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Schwinger's Proper-Time Method and the Eikonal Approximation

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Eikonal techniques are used to present an alternative treatment of the electron's Green's function in the presence of a plane-wave field. After studying the propagation of an electron in a prescribed electromagnetic field we eikonalize the corresponding Green's function. If the external field is chosen to be a laser field, the approximate eikonal propagator is turned into an exact one: Additional assumptions characterizing the eikonal approximation are automatically satisfied. The methods employed are Schwinger's pre-source-theory functional techniques.

I. INTRODUCTION

This paper demonstrates the close resemblance of Schwinger's original proper-time calculations[1] and eikonal techniques.[2] We discuss the specific case where the electron is traveling in a plane-wave field. This particular example allows a closed-form representation of the corresponding electron's Green's function and was solved by Schwinger some time ago. Nevertheless, we regard it as meaningful and useful that the proper-time method can be directly related to eikonal techniques. The connection between the two approaches is exhibited by comparing Schwinger's transformation function with our eikonalized electron's Green's function. Eikonalization is thereby termed the process by which quadratic terms in propagators, which carry low momenta, are neglected. We will demonstrate that this additional assumption is trivially satisfied if the electron is traveling in a laser field. It is in this sense that the eikonalized propagator is turned into an exact one. We present our approach as follows: In Sec. II we give a brief review of Schwinger's functional formalism of quantum electrodynamics (QED).[3] Then we discuss in some detail the electron's Green's function with external field $A_\mu(x)$ in Sec. III. In Sec. IV we perform a chain of variable transformations to produce our final expression for the eikonalized electron's propagation function.

II. FUNCTIONAL FORMULATION OF QED

The Lagrangian that describes electrons in the presence of an arbitrary external electromagnetic field reads, in standard notation,

$$\mathcal{L} = -\bar\psi\gamma^\mu\left(\frac{1}{i}\,\partial_\mu - eA_\mu\right)\psi - m\bar\psi\psi\,.$$

The field equations and commutation relations are given by

$$\left[\gamma^\mu\left(\frac{1}{i}\,\partial_\mu - eA_\mu\right) + m\right]\psi(x) = 0\,,$$

$$\bar\psi(x)\left[\gamma^\mu\left(-\frac{1}{i}\,\overleftarrow{\partial}_\mu - eA_\mu\right) + m\right] = 0\,,$$

$$\{\psi(\vec{x}, x_0), \bar\psi(\vec{x}\,', x_0)\} = \gamma^0\delta(\vec{x} - \vec{x}\,')\,,\quad \bar\psi = \psi^\dagger\gamma_0\,.$$

Here ψ is a second-quantized field operator and $A_\mu(x)$ is an external c-number unquantized field. Defining

$$\mathfrak{z}\{\eta, \bar\eta\} = \left\langle\left(\exp i\int[\bar\psi\eta + \bar\eta\psi]\right)_+\right\rangle,$$

where $\eta, \bar{\eta}$ represent anticommuting c-number sources, we then get a set of coupled functional differential equations:

$$\left(m+\gamma \frac{1}{i} \partial\right)\frac{1}{i} \frac{\delta}{\delta\bar{\eta}(x)} \mathfrak{z}=\eta(x) \mathfrak{z} + e\gamma A(x)\frac{1}{i} \frac{\delta}{\delta\bar{\eta}(x)} \mathfrak{z} ,$$

$$-\frac{1}{i} \frac{\delta}{\delta\eta(x)}\left(m-\gamma \frac{1}{i}\overleftarrow{\partial}\right) \mathfrak{z} =\bar{\eta}(x) \mathfrak{z} + e\gamma A(x)\left(-\frac{1}{i} \frac{\delta}{\delta\eta(x)}\right) \mathfrak{z} ,$$

which can be solved by

$$\mathfrak{z} =\frac{1}{N_v} \exp\left[-i \int \frac{\delta}{\delta\eta}(-ie\gamma A)\frac{\delta}{\delta\bar{\eta}}\right] \exp(i\bar{\eta}G_+\eta)$$

$$=\frac{1}{N_v} \exp[i\bar{\eta}G_+(1-e\gamma AG_+)^{-1}\eta]$$

$$\times \exp[-\mathrm{Tr}\ln(1-e\gamma AG_+)^{-1}] .$$

This solution can also be written in the form

$$\mathfrak{z}=\frac{1}{N_v} \exp(i\bar{\eta}G_+[A]\eta) \exp L[A] , \tag{2.1}$$

where

$$G_+[A] =G_+(1-e\gamma AG_+)^{-1} \tag{2.2}$$

and

$$L[A] =-\mathrm{Tr}\ln(1-e\gamma AG_+)^{-1} . \tag{2.3}$$

Tr indicates the complete diagonal summation in coordinate and spinor space. N_v is a normalization constant denoting the vacuum persistence amplitude; G_+ is the free electron propagator. In short,

$$\langle 0_+ | 0_- \rangle^A_{\eta=\bar{\eta}=0} =N_v , \quad \mathfrak{z}|_{\eta=\bar{\eta}=0}=1 ;$$

therefore

$$N_v = e^{L[A]} .$$

Thus, our generating functional reduces to

$$\mathfrak{z} =\exp\{i\bar{\eta}G_+[A]\eta\} ,$$

which can be used to derive all electron's Green's functions in presence of $A_\mu(x)$. The lowest one is the propagator

$$-\frac{1}{i} \frac{\delta}{\delta\bar{\eta}(x)} \frac{\delta}{\delta\eta(y)} \mathfrak{z}\Big|_0 =G_+(x,y|A) .$$

$G_+(x,y|A)$ satisfies an inhomogeneous differential equation, namely,

$$\left[m+\gamma\left(\frac{1}{i} \partial - eA\right)\right] G_+(x,y|A) = \delta(x-y) . \tag{2.4}$$

Introducing the symbolic operator

$$\Pi_\mu = \frac{1}{i} \partial_\mu - eA_\mu ,$$

the Green's function equation (2.4) can be cast into an algebraic operator equation

$$(m+\gamma\Pi)G_+[A] = 1 . \tag{2.5}$$

Inverting Eq. (2.5) yields

$$G_+[A] = \frac{1}{m+\gamma\Pi} = (m-\gamma\Pi)\frac{1}{m^2-(\gamma\Pi)^2}$$

or

$$G_+[A] = (m-\gamma\Pi)i \int_0^\infty ds\, e^{-im^2 s} e^{is(\gamma\Pi)^2} . \tag{2.6}$$

Schwinger has calculated the coordinate representation of the exponential operator in Eq. (2.6), i.e.,

$$\langle x(s)' | x(0)'' \rangle = \langle x' | U(s) | x'' \rangle , \tag{2.7}$$

where the operator

$$U(s) = e^{-i\mathcal{H}s}$$

describes the development of the system governed by the "Hamiltonian" $\mathcal{H}$,

$$\mathcal{H} = -(\gamma\Pi)^2 ,$$

in the "time" s. The solution of the corresponding equations of motion is obtained in some special cases by merely using the known algebraic properties, i.e., commutation relations of x_μ, Π_μ, etc. Schwinger gives an explicit expression for the transition amplitude in the case of a plane-wave field:

$$F_{\mu\nu} = f_{\mu\nu}F(\xi) , \quad \xi = n\cdot x , \quad n^2 = 0 ,$$

with $F(\xi)$ an arbitrary function. His result is stated in Ref. 1, Eq. (4.26):

$$\langle x(s)' | x(0)'' \rangle = C(x',x'')\frac{1}{s^2} \exp\left[\frac{i}{4s}(x'-x'')^2\right] \exp\left\{-is \frac{1}{\xi'-\xi''} \int_{\xi''}^{\xi'} d\xi [e^2A^2(\xi) - \tfrac{1}{2}eF(\xi)\sigma f]\right\}$$
$$\times \exp\left\{is\left[\frac{1}{\xi'-\xi''} \int_{\xi''}^{\xi'} d\xi\, eA(\xi)\right]^2\right\} , \tag{2.8}$$

with

$$C(x',x'') = -i(4\pi)^{-2} \exp\left[ie\int_{x''}^{x'} d\bar{x}^\mu A_\mu(\bar{x})\right] . \tag{2.9}$$

III. EIKONALIZATION OF THE ELECTRON'S GREEN'S FUNCTION

In this section we want to demonstrate that the expressions (2.8) and (2.9) are a direct consequence of an eikonalized propagator $\Delta^{\text{eik}}[A]$. For this reason we take the Green's function equation (2.4) which can be solved with the ansatz

$$G_+(x, y \mid A) = (m - \gamma\Pi)\Delta(x, y \mid A),$$

where $\Delta(x, y \mid A)$ satisfies

$$(m^2 + \Pi^2 - \tfrac{1}{2}e\sigma F)\Delta(x, y \mid A) = \delta(x-y). \tag{3.1}$$

Under gauge transformation, i.e.,

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x),$$

the Green's function $\Delta[A]$ responds according to

$$\Delta(x, y \mid A + \partial\lambda) = \exp\{ie[\lambda(x) - \lambda(y)]\}\Delta(x, y \mid A).$$

The same behavior holds true for the function $C(x, y)$ as defined in Eq. (2.9):

$$C(x, y) \to C(x, y)\exp\{ie[\lambda(x) - \lambda(y)]\}.$$

If we therefore introduce another expression $\Delta'[A']$ by the transformation

$$\Delta(x, y \mid A) = \exp\left[ie\int_y^x d\bar{x}^\mu A_\mu(\bar{x})\right]\Delta'(x, y \mid A') = \phi(x, y)\Delta'(x, y \mid A'), \tag{3.2}$$

in which the integral is to be performed on a straight line, as parametrized by

$$\bar{x}^\mu = \lambda x^\mu + (1-\lambda)y^\mu,$$

the function $\Delta'[A']$ is gauge-invariant, i.e., depends only on the field strength $F_{\mu\nu}$. This can most easily be seen by looking at the phase transformation (3.2), which induces a gauge transformation on A_μ, replacing it with

$$\begin{aligned}
A'_\mu(x) &= A_\mu(x) - \partial_\mu \int_y^x d\bar{x}^\nu A_\nu(\bar{x}) \\
&= A_\mu(x) - \partial_\mu \int_0^1 d\lambda\,(x-y)^\nu A_\nu(\bar{x}) \\
&= A_\mu(x) - \left(\int_0^1 d\lambda A_\mu(\bar{x}) + \int_0^1 d\lambda\,\lambda(x-y)^\nu[F_{\mu\nu}(\bar{x}) + \bar{\partial}_\nu A_\mu(\bar{x})]\right) \\
&= A_\mu(x) - \left(\int_0^1 d\lambda\,\lambda(x-y)^\nu F_{\mu\nu}(\bar{x}) + \int_0^1 d\lambda\left[A_\mu(\bar{x}) + \lambda\frac{d}{d\lambda}A_\mu(\bar{x})\right]\right) \\
&= -\int_0^1 d\lambda\,\lambda(x-y)^\nu F_{\mu\nu}(\bar{x}).
\end{aligned}$$

The new Green's function equation is given by

$$\left\{m^2 + \left[\gamma\left(\frac{1}{i}\partial - eA'\right)\right]^2\right\}\Delta'(x, y \mid A') = \delta(x-y).$$

Obviously we have chosen a gauge in which the vector potential depends only on the field strength. This is precisely the type of gauge one has in mind when studying the propagation of an electron in a plane-wave field.

In order to solve Eq. (3.1) it is convenient to transcribe it into functional form. This can be done by realizing the translational invariance of $\Delta[A]$, i.e.,

$$(\Delta + \delta\Delta)[A - \delta A] = \Delta[A].$$

With the various arguments in evidence this reads

$$\Delta(x, y \mid A(\xi)) = \Delta(x+h, y+h \mid A(\xi - h)).$$

Expansion of the right-hand side yields

$$\begin{aligned}\Delta(x, y\,|\,A(\xi)) &= \Delta(x+h, y+h\,|\,A(\xi-h))\\ &= \Delta(x+h, y+h\,|\,A(\xi)) - h^\nu\int du\,\frac{\partial}{\partial u^\nu}[A(u)]\,\frac{\delta}{\delta A(u)}\,\Delta(x+h, y+h\,|\,A(\xi)) + \cdots\\ &= \Delta(x, y\,|\,A(\xi)) + h^\nu\left(\frac{\partial}{\partial x^\nu}+\frac{\partial}{\partial y^\nu}\right)\Delta(x, y\,|\,A(\xi)) + \cdots\\ &\quad - h^\nu\int du\,\frac{\partial}{\partial u^\nu}[A(u)]\,\frac{\delta}{\delta A(u)}\left[\Delta(x, y\,|\,A(\xi)) + h^\nu\left(\frac{\partial}{\partial x^\nu}+\frac{\partial}{\partial y^\nu}\right)\Delta(x, y\,|\,A(\xi)) + \cdots\right] + \cdots\\ &= \Delta(x, y\,|\,A(\xi)) + h^\nu\left(\frac{\partial}{\partial x^\nu}+\frac{\partial}{\partial y^\nu} - \int du\,\frac{\partial}{\partial u^\nu}[A(u)]\,\frac{\delta}{\delta A(u)}\right)\Delta(x, y\,|\,A(\xi)) + \cdots\,.\end{aligned}$$

Therefore we obtain

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y} - \int du\,\frac{\partial}{\partial u}[A(u)]\frac{\delta}{\delta A(u)}\right)\Delta(x, y\,|\,A(\xi)) = 0\,. \tag{3.3}$$

This relation allows us to rewrite Eq. (3.1) in the form

$$\left\{m^2 + \left[\left(-\frac{1}{i}\,\partial_y + \frac{1}{i}\int du\,\frac{\partial}{\partial u}[A(u)]\,\frac{\delta}{\delta A(u)}\right) - eA(x)\right]^2 - \tfrac{1}{2}e\sigma F(x)\right\}\Delta(x, y\,|\,A) = \delta(x-y)\,. \tag{3.4}$$

Introducing a Fourier transformation on the initial variable,

$$\Delta(x, p\,|\,A) = \int dy\, e^{ipy}\Delta(x, y\,|\,A) = e^{ipx}g(x, p\,|\,A)\,,$$

we can convert (3.4) into

$$\left\{m^2 + \left[p + \int dk\, k^\nu A(k)\frac{\delta}{\delta A^\nu(k)} - e\int\frac{dk}{(2\pi)^4}\,A(k)e^{ikx}\right]^2 - ie\int\frac{dk}{(2\pi)^4}\,\gamma^\mu\gamma^\nu\, ik_\mu A_\nu(k)e^{ikx}\right\}g(x, p\,|\,A) = 1\,, \tag{3.5}$$

where use has been made of

$$k_\mu A^\mu(k) = 0$$

and the Fourier-transformed quantities

$$A(x) = \int\frac{dk}{(2\pi)^4}\,A(k)e^{ikx}\,,$$

$$\int du\,\frac{\partial}{\partial u}[A(u)]\,\frac{\delta}{\delta A(u)}\,\Delta[A] = \int dk\, ikA(k)\frac{\delta}{\delta A(k)}\,\Delta[A]\,.$$

So far everything is exact. However, Eq. (3.5) can only be solved if certain approximations are performed upon the correct Green's function equation for $g(x, p\,|\,A)$. The ones we want to introduce are of the Bloch-Nordsieck type. That means, first of all, we omit ∂^2 compared to $2p\partial$. In addition it is necessary to assume $eA^\mu(x)\partial_\mu = 0$. Then we obtain the following functional differential equation:

$$\begin{aligned}\Bigg\{p^2 + m^2 + \int dk\, 2p_\mu k^\mu A_\nu(k)\frac{\delta}{\delta A_\nu(k)} - 2e\int\frac{dk}{(2\pi)^4}\,p_\mu A^\mu(k)e^{ikx} + \left[e\int\frac{dk}{(2\pi)^4}\,A(k)e^{ikx}\right]^2&\\ + e\int\frac{dk}{(2\pi)^4}\,\gamma^\mu\gamma^\nu k_\mu A_\nu(k)e^{ikx}\Bigg\}g^{\text{eik}}(x, p\,|\,A) = 1\,.&\end{aligned} \tag{3.6}$$

The appearance of the spin-dependent part forces us to make a particular choice of the vector potential in order that Eq. (3.6) may become solvable. Our original intention was to investigate the structure of the electron's Green's function in the presence of a plane-wave field. So let us introduce

$$A_\mu = a[\epsilon_\mu^1 A_1(\xi) + \epsilon_\mu^2 A_2(\xi)] \equiv a\epsilon_\mu^\lambda A_\lambda(\xi)\,,\quad \xi = nx \tag{3.7}$$

and

$$n^\mu\epsilon_\mu^\lambda = 0 = n_\mu n^\mu\,,\qquad \epsilon_{\lambda\mu}\epsilon_{\lambda'}^\mu = \delta_{\lambda\lambda'}\,,\qquad \lambda, \lambda' = 1, 2\,.$$

Then

$$F_{\mu\nu}=a[f^1_{\mu\nu}F_1(\xi)+f^2_{\mu\nu}F_2(\xi)]\equiv a f^\lambda_{\mu\nu}F_\lambda(\xi),$$

where

$$f^\lambda_{\mu\nu}=(n_\mu\epsilon^\lambda_\nu-n_\nu\epsilon^\lambda_\mu),\quad F_\lambda(\xi)=\frac{dA_\lambda(\xi)}{d\xi},$$

and

$$n^\mu f^\lambda_{\mu\nu}=0,\quad n^\mu F_{\mu\nu}=0=n^\nu F_{\mu\nu},\quad f^\lambda_{\mu\rho}f^{\lambda'}_{\rho\nu}=-n_\mu n_\nu\delta_{\lambda\lambda'}.$$

With this choice of potential, however, our eikonal approximation, characterized by the above-mentioned condition,

$$\left(\frac{1}{i}\partial_\mu\right)^2=0=eA_\mu(x)\partial^\mu,$$

is turned into an exact propagator theory. This can be seen most easily by looking at our basic equation before any functional derivatives have been introduced:

$$\left[m^2+\left(p+\frac{1}{i}\partial-eA\right)^2-\tfrac{1}{2}e\sigma F\right]g(x,p\,|\,A)=1,$$

$$A_\mu(x)=e^1_\mu e^{ikx}+e^2_\mu e^{-ikx},\quad k^2=0=e^\lambda_\mu k^\mu.$$

The consequences of translational invariance can be seen from

$$g(x,p\,|\,A)=e^{-ipx}\Delta(x,p\,|\,A)=e^{-ipx}\int dy\,e^{ipy}\Delta(x,y\,|\,A),$$

where

$$\Delta(x,y\,|\,e^1_\mu,e^2_\mu)\equiv\Delta(x+h,y+h\,|\,e^1_\mu e^{-ikh},e^2_\mu e^{ikh}).$$

If we choose $h=-y$, we obtain on the right-hand side

$$\Delta(x-y,0\,|\,e^1_\mu e^{iky},e^2_\mu e^{-iky}).$$

The corresponding Fourier representation reads

$$\Delta(x,y\,|\,A)=\int\frac{dp}{(2\pi)^4}\,e^{ip(x-y)}\Delta(p\,|\,A)\Big|_{e^1_\mu\to e^1_\mu e^{iky};\,e^2_\mu\to e^2_\mu e^{-iky}}.$$

Now we can perform the derivative $(\partial/\partial x)g(x,p\,|\,A)$, which can be converted into

$$\begin{aligned}\frac{\partial}{\partial x}g(x,p\,|\,A)&=e^{-ipx}\int dy\,e^{ipy}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\Delta(x,y\,|\,A)\\&=e^{-ipx}\int dy\,e^{ipy}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\left[\int\frac{dq}{(2\pi)^4}\,e^{iq(x-y)}\Delta(q\,|\,A)\Big|_{e^1_\mu\to e^1_\mu e^{iky};\,e^2_\mu\to e^2_\mu e^{-iky}}\right]\\&=e^{-ipx}\int dy\,e^{ipy}\left[\int\frac{dq}{(2\pi)^4}\,e^{iq(x-y)}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\Delta(q\,|\,A)\Big|_{e^1_\mu\to e^1_\mu e^{iky};\,e^2_\mu\to e^2_\mu e^{-iky}}\right].\end{aligned}$$

The remaining differentiation acts upon the functional argument of $\Delta[A]$, whose dependence on x and y appears in the form kx and ky. Therefore we can replace

$$\left(\frac{\partial}{\partial x^\mu}+\frac{\partial}{\partial y^\mu}\right)\to k_\mu\left(\frac{\partial}{\partial(kx)}+\frac{\partial}{\partial(ky)}\right),$$

and consequently $[(1/i)\partial_x]^2g=0$, since $k^2=0$; furthermore

$$eA_\mu\partial^\mu g=eA_\mu k^\mu\left(\frac{\partial}{\partial(kx)}+\frac{\partial}{\partial(ky)}\right)g=0,$$

since $e^\lambda_\mu k^\mu=0$. Hence, without any approximations

$$\left[p^2+m^2+2p\left(\frac{1}{i}\partial_x-eA(x)\right)+e^2A^2(x)-\tfrac{1}{2}e\sigma F(x)\right]g(x,p\,|\,A)=1.$$

This type of equation holds true for any kind of vector potential which can be represented by a transverse field,

$$A_\mu(x)=e^1_\mu\left(k,\,i\frac{\partial}{\partial k}\right)e^{ikx}+e^2_\mu\left(k,\,i\frac{\partial}{\partial k}\right)e^{-ikx},\qquad k^2=0=e^\lambda_\mu k^\mu .$$

For instance,

$$\text{(a)}\quad e^1_\mu\left(k,\,i\frac{\partial}{\partial k}\right)=\tfrac{1}{2}\epsilon_\mu(k)\left[1+\exp\left(i2ki\frac{\partial}{\partial k}\right)\right],\qquad e^2_\mu=0$$

yields

$$A_\mu(x)=\epsilon_\mu(k)\cos kx ,$$

or

$$\text{(b)}\quad e^1_\mu\left(k,\,i\frac{\partial}{\partial k}\right)=\tfrac{1}{2}\epsilon^1_\mu(k)\left[1+\exp\left(i2ki\frac{\partial}{\partial k}\right)\right],$$

$$e^2_\mu\left(k,\,i\frac{\partial}{\partial k}\right)=\frac{1}{2i}\,\epsilon^2_\mu(k)\left[1-\exp\left(i2ki\frac{\partial}{\partial k}\right)\right]$$

produces

$$A_\mu(x)=\epsilon^1_\mu(k)\cos kx-\epsilon^2_\mu(k)\sin kx .$$

These are examples of linearly and circularly polarized plane waves.

The introduction of two helicity possibilities for λ leaves open the choice between linearly and circularly polarized light. Apart from the A^2 term, the relevant structure of Eq. (3.6) is exhibited in

$$\exp\left[e\int\frac{dq}{(2\pi)^4}\,\frac{2p^\nu-\gamma^\mu\gamma^\nu q_\mu}{2pq}\,A_\nu(q)e^{iqx}\right]\left[p^2+m^2+\int dk\,2pkA(k)\frac{\delta}{\delta A(k)}\right]\times\exp\left[-e\int\frac{dq}{(2\pi)^4}\,\frac{2p^\nu-\gamma^\mu\gamma^\nu q_\mu}{2pq}\,A_\nu(q)e^{iqx}\right]g=1 . \tag{3.8}$$

Equation (3.8) can be rearranged with the result

$$g^{\text{eik}}(x',p\,|\,A)=i\int_0^\infty d\alpha\,\exp[-i\alpha(p^2+m^2-i\epsilon)]\exp\left[ie\int_0^\alpha d\alpha'\left(2p^\nu+i\gamma^\mu\gamma^\nu\frac{\partial}{\partial x'^\mu}\right)A_\nu(x'-2\alpha'p)\right], \tag{3.9}$$

where use has been made of the formula

$$\exp\left(\int\phi(k)A(k)\frac{\delta}{\delta A(k)}\right)\exp\left(\int\chi(k)A(k)\right)=\exp\left(\int\chi(k)e^{\phi(k)}A(k)\right)$$

and the particular choice of $A_\mu(x)$ as given by (3.7). In terms of the outgoing variable we also can write

$$g^{\text{eik}}(q,x''\,|\,A)=i\int_0^\alpha d\alpha\,\exp[-i\alpha(q^2+m^2-i\epsilon)]\exp\left[i\int_0^\infty d\alpha'[2eqA(x''+2\alpha'q)-e^2A^2(x''+2\alpha'q)+\tfrac{1}{2}e\sigma F(x''+2\alpha'q)]\right]. \tag{3.10}$$

Here we picked up the A^2 term again. Finally the eikonalized propagator reads

$$\begin{aligned}\Delta^{\text{eik}}(x',x''\,|\,A)&=\int\frac{dq}{(2\pi)^4}\,e^{iq(x'-x'')}i\int_0^\infty ds\,e^{-i(q^2+m^2)s}\\&\quad\times\exp\left(i\int_0^s ds'[2eqA(x''+2s'q)-e^2A^2(x''+2s'q)-\tfrac{1}{2}e\sigma F(x''+2s'q)]\right)\\&=\int\frac{dq}{(2\pi)^4}\,e^{iq(x'-x'')}i\int_0^\infty ds\,\exp\left(-i\int_0^s ds'[m^2+\Pi^2(s')-\tfrac{1}{2}e\sigma F(x''+2s'q)]\right),\\&\qquad\Pi^2(s')=[q-eA(x''+2s'q)]^2\\&=i\int_0^\infty ds\,e^{-im^2s}\langle x(s)'\,|\,x(0)''\rangle ,\end{aligned} \tag{3.11}$$

where

$$\langle x(s)'|x(0)''\rangle = \langle x'|e^{-i\mathcal{H}s}|x''\rangle = \int \frac{dq}{(2\pi)^4} e^{iq(x'-x'')} \exp\left(i\int_0^s ds'[\gamma\Pi(s')]^2\right) . \tag{3.12}$$

IV. VARIABLE TRANSFORMATIONS ON $\langle x(s)'|x(0)''\rangle$

It is now our intention to show that Schwinger's proper-time technique is equivalent to our procedure. Armed with the results of the last section, we just have to perform some changes of variables in

$$\langle x(s)'|x(0)''\rangle = \int \frac{dq}{(2\pi)^4} e^{iq(x'-x'')} \exp\left(-i\int_0^s ds'[q^2 - 2eqA(x''+2s'q) + e^2A^2(x''+2s'q) - \tfrac{1}{2}e\sigma F(x''+2s'q)]\right) .$$

As a first step toward the reconstruction of Schwinger's result, let us look at the various terms separately:

(1) $$\exp\left(-i\int_0^s ds'(-2e)qA(x''+2s'q)\right) = \exp\left(2ie\int_0^s ds' q^\mu a\epsilon_\mu^\lambda A_\lambda(\xi''+2s'nq)\right) .$$

Introducing a new variable $\xi = \xi'' + 2s'nq$, and performing a shift in the q variable thereafter, $q \to q' = q - (x'-x'')/2s$, we obtain

$$\exp\left(iea\frac{2sq\epsilon^\lambda + \epsilon^\lambda(x'-x'')}{2sqn + \xi' - \xi''}\int_{\xi''}^{\xi'+2snq} d\xi\, A_\lambda(\xi)\right), \quad \xi' = nx'; \quad \xi'' = nx'' .$$

The same choice of variable transformations leads to

(2) $$\exp\left(-ie^2\int_0^s ds' A^2(x''+2s'q)\right) = \exp\left(-i(ea)^2\frac{s}{2snq + \xi' - \xi''}\int_{\xi''}^{\xi'+2snq} d\xi \sum_\lambda A_\lambda{}^2(\xi)\right) .$$

In order to transform the spin-dependent term, it is convenient to recall the relation

$$\frac{d}{ds'}A_\lambda(\xi''+2s'nq) = 2nq\frac{\partial}{\partial\xi''}A_\lambda(\xi''+2s'nq) .$$

This can be used to show that

(3) $$\exp\left(-e\gamma^\mu\gamma^\nu\int_0^s ds'\partial_\mu A_\nu(x''+2s'q)\right) = \exp\left(-ea\frac{(\gamma n)(\gamma\epsilon^\lambda)s}{2snq + \xi' - \xi''}[A_\lambda(\xi'+2snq) - A_\lambda(\xi'')]\right) .$$

The next stage is reached by using a new coordinate system based on the null vector n, i.e.,

$$q = q' + \beta n, \quad nq' = 0 .$$

This transformation implies a simplification, namely,

(1′) $$\exp\left(iea\frac{2sq\epsilon^\lambda + \epsilon^\lambda(x'-x'')}{\xi' - \xi''}\int_{\xi''}^{\xi'} d\xi\, A_\lambda(\xi)\right),$$

(2′) $$\exp\left(-i(ea)^2\frac{s}{\xi' - \xi''}\int_{\xi''}^{\xi'} d\xi\sum_\lambda A_\lambda{}^2(\xi)\right),$$

(3′) $$\exp\left(ea\frac{(\gamma\epsilon^\lambda)(\gamma n)}{\xi' - \xi''}s[A_\lambda(\xi') - A_\lambda(\xi'')]\right) = \exp\left(\tfrac{1}{2}isae\sigma f^\lambda\frac{1}{\xi' - \xi''}\int_{\xi''}^{\xi'} F_\lambda(\xi)d\xi\right),$$

where the result for the spin-dependent part was derived by employing the decomposition

$$\gamma^\mu\gamma^\nu = -g^{\mu\nu} - i\sigma^{\mu\nu}$$

and using the antisymmetry of $f_{\mu\nu}^\lambda$. We also introduced

$$A_\lambda(\xi') - A_\lambda(\xi'') = \int_{\xi''}^{\xi'}\frac{dA_\lambda(\xi)}{d\xi}d\xi = \int_{\xi''}^{\xi'} F_\lambda(\xi)d\xi .$$

Incidentally, when the remaining q terms in $\langle x(s)'|x(0)''\rangle$ are subjected to the q shift, we simply get

$$\int\frac{dq}{(2\pi)^4}e^{iq(x'-x'')}e^{-iq^2s} \to \int\frac{dq}{2\pi^4}e^{-iq^2s}\exp\left[i\frac{(x'-x'')^2}{4s}\right].$$

Accordingly, we have produced

$$\langle x(s)'|x(0)''\rangle = \int \frac{dq}{(2\pi)^4}\exp(-iq^2s)\exp\left(i\frac{(x'-x'')^2}{4s}\right)\exp\left[iea\frac{2sq\epsilon^\lambda+\epsilon^\lambda(x'-x'')}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi\, A_\lambda(\xi)\right]$$
$$\times\exp\left(-i(ea)^2\frac{s}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi\sum_\lambda A_\lambda{}^2(\xi)\right)\exp\left(\tfrac{1}{2}isae\sigma f^\lambda\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}F_\lambda(\xi)d\xi\right).$$

A final change of variables

$$q'=q-ea\frac{\epsilon^{\lambda'}}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi\, A_\lambda(\xi),$$

and use of

$$\int\frac{dq}{(2\pi)^4}e^{-iq^2s}=\frac{1}{(4\pi)^2}\frac{1}{is^2},$$

then leads us to the following form for the transformation function:

$$\langle x(s)'|x(0)''\rangle=\frac{1}{(4\pi)^2}\frac{1}{is^2}\exp\left(i\frac{(x'-x'')^2}{4s}\right)\exp\left[i(ea)^2s\frac{1}{(\xi'-\xi'')^2}\sum_\lambda\left(\int_{\xi''}^{\xi'}d\xi\, A_\lambda(\xi)\right)^2\right]$$
$$\times\exp\left(-i(ea)^2s\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi\sum_\lambda A_\lambda{}^2(\xi)\right)\exp\left(\tfrac{1}{2}isae\sigma f^\lambda\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi\, F_\lambda(\xi)\right)\exp\left(ie\int_{x''}^{x'}d\bar{x}^\mu A_\mu(\bar{x})\right).$$

Our eikonal propagator then takes the form

$$\Delta^{\text{eik}}(x',x''|A)=i\int_0^\infty ds\, e^{-im^2s}\langle x(s)'|x(0)''\rangle$$
$$=i\int_0^\infty ds\, e^{-im^2s}\frac{1}{i(4\pi)^2s^2}\exp\left(i\frac{(x'-x'')^2}{4s}\right)\exp\left(-i(ea)^2s\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi\sum_\lambda A_\lambda{}^2(\xi)\right)$$
$$\times\exp\left[i(ea)^2s\sum_\lambda\left(\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}A_\lambda(\xi)d\xi\right)^2\right]$$
$$\times\exp\left[\tfrac{1}{2}isae\sigma f^\lambda\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}F_\lambda(\xi)d\xi\right]\exp\left(ie\int_{x''}^{x'}d\bar{x}^\mu A_\mu(\bar{x})\right)$$
$$=\frac{1}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\exp\left[i\left(\frac{(x'-x'')^2}{4s}-[m^2+e^2\delta m^2(A_\lambda)]s\right)\right]$$
$$\times\exp\left[\tfrac{1}{2}isae\sigma f^\lambda\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}F_\lambda(\xi)d\xi\right]\exp\left(ie\int_{x''}^{x'}d\bar{x}^\mu A_\mu(\bar{x})\right),$$

where

$$\delta m^2(A_\lambda)=\frac{a^2}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi\, A_\lambda(\xi)A^\lambda(\xi)-\frac{a^2}{(\xi'-\xi'')^2}\int_{\xi''}^{\xi'}d\xi\, A_\lambda(\xi)\int_{\xi''}^{\xi'}d\eta\, A^\lambda(\eta)\geq 0.$$

If we define

$$M_\lambda=\frac{ea}{2}\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi\, F_\lambda(\xi),$$

we can show, with the aid of the relation

$$\tfrac{1}{2}\{\sigma_{\mu\nu},\sigma_{\lambda\kappa}\}=\delta_{\mu\lambda}\delta_{\nu\kappa}-\delta_{\mu\kappa}\delta_{\nu\lambda}+i\epsilon_{\mu\nu\lambda\kappa}\gamma_5,$$

that

$$\exp(isM_\lambda\sigma^{\mu\nu}f^\lambda_{\mu\nu})=1+isM_\lambda\sigma^{\mu\nu}f^\lambda_{\mu\nu}.$$

Therefore our final answer is

$$\Delta^{\text{eik}}(x',x''|A)=\frac{1}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\exp\left[i\left(\frac{(x'-x'')^2}{4s}-(m^2+e^2\delta m^2)s\right)\right]$$
$$\times\left[1+\tfrac{1}{2}isae\frac{1}{\xi'-\xi''}\int_{\xi''}^{\xi'}d\xi\,\sigma f^\lambda F_\lambda(\xi)\right]\exp\left(ie\int_{x''}^{x'}d\bar{x}^\mu A_\mu(\bar{x})\right)$$

$$= \exp\left(ie\int_{x''}^{x'} d\bar{x}^{\mu} A_{\mu}(\bar{x}) \right) \Delta'^{\text{eik}}(x', x'' \,|\, A_{\mu} = a\epsilon_{\mu}^{\lambda} A_{\lambda}(\xi)),$$

where

$$\Delta'^{\text{eik}}(x', x'' \,|\, a\epsilon_{\mu}^{\lambda} A_{\lambda}(\xi)) = \frac{1}{(4\pi)^2}\int_0^{\infty} \frac{ds}{s^2} \exp\left[i\left(\frac{(x'-x'')^2}{4s} - (m^2 + e^2\delta m^2)s \right)\right]\left[1 + is\,\frac{1}{\xi' - \xi''}\int_{\xi''}^{\xi'} d\xi\, \tfrac{1}{2} e\sigma a f^{\lambda} F_{\lambda}(\xi)\right],$$

which is precisely Schwinger's result, except that we still have the option of summing over the polarization indices $\lambda = 1, 2$.

Note also that, since $nA = 0$, the quantity δm^2 is invariant under gauge transformations

$$A_{\mu}(\xi) \to A_{\mu}(\xi) + \partial_{\mu}\lambda(\xi) = A_{\mu}(\xi) + n_{\mu}\frac{d}{d\xi}\lambda(\xi).$$

So indeed, the entire gauge sensitivity of $\Delta^{\text{eik}}(x', x'' \,|\, A)$ is contained in

$$\phi(x', x'') = ie\int_{x''}^{x'} d\bar{x}^{\mu} A_{\mu}(\bar{x}).$$

It is worth mentioning that the author's results on intense field QED, Ref. 4, can be calculated immediately by assuming the polarization indices to be $\lambda = +, -$, i.e., by introducing circularly polarized light:

$$A_{\mu}(x) = 2a\,\text{Re}\{\epsilon_{\mu} e^{ikx}\}.$$

In that article we analyzed the electron's Green's function in the presence of a circularly polarized laser beam. It was shown that the electron, while traveling in the laser field, experiences a mass shift $\delta m^2 = 2e^2a^2$, which is constant. In the present situation δm^2 is not constant, nor is the laser light circularly polarized. Therefore we conclude that, independent of the kind of polarization, there will be a mass shift $\delta m^2(A_{\lambda})$; however, more significantly, the choice of a laser of arbitrary strength and spectral composition as external field makes it possible to reduce the original eikonal *approximation* to an exact propagator theory.

V. CONCLUSION

The complementarity of proper-time and eikonal methods was displayed in the text. We reviewed Schwinger's work and developed an eikonalization procedure thereafter to obtain a closed-form expression for the electron's Green's function in the presence of a laser field. This particular choice for the external field made it possible to convert the original approximate eikonal propagator into an exact one.

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