SU(3), be valid, the basic fields must be a basis of a *realization* (linear representation included) of the algebra, for then an algebra of transformation vertices can be constructed from this realization. The details for SU(3)×SU(3) will be described in a forthcoming paper.

VIII. SUMMARY AND DISCUSSIONS

The vertices introduced in Sec. II are analogous to products of fields at equal space-time points, but with information contained in the excess-subtraction functions on how subtractions are made in their Feynman amplitudes in Bogoliubov's renormalization scheme.[5-7] They are generalizations of normal products.[4] Instead of formulating a Lagrangian quantum field theory by starting with an ill-defined Lagrangian density operator and deriving from it the Feynman rules for time-ordered products (and hence of the scattering matrix), we have made the following four assumptions:

Assumption (1). Any vertex in $\mathcal{G}_1$ determines a local vertex operator.

Assumption (2). Feynman amplitudes converge as sums over all allowed Feynman graphs, and $\mathcal{L}^{\text{int}}$ is used for constructing Feynman graphs.

Assumption (3). A Feynman amplitude for a set of vertices in $\mathcal{G}_1$ is related to the time-ordered product of their respective vertex operators.

Assumption (4). The Lagrangian vertex belongs to $\mathcal{G}_1$ and satisfies the restrictions of Sec. IV.

From the above assumptions, follow these consequences:

(1) The Euler-Lagrange equations of motion are valid in operator form.

(2) Any nonderivative transformation vertex determines a transformation current operator, which satisfies Ward-Takahashi identities, but with *covariant* Schwinger terms.[27] From this follows the operator form of Noether's theorem.

(3) The energy-momentum tensor and angular momentum current operator can be similarly obtained, and they in turn determine the generators of the Poincaré group.

(4) If $\mathcal{L}^{\text{derivative}}$ has only zero excess-subtraction functions, and is invariant under an algebra of transformation vertices, then the *current-algebra* Ward-Takahashi identities for this algebra are valid.

Note that nowhere are equal-time commutation relations required. Yet without them, we have obtained, by graphical means, interesting consequences that, in the usual formal theory, follow from equal-time canonical commutation relations.

Another significant deviation from the usual theory is that we have used here $i\mathcal{L}^{\text{int}}$, instead of $-i\mathcal{H}^{\text{int}}$, for constructing Feynman graphs, even though $\mathcal{L}^{\text{int}}$ may contain derivative couplings. Our use of $i\mathcal{L}^{\text{int}}$ has two advantages: (1) Lorentz covariance is guaranteed, and (2) we can prove Ward-Takahashi identities, and the operator form of Euler-Lagrange equations of motion and Noether's theorem. Hence if we require the equations of motion to be valid in operator form, we naturally will have to use $i\mathcal{L}^{\text{int}}$, and not $-i\mathcal{H}^{\text{int}}$.[28]

If modifications on Feynman amplitudes can be found, which can be analytically continued to some domain in the space of complex momenta, and which satisfy Steinmann's relations,[18] then these modified Feynman amplitudes define a quantum field theory of which they are the vacuum expectation values of the generalized retarded products. Then assumptions (1) and (3) become consequences, and will be no longer needed as inputs. The Ward-Takahashi identities then may also be expressed in terms of these generalized retarded products.

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APPENDIX A

To prove Lemma 1, it is sufficient to show that for $w=[(f_1,\ldots,f_N),\alpha]$, $v^j=[g^j,\beta^j]$, and a connected graph Γ in $\mathfrak{D}(w,v^1,\ldots,v^n)$,

$$ik_\mu\overline{F}_\Gamma(w,v^1,\ldots,v^n;k,q^1,\ldots,q^{n-1})=\sum_{i=1}^{N}\overline{F}_{\Gamma^i}(w^i,v^1,\ldots,v^n;k,q^1,\ldots,q^{n-1}), \tag{A1}$$

where $w^i=[(f_1,\ldots,\partial_\mu f_i,\ldots,f_N),\alpha]$ and Γ^i is obtained from Γ by replacing f_i by $\partial_\mu f_i$. For this purpose we will first state *Lemma 3:*

Let $u=[(g_1,\ldots,g_r),\alpha]$ be a simple vertex, let γ be a connected graph in $\mathfrak{D}(u,v^1,\ldots,v^n)$ such that $\delta(\gamma)\geq 0$, and let $G(\{l\})$ be a function of all the internal momenta $\{l\}$ of γ. For $i=1,\ldots,r$, define γ^i in $\mathfrak{D}(u^i,v^1,\ldots,v^n)$, where $u^i=[(g_1,\ldots,\partial_\mu g_i,\ldots,g_r),\alpha]$, by replacing g_i in γ by $\partial_\mu g_i$. Then

$$\sum_{i=1}^{r}t^{\delta(\gamma^i)}l_\mu^i G(\{l\})=\left(\sum_{i=1}^{r}l_\mu^i\right)t^{\delta(\gamma)}G(\{l\}),$$

where l_μ^i is the momentum carried by g_i away from u. For the proof, note that $\delta(\gamma^i)=\delta(\gamma)+1$, and that $\sum_{i=1}^{N}l_\mu^i$ is an external momentum to γ.

We will now prove Lemma 1.

Let $U=(U_1,\ldots,U_m)$ be a Γ forest, and define a set R of integers by the relations

$$i\in R \Leftrightarrow f_i\in U_1.$$

Then for $i\in R$ and for $j=1,\ldots,m$, $\delta(U_j^i)$ is independent of i. Therefore in the expression

$$\sum_{i\in R}[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_1^i)}]\,l_\mu^i I_\Gamma\,,$$

we can bring the summation sign $\sum_{i\in R}$ across $[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_2^i)}]$, and then, applying Lemma 3 to the resulting expression, we obtain

$$\sum_{i\in R}[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_1^i)}]\,l_\mu^i I_\Gamma=\sum_{i\in R}[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_2^i)}]\,l_\mu^i[-t^{\delta(U_1)}]I_\Gamma\,. \tag{A2}$$

Since for $i\notin R$, l_μ^i is not an external momentum to U_1^i, and since $\delta(U_1^i)=\delta(U_1)$,

$$t^{\delta(U_1^i)}\,l_\mu^i I_\Gamma=l_\mu^i t^{\delta(U_1)}I_\Gamma \quad \text{for } i\notin R. \tag{A3}$$

From this and (A2), we have

$$\sum_{i=1}^{N}[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_1^i)}]\,l_\mu^i I_\Gamma=\sum_{i=1}^{N}[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_2^i)}]\,l_\mu^i[-t^{\delta(U_1)}]I_\Gamma\,. \tag{A4}$$

Since $(U_2,\ldots,U_m)$ is also a Γ forest, by repeated application of (A4) we find

$$\sum_{i=1}^{N}[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_1^i)}]\,l_\mu^i I_\Gamma=\left(\sum_{i=1}^{N}l_\mu^i\right)[-t^{\delta(U_m)}]\cdots[-t^{\delta(U_1)}]I_\Gamma=\left(\sum_{i=1}l_\mu^i\right)\mathcal{R}I_\Gamma \tag{A5}$$

for a Γ forest $(U_1,\ldots,U_m)$.

Next, we will enlarge the notion of a forest. Let $Z(\Gamma)$ be the set of all one-particle irreducible subgraphs λ of Γ such that (1) $\delta(\lambda)=-1$ and (2) at least one $f_i\in\lambda$. An *extended* Γ *forest* U is a sequence $(U_1,\ldots,U_m)$ satisfying (1) for $a=1,\ldots,m$, U_a either belongs to $Z(\Gamma)$ or is a renormalization part, (2) $a<b\Rightarrow$ either $U_a\subset U_b$ or $U_a\cap U_b=\varnothing$, and (3) there exists an integer C such that (a) $U_C\in Z(\Gamma)$ and (b) $U_a\in Z(\Gamma)\Rightarrow U_a\supset U_C$. Then for any extended Γ forest U, there is a set R_U of integers satisfying $i\in R_U\Leftrightarrow f_i\in U_C$ (where U_C is defined as above); therefore for such $i\in R_U$, $(U_1^i,\ldots,U_m^i)$ is a Γ^i forest.

Consider the expression

$$\sum_{i\in R_U}[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_1^i)}]\,l_\mu^i I_\Gamma$$

for the above extended Γ forest. By the same argument leading to (A5), we find that this is equal to

$$\sum_{i\in R_U}[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_C^i)}]\,l_\mu^i[-t^{\delta(U_{C-1})}]\cdots[-t^{\delta(U_1)}]I_\Gamma\,.$$

Again we find $\delta(U_m^i),\ldots,\delta(U_C^i)$ are independent of i. Therefore we can bring $\sum_{i\in R_U}$ across the factors $[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_C^i)}]$, and, realizing that $\sum_{i\in R_U}l_\mu^i$ is an external momentum to U_C and that $\delta(U_C^i)=0$, we find that the Taylor series $t^{\delta(U_C^i)}$ of one term only is zero; that is,

$$\sum_{i\in R_U}[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_1^i)}]\,l_\mu^i I_\Gamma=0 \tag{A6}$$

for any extended Γ forest $U=(U_1,\ldots,U_m)$.

Finally, $il_\mu^i I_\Gamma$ is the unrenormalized integrand I_{Γ^i} for $(w^i,v^1,\ldots,v^N)$, so that

$$\sum_{i=1}^{N}\mathcal{R}I_{\Gamma^i}=i\sum_{i=1}^{N}\sum_{U\in\mathfrak{U}(\Gamma^i)}[-t^{\delta(U_m)}]\cdots[-t^{\delta(U_1)}]\,l_\mu^i I_\Gamma\,.$$

Defining $\mathfrak{V}_S(\Gamma)$ to be the set of all extended Γ forests U with $R_U=S$, then

$$\sum_{i=1}^{N}\mathcal{R}I_{\Gamma^i}=i\left(\sum_{i=1}^{N}\sum_{U\in\mathfrak{U}(\Gamma)}+\sum_{S\in S_N}\sum_{U\in\mathfrak{V}_S(\Gamma)}\sum_{i\in S}\right)[-t^{\delta(U_m^i)}]\cdots[-t^{\delta(U_1^i)}]\,l^i I_\Gamma\,. \tag{A7}$$

By (A5) and (A6) we therefore obtain

$$\sum_{i=1}^{N} \mathcal{R} I_{\Gamma^i} = i\left(\sum_{i=1}^{N} l_\mu^i\right) \mathcal{R} I_\Gamma ,$$

from which follows (A1).

APPENDIX B

Let

$$\alpha^{\text{kin}} = 0$$

and (B1)

$$\alpha^{\text{mass}}(I_2) = 2 .$$

For a simple vertex

$$w = [(f_1, \ldots, f_N), \eta], \tag{B2}$$

define the notation

$$\begin{aligned} r &\equiv \partial_\mu \left(\frac{\delta \mathcal{L}^{\text{kin}}}{\delta \partial_\mu \phi} \bigg| w \right) - \left(\frac{\delta \mathcal{L}^{\text{kin}}}{\delta \partial_\mu \phi} \bigg| \partial_\mu w \right) \\ &= [(f_1, \ldots, f_N, \partial^2 \phi), \lambda], \end{aligned} \tag{B3}$$

where λ satisfies

$$\lambda(R) = \begin{cases} \eta(R) & \text{if } N+1 \notin R \\ \eta(R - \{N+1\}) & \text{if } N+1 \in R \end{cases} . \tag{B4}$$

Let Γ be a graph in $\mathfrak{D}(r, v^1, \ldots, v^n)$. For generality, we will not distinguish between internal and external simple vertices. So let the simple vertices in Γ be $r, u, y, \ldots, z$. Without loss of generality, assume $\partial^2\phi$ at r is joined by a line L to the simple vertex u. Let k be the external momentum at r and let l be the momentum carried by L away from u. See Fig. 2. Then the unrenormalized integrand of Γ has the form

$$I_\Gamma = -l^2 \frac{i}{l^2 - m^2} I, \tag{B5}$$

where we have omitted the infinitesimal term in the propagator.

The renormalized integrand is

$$\mathcal{R} I_\Gamma = -\sum_{U \in \mathfrak{U}(\Gamma)} \prod_{\gamma \in U} [-t^{\delta(\gamma)}]\, l^2 \frac{i}{l^2 - m^2} I. \tag{B6}$$

Let

$$s \equiv \left(\frac{\delta \mathcal{L}^{\text{mass}}}{\delta \phi} \bigg| w \right) = -m^2 [(f, \phi), \lambda'], \tag{B7}$$

where λ' satisfies

$$\lambda'(R) = \begin{cases} \eta(R) & \text{if } N+1 \notin R \\ \eta(R - \{N+1\}) + 2 & \text{if } N+1 \in R \end{cases} . \tag{B8}$$

From Γ, define[29] $\Gamma' \in \bar{\mathfrak{D}}(s, u, y, \ldots, z)$ by changing $\partial^2\phi$ to $m^2\phi$ in Γ. See Fig. 3. For $\gamma \in \Gamma$, define γ' similarly. So

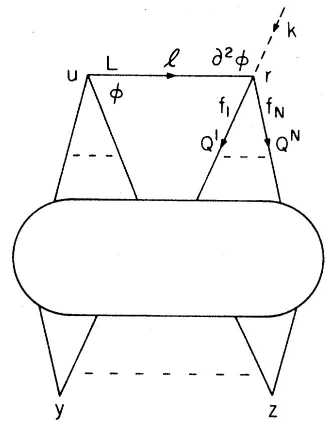

FIG. 2. A graph Γ in $\bar{\mathfrak{D}}(r, u, y, \ldots, z)$.

$$\mathcal{R} I_{\Gamma'} = -m^2 \sum_{U \in \mathfrak{U}(\Gamma')} \prod_{\gamma \in U} [-t^{\delta(\gamma)}] \frac{i}{l^2 - m^2} I. \tag{B9}$$

Now (B4) and (B8) imply

$$\delta(\gamma) = \delta(\gamma') \quad \forall \gamma \in \Gamma .$$

It can easily be seen that

$$\mathfrak{U}(\Gamma') = \{U' : U \in \mathfrak{U}(\Gamma)\},$$

whereas $t^{\delta(\gamma)}$ and $t^{\delta(\gamma')}$ act on the same momentum variables. Therefore

$$\mathcal{R} I_\Gamma - \mathcal{R} I_{\Gamma'} = -i \sum_{U \in \mathfrak{U}(\Gamma)} \sum_{\gamma \in U} [-t^{\delta(\gamma)}] I. \tag{B10}$$

Before proceeding further, we need two lemmas.

Lemma 4: Let Γ and $L \in \Gamma$ be defined as before, and let a renormalization part μ of Γ be such that $\mu \cap r \neq \varnothing$ and $\mu \cap u \neq \varnothing$, but $L \notin \mu$. See Fig. 4. Then if $Z_\mu(\Gamma)$ is the set of all Γ forests each containing μ,

$$\sum_{U \in Z_\mu(\Gamma)} \prod_{\gamma \in U} [-t^{\delta(\gamma)}] I = 0 . \tag{B11}$$

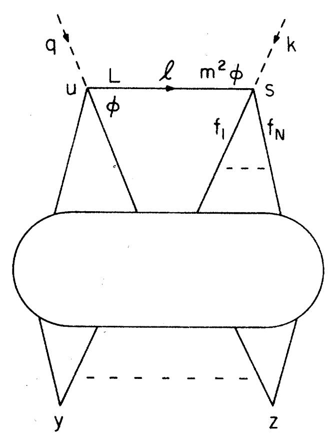

FIG. 3. Derived graph Γ' in $\bar{\mathfrak{D}}(s, u, y, \ldots, z)$.

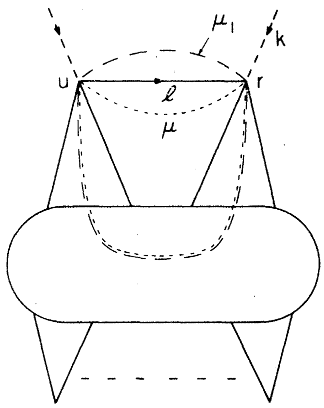

FIG. 4. Particular renormalization parts μ and μ_1 of Γ.

Proof. $\mu_1 \equiv \mu \cup \{L\}$ is also a renormalization part. The forests of $Z_\mu(\Gamma)$ can be partitioned into disjoint pairs of the form (U, U_1) where $\mu_1 \notin U$ but $\mu_1 \in U_1$ and otherwise U is identical to U_1. Then the proof consists of showing that the contribution of such a pair to the left-hand side of Eq. (B11) vanishes. In fact this contribution is

$$[-t^{\delta(\gamma_m)}]\cdots[-t^{\delta(\gamma_{i+1})}][1-t^{\delta(\mu_1)}][-t^{\delta(\mu)}]\times[-t^{\delta(\gamma_{i-1})}]\cdots[-t^{\delta(\gamma_1)}]I, \quad \text{(B12)}$$

where $\gamma_i = \mu$ and $(\gamma^1, \ldots, \gamma^m) = U$. $k+l$ is an external momentum to μ. Let the rest of the external momenta be $p_1, \ldots, p_t$. Then

$$t^{\delta(\mu)}[-t^{\delta(\gamma_{i-1})}]\cdots[-t^{\delta(\gamma_1)}]I$$

is a polynomial in $k+l$, $p_1, \ldots, p_t$ of degree $\delta(\mu)$, that is, a polynomial in k, $p_1, \ldots, p_t$ of degree $\delta(\mu)$. Now the external momenta to μ_1 is k, $p_1, \ldots, p_t$ and $\delta(\mu_1) \geq \delta(\mu)$ (because of the nature of excess-subtraction function). Therefore,

$$[1-t^{\delta(\mu_1)}][-t^{\delta(\mu)}][-t^{\delta(\gamma_{i-1})}]\cdots[-t^{\delta(\gamma_1)}]I = 0,$$

and so (B12) vanishes. Q.E.D.

Lemma 5: Let Γ and $L \in \Gamma$ be defined as before, but under the restriction that there is another line $L_1 \in \Gamma$ also joining r and u. Then

$$\sum_{U \in \mathfrak{U}(\Gamma)} \prod_{\gamma \in U} [-t^{\delta(\gamma)}] I = 0. \quad \text{(B13)}$$

Proof. Let σ be the subgraph consisting of L and L_1 only. Then its degree $\delta(\sigma) \geq 0$. $\mathfrak{U}(\Gamma)$ can be partitioned into disjoint pairs of the form (U, U_1) where $\sigma \in U_1$ but $\notin U$ and otherwise U is identical to U_1. Like the proof of the previous lemma, we will prove this one by showing that the contribution from any such pair to the left-hand side of (B13) vanishes. In fact, writing

$$I = \frac{i}{l_1^2 - m^2} G, \quad \text{(B14)}$$

where l_1 is the momentum carried in L_1 and G is a function of momentum variables other than l_1, this contribution is

$$[-t^{\delta(\gamma_m)}]\cdots[-t^{\delta(\gamma_2)}]G[1-t^{\delta(\sigma)}]\frac{i}{l_1^2-m^2}. \quad \text{(B15)}$$

Now $t^{\delta(\sigma)}$ acts on the external momenta of σ; therefore

$$t^{\delta(\sigma)}\frac{i}{l_1^2-m^2} = \frac{i}{l_1^2-m^2},$$

and hence (B15) vanishes. Q.E.D.

Returning to our original objective, we find that if Γ satisfies the restriction of Lemma 5, then (B10) reduces to

$$\mathfrak{R}I_\Gamma - \mathfrak{R}I_{\Gamma'} = 0. \quad \text{(B16)}$$

For Γ not satisfying the restriction of Lemma 5, then Lemma 4 and (B10) yield

$$\mathfrak{R}I_\Gamma - \mathfrak{R}I_{\Gamma'} = -i \sum_{U \in \mathfrak{U}_1(\Gamma)} \prod_{\gamma \in U} [-t^{\delta(\gamma)}] I, \quad \text{(B17)}$$

where $\mathfrak{U}_1(\Gamma)$ is the set of all Γ forests, each not containing any renormalization part μ as described previously and shown in Fig. 4. For each $\gamma \subseteq \Gamma$, define γ'' by removing the line L and combining r and u into one simple vertex. See Fig. 5. Let us consider the following restrictions of $\mathfrak{L}^{\text{int}}$ and $v^1, \ldots, v^n$, in order of increasing generality.

(1) If there is only one ϕ and no $\partial_\lambda \phi$ in u, then for Γ not satisfying the restriction of Lemma 5,

$$\Gamma'' \in \overline{\mathfrak{D}}\left(\left(\frac{\delta u}{\delta \phi}\middle| w\right), y, \ldots, z\right),$$

$$I = I_{\Gamma''}, \quad \text{(B18)}$$

and

$$\delta(\gamma) = \delta(\gamma''). \quad \text{(B19)}$$

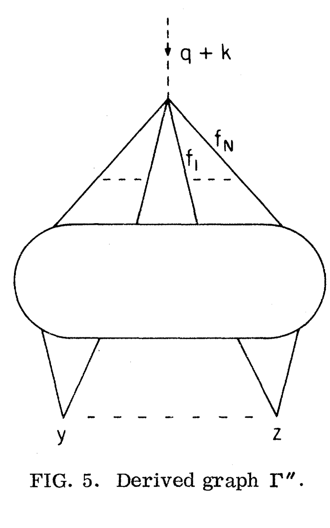

FIG. 5. Derived graph Γ''.

The last equation is valid because of the excess-subtraction function of a substitution. Again $t^{\delta(\gamma)}$ and $t^{\delta(\gamma'')}$ act on the same momentum variables. Also, $\mathfrak{U}(\Gamma'')=\{U'': U\in\mathfrak{U}_1(\Gamma)\}$. Therefore the right-hand side of (B17) is simply

$$-i\sum_{U\in\mathfrak{U}_1(\Gamma)}\prod_{\gamma\in U}[-t^{\delta(\gamma)}]I=-i\,\mathcal{R}I_{\Gamma''}; \tag{B20}$$

that is,

$$\mathcal{R}I_\Gamma-\mathcal{R}I_{\Gamma'}=-i\,\mathcal{R}I_{\Gamma''}. \tag{B21}$$

If $\mathcal{L}^{\text{int}}$ and $v^1,\ldots,v^n$ do not contain derivatives of ϕ (but each may have more than one ϕ), then for each graph $\Gamma\in\mathfrak{D}(r,v^1,\ldots,v^n)$ either (B16) or (B21) is valid. On summing over all graphs in $\mathfrak{D}(r,v^1,\ldots,v^n)$, we therefore obtain

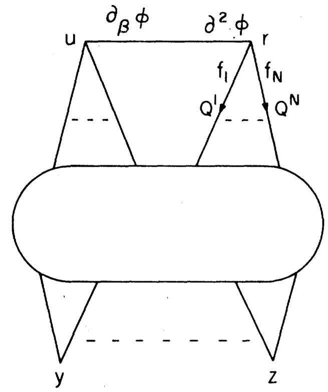

FIG. 6. Graph Γ for u containing $\partial_\beta\phi$.

$$\bar{\mathfrak{F}}(r-s,k;v^1,q^1;\ldots;v^n,q^n)=\bar{\mathfrak{F}}\left(\left(\frac{\delta\mathcal{L}^{\text{int}}}{\delta\phi}\,\middle|\,w\right),k;v^1,q^1;\ldots;v^n,q^n\right)$$
$$-i\sum_{i=1}^{n}\bar{\mathfrak{F}}\left(v^1,q^1;\ldots;\left(\frac{\delta v^i}{\delta\phi}\,\middle|\,w\right),k+q^i;\ldots;v^n,q^n\right). \tag{B22}$$

(2) If there is only one derivative $\partial_\beta\phi$ but no ϕ in u, then the line L connects $\partial_\beta\phi$ of the simple vertex u to $\partial^2\phi$ of r (Fig. 6), and I_Γ has the form

$$I_\Gamma=-l^2\frac{i}{l^2-m^2}\,il_\beta A^\beta=-l^2\frac{i}{l^2-m^2}\left(\sum_{t=1}^{N}iQ^t_\beta-ik_\beta\right)A^\beta, \tag{B23}$$

where $Q^1,\ldots,Q^N$ are the momenta carried by $f_1,\ldots,f_N$ away from r. See Fig. 2. Now

$$-l^2\frac{i}{l^2-m^2}\,iQ^t_\beta A^\beta$$

is the unrenormalized integrand of a graph $\Lambda^{t\beta}$ (Fig. 7) in

$$\bar{\mathfrak{D}}\left([(f_1,\ldots,\partial_\beta f_t,\ldots,f_N,\partial^2\phi),\lambda],\left(\frac{\delta u}{\delta\partial_\beta\phi}\,\middle|\,\phi\right),y,\ldots,z\right),$$

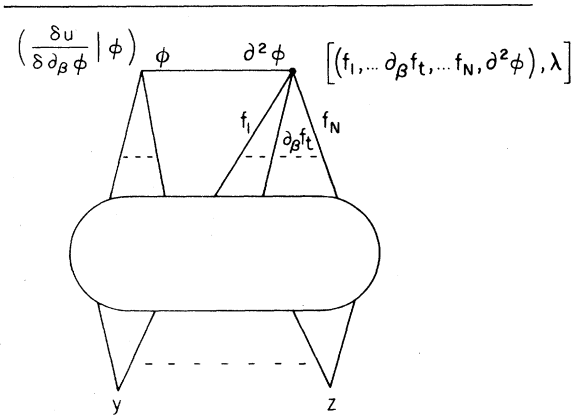

FIG. 7. Graph $\Lambda^{t\beta}$ in $\bar{\mathfrak{D}}\left([(f_1,\ldots,\partial_\beta f_t,\ldots,f_N,\partial^2\phi),\lambda],\left(\frac{\delta u}{\delta\partial_\beta\phi}\middle|\phi\right),y,\ldots,z\right)$.

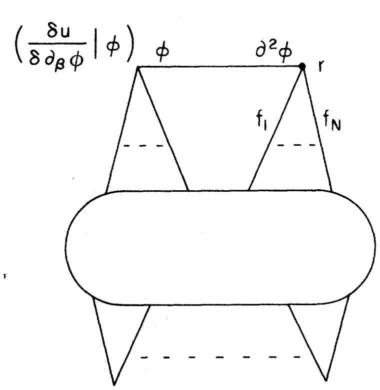

FIG. 8. Graph π^β in $\bar{\mathfrak{D}}\left(r,\left(\frac{\delta u}{\delta\partial_\beta\phi}\middle|\phi\right),y,\ldots,z\right)$.

and

$$-l^2 \frac{i}{l^2 - m^2} A^\beta$$

is the unrenormalized integrand of a graph Π^β (Fig. 8) in

$$\overline{\mathfrak{D}}\left(r, \left(\frac{\delta u}{\delta \partial_\beta \phi} \middle| \phi\right), y, \ldots, z\right).$$

So by applying (B21) to $I_{\Lambda^{t\beta}}$ and I_{Π^λ} separately, we obtain

$$\mathcal{R}I_\Gamma - \mathcal{R}I_{\Gamma'} = -i \sum_{t=1}^{N} \sum_\beta [\mathcal{R}I_{\Lambda^{t\beta''}} - ik_\beta \mathcal{R}I_{\Pi^{\beta''}}]. \tag{B24}$$

If $\mathfrak{L}$ and $v^1, \ldots, v^n$ contain at most first-order derivatives of ϕ, there will be additional terms in (B22) due to graphs in which $\partial^2\phi$ is joined to $\partial_\beta\phi$ of another simple vertex, and these additional terms are given by our last equation. Therefore, on summing up all graphs, we have a generalization of (B22):

$$\overline{\mathfrak{F}}(r-s, k; v^1, q^1; \ldots; v^n, q^n) = \overline{\mathfrak{F}}\left(\left(\frac{\delta \mathfrak{L}^{\text{int}}}{\delta \phi} \middle| w\right) + \left(\frac{\delta \mathfrak{L}^{\text{int}}}{\delta \partial_\beta \phi} \middle| (\partial_\beta - ik_\beta) w\right), k; v^1, q^1; \ldots; v^n, q^n\right)$$
$$-i\,\overline{\mathfrak{F}}\left(v^1, q^1; \ldots; \left(\frac{\delta v^i}{\delta \phi} \middle| w\right) + \left(\frac{\delta v^i}{\delta \partial_\beta \phi} \middle| (\partial_\beta - ik_\beta) w\right), q^i + k; \ldots; v^n, q^n\right). \tag{B25}$$

(3) In general, where the order of derivatives of ϕ in $\mathfrak{L}^{\text{int}}$ and $v^1, \ldots, v^n$ are not restricted, Eq. (B25) is easily further generalized to

$$\sum_{s=0}^{\infty} \left[\overline{\mathfrak{F}}\left(\left(\frac{\delta \mathfrak{L}}{\delta \partial_{\mu_1} \cdots \partial_{\mu_s} \phi} \middle| (\partial_{\mu_1} - ik_{\mu_1}) \cdots (\partial_{\mu_s} - ik_{\mu_s}) w\right), k; v^1, q^1; \ldots; v^n, q^n\right)\right.$$
$$\left. -i \sum_{i=1}^{n} \overline{\mathfrak{F}}\left(v^1, q^1; \ldots; \left(\frac{\delta v^i}{\delta \partial_{\mu_1} \cdots \partial_{\mu_s} \phi} \middle| (\partial_{\mu_1} - ik_{\mu_1}) \cdots (\partial_{\mu_s} - ik_{\mu_s}) w\right), k + q^i; \ldots; v^n, q^n\right)\right] = 0, \tag{B26}$$

whose Fourier transform is the first identity (4.6).

APPENDIX C

We present here the proof of Lemma 2.

Assumptions (1) to (3) of Sec. VII imply that $\mathfrak{L}^{\text{derivative}}$ has the form

$$\mathfrak{L}^{\text{derivative}} = \sum_{n=2}^{N} \partial_{\mu_1}\phi^{a_1} \cdots \partial_{\mu_n}\phi^{a_n} F^{\mu_1 \cdots \mu_n}_{n; a_1 \cdots a_n}(\phi), \tag{C1}$$

where F's are polynomials of basic fields ϕ. [Because of assumption (3), a vertex can be written as a polynomial of fields without confusion.] For integers i and n satisfying $1 \leq i \leq n \leq N$, define

$$G^{i\mu}_{na} = F^{\nu_1 \cdots \nu_{i-1} \mu \nu_i \cdots \nu_{n-1}}_{n; b_1 \cdots b_{i-1} a b_i \cdots b_{n-1}}(\phi) \partial_{\nu_1}\phi^{b_1} \cdots \partial_{\nu_{n-1}}\phi^{b_{n-1}}. \tag{C2}$$

Then for any set of integers $I_2, \ldots, I_N$ such that $1 \leq I_n \leq n$,

$$\mathfrak{L}^{\text{derivative}} = \sum_{n=2}^{N} \partial_\mu \phi^a G^{I_n \mu}_{na}. \tag{C3}$$

Hence assumption (4) implies

$$\delta_\Lambda \mathfrak{L}^{\text{derivative}} = \sum_{n=2}^{N} \left[\frac{\delta \Lambda^a}{\delta \phi^b} \partial_\mu \phi^b G^{I_n \mu}_{na} + \partial_\mu \phi^a \delta_\Lambda G^{I_n \mu}_{na}\right].$$

Combining this with $\delta_\Lambda \mathfrak{L}^{\text{derivative}} = 0$ [assumption (5)],

$$\delta_\Lambda G^{i\mu}_{na} = -\frac{\delta \Lambda^b}{\delta \phi^a} G^{i\mu}_{nb} \quad \text{for } 1 \leq i \leq n \leq N. \tag{C4}$$

By assumption (2) and the definition of transformation current vertex,

$$(J^{\Gamma})^{\mu}=\sum_{n=2}^{N}\sum_{i=1}^{n}G^{i\mu}_{na}\Gamma^{a}, \tag{C5}$$

and we find

$$\delta_{\Lambda}(J^{\Gamma})^{\mu}=\sum_{n=2}^{N}\sum_{i=1}^{n}[(\delta_{\Lambda}G^{i\mu}_{na})\Gamma^{a}+G^{i\mu}_{na}\delta_{\Lambda}\Gamma^{a}].$$

Using (C4), this becomes

$$\delta_{\Lambda}(J^{\Gamma})^{\mu}=\sum_{n=2}^{N}\sum_{i=1}^{n}\left[-\frac{\delta\Lambda^{b}}{\delta\phi^{a}}\,G^{i\mu}_{nb}\Gamma^{a}+G^{i\mu}_{na}\,\frac{\delta\Gamma^{a}}{\delta\phi^{b}}\,\Lambda^{b}\right]=\sum_{n=2}^{N}\sum_{i=1}^{n}G^{i\mu}_{na}[\Gamma,\Lambda]^{a}=(J^{[\Gamma,\Lambda]})^{\mu}. \tag{7.3}$$

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[1]For example, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[2]S. Weinberg, in *Lectures on Elementary Particles and Quantum Field Theory* (1970 Brandeis Lectures), edited by S. Deser *et al*. (MIT Press, Cambridge, Mass., 1970).

[3]M. Gell-Mann, Physics 1, 63 (1964).

[4]W. Zimmermann, in *Lectures on Elementary Particles and Quantum Field Theory* (1970 Brandeis Lectures), edited by S. Deser *et al*. (MIT Press, Cambridge, Mass., 1970).

[5]N. N. Bogoliubov and O. Parasiuk, Acta Math. 97, 227 (1957).

[6]K. Hepp, Commun. Math. Phys. 2, 301 (1966).

[7]W. Zimmermann, Commun. Math. Phys. 15, 208 (1969).

[8]J. Lowenstein, Phys. Rev. D 4, 2281 (1971).

[9]J. C. Ward, Phys. Rev. 78, 1824 (1950); Y. Takahashi, Nuovo Cimento 6, 370 (1957).

[10]A formal derivative of nth order is defined as follows. Let $\Delta\equiv\{\partial_0,\partial_1,\partial_2,\partial_3\}$ and let $Z\equiv\Delta\times\cdots\times\Delta\times\mathcal{B}$ be the Cartesian product set of n Δ's with $\mathcal{B}$. Let two elements $(\partial_{\mu_1},\partial_{\mu_2},\ldots,\partial_{\mu_n},\phi)$ and $(\partial_{\nu_1},\ldots,\partial_{\nu_n},\chi)$ of Z be said to be equivalent if $\phi=\chi$ and if there exists a permutation γ over the integers $1,\ldots,n$, such that $\mu_i=\nu_{\gamma_i}$. Then a formal derivative is an equivalence class of Z under the above equivalence relation.

[11]The zero excess-subtraction function of any order N will be denoted by 0: $0(R)=0$ for all $R\in S_N$.

[12]The vigorous meaning of a finite formal linear combination of simple vertices will be given here. Two simple vertices $[f,\alpha]$ and $[g,\beta]$ in $\mathcal{V}$ are said to be $\mathcal{E}$-equivalent if (1) they have the same order N and (2) there is a permutation π over the integers $1,\ldots,N$ such that $\alpha(R)=\beta(\pi(R))$ for $R\subset I_N$, and $f_i=g_{\pi(i)}$ for $i\in I_N$. Let $\mathcal{K}$ be the set of all pairs of finite sequences $(a_1,\ldots,a_n;\ \omega_1,\ldots,\omega_n)$ of equal "length," where the a's are polynomially bounded infinitely differentiable functions of space-time coordinates and the ω's are elements (equivalent classes) in $\mathcal{V}/\mathcal{E}$. Two elements (a,ω) and (b,λ) in $\mathcal{K}$ are said to be $\mathcal{E}_1$-equivalent if (1) they have the same length n and (2) there is a permutation γ over the integers $1,\ldots,n$ such that $a_i=b_{\gamma(i)}$ and $\omega_i=\lambda_{\gamma(i)}$. If the element $(a_1,\ldots,a_n;\ \omega_1,\ldots,\omega_n)$ in $\mathcal{K}$ is such that $\omega_{n-1}=\omega_n$, then it is said to be $\mathcal{E}_2$-equivalent to the pair $(a_1,\ldots,a_{n-2},a_{n-1}+a_n;\ \omega_1,\ldots,\omega_{n-1})$. We now define a third equivalence relation $(\mathcal{E}_1\mathcal{E}_2)$ on $\mathcal{K}$ by the property that if two elements in $\mathcal{K}$ are either $\mathcal{E}_1$-equivalent or $\mathcal{E}_2$-equivalent then they are also $(\mathcal{E}_1\mathcal{E}_2)$-equivalent. Then a finite formal linear combination of simple vertices is a member of $\mathcal{G}\equiv\mathcal{K}/(\mathcal{E}_1\mathcal{E}_2)$.

[13]W. Zimmermann, Commun. Math. Phys. 11, 1 (1968).

[14]A subgraph γ of Γ is a subset of lines of Γ and their end points.

[15]Zimmermann's normal product is of the form $N_\delta(f_1\cdots f_N)$ where δ is an integer greater or equal to the dimension d of the product $f_1\cdots f_N$, and where $f_1,\ldots,f_N$ are fields. With this normal product, the subtraction number for a subgraph γ is then defined to be $\delta(\gamma)=d(\gamma)+(\delta-d)$. Comparing this with the properties of simple vertices, we see that a simple vertex $[(f_1,\ldots,f_N),\alpha]$, with the excess-subtraction function satisfying $\alpha(R)=\delta-d$ when R contains more than one integer, is equivalent to $N_\delta(f_1\cdots f_N)$.

[16]If there is more than one external line going into a point of γ, the external momentum at that point is taken to be the sum of the momenta of these lines. See Ref. 7 on how the momenta of internal lines of γ depend on the external momenta.

[17]D. Ruelle, Nuovo Cimento 16, 356 (1961).

[18]O. Steinmann, Commun. Math. Phys. 10, 245 (1968).

[19]H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 425 (1955).

[20]Equal-time commutators do not exist in general, although it is a well-defined concept in free-field theories.

[21]J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

[22]There is one notable exception to this. Theories of field-current identity specify the Schwinger terms completely. See T. D. Lee and B. Zumino, Phys. Rev. 163, 1667 (1967); T. D. Lee, S. Weinberg and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).

[23]For a review on properties of conserved currents, see C. A. Orzalesi, Rev. Mod. Phys. 42, 381 (1970).

[24]F. J. Belinfante, Physica (Utrecht) 7, 449 (1940).

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[26]The definition of commutator here is analogous to that of a Lie algebra of transformation group on a coordinate space of finite dimension.

[27]W.-K. Tung [Phys. Rev. 188, 2404 (1969)] has shown that such terms exist in a Ward-Takahashi identity for the simple current $\bar\psi\gamma_\mu\psi$ in the Feynman graph theory of the gluon model.

[28]In some theories [e.g., I. S. Gerstein, R. Jackiw, B. Lee, and S. Weinberg, Phys. Rev. D 3, 2486 (1971)], it can be shown that the use of $i\mathcal{L}^{\text{int}}$ in evaluating Feynman amplitudes is equivalent to the use of $-i\mathcal{H}^{\text{int}}$ together with some appropriate noncovariant propagators. ($\mathcal{H}^{\text{int}}$ means the interaction Hamiltonian density.)

[29]The notation $\mathfrak{D}(s,u,\ldots z)$ is the set of all Feynman graphs with simple vertices (internal or external) $s,u,\ldots,z$.

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Perturbation Lagrangian Theory for Dirac Fields–Ward-Takahashi Identity and Current Algebra*

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In a class of Lagrangian field theories for Dirac spin-$\frac{1}{2}$ particles, the Bogoliubov-Parasiuk-Hepp renormalization scheme provides a proof of the operator forms of Euler-Lagrange equations of motion, Noether's theorem, and Ward-Takahashi identities. Time-ordered products for some derivatives of Dirac fields can only be defined with special care in terms of Feynman graphs. Current-algebra Ward-Takahashi identities are obtained if $\mathcal{L}^{\text{derivative}}$ is invariant under the algebra; however, Schwinger terms are absent from these identities.

I. INTRODUCTION

In the preceding paper,[1] we described perturbation theory for scalar fields in terms of objects called *vertices* and employed the Bogoliubov-Parasiuk-Hepp[2] (BPH) renormalization scheme to prove Ward-Takahashi identites,[3] which led to operator forms of Euler-Lagrange equations of motion and Noether's theorem. They also enabled us to construct the energy-momentum tensor, angular momentum current operators, and internal-symmetry currents and generators. Assuming that the derivative part, $\mathcal{L}^{\text{derivative}}$, of a Lagrangian was invariant under a symmetry algebra, we could prove current-algebra Ward-Takahashi identities. This is rather interesting, as the main bulk of current-algebra[4] results that agree with experiment comes from these identities.

The purpose of this paper is to extend these results to Dirac fields, which call for some modification. We will assume that the reader is familiar with Ref. 1, on which we will rely extensively.

The basic fields here are basis vectors of Dirac's representation[5] of the Lorentz group. They are ψ^a ($a=1,\ldots,4$) and their complex conjugate ψ^{a*} for each Dirac spin-$\frac{1}{2}$ particle. The generator of Lorentz rotation in the α-β plane is the spin matrix

$$s_{\alpha\beta}\equiv\tfrac{1}{4}[\gamma_\alpha,\gamma_\beta], \tag{1.1}$$

satisfying the commutation relation

$$[s_{\alpha\beta},s_{\mu\nu}]=g_{\alpha\nu}s_{\beta\mu}+g_{\beta\mu}s_{\alpha\nu}-g_{\alpha\mu}s_{\beta\nu}-g_{\beta\nu}s_{\alpha\mu}\,. \tag{1.2}$$

Repeating from Ref. 1, a *simple vertex* is the ordered pair $[f,\alpha]$, where f is a (possibly empty) sequence of fields[6] and α is an *excess-subtraction function* (see Ref. 1 for definition). A *vertex* is a finite formal linear combination of simple vertices:

$$W=a^1w^1+\cdots+a^nw^n, \tag{1.3}$$

where the coefficients a^i are polynomially bounded infinitely differentiable functions of space-time coordinates. Because of Fermi statistics, the notion of substitution needs a little modification which consists of inserting appropriate Fermi signature factors to every term in Eq. (2.4) of Ref. 1. Thus for simple vertices $w\equiv[f,\alpha]$ and $v\equiv[g,\beta]$ and any field ϕ, the *substitution* is

$$\left(\frac{\delta w}{\delta\phi}\middle|v\right)\equiv\sum_{j\in U_\phi^w}\begin{pmatrix}g,\phi,f_1,\ldots,f_{j-1}\\ \phi,f_1,\ldots,f_{j-1},g\end{pmatrix}[h^{(j)},\gamma^{(j)}]. \tag{1.4}$$

Denoting the sequences of Dirac fields in the two rows above by A and A', the signature factor[7] is defined by

$$\begin{pmatrix}A\\A'\end{pmatrix}\equiv(-)^r, \tag{1.5}$$

where r is the number of transpositions of Dirac fields required to permute the sequence A to A'.