K^0-$\bar{K}^0$ States in a K_L-K_S Phenomenological S Matrix

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(Received 27 April 1972)

It is shown that for a CPT-invariant S matrix which treats K_S and K_L states as overlapping resonances, one can always define K^0 and $\bar{K}^0$ states which are consistent with the usual phenomenology. In particular the unitarity and CPT analysis of the $K_L\to\mu^+\mu^-$ puzzle are contained in such a phenomenological S matrix.

I. INTRODUCTION

In the past few years there has been some discussion on treating K_S and K_L states as overlapping resonances in an S-matrix theory.[1] Many authors[2,3] have addressed themselves to the problem of relating theoretically the weakly decaying states K_S and K_L to the states K^0 and $\bar{K}^0$ produced in strong interactions. In particular Stodolsky[2] and Gien[3] have shown that starting from a Hermitian potential V, which connects the degenerate, discrete states, say K^0 and $\bar{K}^0$, to the continuum states, a Lippmann-Schwinger type of potential theory of scattering gives an S matrix which can be reduced easily to the form of a phenomenological S matrix such as the one given by Durand and McVoy[1] which identifies K_S and K_L states to be those which diagonalize the corresponding mass matrix. The author[4] himself in a recent note has shown that even the converse of the above statement is true, i.e., given a phenomenological S matrix which treats K_S and K_L states as overlapping resonances, one can construct the matrix elements of an effective Hermitian potential in terms of the given (i.e., of K_S and K_L) parameters such that the given arbitrary S matrix can always be written in a form derivable from a potential theory of scattering. The matrix elements of this effective Hermitian potential so constructed are between the continuum and discrete orthogonal states called $|1\rangle$ and $|2\rangle$. These states are related to the standard $|S\rangle$ and $|L\rangle$ states representing K_S and K_L states, respectively, via two real parameters, r and s, which are also given in terms of the decay parameters of the S matrix by the Bell-Steinberger sum rule.[5]

In the present article we extend these previous results by including a background S matrix and showing that for a CPT-invariant S matrix one can always define a set of states K^0 and $\bar{K}^0$ such that the mass matrix in the S matrix has equal diagonal elements and such that the decay amplitudes for K^0 and $\bar{K}^0$ are related in a way usually derived from CPT arguments assuming $CPT|K^0\rangle=|\bar{K}^0\rangle$.

That such "resonance" K^0-$\bar{K}^0$ states can always be defined we find particularly gratifying in light of the essential "unitarity" and "CPT" arguments applied to decay amplitudes that are used in the discussion of the $K_L\to\mu^+\mu^-$ puzzle.[6] That is, we find the conventional discussion of these unitarity-CPT relations obtainable from an S-matrix phenomenology.

II. UNITARITY, CPT, AND K^0-$\bar{K}^0$ BASES

We start with a phenomenological S matrix of the Durand-McVoy[1] type:

$$S(E)=B-i\,\frac{\Gamma_S g_S\tilde{h}_S}{E-M_S}-i\,\frac{\Gamma_L g_L\tilde{h}_L}{E-M_L}\,, \tag{1}$$

where $M_{S,L}=m_{S,L}-i\Gamma_{S,L}/2$ and $g_{S,L}$ and $\tilde{h}_{S,L}$ are column and row vectors, the components of which represent the partial decay and production amplitudes of K_S and K_L resonances, respectively. B is the background scattering term.

The unitarity conditions for the above S matrix,

$$S^\dagger(E^*)S(E)=1\,,$$

implying $B^\dagger B=1$ throughout, can be written exactly as

$$\tilde{h}_S=N[g_S^\dagger-(\Gamma_L/\Gamma_S)^{1/2}\alpha g_L^\dagger]B\,, \tag{2a}$$

$$\tilde{h}_L=N[g_L^\dagger-(\Gamma_L/\Gamma_S)^{1/2}\alpha^* g_S^\dagger]B\,, \tag{2b}$$

where

$$N=(1-|\alpha|^2)^{-1}\,, \tag{3}$$

$$\alpha=i(\Gamma_S\Gamma_L)^{1/2}(g_S^\dagger g_L)/M_S^*-M_L \tag{4}$$

and we have used the normalization for decay amplitudes as

$$g_S^\dagger g_S=g_L^\dagger g_L=1\,. \tag{5}$$

Since in Eqs. (2) the background term B is separated, B can be factored out of the full S matrix and hence the results of Ref. 4 apply to the present case as well. In other words, by using the same arguments as in Ref. 4, we can show that the above S matrix can be written in an orthonormal basis as

$$S_{\alpha\beta}(E)=B_{\alpha\beta}-i\left\langle \alpha,\text{out}\,\middle|\, V\left(\frac{1}{E-\underline{\mu}}\right)V\,\middle|\,\beta,\text{in}\right\rangle, \tag{6}$$

where in the present paper we have used the identifications

$$g_i^\alpha \equiv \frac{1}{\sqrt{\Gamma_i}}\langle \alpha,\text{out}|V|i\rangle, \qquad h_i^\alpha \equiv \frac{1}{\sqrt{\Gamma_i}}\langle i|V|\alpha,\text{in}\rangle, \quad i=S,L \tag{7}$$

instead of the ones used in Ref. 4. Also

$$B_{\alpha\beta}=\langle \alpha,\text{out}|\beta,\text{in}\rangle, \qquad \langle\alpha|V|i\rangle^*=\langle i|V|\alpha\rangle, \tag{8}$$

i.e., V is Hermitian. The unitarity condition for Eq. (6) is still the Bell-Steinberger sum rule:

$$\alpha=\langle S|L\rangle \quad (\text{written in } |S\rangle, |L\rangle \text{ basis}) \tag{9a}$$

or

$$i(\underline{\mu}-\underline{\mu}^\dagger)_{ij}=\sum_\alpha \langle i|V|\alpha,\text{out}\rangle\langle\alpha,\text{out}|V|j\rangle, \quad i,j=1,2 \tag{9b}$$

where $|1\rangle, |2\rangle$ basis is as defined below and originally in Ref. 4, and the 2×2 complex mass matrix $\underline{\mu}$ is the same as given in Ref. 4:

$$\underline{\mu}=\frac{1}{1-irs}\begin{pmatrix} M_S-irsM_L & is(M_L-M_S)\\ r(M_S-M_L) & M_L-irsM_S\end{pmatrix}, \tag{10}$$

$$|S\rangle=\frac{1}{(1+r^2)^{1/2}}(|1\rangle+r|2\rangle), \qquad |L\rangle=\frac{1}{(1+s^2)^{1/2}}(|2\rangle+is|1\rangle) \tag{11}$$

[r, s being two real parameters in Eq. (10)].

For the purpose of identifying $|K^0\rangle$ and $|\bar{K}^0\rangle$ basis with an orthonormal basis, the transformation (11) is obviously not general enough and we therefore introduce another orthonormal basis, $|1'\rangle, |2'\rangle$ related to $|1\rangle, |2\rangle$ by a unitary transformation $\underline{U}$:

$$\begin{pmatrix}|1\rangle\\ |2\rangle\end{pmatrix}=\underline{U}\begin{pmatrix}|1'\rangle\\ |2'\rangle\end{pmatrix}, \tag{12a}$$

where

$$\underline{U}=\begin{pmatrix} a & b\\ -b^* & a^*\end{pmatrix}, \tag{12b}$$

with

$$|a|^2+|b|^2=1. \tag{12c}$$

Here a and b are two complex parameters. In this new basis, Eq. (6) can be written as

$$S_{\alpha\beta}(E)=B_{\alpha\beta}-i\left\langle \alpha,\text{out}\,\middle|\, V\left(\frac{1}{E-\underline{G}}\right)V\,\middle|\,\beta,\text{in}\right\rangle. \tag{13}$$

How are we to identify the $|1'\rangle$ and $|2'\rangle$ states with the conventional $|K^0\rangle$ and $|\bar{K}^0\rangle$ states? Clearly CPT properties must be used. We will show that the following two properties derived in the conventional formalism can be used to define the K^0-$\bar{K}^0$ states:

(i) The diagonal matrix elements of the mass matrix are equal in the K^0-$\bar{K}^0$ basis, i.e.,

$$\langle 1'|G|1'\rangle=\langle 2'|G|2'\rangle=\tfrac{1}{2}(M_S+M_L). \tag{14}$$

(ii) The decay amplitudes of the K^0 and $\bar{K}^0$ states are related in the following manner [expressed in the notation defined by Eq. (7)]:

$$\langle\alpha,\text{out}|V|1'\rangle=\langle\hat{\alpha},\text{in}|V|2'\rangle^*, \qquad \langle\alpha,\text{out}|V|2'\rangle=\langle\hat{\alpha},\text{in}|V|1'\rangle^*, \tag{15}$$

where the caret denotes the CPT conjugate of the corresponding state.[7]

It is easy to see that if there exist basic states such that (i) and (ii) are true and if the background matrix is CPT-invariant, i.e.,

$$B_{\alpha\beta}=B_{\hat{\beta}\hat{\alpha}}, \tag{16}$$

then $S(E)$ is CPT-invariant, i.e.,

$$S_{\alpha\beta}(E)=S_{\hat{\beta}\hat{\alpha}}(E). \tag{17}$$

We now proceed to show the converse, i.e., if Eq. (17) is valid there exists a basis $|1'\rangle, |2'\rangle$ such that (i) and (ii) are satisfied. (i) is implied by a complex condition on the parameters,

$$(1+irs)(|a|^2-|b|^2)+2isab-2ra^*b^*=0, \tag{18}$$

and Eq. (17) implies Eq. (16) and

$$g_i^\alpha \tilde{h}_i^\beta=g_i^{\hat{\beta}}\tilde{h}_i^{\hat{\alpha}}, \quad i=S,L \tag{19}$$

or

$$\frac{h_i^\beta}{g_i^{\hat{\beta}}}=\frac{h_i^{\hat{\alpha}}}{g_i^\alpha}\equiv\lambda_i, \tag{20}$$

implying

$$h_i^\alpha=\lambda_i g_i^{\hat{\alpha}}. \tag{21}$$

We notice that due to Eqs. (2) and (5)

$$h_S^\dagger h_S=h_L^\dagger h_L=N \tag{22}$$

and hence[8]

$$|\lambda_S|=|\lambda_L|=\sqrt{N}\ . \tag{23}$$

For later convenience we choose the following phases of λ_S and λ_L:

$$\lambda_S=-\lambda_L=\sqrt{N}\ e^{i\phi}, \tag{24}$$

where $\phi=\arctan(rs)$.

Now by putting Eq. (21) in Eq. (2) and using Eq. (8), one obtains

$$\langle 1'|V|\hat{\alpha},\text{in}\rangle=\frac{1}{\sqrt{N}(1+irs)^2}\{[(1+r^2)(a-isb^*)^2-(1+s^2)(b^*+ar)^2]\langle\alpha,\text{out}|V|1'\rangle$$
$$+[(1+r^2)(a-isb^*)(b+isa^*)+(1+s^2)(b^*+ra)(a^*-rb)]\langle\alpha,\text{out}|V|2'\rangle\}, \tag{25a}$$

$$\langle 2'|V|\hat{\alpha},\text{in}\rangle=\frac{1}{\sqrt{N}(1+irs)^2}\{[(1+r^2)(b+ia^*s)^2-(1+s^2)(a^*-rb)^2]\langle\alpha,\text{out}|V|2'\rangle$$
$$+[(1+r^2)(b+ia^*s)(a-isb^*)+(1+s^2)(a^*-rb)(ar+b^*)]\langle\alpha,\text{out}|V|1'\rangle\}. \tag{25b}$$

The condition

$$(1+r^2)(a-ib^*s)^2=(1+s^2)(b^*+ar)^2 \tag{26}$$

along with Eq. (18) imply that the coefficients of $\langle\alpha,\text{out}|V|1'\rangle$ and $\langle\alpha,\text{out}|V|2'\rangle$ in Eqs. (25a) and (25b), respectively, vanish, and the coefficients of $\langle\alpha,\text{out}|V|2'\rangle$ and $\langle\alpha,\text{out}|V|1'\rangle$ in Eqs. (25a) and (25b), respectively, become unity.

We can now, through Eqs. (10) and (11), write down the states $|S\rangle$ and $|L\rangle$ in terms of $|K^0\rangle$ and $|\bar{K}^0\rangle$ by identifying

$$\begin{aligned}
&|1'\rangle\equiv|\bar{K}^0\rangle,\\
&|2'\rangle\equiv|K^0\rangle,\\
&|S\rangle=\frac{1}{[2(1+\text{Re}\alpha)]^{1/2}}[(1+\alpha^*)|K^0\rangle+(1-|\alpha|^2)^{1/2}e^{-i\phi}|\bar{K}^0\rangle],\\
&|L\rangle=\frac{1}{[2(1+\text{Re}\alpha)]^{1/2}}[(1+\alpha)|K^0\rangle-(1-|\alpha|^2)^{1/2}e^{-i\phi}|\bar{K}^0\rangle].
\end{aligned} \tag{27}$$

A careful analysis of the phase arbitrariness in the above two equations reveals that they are completely consistent with the standard results of CP phenomenology.[9]

III. PHASES

In Eq. (1) there are two kinds of phase freedoms involved. First is freedom in the relative phase of g_S and g_L which can be fixed by choosing a phase of α and second is the freedom in the phase of g_S or g_L itself. Because of this second-phase arbitrariness and some freedom in choosing the phase in CPT operation on $|\alpha,\text{out}\rangle$ or $|\alpha,\text{in}\rangle$ the phase of λ_S or λ_L remains completely arbitrary. We have already chosen by Eq. (24) the relative phase of λ_S and λ_L. A motivation for such a choice is the physical requirement that in the limit of CP invariance (i.e., r, s tend to zero and a, b tend to $1/\sqrt{2}$) Eqs. (25) should imply Eq. (15). Therefore, since the phase of λ_S or λ_L is arbitrary, the phase ϕ, although known in terms of Reα and Imα, is arbitrary.

To fix the first arbitrariness in the relative phase of g_S and g_L we choose α to be real (and also to agree with Wu-Yang phase convention) and by making the following transformation in α:

$$\epsilon=\left(\frac{1-\alpha}{1+\alpha}\right)^{1/2}e^{-i\phi} \tag{28}$$

obtain

$$\begin{aligned}
&|S\rangle=\frac{1}{(1+|\epsilon|^2)^{1/2}}[|K^0\rangle+\epsilon|\bar{K}^0\rangle],\\
&|L\rangle=\frac{1}{(1+|\epsilon|^2)^{1/2}}[|K^0\rangle-\epsilon|\bar{K}^0\rangle],
\end{aligned} \tag{29}$$

which is just the result of Wu-Yang phase convention in CP phenomenology.

IV. DISCUSSION

We have seen that starting from an arbitrary CPT-invariant S matrix involving two overlapping resonances, it is possible to define K^0 and $\bar{K}^0$ states such that the standard CP phenomenology results.

In particular, the "unitarity" and CP conditions used in the analysis of the $K_L \to \mu^+\mu^-$ puzzle are contained in such an S matrix. Thus, any results relating K_L decays to K_S decays are valid in a pure S-matrix formulation.

V. ACKNOWLEDGMENT

The author wishes to express his deep gratitude to Professor William D. McGlinn for suggesting the problem and various helpful discussions.

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[6]See, for example, M. K. Gaillard, Phys. Letters 36B, 114 (1971), and references cited therein.

[7]Equation (15) is precisely the condition necessary to show the validity of the unitarity condition used by Gaillard (Ref. 6), $-i(T_{if} - T_{if}^*) = \sum_n T_{in} T_{fn}^* = \sum_n T_{in}^* T_{nf}$.

[8]It may be worthwhile to note that McVoy's choice (Ref. 1) of normalization, i.e., $g_i^\dagger g_i = h_i^\dagger h_i = 1$ implying $|\lambda_i| = 1$ is not consistent with the exact unitarity conditions (2). This inconsistency, however, seems to disappear in the later choice of Durand and McVoy (Ref. 1) which, by choosing $g_i^\dagger g_i = h_i^\dagger h_i$, implies $g_i^\dagger g_i = (1 + |\alpha|^2)^{1/2}$ and $|\lambda_i| = 1$. The unitarity conditions, Eqs. (12) and (13) of Durand and McVoy (Ref. 1), are therefore exact and, contrary to their remarks, they are independent of any CPT assumption.

[9]See, for example, R. G. Sachs, Ann. Phys. (N.Y.) 22, 239 (1963), and P. K. Kabir, *The CP Puzzle* (Academic, London and New York, 1968).

PHYSICAL REVIEW D VOLUME 6, NUMBER 7 1 OCTOBER 1972

Why Do Resonances Scale?

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(Received 31 March 1972)

Scaling of hadronic resonance excitation is discussed within the framework of a simple potential-theoretic parton model.

I. INTRODUCTION

The experimental scaling behavior of deep-inelastic electroproduction cross sections[1,2] and diffractive resonance excitations[3] raises a number of questions which we will consider in this paper:

(1) What is the scale of energies and momentum transfers which determines the onset of scaling?

(2) What is the role of resonances in the scaling phenomena?

(3) In terms of which kinematic variables does the scaling work best?

The experimental answer to question (1) is that scaling sets in for values of ν and Q^2 in excess of 1 GeV^2. This suggests that the important scale in the problem is the hadronic level spacing[4] (inverse Regge slope).

Questions (2) and (3) were answered by Bloom and Gilman,[5] who observed that in the scaling variable

$$\omega' = \frac{2M\nu}{Q^2} + \frac{M^2}{Q^2}$$
$$= 1 + \frac{s}{Q^2} \qquad (1)$$