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Four-Photon Decay of the Neutral Pion

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The decay rate for $\pi^0 \to 4\gamma$ is estimated to order α^4 in several models and found to be considerably suppressed by the angular momentum barriers required by gauge invariance and even further suppressed in models satisfying a partially-conserved-axial-vector-current condition. Typical values for $\Gamma(\pi^0 \to 4\gamma)/\Gamma(\pi^0 \to 2\gamma)$ range from 4×10^{-14} to 10^{-16}. We conclude that the dominant contribution is likely to come from an electron-loop diagram contributing in order α^6 but without angular momentum barriers.

I. INTRODUCTION

The very rare decay of the π^0 into four photons is one possible source of background in experiments looking for the C-violating three-photon decay mode.[1] There does not seem to be any estimate of the rate for this mode in the literature although the crudest estimate, $\Gamma(\pi^0 \to 4\gamma)/\Gamma(\pi^0 \to 2\gamma) \cong (\alpha/\pi)^2 \cong 10^{-5}$, is comparable to the experimental limit on the 3γ mode.[2] We have calculated the decay rate to order α^4 using several models which include the requirements of gauge invariance and Bose-Einstein statistics and find branching ratios in the range 4×10^{-14} to 10^{-16}. The smaller rates are for models which satisfy the PCAC (partial conservation of axial-vector current) condition. We show that this is a general feature of PCAC-satisfying models and demonstrate how this extra suppression comes about for a simplified σ model. Other contributions to $\pi^0 \to 4\gamma$ are mentioned briefly.

II. RATE SUPPRESSION DUE TO GAUGE INVARIANCE AND PCAC REQUIREMENTS

Aside from the powers of the fine-structure constant α, the main reasons that the $\pi^0 \to 4\gamma$ decay rate is small is that the lowest orbital angular momentum states are forbidden by the requirements of gauge invariance and PCAC. The centrifugal barriers of the higher states inhibit the decay by introducing extra powers of (KR) where K is the momentum of a photon and R is a radius characteristic of the interaction (e.g., $R = 1/M$ where M is the mass of one of the virtual particles effecting the decay).

From gauge invariance, the low-energy theorem for photon emission[3] says that the amplitude for $\pi^0 \to 4\gamma$ vanishes when any one of the photon momenta goes to zero (i.e., there are no inner-bremsstrahlung graphs for this process). This means that, since all four photon momenta are independent variables, the decay amplitude must contain at least one power of each photon momentum and thus must be at least of order $(KR)^4$.

To demonstrate the effects of the PCAC condition we consider the matrix element

$$T_\mu^5 = \langle 4\gamma | A_\mu(0) | 0 \rangle \quad , \tag{1}$$

where A_μ is the axial-vector current. Again we have the gauge-invariance requirement of at least one power of each photon momentum; however, since T_μ^5 has a vector index, there must be an odd number of powers of momentum. Thus T_μ^5 contains at least 5 powers of momentum. The PCAC equation[4]

$$\partial_\mu A_\mu = C_\pi \phi_\pi \tag{2}$$

then indicates that the amplitude for $\pi^0 \to 4\gamma$ has one power of momentum more than T_μ^5, so that it is not of order $(KR)^4$ but actually order $(KR)^6$.

This conclusion is an extension of the older argument which showed that $\pi^0 \to 2\gamma$ would be suppressed by PCAC if there were no anomalous commutators.[5] It therefore depends on the absence of anomalous commutators for the $\pi^0 \to 4\gamma$ reaction. Bardeen and Brown, Shih, and Young[4] have investigated this question and we find, using their results, no anomalous term should arise here.

III. MODEL AMPLITUDES FOR $\pi^0 \to 4\gamma$

To get a quantitative idea of the size of the effective radius R and to estimate the coefficient of $(KR)^4$ we have calculated the branching ratio

$$\mathcal{R} = \Gamma(\pi^0 \to 4\gamma)/\Gamma(\pi^0 \to 2\gamma) \tag{3}$$

in several models. No attention is paid to the

PCAC constraint which will be considered in the following section.

A. Vector-Meson-Dominance Model

In the first model, we assume that the important graphs are those with vector mesons in the intermediate state, as represented by the diagrams of Fig. 1. Here V and V' represent the ρ and ω mesons (the ϕ meson is left out because the $\pi\rho\phi$ coupling, and hence the $\pi\gamma\phi$ coupling, is believed to be small) and P could be the pion, η, or X^0 mesons. We have included only the pion in our order-of-magnitude calculation because we expect it to have the largest contribution due to its smaller mass. (The pole of the pion propagator is on the edge of the 4γ phase space.)

With the usual notation ϵ_i and k_i for the polarization and momentum vectors of the ith photon, the amplitude represented by the diagram of Fig. 1 is

$$T=\frac{F_{\pi^0\gamma\gamma}}{m_\pi}\left[\frac{g_{\rho\pi\gamma}{}^2}{(k_2+k_3+k_4)^2-m_\rho{}^2}+\frac{g_{\omega\pi\gamma}{}^2}{(k_2+k_3+k_4)^2-m_\omega{}^2}\right]\frac{1}{(k_3+k_4)^2-m_\pi{}^2}\,\epsilon_{\mu\nu\sigma\lambda}\epsilon_3^\mu k_3^\nu\epsilon_4^\sigma k_4^\lambda\,\epsilon_{\alpha\beta\gamma\delta}(k_2+k_3+k_4)^\beta\epsilon_1^\gamma k_1^\delta$$
$$\times\epsilon_{\alpha\beta'\gamma'\delta'}(k_2+k_3+k_4)^{\beta'}\epsilon_2^{\gamma'}k_2^{\delta'}+11\text{ terms obtained by permutations}\ . \qquad (4)$$

Evaluation of the $12\times12=144$ terms in $|T|^2$ resulting from this decay amplitude is very tedious. For the total rate the permutation symmetry of phase space can be used to classify the 144 terms into 7 types. If we denote the term written explicitly in Eq. (4) by $T(1,2;3,4)$, then the seven types are

(i) $|T(1,2;3,4)|^2$ for the 12 diagonal terms,

(ii) $T(1,2;3,4)T^*(2,1;3,4)$ for 12 terms,

(iii) $T(1,2;3,4)T^*(1,3;2,4)$ for 24 terms,

(iv) $T(1,2;3,4)T^*(2,3;1,4)$ for 24 terms,

(v) $T(1,2;3,4)T^*(3,1;2,4)$ for 24 terms,

(vi) $T(1,2;3,4)T^*(3,2;1,4)$ for 24 terms,

(vii) $T(1,2;3,4)T^*(3,4;1,2)$ for 24 terms.

Of these seven types we have only evaluated three and set an upper bound on the rest. Terms of type (vii) give zero upon doing the spin sum. Terms of types (i) and (ii) we have evaluated with numerical integration over phase space. To evaluate the coupling constants $g_{\pi\gamma v}$ we use vector dominance,

$$g_{\pi\gamma v}=\frac{g_{\pi\omega\rho}\lambda_v}{m_\pi{}^3}\ , \qquad (5)$$

with $\lambda_\rho=g_\rho$, $\lambda_\omega=(1/\sqrt{2})g_\omega$, $g_\rho{}^2=2F_\pi{}^2m_\rho{}^2$, and g_ω as found either by Das, Mathur, and Okubo (DMO),[6] or by Oakes and Sakurai (OS)[7]:

$$g_\omega{}^2=\begin{cases}0.43g_\rho{}^2 & \text{(DMO)}\\ 0.23g_\rho{}^2 & \text{(OS)}\ .\end{cases} \qquad (6)$$

The Gell-Mann–Sharp–Wagner model[8] for $\omega\to3\pi$ is used to obtain $g_{\rho\omega\pi}{}^2/4\pi=0.51$.

The total contribution, $\mathfrak{R}'$, to $\mathfrak{R}$ from the terms of types (i), (ii), and (vii) is

$$\mathfrak{R}'=\begin{cases}1.5\times10^{-16} & \text{(OS)}\\ 1.7\times10^{-16} & \text{(DMO).}\end{cases} \qquad (7)$$

Since each remaining term must contribute to $\mathfrak{R}$ less than is contributed by a diagonal term [type (i)] we have an upper limit for this model of

$$\mathfrak{R}\leq\begin{cases}7\times10^{-16} & \text{(OS)}\\ 8.6\times10^{-16} & \text{(DMO).}\end{cases} \qquad (8)$$

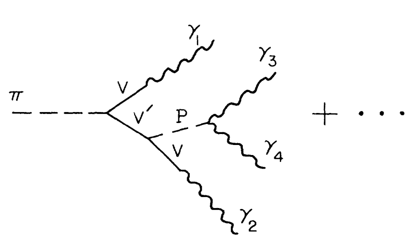

FIG. 1. Four-photon decay of the π^0 in the vector-meson-dominance model. V and V' are vector mesons (ρ or ω) and P is a pseudoscalar (π, η, or X^0).

B. Nucleon-Loop Diagrams

In this model, which is a generalization of the $\pi^0\to2\gamma$ calculation done by Steinberger,[9] the interaction between the π^0 and the photons is mediated through the nucleons. The simplest diagram is the pentagon of Fig. 2. However the contribution from this diagram to the decay amplitude is expected to be of order $(K/M)^6$ because of PCAC as discussed in Sec. II. More important diagrams are the combination box-triangle diagrams of Fig. 3. Calculation of the triangle part of these diagrams is the same as done by Steinberger. To lowest order in reciprocal powers of the nucleon mass, it is

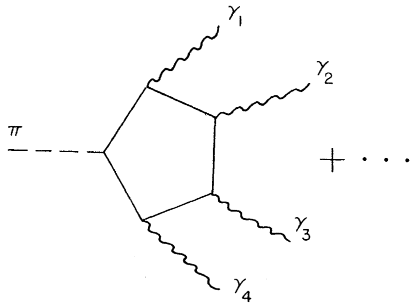

FIG. 2. The simplest nucleon loop diagram for $\pi^0 \to 4\gamma$.

$$\mathfrak{M}_T = \alpha g \frac{4}{M} \epsilon_{\alpha\beta\gamma\delta} \epsilon_3^\alpha k_3^\beta \epsilon_4^\gamma k_4^\delta / 4\pi \ . \tag{9}$$

The loop integral for the box part of these diagrams is logarithmically divergent. The divergence can be removed, however, either by regularizing with the Pauli-Villars regularization method[10] or by requiring the divergent term to satisfy gauge invariance (i.e., requiring the divergent, and therefore undefined, coefficient of $\epsilon_1 \cdot \epsilon_2$ to be properly related to the convergent coefficient of $k_1 \cdot \epsilon_2 k_2 \cdot \epsilon_1$). Both methods give the same result. Expanding in inverse powers of the nucleon mass (because $m_\pi/M_N \ll 1$) we obtain for the $\pi^0\pi^0\gamma\gamma$ box amplitude

$$\mathfrak{M}_B = \alpha \frac{g^2}{4\pi} (\tfrac{8}{3})(k_1 \cdot k_2 \epsilon_1 \cdot \epsilon_2 - k_1 \cdot \epsilon_2 k_2 \cdot \epsilon_1)\left(\frac{1}{M}\right)^2 \ . \tag{10}$$

The total amplitude for the box-triangle contribution to $\pi^0 \to 4\gamma$ is then

$$\begin{aligned} \mathfrak{M} = \alpha^2 \frac{g^3}{(4\pi)^2} \frac{32}{3} \left(\frac{1}{M}\right)^3 \frac{1}{(k_3+k_4)^2 - m_\pi^2} \\ \times (\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 - \epsilon_1 \cdot \epsilon_2 k_1 \cdot k_2) \epsilon_{\mu\sigma\nu\lambda} \epsilon_3^\mu k_3^\sigma \epsilon_4^\nu k_4^\lambda \\ + 5 \text{ permutations of } 1, 2, 3, 4 \ . \end{aligned} \tag{11}$$

The 36 terms in $|\mathfrak{M}|^2$ can be classified into three types [exemplified by types (i), (iii), and (vii) of Sec. III A. The extra symmetry of the basic box-triangle amplitude under exchange of particles 1 and 2 reduces the number of distinct cross terms.]

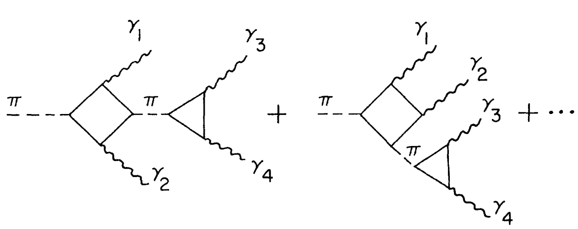

FIG. 3. Nucleon loop diagrams which lead to amplitudes of order $(K/M)^4$.

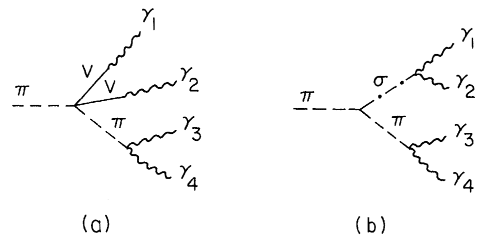

FIG. 4. Additional diagrams for $\pi^0 \to 4\gamma$ in the phenomenological Lagrangian model.

The phase-space integration can be carried out numerically and the resulting branching ratio (the $\pi^0 \to 2\gamma$ rate is taken here to be that of the triangle graph, 13.7 eV) is

$$\mathfrak{R} = 4.6 \times 10^{-14} \ . \tag{12}$$

C. A Phenomenological Lagrangian Model

A phenomenological Lagrangian given by Gasiorowicz and Geffen[11] and by Gounaris[12] allows the $\pi^0 \to 4\gamma$ decay to occur through a contact term [Fig. 4(a)] and a σ-π intermediate state [Fig. 4(b)]. Here the V's are neutral vector mesons and σ is an isoscalar scalar meson. The relevant couplings are

$$\begin{aligned} \mathfrak{L}_{\text{contact}} &= \frac{\xi}{4} \frac{Z_\pi}{2} \vec{\pi}^2 [Z_\omega(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)^2 + Z_\rho(\partial_\mu \rho_\nu - \partial_\nu \rho_\mu)^2] \ , \\ \mathfrak{L}_{\sigma\pi\pi} &= Z_\pi Z_{\sigma X}{}^{1/2} g_s \vec{\pi}^2 (\sin\Psi_\sigma \sigma_\eta - \cos\Psi_\sigma \sigma_X) \ , \\ \mathfrak{L}_{\sigma VV} &= \frac{\xi a}{2} \frac{Z_{\sigma\pi}{}^{1/2}}{\sqrt{2}} (\cos\Psi_\sigma \sigma_\eta + \sin\Psi_\sigma \sigma_X) \\ &\quad \times [Z_\omega(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu)^2 + Z_\rho(\partial_\mu \rho_\nu - \partial_\nu \rho_\mu)^2] \ , \end{aligned} \tag{13}$$

where ρ, ω, and $\vec{\pi}$ are field operators for the ρ, ω, and π mesons, σ_X and σ_η are the scalar counterparts of the X^0 and η mesons, and Ψ_σ is the σ_X-σ_η mixing angle. The various parameters were determined by Gounaris and are summarized in Table I.

The tensor structure of the matrix elements is similar to that of the box-triangle graph, so the

TABLE I. The parameters of the phenomenological Lagrangian [Eq. (13)] as given by Gounaris (private communication).

$\xi = 1.31 \times 10^{-4}$ MeV^{-2}	$Z_\pi = 0.364$
$g_s a = -33.4 \times 10^4$ MeV^2	$Z_\rho = 1.8$
$\Psi_\sigma \cong -36°$ or $-42°$	$Z_{\sigma\pi} = 0.58$
$Z_\omega/m_\omega^2 = Z_\rho/m_\rho^2$	$Z_{\sigma X} = 0.245$

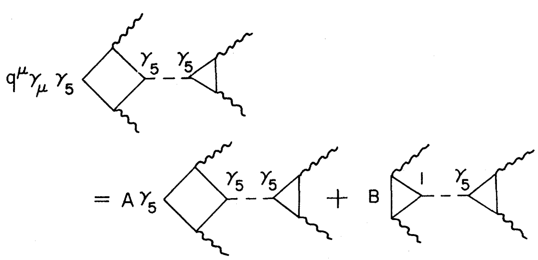

FIG. 5. A schematic representation of the diagrams contributing to the divergence of the axial-vector current. A and B are known nonzero functions of masses and coupling constants.

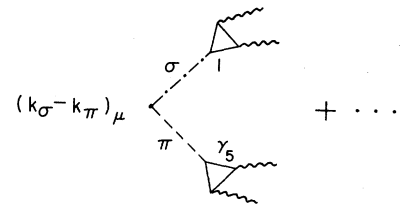

FIG. 6. An additional contribution to the 4γ matrix element of the axial-vector current in the σ model.

remainder of the calculation proceeds as in Sec. III B. The resulting branching ratios are

$$\mathcal{R}_{\text{contact}} \cong \begin{cases} 1.8\times10^{-14} & (\text{DMO}) \\ 1.2\times10^{-14} & (\text{OS}), \end{cases} \tag{14}$$

$$\frac{\mathcal{R}_{\sigma\pi}}{\mathcal{R}_{\text{contact}}} \cong 0.1 \text{ for either value of } \Psi_\sigma \,. \tag{15}$$

IV. AN EXAMPLE OF THE SUPPRESSION BY PCAC: A σ MODEL

Of the models calculated in Sec. III, the box-triangle (BT) graph produces the largest decay rate. However, a detailed examination of the BT amplitude shows that the PCAC condition is not satisfied. This is represented diagrammatically in Fig. 5 where the peculiar additional term with coefficient B is not present in the PCAC equation. The reason for this extra term is that since the theory explicitly contains pions, they should have been included in the axial-vector current operator. As an example of a PCAC-satisfying theory which includes the pions, let us look at the isoscalar version[13] of the σ model of Gell-Mann and Lévy.[14] In this model, the Lagrangian contains protons, which alone interact with the electromagnetic current, neutral pions, and neutral scalar mesons called σ mesons. We calculate in lowest-order perturbation theory and find that, in addition to BT diagrams with the pion intermediate state, we should also have BT graphs with a σ pole. The peculiar term noted in Fig. 5 for the BT graph, together with the corresponding term with a σ-meson propagator, are canceled by a term coming from the divergence of an addition contribution to the axial current shown in Fig. 6. Suppression of the $\pi^0 \to 4\gamma$ rate comes about because the two box-triangle graphs (one with π pole and one with σ pole) have opposite sign. The amplitude of Eq. (11) becomes

$$\mathfrak{M} = \alpha^2 \frac{g_{\pi NN}}{(4\pi)^2}\frac{32}{3}\left(\frac{1}{M}\right)^3 (\epsilon_1\cdot k_2\epsilon_2\cdot k_1 - \epsilon_1\cdot\epsilon_2 k_1\cdot k_2)\epsilon_{\mu\sigma\lambda\nu}\epsilon_3^\mu k_3^\sigma \epsilon_4^\lambda k_4^\nu \left\{\frac{g_{\pi NN}{}^2}{(k_3+k_4)^2 - m_\pi{}^2} - \frac{g_{\sigma NN}{}^2}{(k_1+k_2)^2 - m_\sigma{}^2}\right\} + \text{perm.} \tag{16}$$

To the lowest order of perturbation theory in the σ model, we have symmetry, $g_{\pi NN} = g_{\sigma NN}$ and $m_\pi = m_\sigma$. Thus $\mathfrak{M}$ is of order $(KR)^6$ as required by PCAC. If instead of full symmetry we use an estimated physical mass and coupling for the σ meson (we take $m_\sigma \gg m_\pi$ and $g_{\sigma NN}{}^2/m_\sigma{}^2 = g_{\pi NN}{}^2/m_\pi{}^2$), the $\pi \to 4\gamma$ branching ratio becomes

$$\mathcal{R} \cong 3.4\times10^{-15}\,, \tag{17}$$

which is an order of magnitude smaller than the value obtained with the π^0 pole only.

V. CONCLUSIONS

In the models we have studied, the effective radius R is smaller than $1/M_\pi$ because each graph has a heavier particle (nucleon, vector meson, etc.) in an intermediate state. Since the typical photon energy K is about $\frac{1}{4}M_\pi$ the suppression factors $(KR)^4$ or $(KR)^6$ are very effective even for those graphs which have an intermediate-state pion pole at the edge of the physical region (Figs. 1, 3, and 4) and thus might be expected to have a large

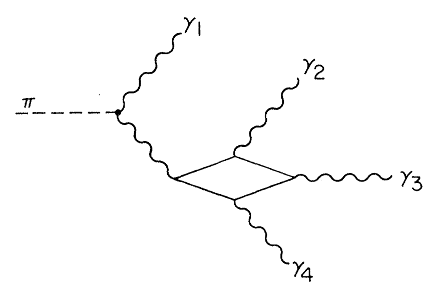

FIG. 7. An electron-loop contribution to $\pi^0 \to 4\gamma$.

effective radius. We have not found any graphs which do not have these large mass ($M \geq M_\pi$) intermediate-state particles except those involving electrons and additional photons, as in Fig. 7.

For the electron-loop graph of Fig. 7 an expansion in powers of photon energy is not valid because of the small electron mass, so there is no suppression by powers of (KR). However, it has one extra power of α (α^2 in the rate) and thus is comparable or only slightly larger than those we have calculated. A crude estimate yields

$$\mathcal{R} \cong \left(\frac{\alpha}{\pi}\right)^4 \frac{1}{4!} \approx 10^{-12}$$

with most of the contribution coming from the region where the intermediate photon has a virtual mass on the order of the electron mass (i.e., one photon has energy $\frac{1}{2}M_\pi$; the other three each have $\frac{1}{6}M_\pi$). This graph could presumably be calculated with known techniques[15] if a more accurate number were needed.

On the basis of the models we have calculated and the general arguments concerning angular momentum barriers presented in Sec. II, we feel that any reasonable estimate of the $\pi^0 \to 4\gamma$ branching ratio to order (α^4/α^2) will be in the range 10^{-13} to 10^{-16} and therefore the dominant contribution will be the purely electromagnetic (α^6/α^2) photon-splitting graph of Fig. 7.

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