Scalar-Tensor Theory and General Relativity

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The various versions of the scalar-tensor theory (e.g., the theories of Jordan, Hoyle, and Brans-Dicke) are derived from a general variational principle. It is shown that scalar-conformal transformations not only interconvert the various current versions of the scalar-tensor theory (i.e., Brans-Dicke theory ⇄ Hoyle steady-state theory), but also convert the scalar-tensor variational principle into the variational principle of general relativity. The scalar-tensor formalism is therefore implicitly embodied in the theory of general relativity, thus illustrating the considerable freedom available in specifying the nature and physical content of the "matter tensor" in the Einstein equation.

Various versions of the scalar-tensor theory of the gravitational field have been suggested and widely discussed. The theory is attractive to many mainly because fundamental physical quantities become space-time dependent. The scalar-tensor theory, as first proposed by Jordan,[1] was inspired by Dirac's principle.[2] One interpretation of this principle, adopted by Jordan, is that the gravitational constant varies with time. The Brans-Dicke[3] theory, motivated by similar considerations,[4] was inspired by the principles of both Mach and Dirac. The steady-state theory of the universe, proposed by Bondi and Gold,[5] also conforms to the Mach and Dirac principles and was formulated by Hoyle[6] as a version of the scalar-tensor theory. The Mach and Dirac principles, insofar as we understand them, can be expressed in crude form by the relation

$$GNm \propto \text{size of observable universe.} \qquad (1)$$

According to this cosmological "law" the gravitational constant G, the total number of particles N, and the particle mass[7] m, either separately or in ∞^2 possible combinations, are time-varying as the universe expands.

The scalar-tensor theories of Jordan, Hoyle, and Brans-Dicke all conform to the Mach-Dirac law (1). The basic physics involved in these and other scalar-tensor theories is still obscure, and except by appeal to observation we are at present unable to determine which, if any, of the ∞^2 possibilities represents the physical world. We show below that all versions of the general scalar-tensor variational principle are conformally equivalent to each other and to the Einstein equation, and as a consequence the law (1) and similar laws are embodied in the theory of general relativity.

In the various versions of the scalar-tensor theory the equations of motion and the field equations are derived from the general variational principle

$$\delta\int (R\psi^A + \lambda\psi^i\psi_i\psi^{A-2} + \kappa\bar{L}\psi^B)(-g)^{1/2}d^4x = 0 \ , \qquad (2)$$

where R is the scalar curvature, ψ is the scalar function (where $\psi_i \equiv \partial\psi/\partial x^i$), λ and κ are coupling constants, and A and B are constants assigned by different authors[1,3,4,7-14] (values of A and B are displayed in Table I). The matter-Lagrangian density $\bar{L}$ in the various versions of the theory is often expressed as a density of particles moving along trajectories $x^i = z_p^i(\tau_p)$:

$$\bar{L}(x) = 2(-g)^{1/2}\sum_p \int m_p[g_{ik}(x)\dot{z}^i\dot{z}^k]^{1/2}\delta^4(x - z_p)d\tau_p \ , \qquad (3)$$

where $\dot{z}^i = dz_p^i/d\tau_p$ and τ_p is a path parameter of the pth particle. The inclusion of electromagnetic interactions introduces only slight modifications. Equation (2) is invariant under coordinate transformations but not under conformal transformations.[7,11,15,16] Applying in succession to (2) the scalar and conformal transformations

$$\psi \to \phi = \psi^{-\mu} \ , \qquad (4)$$

$$g_{ik} \to g'_{ik} = \phi^F g_{ik} \ , \qquad (5)$$

and also using[17]

$$\bar{L}(g_{ik}) \to L'(g'_{ik}) = \phi^{-(3/2)F-H}\,\bar{L}(g_{ik}) \ , \qquad (6)$$

in which particle masses transform as[18]

$$m(g_{ik}) \to m'(g'_{ik}) = \phi^{-H}m(g_{ik}) \ , \qquad (7)$$

gives, after discarding a divergence term,

$$\delta\int (R'\phi^C + \lambda'\phi^i\phi_i\phi^{C-2} + \kappa L'\phi^D)(-g')^{1/2}d^4x = 0 \ , \qquad (8)$$

in which

TABLE I. Values of constants A and B used by authors of different versions of the scalar-tensor theory.

References	A	B
8	2	$\bar{L}=0$
9, 10, 11	2	0
1, 12	1	2
13	1	1
3, 4	1	0
7	0	0
14	−1	0

$$C=\mu A-F\ ,$$
$$D=\mu B+H-\tfrac{1}{2}F\ , \qquad (9)$$
$$\lambda'=\mu^2\lambda+3F(\mu A-\tfrac{1}{2}F)\ .$$

In the variational equation (8) uncharged particles follow geodesics when $H=0$; world lines intersecting a hypersurface are conserved (i.e., no creation) when $\mu B=\frac{1}{2}F$; and the gravitational "constant" is then constant when $\mu A=F$. Equations (2) and (8) are identical in form, and it is evident that an appropriate choice in the parameters μ, F, and H interconverts any two versions of the scalar-tensor theory. The arbitrariness in the value of the coupling constant, owing to the scalar transformation (4), means that there are only two disposable constants, A and B (or C and D), thus affording the ∞^2 versions required by (1).

The conformal equivalence of scalar-tensor models of the universe is illustrated by the interconversion of the Brans-Dicke ($A=1$, $B=0$) and the continuous-creation models. Particle masses are constant in both, and the transformation $\mu=\frac{1}{2}F$, $H=0$ converts the Brans-Dicke model of $\lambda=-2$ into a steady-state continuous creation model[13] ($C=D$, $\lambda'=-2C^2$). Other values of the Brans-Dicke coupling constant correspond to continuous-creation models not in a steady state. Similarly the Jordan[15] ($A=-1$, $B=0$) models convert to Brans-Dicke models with $\mu=-1$, $F=0$, and become the steady-state model with $\lambda=-2$, $\mu=-\frac{1}{2}F$.

Independent variations in g'_{ik} and ϕ of (8) give the field equations

$$(R_k'^{\,j}-\tfrac{1}{2}\delta_k^j R')\phi^C+D_k^j(\phi)+\kappa T_k'^{\,j}\phi^D=0\ , \qquad (10)$$

$$CR'\phi^C-\lambda'[2\phi\Box'\phi-(2-C)\phi^i\phi_i]\phi^{C-2}+\kappa DL'\phi^D=0\ , \qquad (11)$$

in which[19]

$$D_k^j(\phi)=(\phi^C)_k^j-\delta_k^j\Box'\phi^C+\lambda'(\phi^j\phi_k-\tfrac{1}{2}\delta_k^j\phi^i\phi_i)\phi^{C-2}\ ,$$

and $T_k'^{\,j}$ is the energy-momentum tensor. From (10) and (11) it follows (pressure = 0) that

$$(3+2\lambda'/C^2)\Box'\phi^C-\kappa T'(1+2D/C)\phi^D=0\ , \qquad (12)$$

$$(T_k'^{\,j}\phi^D)_{;j}+T'\phi_k^D=0\ . \qquad (13)$$

The equations of general relativity are derived from the variational principle

$$\delta\int(R+\kappa L)(-g)^{1/2}\,d^4x=0\ , \qquad (14)$$

where L is a function of g_{ik} and any number of scalar functions $\psi_{(1)}, \psi_{(2)}, \ldots, \psi_{(q)}, \ldots$ and their first derivatives.[20] From (14) we obtain the Einstein equation

$$R_{ik}-\tfrac{1}{2}g_{ik}R=-\kappa T_{ik}\ , \qquad (15)$$

where

$$T_{ik}=(-g)^{-1/2}\,\frac{\partial((-g)^{1/2}L)}{\partial g^{ik}},\quad T_{k;j}^j=0\ , \qquad (16)$$

$$(-g)^{-1/2}\left((-g)^{1/2}\frac{\partial L}{\partial\psi_{(q)i}}\right)_i-\frac{\partial L}{\partial\psi_{(q)}}=0\ . \qquad (17)$$

For our purpose a suitable and relatively simple Lagrangian density of matter, incorporating a single scalar function ψ, is

$$L=\lambda\psi^i\psi_i\psi^{-2}+\kappa\bar{L}\psi^B\ , \qquad (18)$$

where $\bar{L}$ is a function only of g_{ik}, and B is an arbitrary constant. By comparing (2) and (14), and using the special form of L in (18), it is seen that the transformations (4)–(7) convert the Einstein equations (15)–(17) into the scalar-tensor equations (10)–(13).

If follows that the physically inequivalent but conformally equivalent versions of the scalar-tensor formalism are all implicitly embodied in the field equations of general relativity. They constitute in fact a limited and particular class of equations that derive from general relativity and are of lesser generality. In the Einstein equation (15) geometry is coupled to matter and imposes minimal physical constraint on the nature of matter apart from the zero divergence of the "matter tensor." According to this interpretation the Einstein equation represents a diversity of universes whose range in physical properties is quite capable of accommodating the modest requirements of the Mach-Dirac law. The elementary energy-momentum tensor that is commonly used is not an essential feature of general relativity. McCrea[21] has shown, for example, that a matter tensor containing a negative stress term not only accounts for the continuous-creation universes[22] but also offers physical insight into the creation process. The actual nature of the matter tensor in our universe must be determined either by observation and experiment or by appeal to the theories of other branches of physics. The virtue of the scalar-tensor formalism is that it displays explicitly in differential equations our implicit assumptions concerning the content of T_{ik}. It is a debatable matter, however, whether these equations provide physical understanding of the Mach-Dirac law.

The transformations we have discussed are simple, but the physical meaning of the transformations is evidently far from trivial. The scalar-tensor formalism offers us an overwhelming wealth of possible universes.[23] By recognizing that the formalism is not a fundamentally new departure from the Einstein equation, and by paying attention to the physical nature of the matter tensor, it is possible that we may eventually single out an acceptable and convincing model of the universe.

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[13]The steady-state cosmological model comes from the substitution of $C=D$, $\lambda' = -2C^2$ in (10)–(13), using a Robertson-Walker metric of zero curvature constant. See Ref. 6.

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[17]Maxwell's equations are conformally invariant, and the addition of electromagnetic interactions introduces a term in the matter-Lagrangian density of $H=0$. The omission of this term does not affect the general nature of the argument.

[18]A time variation in m produces a nonrecessional cosmological red shift which might be difficult to identify observationally (assuming $\hbar$ constant). Dicke (Ref. 7) uses $H=\frac{1}{2}$.

[19]$D_k^i(\phi)$ is the generalized covariant stress tensor (of $C=2$, $\lambda' = -\frac{3}{2}C^2$) adopted by C. G. Callan, S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).

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[23]The "trembling universe" is an example of the rich choice available. From (10)–(13), using a Robertson-Walker metric, we find that it is possible to have a finite amplitude fluctuation superimposed on the general expansion of the universe induced by a fluctuating creation rate. Such a model might conceivably be used to explain the clustering of quasar red shifts.