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Simple Bound for K_{l3} Decay Parameters Using the πK ($s=0$, $T=\frac{1}{2}$) Phase Shift

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Taking as known the value of the propagator of the divergence of the weak strangeness-changing vector current at zero momentum transfer ($t=0$) and the phase of the K_{l3} form factor $d(t)$ in the elastic region (where it is given by the phase of the $s=0$, $T=\frac{1}{2}$ πK scattering), one finds a simple bound for $d'(0)$ which gives a stronger restriction on the K_{l3} parameters than the one originally obtained by Li and Pagels.

The work of Li and Pagels[1] stimulated the search for bounds on K_{l3} decay parameters, in terms of the propagator $\Delta(t)$ of the divergence of the weak strangeness-changing vector current $V_\mu^{(K)}$, taken as a known quantity at zero momentum transfer ($t=0$). For details concerning the methods used and for references on the existing literature in this field, we refer the reader to the papers cited in Refs. 1–4.

In this note, using the simple method of Li and Pagels,[1] and enlarging the information accepted as known, by taking the phase $\delta(t)$ of the form factor $d(t)$,

$$\frac{1}{2}d(t)=\langle \pi^0(p)|i\partial_\mu V_\mu^{(K^-)}(0)|K^+(k)\rangle = \frac{1}{2}[(m_K{}^2-m_\pi{}^2)f_+(t)+tf_-(t)] \tag{1}$$

(where m_K and m_π are the kaon and pion masses, respectively) as a given function of $t=(p-k)^2$ on the unitarity cut from the elastic threshold $m_+{}^2=(m_K+m_\pi)^2$ up to the inelastic one $m_1{}^2=(m_K+3m_\pi)^2$ [where $\delta(t)$ is just the πK ($s=0, T=\frac{1}{2}$) phase shift], we shall obtain in a simple way a simple bound on the derivative of $d(t)$ at $t=0$. The derivative $d'(0)$ is related to the parameters $\xi\equiv f_-(0)/f_+(0)$ and $\lambda_+\equiv m_\pi{}^2 f_+{}'(0)/f_+(0)$ through the known relation

$$d'(0)=f_+(0)\left(\xi+\frac{m_K{}^2-m_\pi{}^2}{m_\pi{}^2}\lambda_+\right). \tag{2}$$

We start by considering the Cauchy representation for $d(t)$,

$$d(t)=\frac{1}{\pi}\int_{m_+{}^2}^\infty \frac{\mathrm{Im}\,d(t')}{t'-t}\,dt', \tag{3}$$

which gives

$$d'(0)=\frac{1}{\pi}\int_{m_+{}^2}^\infty \frac{\mathrm{Im}\,d(t)}{t^2}\,dt. \tag{4}$$

Denoting by $\varphi(t')$ the phase of $d(t')$ all over the cut ($m_+{}^2\le t'<\infty$), noting that for t' in the elastic region $\varphi(t')\equiv\delta(t')$, $\delta(t')$ being the πK ($s=0, T=\frac{1}{2}$) phase shift taken here as given, and using the relation[1]

$$|d(t')|\le\frac{8\pi}{\sqrt{3}}\frac{\sqrt{t'}\,\rho^{1/2}(t')}{[(t'-m_+{}^2)(t'-m_-{}^2)]^{1/4}},\qquad m_-=m_K-m_\pi, \tag{5}$$

where $\rho(t')$ is the spectral function of $\Delta(t)$,

$$\Delta(t)=\int d^4x\,e^{iqx}\langle 0|T(\partial_\mu V_\mu^{(K^+)}(x)\partial_\nu V_\nu^{(K^-)}(0))|0\rangle \equiv\int_{m_+{}^2}^\infty\frac{\rho(t')}{t'-t}\,dt\qquad(q^2=t), \tag{6}$$

we have successively for $|d'(0)|$

$$\begin{aligned}|d'(0)|&=\frac{1}{\pi}\left|\int_{m_+{}^2}^\infty\frac{|d(t)|\sin\varphi(t)}{t^2}\,dt\right|\\&\le\frac{8}{\sqrt{3}}\int_{m_+{}^2}^\infty\frac{\rho^{1/2}(t)|\sin\varphi(t)|\,dt}{[(t-m_+{}^2)(t-m_-{}^2)]^{1/4}t^{3/2}}\\&\le\frac{8}{\sqrt{3}}\sqrt{\Delta(0)}\left(\int_{m_+{}^2}^\infty\frac{\sin^2\varphi(t)dt}{[(t-m_+{}^2)(t-m_-{}^2)]^{1/2}t^2}\right)^{1/2}\\&\le\frac{8}{\sqrt{3}}\sqrt{\Delta(0)}\left(\int_{m_+{}^2}^{m_1{}^2}\frac{\sin^2\delta(t)dt}{[(t-m_+{}^2)(t-m_-{}^2)]^{1/2}t^2}+\int_{m_1{}^2}^\infty\frac{dt}{[(t-m_+{}^2)(t-m_-{}^2)]^{1/2}t^2}\right)^{1/2}.\end{aligned} \tag{7}$$

In the above sequence of inequalities, we have used in the first step Eq. (5), in the second one the Schwarz integral inequality by means of which one succeeds to factorize $\sqrt{\Delta(0)}$, and in the third step the relation $\varphi(t)=\delta(t)$ for $m_+{}^2\le t\le m_1{}^2$ and the replacement of $\sin^2\varphi(t)$ by unity for $t\ge m_1{}^2$.

In this way, a simple bound for $|d'(0)|$ is found in terms of $\Delta(0)$ and $\delta(t)$:

$$|d'(0)|\le\frac{8}{\sqrt{3}}\frac{\sqrt{\Delta(0)}}{m_+{}^2}(A+B)^{1/2}, \tag{8}$$

where

$$A=\int_1^{m_1{}^2/m_+{}^2}\frac{\sin^2\delta(xm_+{}^2)dx}{x^2[(x-1)(x-\omega^2)]^{1/2}} \tag{9}$$

and B is the known number [obtained after a simple computation of the last integral from Eq. (7)]:

$$B=\frac{1+\omega^2}{2\omega^3}\ln\left(\frac{y-\omega}{y+\omega}\,\frac{1+\omega}{1-\omega}\right)+\frac{1}{\omega^2}\left(\frac{(1-\omega^2)y}{y^2-\omega^2}-1\right),$$

$$\omega=\frac{m_-}{m_+},\quad y=\left(\frac{m_1{}^2-m_-{}^2}{m_1{}^2-m_+{}^2}\right)^{1/2}. \qquad (10)$$

In the following we shall give a simple evaluation of the integral A from Eq. (9) in a scattering-length approximation[5]:

$$K\cot\delta=\frac{1}{a}+\text{neglected terms},$$

where a is the $\pi K\,(s=0, T=\frac{1}{2})$ scattering length and K is the c.m. momentum. Then

$$\sin^2\delta=\frac{a^2K^2}{1+a^2K^2}\,. \qquad (11)$$

For simplicity we shall take for K the usual approximation[5]

$$K=[m_K m_\pi(x-1)]^{1/2}\quad (t=xm_+{}^2)\,. \qquad (12)$$

Under the above approximations the integral A can be easily computed and the result [dependent on the $\pi K\,(s=0, T=\frac{1}{2})$ scattering length a] is

$$A=\frac{1-\omega^2}{(1-\gamma)(\sigma^2-\omega^2)}\left[\left(\frac{1-\sigma^2}{\omega(\sigma^2-\omega^2)}+\frac{1-\omega^2}{2\omega^3}\right)\ln\frac{y+\omega}{y-\omega}+\frac{1-\sigma^2}{\sigma(\sigma^2-\omega^2)}\ln\frac{y+\sigma}{y-\sigma}-\frac{y(1-\omega^2)}{\omega^2(y^2-\omega^2)}\right],$$

$$\gamma=(a^2m_Km_\pi)^{-1}-1\,,$$

$$\sigma^2=\frac{\omega^2+\gamma}{1+\gamma}\,,\quad y^2=\frac{4}{3-\omega}\,. \qquad (13)$$

Equations (8), (10), and (13) allow us to obtain a restriction on $|d'(0)|$ in terms of the scattering length a and $\Delta(0)$.

In order to have a comparison with the results of Li and Pagels,[1] we take (as these authors did)

$$\sqrt{\Delta(0)}=1.006f_\pi m_\pi\,. \qquad (14)$$

Numerically this implies the results

$$|12.3\lambda_+ +\xi|\leqslant 0.10\quad \text{for}\quad a=1/2m_K$$

and (15)

$$|12.3\lambda_+ +\xi|\leqslant 0.17\quad \text{for}\quad a=1/m_K,$$

which are to be compared with the one found in Ref. 1,

$$|12.3\lambda_+ +\xi|\leqslant 0.29\,, \qquad (16)$$

obtained without including the phase in the amount of information accepted as known.

The above evaluation of the integral containing the phase has only an orientative value to show how strong could become the restrictions on the ξ and λ_+ parameters by supplementing the input data with the phase of the form factor in the elastic region. The scattering length used by us is taken anyway to be in agreement with the theoretical estimation $a\approx 0.6m_K{}^{-1}$ based on current-algebra calculations.[6] We note also that, for instance, the values obtained in Ref. 7 for a lie all in the interval $0.44m_K{}^{-1}-0.66m_K{}^{-1}$.

Equations (8), (9), and (10) represent a rigorous bound on $|d'(0)|$ in terms of $\Delta(0)$ and the $\pi K\,(s=0, T=\frac{1}{2})$ phase shift. The numerical form of the bound obtained above in the specified approximations, could, perhaps, seem too stringent. It should be kept in mind that it is affected not only by the poor knowledge of the phase shift, but also by the fact of being based on the, after all, not so certain value of $\Delta(0)$.

Finally, we mention that even from the point of view of the mathematical method used, what has been done before has mainly an illustrative character and does not represent the best way to exploit the supplementary information (expressed by the phase of the form factor in the elastic region), when one tries to obtain restrictions on the parameters appearing in K_{l3} decays. New work in this direction is deferred to further communications.

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