frame and the π^+ momentum in the dipion rest frame. As can be seen from the figures, this model gives essentially the same curves as Moffat's model, and we would expect that in other reactions we would also be able to get reasonable results.

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Calculation of the Second-Order Weak Amplitude for $K_2 \to \pi^0 e^+ e^-$*

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The second-order weak amplitude for $K_2 \to \pi^0 e^+ e^-$ has a finite imaginary part arising from the presence of a $\pi\nu$ intermediate state in the channels $K_2 + e^\pm \to \pi^0 + e^\pm$, which we calculate in terms of the amplitudes for $K_2 \to \pi^\pm e^\mp \nu$ and $\pi^\pm \to \pi^0 e^\pm \nu$. The real part is determined by means of a dispersion relation, and the result converges for a simple choice of the K_{2e3} and π_{e3} form factors.

A major unknown in weak-interaction physics is the magnitude and structure of amplitudes that are second-order in the Fermi constant G. In a recent survey of the experimental possibilities in this field,[1] the decay $K_2 \to \pi^0 e^+ e^-$ has been spotlighted as a promising reaction in which to look for such higher-order effects. We describe here a calculation of the second-order weak amplitude for $K_2 \to \pi^0 e^+ e^-$. In the limit in which the effects of strong interaction are ignored, the amplitude diverges logarithmically. Inclusion of the strong-interaction effects in a reasonable way leads to a finite result.

We denote the momenta of the reaction by

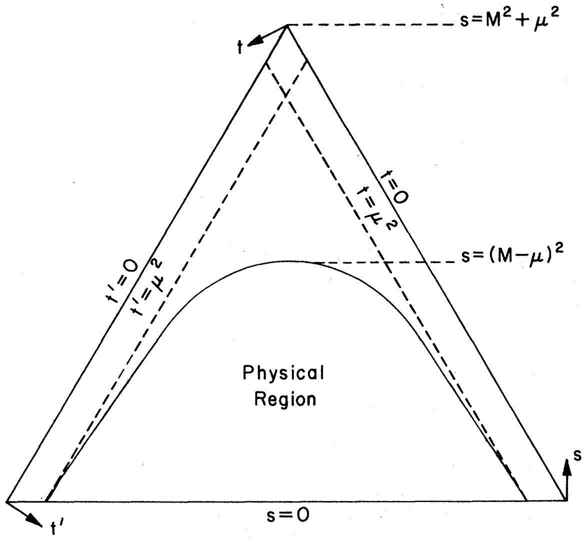

FIG. 1. Dalitz plot for $K_2 \to \pi^0 e^+ e^-$.

$$K_2(Q) \to \pi^0(p) + e^-(k) + e^+(k'),$$

and define the three invariants

$$s=(Q-p)^2, \quad t=(Q-k)^2, \quad t'=(Q-k')^2, \qquad (1)$$
$$s+t+t' = M^2+\mu^2,$$

where M and μ are the masses of the K_2 and π^0, and the mass of the electron is neglected. The physical region of these invariants is shown in the Dalitz diagram of Fig. 1, and is bounded by the curves

$$s=0, \qquad (2)$$
$$(t'-t)^2 = (M^2+\mu^2-s)^2 - 4M^2\mu^2.$$

It is convenient to define two further quantities

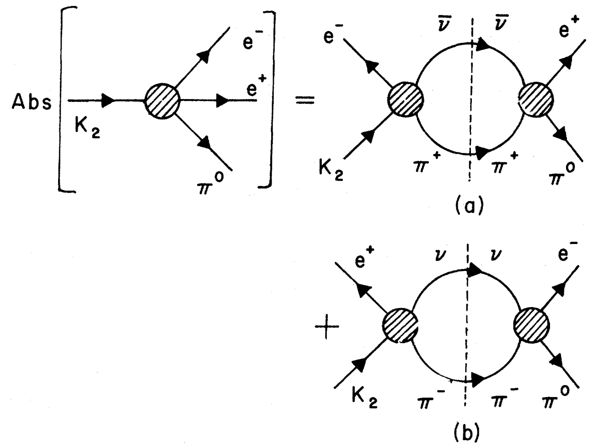

FIG. 2. Unitarity relation for the absorptive part of the $K_2 \to \pi^0 e^+ e^-$ amplitude.

$$\Delta = t' - t, \qquad (3)$$
$$w = t' + t = M^2 + \mu^2 - s,$$

which are related in a simple way to the horizontal and vertical coordinates of the Dalitz plot.

The most general form of the second-order weak amplitude for $K_2 \to \pi^0 e^+ e^-$, in the limit of zero electron mass, is[2]

$$\mathfrak{M} = F(w, \Delta)\bar{u}(k)\not{Q}(1+\gamma_5)v(k'). \qquad (4)$$

CP invariance requires that $F(w,\Delta)$ be an odd function of Δ, so that the amplitude vanishes along the vertical axis $t=t'$. To calculate the form factor $F(w,\Delta)$, we observe that the amplitude $\mathfrak{M}$ possesses a nonvanishing absorptive part arising from the presence of a $\pi^+\bar{\nu}_e$ ($\pi^-\nu_e$) intermediate state in the t (t') channel for values of the invariant t (t') $> \mu^2$. This absorptive part can be calculated by means of a unitarity relation (Fig. 2),

$$\mathrm{Im}\mathfrak{M} = \tfrac{1}{2}\sum \langle \pi^+(q)\,\bar{v}(l)\,|\mathfrak{M}|\,\pi^0(p)e^+(k')\rangle^* \langle \pi^+(q)\bar{v}(l)\,|\mathfrak{M}|\,K_2(Q)e^+(-k)\rangle$$
$$\times\left(\frac{m_\nu}{2q_0 l_0}\right)(2\pi)^4\delta^4(Q-k-q-l) + \text{crossed term}, \qquad (5)$$

where we have written the contribution of graph (a) and the crossed term is the contribution of graph (b). In Eq. (5), q and l denote the momenta of the intermediate π and ν and $\sum$ stands for a sum over the intermediate spin and an integration over the intermediate momenta.[3] The first-order ampltitudes may be parametrized as

$$\langle \pi^+(q)\bar{v}(l)\,|\mathfrak{M}|\,K_2(Q)e^+(-k)\rangle = -\tfrac{1}{2}G\sin\theta f_+(R^2)(Q+q)_\mu \bar{u}(k)\gamma_\mu(1+\gamma_5)v(l), \qquad (6)$$
$$\langle \pi^+(q)\bar{v}(l)\,|\mathfrak{M}|\,\pi^0(p)e^+(k')\rangle = G\cos\theta f(N^2)(p+q)_\mu \bar{v}(k')\gamma_\mu(1+\gamma_5)v(l),$$

where $R=Q-q$, $N=p-q$, and θ is the Cabibbo angle. The form factor f is normalized to $f(0)=1$ (by the isotriplet vector current hypothesis) while f_+ is normalized to $f_+(0)\approx 1$ [the deviation from unity being second order in SU(3) symmetry breaking]. Carrying out the spin summation, we obtain

$$\mathrm{Im}\mathfrak{M} = -\frac{G^2}{32\pi^2}\cos\theta\sin\theta\int\frac{d^3q\,d^3l}{q_0 l_0}f_+(R^2)f(N^2)\bar{u}(k)\Gamma(1+\gamma_5)v(k')\theta(t-\mu^2) + \text{crossed term,} \tag{7}$$

where

$$\Gamma = \not{Q}\not{l}\not{p} + \not{Q}\not{l}\not{q} + \not{q}\not{l}\not{p} - \mu^2\not{l} + 2l\cdot q\not{q}\,. \tag{8}$$

We examine first the situation when the form factors f_+ and f are taken to be constant, that is, when the strong-interaction effects are neglected. The integral in Eq. (7) is then equal to

$$[4\pi(t-\mu^2)^2/t]\theta(t-\mu^2)\bar{u}\not{Q}(1+\gamma_5)v\,,$$

and we obtain for the imaginary part of $F(\omega, \Delta)$ the simple form

$$\mathrm{Im}F(w,\Delta) = -\frac{G^2}{8\pi}\cos\theta\sin\theta\left[\frac{(t-\mu^2)^2}{t}\theta(t-\mu^2) - \frac{(t'-\mu^2)^2}{t'}\theta(t'-\mu^2)\right] \tag{9}$$

[where, by definition, $t=\frac{1}{2}(w-\Delta)$, $t'=\frac{1}{2}(w+\Delta)$]. To obtain the real part of $F(w,\Delta)$, we write an unsubtracted dispersion relation in the variable s (or, equivalently, in w), keeping Δ fixed:

$$\mathrm{Re}F(w,\Delta) = -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\mathrm{Im}F(w',\Delta)dw'}{w'-w}\,. \tag{10}$$

An examination of the behavior of $\mathrm{Im}F(w,\Delta)$ in the limit $w\to\infty$ shows that the integral diverges logarithmically. Explicitly, we find

$$\mathrm{Re}F(w,\Delta) = \frac{1}{\pi}\left(\frac{G^2}{8\pi}\cos\theta\sin\theta\right)\left\{-\Delta\left[\ln\frac{\Lambda^2}{w} + 1 + \left(1-\frac{2\mu^4}{w^2-\Delta^2}\right)\ln\frac{w^2}{(w-2\mu^2)^2-\Delta^2}\right]\right.$$
$$\left. + \left(\frac{w}{2} - 2\mu^2 + \frac{2\mu^4 w}{w^2-\Delta^2}\right)\ln\frac{w-2\mu^2+\Delta}{w-2\mu^2-\Delta}\right\}, \tag{11}$$

where Λ^2 is a "cutoff." For small values of Δ, and in the limit $\mu^2=0$, the expression in curly brackets reduces to $-\Delta\ln(\Lambda^2/w)$.

Because of the mild nature of the divergence obtained above, one may expect that inclusion of the form factors $f_+(R^2)$ and $f(N^2)$ will lead to a convergent result. We are able to demonstrate this in the case where the form factors have a pole-type behavior, that is, when

$$f_+(R^2) = M_1^2/(M_1^2 - R^2), \quad f(N^2) = M_2^2/(M_2^2 - N^2), \tag{12}$$

where M_1 and M_2 are mass parameters (which could be of the order of the K^* and ρ-vector-meson masses, respectively). The integration in Eq. (7) now leads to the following result,[4] which is a simple modification of Eq. (9):

$$\mathrm{Im}F(w,\Delta) = -\frac{G^2}{8\pi}\cos\theta\sin\theta\left[\frac{(t-\mu^2)^2}{t}I(t)\theta(t-\mu^2) - \frac{(t'-\mu^2)^2}{t'}I(t')\theta(t'-\mu^2)\right], \tag{13}$$

where

$$I(t) = \int_{-1/2}^{+1/2}\frac{dy}{A+By+Cy^2}\,,$$
$$A = 1 + \frac{t-\mu^2}{2t}\left(\frac{t-\mu^2}{M_2^2} - \frac{t-M^2}{M_1^2}\right) - \frac{1}{4M_1^2M_2^2}\frac{(t-\mu^2)^2 s}{t}\,, \tag{14}$$
$$B = -\frac{t-\mu^2}{t}\left(\frac{t-\mu^2}{M_2^2} - \frac{t-M^2}{M_1^2}\right), \quad C = \frac{(t-\mu^2)^2 s}{tM_1^2M_2^2}\,.$$

In the limit $w\to\infty$, for fixed Δ, $I(t)$ goes to zero like $1/w^2$, thus showing that $\mathrm{Re}F(w,\Delta)$ derived from Eq. (13) converges.

For the purpose of estimating the decay rate, we work with the expressions for $\mathrm{Im}F(w,\Delta)$ and $\mathrm{Re}F(w,\Delta)$ given in Eqs. (9) and (11), using in the latter a value of Λ^2 equal to 0.7 GeV^2, which is the average of the $(\mathrm{mass})^2$ for the ρ and K^*. The precise choice of Λ^2 is unimportant for our conclusions, as the dependence of $\mathrm{Re}F(w,\Delta)$ on Λ^2 is weak, and, further, because the real part of the amplitude makes a minor contribution to the decay rate. To a good approximation, it is sufficient to keep in $F(w,\Delta)$ only the terms linear in Δ and to neglect those which vanish when $\mu^2=0$. We then have

$$\mathrm{Im}F(w,\Delta)\simeq\left(\frac{G^2}{8\pi}\cos\theta\sin\theta\right)\Delta\,,\quad \mathrm{Re}F(w,\Delta)\simeq-\frac{1}{\pi}\left(\frac{G^2}{8\pi}\cos\theta\sin\theta\right)\Delta\ln\frac{\Lambda^2}{w}\,, \tag{15}$$

and the decay rate is

$$\Gamma=\frac{1}{2^8\pi^3M^3}\left(\frac{G^2}{8\pi}\cos\theta\sin\theta\right)^2\int_{2M\mu}^{M^2+\mu^2}dw\left(1+\frac{1}{\pi^2}\ln^2\frac{\Lambda^2}{w}\right)\int_{-(w^2-4M^2\mu^2)^{1/2}}^{(w^2-4M^2\mu^2)^{1/2}}d\Delta\,\Delta^2(w^2-4M^2\mu^2-\Delta^2)$$

$$=(0.38\times10^{-3})\frac{M}{\pi}\left(\frac{GM^2}{4\pi}\right)^4\cos^2\theta\sin^2\theta=0.9\times10^{-8}\ \mathrm{sec}^{-1}\,, \tag{16}$$

where $\sin\theta$ has been taken as 0.2. The bulk of the decay rate is absorptive, the dispersive part being about 10%. The branching ratio $\Gamma(K_2\to\pi^0e^+e^-)/\Gamma(K_2\to\mathrm{all})$ is 4.5×10^{-16}.

Our calculation neglects the effects of the mass-shell intermediate states $\pi\pi\nu$ and $\pi\pi\pi\nu$ which can also appear in the t and t' channels. The correction to ImF is likely to be small because of the low phase space available. The correction to ReF is hard to estimate, because little is known about the form factors involved.

The low value of the second-order weak contribution to $K_2\to\pi^0e^+e^-$ found in Eq. (16) indicates that this effect will probably be much smaller than the weak-electromagnetic amplitude of order Ge^4 or the CP-violating amplitude of order ϵGe^2 (ϵ being the amplitude of K_1 present in K_L). Estimates of the decay rate of[5] $K_1\to\pi^0e^+e^-$ suggest that the latter effect alone should give a branching ratio for $K_L\to\pi^0e^+e^-$ of 10^{-11}–10^{-12}. The hope that the $O(G^2)$ amplitude might be enhanced by the presence of some large cutoff parameter does not receive support from the present calculation.

Some remarks may be made about other decays of the type $K\to\pi l\bar{l}\,(l=e,\mu,\nu)$. The result obtained in Eq. (16) holds, for obvious reasons, also for the decay $K_2\to\pi^0\nu_e\bar{\nu}_e$. Since the weak-electromagnetic effect in this case is even smaller than the $O(G^2)$ effect, this decay should be forbidden to a very high degree. The same comments apply to the decay $K_2\to\pi^0\nu_\mu\bar{\nu}_\mu$; however, since the mass of the muon in the intermediate state is not negligible, the numerical result may differ slightly from Eq. (16). For the decay $K_2\to\pi^0\mu^+\mu^-$, the second-order weak matrix element will be more general than that for $K_2\to\pi^0e^+e^-$ [given by Eq. (4)] and will have the form

$$\mathfrak{M}=F_1(w,\Delta)\bar{u}\not{q}(1+\gamma_5)v+F_2(w,\Delta)\bar{u}\gamma_5v\,, \tag{17}$$

where the second form factor F_2 vanishes in the limit of vanishing muon mass. This complication should not alter the order of magnitude of the $O(G^2)$ decay rate from that found above. Finally, the charged-K-meson decays $K^\pm\to\pi^\pm l\bar{l}$ differ from $K_2\to\pi^0l\bar{l}$ in one essential respect: The $O(G^2)$ amplitude is *quadratically*, rather than logarithmically, divergent. This comes about because CP invariance puts no restriction on the individual amplitudes of $K^\pm\to\pi^+l\bar{l}$, and the analog of the unitarity relation (5) for these processes does not contain a "crossed term."

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