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Symmetries and Nonsymmetries of the Relativistic Quark Model*

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The dynamical successes of the relativistic quark model are separated from those due to symmetry arguments. Particular attention is paid to the example of the $q\bar{q}$, $L=1$ meson decays. A relativistically invariant coplanar form of $U(3)\otimes U(3)$ is found to explain all predictions of the relativistic quark model for decays within a given multiplet except (a) the absence of spin-orbit effects and (b) relations between different partial waves, e.g., in $B\rightarrow\omega\pi$. The coplanar symmetry is expected to be as good as the quark-model classification scheme itself. It preserves most of the better predictions of $SU(6)_W$, while avoiding some that are questionable. Some of the "good" predictions preserved are $\Gamma(A_2\rightarrow\rho\pi)/\Gamma(A_2\rightarrow K\bar{K})=6$, $(f/d)_{\underline{8}(5/2^-)\rightarrow\underline{8}(1/2^+)\underline{8}(0^-)}=-\frac{1}{3}$, and $(f/d)_{\underline{8}(5/2^+)\rightarrow\underline{8}(1/2^+)\underline{8}(0^-)}=\frac{2}{3}$. Questionable predictions of $SU(6)_W$ avoided by the coplanar symmetry include the selection rules $A_1\not\rightarrow\rho(\lambda=0)\pi$, $B\not\rightarrow\omega(\lambda=1)\pi$. A model of hadron decays which assumes them to occur via production of a $^3P_0$ $q\bar{q}$ pair is consistent with (but more specific than) the coplanar symmetry.

I. INTRODUCTION

The quark model has had considerable success in classifying mesons and baryons, and also describes certain features of their interactions with one another fairly well. Many of the predictions of the quark model can be stated more abstractly in terms of various forms of SU(6).[1]

More recently it has been found that certain quark models which violate $SU(6)_W$ describe hadronic interactions quite a bit more accurately.[2-4] The relativistic quark model,[3,4] for example, has specific $SU(6)_W$-violating interactions which correspond to quark recoil during meson emission. Another picture views hadron decays as proceeding via the creation of a $^3P_0$ $q\bar{q}$ pair.[2,5-7] Matrix elements in this picture, which we shall call the $^3P_0$ picture, bear striking similarity to those in the relativistic quark model.

One is thus led to ask whether there is a symmetry intermediate between $SU(6)_W$ and SU(3) which is characteristic of the relativistic quark model and the $^3P_0$ picture. Such a symmetry in fact exists. It is a coplanar form of $U(3)\otimes U(3)$,[8] originally proposed as a possible description of any process involving two directions. Its specific applicability to hadron decays has been suggested previously.[9]

The model presented in Ref. 2 is sufficiently general that it serves primarily as a parametrization of this symmetry. Our purpose here is mainly an expository one. We wish to present the rules for calculating directly using this symmetry (without the reference to additional parameters of Ref. 2) and to show its relation to other schemes. This will be done with particular reference to decays of the $q\bar{q}$, $L=1$ mesons. We find, in order of increasing predictive power, the sets of predictions given in Eq. (1):

$$\{SU(3)\} \subset \{\text{coplanar } U(3)\otimes U(3)\} \subset \{{}^3P_0 \text{ picture}^{2,5-7}\} \subset \{\text{relativistic quark model}^3 \text{ or } SU(6)_W\}\,. \quad (1)$$

(As mentioned, the last two are inconsistent with one another.) We shall give the physical assumptions that allow one to move to successively more predictive schemes in Eq. (1).

The symmetry $SU(6)_W$ is generated by those generators of $[U(6)\otimes U(6)]_\beta$ (Refs. 1, 8, 10) which commute with the generator α_z of Lorentz boosts in a given direction.[11] The coplanar $U(3)\otimes U(3)$ symmetry is generated by the subset which commutes with Lorentz boosts in a *plane*. The motivation for applying such a symmetry to hadron decays is discussed in Sec. II. Section III deals with the specific example of the decays of the lowest 2^+, 1^+, and 0^+ mesons, classified as $q\bar{q}$, $L=1$ states. The predictions of the coplanar symmetry are compared there with those of other approaches. A brief discussion of more general predictions and of the physical differences between the various schemes is presented in Sec. IV. Appendix A gives rules for classifying states under coplanar $U(3)\otimes U(3)$, while Appendix B deals with transformations between spin quantization axes. Appendix C shows how angular momentum conservation constrains various amplitudes.

II. $SU(6)_W$ VERSUS COPLANAR SYMMETRY

Consider, to begin, quarks and antiquarks at rest. The projection operator for a quark is $(1+\beta)/2$, that for an antiquark is $(1-\beta)/2$. These projections times the spin operators $(\underline{1}, \vec{\sigma})$ times the nine λ_i matrices[12] of U(3) then generate an algebra of $[U(6)\otimes U(6)]_\beta$. Particles may be classified according to this symmetry by building them up from quarks q and antiquarks $\bar{q}$:

$$q \in (\underline{6}, \underline{1}), \quad \bar{q} \in (\underline{1}, \underline{\bar{6}}), \tag{2}$$

so that (for example) π and $\rho \in (\underline{6}, \underline{\bar{6}})$, N and $\Delta \in (\underline{56}, \underline{1})$, $\bar{N}$ and $\bar{\Delta} \in (\underline{1}, \underline{\overline{56}})$, etc. One can imagine the quarks to have some relative orbital angular momentum L, or introduce an abstract quantity with properties of L.[10] The resulting scheme, $[U(6)\otimes U(6)]_\beta \otimes O(3)$, then describes the low-lying hadrons fairly well.[13]

Note at this point that a meson need not be *purely* $q\bar{q}$, L nor a baryon qqq, L for the classification just mentioned to work. The same results will hold if the wave functions have admixtures of SU(3) singlet $q\bar{q}$ pairs, for example a $^3S_1$ pair replacing a unit of L. Experiments on deep-inelastic scattering of electrons indicate that the quark structure (if any) of protons and neutrons is more complicated[14] than the naive picture.[13] Hence we must be wary of any attempt to ascribe the success of $[U(6)\otimes U(6)]_\beta \otimes O(3)$ to any too-realistic quark model. We proceed further bearing such caution in mind.

The symmetry $[U(6)\otimes U(6)]_\beta \otimes O(3)$ must be violated if hadrons are to decay, as one can easily see by taking the example of $\Delta \to N\pi$. The initial state belongs to $(\underline{56}, \underline{1})$. The final state would belong to $(\underline{56}\otimes\underline{6}, \underline{\bar{6}})$ if all the quarks were at rest. This is not the case, of course, since the N and π must be in a relative P wave. Fortunately, a moving quark is not a member of a pure representation of $[U(6)\otimes U(6)]_\beta$. Of the generators of this "rest symmetry," only a subset commute with the "boost" operator $\exp\{\alpha_z\chi/2\}$ [where $\cosh\chi = (1-v^2/c^2)^{-1/2}$]. This latter set[15] forms the algebra of $SU(6)_W$:

$$SU(6)_W: \left\{ \begin{matrix} \lambda_i \\ \lambda_i\sigma_z \\ \lambda_i\beta\sigma_x \\ \lambda_i\beta\sigma_y \end{matrix} \right\} \text{commute with } \alpha_z . \tag{3}$$

A quark moving along the z axis is then to be classified according to the 6-dimensional representation of $SU(6)_W$. Its "W spin" (generated by σ_z, $\beta\sigma_x$, $\beta\sigma_y$) is the same as its quark spin. An antiquark moving along the z axis belongs to $\underline{\bar{6}}$, and its x, y components of W spin are opposite to those of its quark spin. Because of Eq. (3), these classifications are invariant under change of frame along the z axis.

What has happened to the moving quark under the "boost"? The quark number is the eigenvalue of $(1+\beta)/2$, and the antiquark number is the eigenvalue of $(1-\beta)/2$. Then, for a quark at rest, "boosted" by $e^{\alpha_z\chi/2}$,

$$\begin{aligned} N_q &= q^\dagger_{\text{rest}}\; e^{-\alpha_z\chi/2}\frac{1+\beta}{2}e^{\alpha_z\chi/2}q_{\text{rest}} \\ &= \tfrac{1}{2}\,\mathrm{Tr}\left\{\frac{1+\beta}{2}e^{-\alpha_z\chi/2}\frac{1+\beta}{2}e^{\alpha_z\chi/2}\right\} \\ &= \cosh^2(\tfrac{1}{2}\chi), \end{aligned} \tag{4}$$

while

$$\bar{N}_q = \sinh^2(\tfrac{1}{2}\chi). \tag{5}$$

The quark has thus acquired a "cloud" of quark-antiquark pairs [in an SU(3) singlet state] as we move past it. Hence the decay of a hadron into two moving ones is under no obligation to conserve the number of quarks plus antiquarks.

Under $SU(6)_W$, since $W=S$ for quarks, the N and Δ belong to a $\underline{56}$, the negative-parity baryons below ~1800 MeV belong to a $\underline{70}$, and so on. For mesons, the fact that $W_{x,y} = -S_{x,y}$ for antiquarks makes the classification less straightforward. The ρ and π have quark spin 1 and 0, respectively. However, the helicity-zero ρ has $W=0$ while the π has $W=1$. This phenomenon is known as W-S flip.[15, 16] The longitudinal ρ, with $W=0$, can decay without trouble to two pions, each with $W=1$. Similarly the Δ $(W=\frac{3}{2})$ can decay to a π $(W=1)$ and a nucleon $(W=\frac{1}{2})$. If quark spin conservation had been demanded instead, neither of these two decays could occur.

Let us now take L of the initial hadron to be not necessarily zero, while the final hadrons will still be taken to have $L=0$. This situation describes most of the well-studied decays; the exceptions are few enough that a systematic treatment of them would be premature at present. $SU(6)_W$ can still be applied, if one assumption is made. Since the z component of quark spin, S_z, is equal to W_z (which is conserved), and since $J_z = S_z + L_z$ is conserved, L_z must also be conserved. Since the final state has $L_z=0$, so must the initial state. The resulting symmetry[17] has sometimes been called $SU(6)_W \otimes O(2)_{L_z}$,[18] since it entails L_z conservation as well as W-spin conservation. These are not independent, of course.

The symmetry $SU(6)_W \otimes O(2)_{L_z}$ and models equivalent to it have been applied to the decays of mesons,[19] to the decays of baryons,[20] and to photoproduction of baryon resonances.[21] Some of the results indeed agree with experiment fairly well. Typical predictions of this scheme which work qualitatively include the following:

(1) Predictions for decays of $L=0$ hadrons. For $L=0$, the question of what to do with L_z never arises. One then obtains relations such as[22]

$$\Gamma(\Delta\to N\pi)=\frac{12}{25}\frac{g_{N\bar{N}\pi}^{\;2}}{4\pi}\frac{p^3}{m_\Delta^{\;2}}\ . \tag{6}$$

Symmetry-breaking corrections due to masses[22] have been neglected here. Equation (6) then predicts a partial width of about 60 MeV, low by a factor of 2.[23] We will take this as an index of the quality of the "good" predictions of $SU(6)_W$. The "bad" ones turn out to be considerably worse.[2,3]

A prediction analogous to Eq. (6) for mesons, which makes use of vector dominance, is based on an $SU(6)_W$ relation between $g_{\rho\pi\pi}$ and $g_{\omega\rho\pi}$.[24] It reads

$$\frac{\tilde{\Gamma}(\omega\to\gamma\pi^0)}{\tilde{\Gamma}(\rho\to\pi\pi)}=\frac{2e^2}{g_{\rho\pi\pi}^{\;2}}\simeq 8\times10^{-3}\,, \tag{7}$$

whereas the present experimental ratio is about[23] (1 MeV)/(150 MeV) $\simeq(7\times10^{-3})$. Here $\tilde{\Gamma}$ denotes the partial width corrected for phase-space and centrifugal-barrier factors.

(2) Ratios $\Gamma(2^+\to1^-0^-)/\Gamma(2^+\to0^-0^-)$ *for mesons.* A typical prediction is

$$\frac{\Gamma(A_2\to\rho\pi)}{\Gamma(A_2\to K\bar{K})}\simeq\frac{\tilde{\Gamma}(A_2\to\rho\pi)}{\tilde{\Gamma}(A_2\to K\bar{K})}=6\,, \tag{8a}$$

which agrees very well with experiment.[23,25] Such a relation can be generalized to arbitrary even J along the A_2 Regge trajectory, and reads[26]

$$\frac{\tilde{\Gamma}(A_2^*(J)\to\rho\pi)}{\tilde{\Gamma}(A_2^*(J)\to K\bar{K})}=4\frac{J+1}{J}\ . \tag{8b}$$

(3) Baryon f/d ratios. The $SU(6)_W$ predictions for baryons,

$$(f/d)_{5/2^-}=-\tfrac{1}{3}\,, \tag{9}$$

$$(f/d)_{1/2^+}=(f/d)_{5/2^+}=\tfrac{2}{3}\,, \tag{10}$$

seem to be fairly consistent with the data.[27-29] An example of this is the extreme inelasticity of the $\Lambda(1830)\ \frac{5}{2}^-$ resonance. The $N\bar{K}$ branching ratio $x_{N\bar{K}}$ of this resonance is not quoted consistently from experiment to experiment, but most analyses agree that $x_{N\bar{K}}$ is no more than ~10%. On the other hand $(x_{N\bar{K}}x_{\Sigma\pi})^{1/2}$ is significantly larger than this. For $f/d=-\frac{1}{3}$, the $\Lambda(1830)$ would decouple altogether from $N\bar{K}$.

(4) Various photoproduction selection rules.[21] These include the fact that the transition $\gamma N\to\Delta$ seems to be $M1$, the failure of $N^+(\frac{5}{2}^-)$ to be photoproduced from protons, and the failure of the $N^0(\frac{5}{2}^+)$ to be photoproduced in the $\lambda=\frac{3}{2}$ state from neutrons.[30]

On the other hand, there are predictions of $SU(6)_W$ which are poorer than those just mentioned. As an example, we consider the decays of $\underline{35}$, $L=1$ mesons into 1^-0^- and 0^-0^-. Since the $\underline{35}$, $L=1$ mesons have $J^P=2^+,\ 1^+$, and 0^+, the allowed partial waves in the final state are $l=0$ (S) and $l=2$ (D). *$SU(6)_W$ relates these two.* This effect is most strongly felt in the decays $1^+\to1^-0^-$ where both partial waves can occur. $SU(6)_W$ predicts that they will interfere in such a way as to produce vector mesons of a given helicity. For example, $SU(6)_W$ predicts $A_1\not\to\rho(\lambda=0)\pi$ and $B\not\to\omega(\lambda=1)\pi$. Experimentally these decays are very much present and the latter indeed appears to be dominant.[31] In this case, the admixture of D wave seems to have the *opposite sign,* relative to the S wave, from the $SU(6)_W$ prediction.[2]

Similar shortcomings are apparent in baryon decays into $\frac{3}{2}^+\ 0^-$. For example, $SU(6)_W$ predicts that the $\frac{5}{2}^+\ F_{15}\ N(1690)$ resonance should decay into $\Delta\pi$ only in the $\lambda=\frac{1}{2}$ state. (This follows trivially from S_z conservation.) The suppression of $\lambda=\frac{3}{2}$ decays requires the P-wave and F-wave amplitudes to interfere in a very definite way. No evidence for such interference is seen.[32]

If one treats photoproduction using vector dominance, so that the photon is equivalent to a transversely polarized $^3S_1\ q\bar{q}$ pair, S_z conservation means that upon absorbing a photon a quark *must* flip its spin. Put another way, the quark cannot be given any L_z by the photon. This is very far from true. Such a rule would forbid all $\lambda=\frac{3}{2}$ excitations of $S_q=\frac{1}{2}$ resonances, whereas these seem to *dominate* in many instances.[33]

These problems with $SU(6)_W$ are not characteristic of various recent quark models.[2-7] We shall show that such models possess a well-defined *symmetry,* weaker than $SU(6)_W\otimes O(2)_{L_z}$ but stronger than SU(3). The symmetry may be motivated on rather general grounds independent of the models.

As the reader may have suspected, the weak assumption in applying $SU(6)_W$ to decays of L excited hadrons is the restriction $L_z=0$. Physically this corresponds to saying the quarks in the initial hadron have no transverse momentum relative to the eventual decay axis. In our opinion, there is no *a priori* reason for this assumption. We shall now see what happens when it is relaxed.

Consider a hadron with $L\neq0$. Its quarks with spin S are coupled with L to total angular momentum J:

$$|JJ_z\rangle=\sum_{L_zS_z}(SS_zLL_z|JJ_z)\,|LL_z\rangle\,|SS_z\rangle\,. \tag{11}$$

In $SU(6)_W$, matrix elements from states with $L_z\neq0$

are assumed to vanish, as mentioned, so that $S_z = W_z$ may be conserved. Even when the matrix elements for $L_z \neq 0$ do *not* vanish, we shall continue to apply $SU(6)_W$ to the case $L_z = 0$, projecting out the appropriate piece of the initial-state wave function using Eq. (11).

The decay helicity amplitude for $A(L_z=0, \lambda) \to B(\lambda')C$ may then be written

$$\mathfrak{M}[A(L_z=0;\lambda)\to B(\lambda')C] = \sum_i \left(\frac{A}{\alpha, a}\,\middle|\,\frac{B}{\beta, b}\;\frac{C}{\gamma, c}\right)_i \left(\begin{matrix}\alpha\\A\end{matrix}\middle|\begin{matrix}\beta & \gamma\\ B & C\end{matrix}\right)_i \times X[A(L_z=0,\lambda)\to B(\lambda')C]\,, \quad (12)$$

where

$$X[A(L_z=0,\lambda)\to B(\lambda')C] = (W^B\lambda' W^C \lambda-\lambda' \,|\, W^A\lambda) \times (S^A \lambda L 0 \,|\, J\lambda) a^{(0)}[\underline{A}(L)\to \underline{BC}]\,. \quad (13)$$

The z axis is taken to be the direction of B. λ and λ' are the respective projections of J^A and J^B along this axis. The first term on the right-hand side of Eq. (12) is an $SU(6)_W$ Clebsch-Gordan coefficient, with the Clebsch-Gordan coefficients of SU(3) and $SU(2)_W$ factored out.[34] A, B, and C denote $SU(6)_W$ multiplets and α, a; β, b; and γ, c label the dimensionality of $SU(3)\otimes SU(2)_W$ multiplets. The second term is an isoscalar factor[35] with A, B, and C labeling specific isomultiplets. The sum over i corresponds to d and f couplings when α, β, γ are all octets. In the case of mixed SU(3) representations (such as decays involving ω, ϕ, f, and f') two or more expressions (12) must be summed with appropriate mixing coefficients. For decays into specific charge states, $\mathfrak{M}$ in Eq. (12) is to be multiplied by an appropriate isospin Clebsch-Gordan coefficient. In Eq. (13) the first factor expresses the coupling of W spin, while the second is the term coming from Eq. (11) that projects out $L_z = 0$. The reduced matrix element $a^{(0)}[A(L)\to \underline{BC}]$ depends only on the initial and final multiplets. For example, the decays of the 2^+, 1^+, and 0^+ mesons $[\underline{35},\ L=1]$ into 1^-0^- or 0^-0^- pairs $[\underline{35},\ L=0 \otimes \underline{35},\ L=0]$ are described by a *single* reduced matrix element of $SU(6)_W \times O(2)_{L_z}$.

As mentioned previously, decays from $L_z \neq 0$ substates into two $L=0$ hadrons cannot conserve $SU(6)_W$. In certain circumstances they can, however, conserve a "coplanar" symmetry defined by a subset of the generators in Eq. (3). To see this, consider the combinations

$$|\vec{L}=\hat{n}_x\rangle \equiv (|L=1, L_z=-1\rangle - |L=1, L_z=1\rangle)/\sqrt{2}\,, \quad (14)$$

$$|\vec{L}=\hat{n}_y\rangle \equiv i(|L=1, L_z=-1\rangle + |L=1, L_z=1\rangle)/\sqrt{2}\,, \quad (15)$$

$$|\vec{L}=\hat{n}_z\rangle \equiv |L=1, L_z=0\rangle\,. \quad (16)$$

Together these span the states of $L=1$; they correspond to "linear polarization" states. Now take a hadron with $\vec{L}=\hat{n}_x$. The orbital part of the quark wave function is proportional to $Y_1^{-1}(\theta,\phi) - Y_1^1(\theta,\phi) \sim \sin\theta\cos\phi$, corresponding to quarks which tend to move in the x-z plane. When a hadron containing such quarks decays to two $L=0$ hadrons, these quarks must end up traveling essentially along the z axis. The decay Hamiltonian must then give them a "kick" in the x-z plane. We seek a symmetry invariant under such a "kick," i.e., under boosts in the x-z plane, as compared with $SU(6)_W$ which is defined by invariance under boosts in z alone.

A subset of the generators (3) commutes with both α_z and α_x. It forms an algebra of $U(3)\otimes U(3)$ defined by the y component of W spin, W_y:

$$[U(3)\otimes U(3)]_{\beta\sigma_y}:\ \left\{\begin{matrix}\lambda_i\\ \lambda_i\beta\sigma_y\end{matrix}\right\} \text{ commute with } \alpha_z, \alpha_x\,. \quad (17)$$

This is a *coplanar*[8,9] $U(3)\otimes U(3)$ whose applicability to decays of L excited hadrons has been noted previously.[9] Our purpose is to carry out such applications, and to show that the relativistic quark model and $^3P_0$ picture possess this symmetry.

In Appendix A we shall classify hadrons according to multiplets of $[U(3)\otimes U(3)]_{\beta\sigma_y}$: for example,

$$\Delta(W_y=\tfrac{3}{2}) \in (\underline{10}, \underline{1})\,. \quad (18)$$

The first and second numbers here denote the dimension of the representation dealing with W_y up and down, respectively. These classifications of course, are relevant also for $[U(3)\otimes U(3)]_{\beta\sigma_x}$ when W is quantized along the x axis. Since

$$|L=1, L_z=\pm 1\rangle = [\mp|\vec{L}=\hat{n}_x\rangle - i|\vec{L}=\hat{n}_y\rangle]/\sqrt{2}\,, \quad (19)$$

one needs both of the above coplanar groups, but matrix elements associated with the two will turn out to be related by angular momentum conservation. One must transform states of definite S_z to those of definite S_y and S_x in order to apply the symmetry to physical processes. This involves matrices M:

$$|Sm\rangle_z = M_{mm'}^{(S;\,zy)}|Sm'\rangle_y \quad (20a)$$

$$= M_{mm'}^{(S;\,zx)}|Sm'\rangle_x\,, \quad (20b)$$

which are derived in Appendix B. [In Eq. (20) the subscript denotes the quantization axis, while S denotes the total spin.] Appendix C deals with constraints due to angular momentum conservation.

III. DECAYS OF $q\bar{q}$, $L=1$ MESONS

To begin with, we present the matrix elements for decays of the 2^+, 1^+, and 0^+ mesons (A) into 1^-0^- and 0^-0^- (B, C) pairs based on $SU(6)_W$ invariance. These matrix elements are taken to describe the decays of the $L=1$ mesons from the $L_z=0$ state.

Most of the transitions of interest describe the coupling $\underline{35}\to\underline{35}\otimes\underline{35}$. In order that the constraints of charge-conjugation invariance be obeyed, this coupling must be symmetric. This will ensure[34] that the SU(3) couplings of octets be pure F when the product of charge parities of A, B, and C is $-$, and pure D when it is $+$.

Since particle C is always taken to be a member of the pseudoscalar octet, it will always belong to the $\underline{35}$ of $SU(6)_W$. On the other hand, both A and B can involve $SU(6)_W$ singlet states, e.g., in $f\to\pi\pi$ and $B\to\omega(\lambda=0)\pi$. Hence, one would expect – *a priori* – three independent amplitudes, corresponding to $\underline{35}\to\underline{35}\otimes\underline{35}$, $1\to\underline{35}\otimes\underline{35}$, and $\underline{35}\to\underline{1}\otimes\underline{35}$. The second and third are related to the first if we make the quark-model assumptions $f'\not\to\pi\pi$ and $B\not\to\phi\pi$, rules which are certainly well in agreement with present data. As our main concern in what follows is the *spin structure* of decay amplitudes, we shall concentrate on a decay which is pure $\underline{8}\to\underline{8}\otimes\underline{8}$ in SU(3). The most convenient such decay is

$$\text{“}K\text{”}\to\text{“}K\text{”}\pi, \tag{21}$$

where the particle symbols in quotes denote the appropriate octet member. One can then obtain any matrix element involving pure octets in terms of those in Eq. (21). Some of the more useful are summarized in Table I(a). With the additional assumptions $f'\not\to\pi\pi$, $B\not\to\phi\pi$ one can obtain as well decay amplitudes involving octet-singlet mixtures, some of which are compared to those of Eq. (21) in Table I(b).

The helicity amplitudes for “K”$\to$“K”π are now taken from Eqs. (12) and (13), and summarized in the $a^{(0)}$ column of Table II. The W-spin assignments used are based on using $W^-=W_q^-+W_{\bar{q}}^-=S_q^- - S_{\bar{q}}^-$ to construct multiplets, so that

$$\begin{aligned}&|11\rangle_S=|11\rangle_W,\\ &|10\rangle_S=-|00\rangle_W \text{ and } |00\rangle_S=-|10\rangle_W,\\ &|1-1\rangle_S=-|1-1\rangle_W.\end{aligned} \tag{22}$$

The partial widths are related to the values of $\mathfrak{M}_\lambda$ by

$$\Gamma(A\to BC)=\frac{1}{2J_A+1}\frac{p}{M_A{}^2}\sum_\lambda|\mathfrak{M}_\lambda|^2, \tag{23}$$

where p is the magnitude of the final c.m. 3-momentum.

At this point we remind the reader that the only *experimental* reason for doubting the predictions of $SU(6)_W$ comes from helicity distributions in $1^{+\pm}\to1^-0^-$ decays (notably $B\to\omega\pi$).[2] These data would not justify in themselves an attempt to “repair” $SU(6)_W$ at their present level of accuracy. Reliable studies of the contamination due to nonresonant effects in $\pi^\pm p\to\pi^\pm\omega p$ would be very helpful in clearing up this question.

Our prime interest here is a theoretical one: *We have no compelling reason to restrict L_z to be zero.* Hence, we should be surprised if all the predictions of $SU(6)_W$ held true. As we shall see, some of them are preserved even when $L_z\neq0$ decays are admitted.

The initial ($L=1$) state of definite J, J_z may be written as

$$|JJ_z\rangle=\sum_{L_z}(SS_z1L_z|JJ_z)|SS_z\rangle|1L_z\rangle \tag{24}$$

or, using Eqs. (14)–(16), as

$$\begin{aligned}|JJ_z\rangle=&|\vec{L}=\hat{n}_x\rangle\{(SJ_z+1\,1\,-1|JJ_z)|SJ_z+1\rangle-(SJ_z-1\,1\,1|JJ_z)|SJ_z-1\rangle\}/\sqrt{2}\\ &+|\vec{L}=\hat{n}_y\rangle\{(SJ_z+1\,1\,-1|JJ_z)|SJ_z+1\rangle+(SJ_z-1\,1\,1|JJ_z)|SJ_z-1\rangle\}/i\sqrt{2}\\ &+|\vec{L}=\hat{n}_z\rangle(SJ_z\,1\,0|JJ_z)|SJ_z\rangle.\end{aligned} \tag{25}$$

We have already calculated the decay from the $|\vec{L}=\hat{n}_z\rangle$ ($L_z=0$) substate. Now it remains to treat the decays from $|\vec{L}=\hat{n}_{x,y}\rangle$ using $[U(3)\otimes U(3)]_{\beta\sigma_{y,x}}$, respectively, adding the resulting matrix elements in such a way as to ensure conservation of total angular momentum. Rather than presenting all results at every intermediate stage, we prefer to limit the discussion to some examples of how constraints on various amplitudes arise. This discussion is presented in Appendix C. The net result is that there are two independent amplitudes describing $L_z\neq0$ decays into $K^*\pi$ and $K\pi$, which we shall call $a^{(3)}$ and $a^{(8)}$, for reasons stated in Appendix C. Table II gives the amplitudes for “K”$\to$“K”π in terms of these, as well as the $SU(6)_W$-invariant amplitude $a^{(0)}$. The reader is re-

TABLE I. Ratios of isoscalar factors and amplitudes for some meson decays.

(a) Ratios of isoscalar factors			
Decay [a] $A \to BC$	F or D	Isoscalar factor $\begin{pmatrix} 8 & 8 & 8 \\ A & B & C \end{pmatrix}$	Ratio to $\begin{pmatrix} 8 & 8 & 8 \\ K & K & \pi \end{pmatrix}$ $[F: \frac{1}{2};\ D: 3/2\sqrt{5}]$
"π" $\to$ "π" $\pi (A_{1,2} \to \rho\pi)$	F	$(\frac{2}{3})^{1/2}$	$2(\frac{2}{3})^{1/2}$
"π" $\to K\bar{K}$ (e.g. $A_2 \to K\bar{K}$)	D	$-(\frac{3}{10})^{1/2}$	$-(\frac{2}{3})^{1/2}$
"π" $\to \eta\pi$ (e.g. $A_2 \to \eta\pi$)	D	$1/\sqrt{5}$	$\frac{2}{3}$

(b) Ratios of amplitudes to those for "K" $\to$ "K" π for some decays involving octet-singlet mixtures [b]

Decay [a] $A \to BC$	Ratio to $M(K \to K\pi)$
$f \to \pi\pi$ [b]	$-\sqrt{2}$
$f \to K\bar{K}$ [b]	$-\sqrt{2/3}$
$f' \to K\bar{K}$ [b]	$2/\sqrt{3}$
$B \to \omega\pi$ [b]	$2/\sqrt{3}$

[a] For specific charge states, multiply by the appropriate isospin Clebsch-Gordan coefficient.
[b] We take $f = (\frac{1}{3})^{1/2}|\underline{1}\rangle + (\frac{2}{3})^{1/2}|\underline{8}\rangle$; $f' = (\frac{2}{3})^{1/2}|\underline{1}\rangle - (\frac{1}{3})^{1/2}|\underline{8}\rangle$, etc., and have assumed $f' \not\to \pi\pi$, $B \not\to \phi\pi$. Octet couplings in this subtable are all D type.

minded that other physical amplitudes are to be constructed using Table I with isospin Clebsch-Gordan coefficients for specific charge states.

As Table II gives seven amplitudes in terms of three parameters, it involves four relations among these amplitudes. Three of these allow direct comparison of partial widths. They are, for "K" $\to$ "K"π,

$$\mathfrak{M}_1(2^{++} \to 1^-0^-) = (\sqrt{3}/2)\mathfrak{M}_0(2^{++} \to 0^-0^-)\,, \tag{26a}$$

$$\mathfrak{M}_0(1^{++} \to 1^-0^-) = \sqrt{2}\,\mathfrak{M}_1(1^{+-} \to 1^-0^-)\,, \tag{26b}$$

and

$$\tfrac{1}{2}|\mathfrak{M}_0(2^{++} \to 0^-0^-)|^2 + 2|\mathfrak{M}_1(1^{++} \to 1^-0^-)|^2 = |\mathfrak{M}_0(0^{++} \to 0^-0^-)|^2 + |\mathfrak{M}_0(1^{+-} \to 1^-0^-)|^2\,. \tag{26c}$$

A fourth relation can be tested via a triangle inequality: It can be written, for example,

$$\mathfrak{M}_1(2^{++} \to 1^-0^-) + \mathfrak{M}_1(1^{++} \to 1^-0^-) = \sqrt{2}\,\mathfrak{M}_0(1^{+-} \to 1^-0^-)\,. \tag{26d}$$

Let us discuss these relations briefly.

Relation (26a) is one of the better $SU(6)_W$ predictions that we see *continues to hold* when coplanar symmetry is involved. It relates all the pure D-wave decays to one another, and has been shown to agree with experiment when a D-wave centrifugal barrier is used.[25]

Relation (26b) involves $SU(6)_W$-violating decays. It predicts, for example,

$$\frac{\tilde{\Gamma}(A_1 \to \rho(\lambda=0)\pi)}{\sum_{\lambda=\pm1}\{\tilde{\Gamma}(B \to \omega(\lambda)\pi)\}} = 2 \tag{27}$$

when one takes account of Table II. The quantities $\tilde{\Gamma}$ are partial widths uncorrected for any symmetry breaking due to masses, and without any phase-space factors. As the amplitudes $\mathfrak{M}_0(1^{++} \to 1^-0^-)$ and $\mathfrak{M}_1(1^{+-} \to 1^-0^-)$ contain both S- and D-wave contributions, an unambiguous prescription for dealing with centrifugal-barrier effects is hard to find. Similar objections apply to the direct comparison of Eqs. (26c) and (26d) with experiment. On the other hand, there is a parametrization of the symmetry presented here for which comparison with experiment becomes notably easier. Let us decompose the final 1^-0^- states in 1^+ decays into partial-wave amplitudes $a^{(l)}$:

TABLE II. Amplitudes for "K" $\to$ "K" π decays. Coefficients of $a^{(0)}$, $a^{(3)}$, $a^{(8)}$ shown. Other amplitudes may be obtained using Table I. For specific charge states, multiply by the appropriate Clebsch-Gordan coefficient.

$J^{PC}(A)$	$J^{PC}(B, C)$	λ	$a^{(0)}$	$a^{(3)}$	$a^{(8)}$
2^{++}	1^-0^-	1	$-\frac{3}{16}$	0	$-1/2\sqrt{2}$
	0^-0^-	0	$-\sqrt{3}/8$	0	$-1/\sqrt{6}$
1^{++}	1^-0^-	1	$-\frac{3}{16}$	0	$1/2\sqrt{2}$
		0	0	$1/\sqrt{2}$	0
1^{+-}	1^-0^-	1	0	$\frac{1}{2}$	0
		0	$-3/8\sqrt{2}$	0	0
0^{++}	0^-0^-	0	$(\frac{1}{8})(\frac{3}{2})^{1/2}$	0	$-1/\sqrt{3}$

$$\mathfrak{M}_\lambda = \sum_l N_l (l 0 1 \lambda | 1 \lambda) a^{(l)},$$

where N_l are normalization factors chosen for convenience. Specifically,

$$\mathfrak{M}_1(1^{++} \to 1^-0^-) = S_A + D_A/2,$$
$$\mathfrak{M}_0(1^{++} \to 1^-0^-) = S_A - D_A, \tag{28}$$

$$\mathfrak{M}_1(1^{+-} \to 1^-0^-) = (S_B - D_B)/\sqrt{2},$$
$$\mathfrak{M}_0(1^{+-} \to 1^-0^-) = (S_B + 2D_B)/\sqrt{2}. \tag{29}$$

(These represent S-wave and D-wave contributions.) Equations (28) and (29) are completely general. We have chosen the normalizations in such a way that in the $SU(6)_W$ limit, $S_A = D_A = S_B = D_B$.

Now, Eq. (26b) implies that

$$S_A - D_A = S_B - D_B; \tag{30}$$

Eq. (26d) implies that

$$\mathfrak{M}_1(2^{++} \to 1^-0^-) = S_B + 2D_B - (S_A + D_A/2). \tag{31}$$

Together Eqs. (30) and (31) imply that

$$\mathfrak{M}_1(2^{++} \to 1^-0^-) = 3(D_B - D_A/2), \tag{32}$$

whereby this amplitude is expressed (as it should be) in terms of purely D-wave amplitudes. Similarly one can write an expression for $\mathfrak{M}_0(2^{++} \to 0^-0^-)$ expressed in terms of D waves:

$$\mathfrak{M}_0(2^{++} \to 0^-0^-) = 2\sqrt{3}\,(D_B - D_A/2) \tag{33}$$

and an expression for $\mathfrak{M}_0(0^{++} \to 0^-0^-)$ expressed in terms of S waves:

$$\mathfrak{M}_0(0^{++} \to 0^-0^-) = (3/2)^{1/2}(S_B - 2S_A). \tag{34}$$

Equations (28)–(34) are equivalent to Table II. One may then describe the symmetry breaking in a phenomenological way by setting

$$S_A = a_A + b_A(p/p_0)^2,$$
$$S_B = a_B + b_B(p/p_0)^2,$$
$$D_A = d_A(p/p_0)^2, \tag{35}$$
$$D_B = d_B(p/p_0)^2,$$

where p_0 is a suitably chosen scale factor which we find convenient to take equal to 0.5 GeV. Equation (30) then implies

$$a_A = a_B \equiv a \tag{36}$$

and

$$b_A - b_B = d_A - d_B \tag{37}$$

so that four free parameters are involved. In $SU(6)_W$ $a_A = a_B = 0$ and $b_A = b_B = d_A = d_B$.

Present data are inadequate for fixing the parameters of Eq. (35) satisfactorily. A particular case seems to fit what data are available adequately. This is the case $b_A = b_B = 0$, for which the quality of the resulting fit has already been discussed in Ref. 2.

It is worth noting the types of improvement in data that would enable one to check the full range of possibilities suggested by Eqs. (35)–(37).

(a) $0^{++} \to 0^-0^-$ decays [Eq. (34)]. One must ask whether the 0^+ mesons form a true nonet. This requires, in particular, determination of the $\pi\pi/K\bar{K}$ coupling ratios of the $I = Y = 0$ members. Otherwise, one can deal only with the π-like member ("δ"?) and the K-like member (seen at various masses in different $K\pi$ phase shift analyses). Then, *assuming* SU(3) to hold in relating these two, one could solve for both the constant and the momentum-dependent term given sufficiently accurate partial widths. This is not yet possible.

(b) $1^{+-} \to 1^-0^-$ decays [Eq. (29)]. At present, the decay $B \to \omega\pi$ is the only candidate. The mass and width of the B vary somewhat from experiment to experiment.[23] The other nonet members are either missing ($I = Y = 0$ ones) or mix with the 1^{++} nonet ($|S| \neq 0$ ones). Agreement on the exact values of $|\mathfrak{M}_\lambda(B \to \omega\pi)|$ (and their relative phase, if possible) then becomes of prime importance.

(c) $1^{++} \to 1^-0^-$ decays [Eq. (28)]. Here the situation is similar to that just mentioned, with the decay $A_1 \to \rho\pi$ the only candidate. The added problems of a large nonresonant Deck effect in $\pi p \to A_1 p$ and a possibly substantial $\sigma\pi$ mode make the A_1 parameters difficult to determine. It is hoped that experiments on backward production of A_1 will clarify the situation. A Dalitz-plot analysis to determine $\mathfrak{M}_1/\mathfrak{M}_0$ is of particular importance in such cases.

(d) 2^{++} decays [Eqs. (32), (33)]. Improvements possible here include such partial widths as $K^{**} \to \rho K$, $f' \to \bar{K}^*K + K^*\bar{K}$, and $f' \to K\bar{K}$. By and large, however, properties of these mesons are much better known than those of the 1^+ and 0^+ mesons at present.

(e) Interference between 2^+ and 0^+ amplitudes. It has been suggested[36] that the phase of any S-wave $\pi\pi$ resonance under the f_0 would provide a test of the $\Delta L_z = 0$ assumption. Let us assume, with Ref. 36, that there existed an S wave $\pi\pi$ resonance exactly under the f_0 and with *identical nonet properties*. Then, in the narrow-width approximation, the expected $\pi\pi$ angular distribution would take the form

$$W(\cos\theta) \sim \left[\cos^2\theta - \frac{4\sqrt{2}}{3}\frac{a^{(8)}}{a^{(0)}}\sin^2\theta\right]^2. \tag{38}$$

For $a^{(8)}/a^{(0)} = 0$, this would give an angular distribution vanishing at $\theta = 90°$, in accord with observation.[37] To determine if the nonet structure of the

S-wave resonance claimed in Ref. 37 is really the same as that of the f_0, one should perform similar analyses on K^+K^- and K_1K_1 final states, with truly adequate statistics.

It is interesting to see how Table II relates to other approaches intermediate between $SU(6)_W$ and SU(3). To do this, we may consider a relation derived in Ref. 2, which is also true in the relativistic quark model[3]:

$$2(\mathfrak{M}_1/\mathfrak{M}_0)_{A_1\to\rho\pi} = (\mathfrak{M}_0/\mathfrak{M}_1)_{B\to\omega\pi} + 1\,. \qquad (39a)$$

This expression relates the helicity distribution in $A_1 \to \rho\pi$ to that in $B \to \omega\pi$. Instead of (38), Table II implies

$$2(\mathfrak{M}_1/\mathfrak{M}_0)_{A_1\to\rho\pi} = (\mathfrak{M}_0/\mathfrak{M}_1)_{B\to\omega\pi} + a^{(8)}/a^{(3)}\,. \qquad (39b)$$

The ratio $a^{(8)}/a^{(3)}$ is thus an extra degree of freedom as compared with the model giving Eq. (39a)[2] (the "$^3P_0$ model"[2, 5-7]). In the $^3P_0$ model there is only *one* amplitude describing decays from the $L_z \neq 0$ substate. It corresponds to the additional $^3P_0$ $q\bar{q}$ pair being produced with $L_z \neq 0$.

To interpret the extra degree of freedom of the coplanar symmetry we return to the graphical language of Ref. 2. There, the $L=1$ mesons were described by 4×4 Dirac matrices with a Lorentz index μ for orbital angular momentum. This Lorentz index could be saturated in *three* distinct ways to form a Lorentz-invariant coupling as described in Fig. 1:

(a) Contraction with a final momentum: coupling c_0.

(b) Contraction with γ^μ on the final $q\bar{q}$ pair: coupling c_1.

(c) Contraction with γ^μ one of the initial quark lines (spin-orbit effect): coupling c_2.

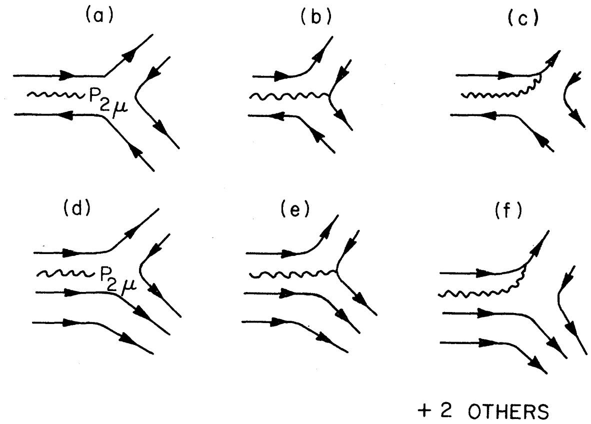

FIG. 1. Graphs describing independent decay amplitudes in the model of Ref. 2. (a)–(c): $L=1$ mesons; (d)–(f): $L=1$ baryons. (a) coupling constant c_0; (b) coupling constant c_1; (c) coupling constant c_2.

Table III contains helicity amplitudes calculated in terms of c_0, c_1, and c_2. By comparing it with Table II, one finds the two descriptions are equivalent to one another with

$$\begin{bmatrix} a^{(3)} \\ a^{(8)} \\ a^{(0)} \end{bmatrix} = \begin{bmatrix} A_1^{(3)} & 0 & 0 \\ A_1^{(3)} & A_2^{(8)} & 0 \\ A_1^{(0)} & A_2^{(0)} & A_3^{(0)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_0 \end{bmatrix}. \qquad (40)$$

(The exact values of the $A_j^{(i)}$ do not matter here.) Thus, $a^{(i)}$ and c_j are related so that when $c_2=0$, $a^{(3)}=a^{(8)}$, and conversely. *The results of the $^3P_0$ model then follow from coplanar symmetry when spin-orbit effects are neglected.* When *both* c_2 and c_1 vanish, $a^{(3)}$ and $a^{(8)}$ vanish, leaving the $SU(6)_W$-conserving term $a^{(0)}$.

TABLE III. Parametrization of $q\bar{q}$, $L=1$ decays in terms of the quark graphs of Fig. 1.[2] This formalism is equivalent to coplanar symmetry. Here m_1, μ are the mean masses of the $q\bar{q}$, $L=1$ and 0 multiplets, respectively, and $p^2 = m_1^2/4-\mu^2$. These are amplitudes for "K"→"K"π decays. Other amplitudes may be obtained using Table I. For specific charge states, multiply by the appropriate isospin Clebsch-Gordan coefficient.

Decay	Coefficient of c_0 [Fig. 1(a)]	c_1 [Fig. 1(b)]	c_2 [Fig. 1(c)]
$2^{++}\to 1^-0^-(\lambda=1)$	$2\sqrt{2}(p/\mu)^2(1+2\mu/m_1)$	$-4\sqrt{2}(p/\mu)^2$	0
$0^-0^-(\lambda=0)$	$4\sqrt{2/3}\,(p/\mu)^2(1+2\mu/m_1)$	$-8(\frac{2}{3})^{1/2}(p/\mu)^2$	0
$1^{++}\to 1^-0^-(\lambda=1)$	$2\sqrt{2}\,(p/\mu)^2(1+2\mu/m_1)$	$4\sqrt{2}(1+m_1/2\mu)^2$	$16\sqrt{2}\,(p/\mu)^2$
$(\lambda=0)$	0	$2\sqrt{2}\,m_1(m_1+2\mu)/\mu^2$	0
$1^{+-}\to 1^-0^-(\lambda=1)$	0	$2m_1(m_1+2\mu)/\mu^2$	0
$(\lambda=0)$	$4\,(p/\mu)^2(1+2\mu/m_1)$	$4(m_1+2\mu)/\mu$	$16(p/\mu)^2$
$0^{++}\to 0^-0^-$	$-(4/\sqrt{3})\,(p/\mu)^2(1+2\mu/m_1)$	$-(4/\sqrt{3})\left(2+\frac{m_1}{\mu}\right)\left(1+\frac{m_1}{\mu}\right)$	$-16\sqrt{3}\,(p/\mu)^2$

IV. DISCUSSION

Our experience with the mesons points to certain $SU(6)_W$ predictions that will continue to hold in the $^3P_0$ model and under (even less restrictive) coplanar symmetry. Consider hadrons with a certain L, S such that $J = L + S$, and let them decay to states for which only one partial wave is allowed. Examples are

$$\underline{8}(2^+) \to 1^-0^-, 0^-0^- \ (L = S = 1; D \text{ wave}), \tag{41}$$

$$\underline{8}(\tfrac{5}{2}^-) \to \tfrac{1}{2}^+ 0^- \ (L = 1, S = \tfrac{3}{2}; D \text{ wave}), \tag{42}$$

$$\underline{8}(\tfrac{5}{2}^+) \to \tfrac{1}{2}^+ 0^- \ (L = 2, S = \tfrac{1}{2}; \ F \text{ wave}), \tag{43}$$

and the Regge recurrences of these. $SU(6)_W$ predicts Eqs. (8)–(10) and relations (9), (10) for Regge recurrences as well. First, we relax $SU(6)_W$ to the $^3P_0$ model, allowing for $L_z \neq 0$ decays. Since the final state contains only one l ($l = L + 1$ here), it turns out that this relaxation does not affect anything. This is because it has been shown[7] that *the* $^3P_0$ *model preserves* $SU(6)_W$ *relations among processes with the same final* l.

Next, we relax the $^3P_0$ model to allow for spin-orbit effects. A unit of L gets "absorbed" by a quark as in Fig. 1(c). We can see that this is impossible for the sequences $A_2(J=2)$, $A_2^*(J=4), \ldots,$ and $\underline{8}(\frac{5}{2}^-, \frac{9}{2}^-, \ldots)$, since all the quark spins and units of L are coupled up to the maximum possible J. The absorption of a unit of L by a quark is impossible since each unit of L and each quark form a state of $J = \frac{3}{2}$. Hence, we expect the predictions (8), (9), as well as

$$(f/d)_{9/2^-,\, L=3} = (f/d)_{13/2^-,\, L=5} = \cdots = -\tfrac{1}{3} \tag{44}$$

to *remain valid when spin-orbit effects of the type shown in Fig. 1(c) are admitted.* This has already been shown for Eq. (8a). Furthermore, explicit calculation using results of the appendices allows one to conclude that *coplanar symmetry implies Eq. (9)* as well. This happens for a rather pretty reason. By virtue of angular momentum Clebsch-Gordan coefficients, one finds that an octet member with $L = 1$, $J^P = \frac{5}{2}^-$, helicity $\pm\frac{1}{2}$ belongs only to $(\underline{8}, \underline{1})$ or $(\underline{1}, \underline{8})$. The decay of such a particle to a $\frac{1}{2}^+$, $L = 0$ baryon and a 0^-, $L = 0$ meson, belonging respectively to $(\underline{6}, \underline{3})$ or $(\underline{3}, \underline{6})$ and $(\underline{3}, \underline{\bar{3}})$ or $(\underline{\bar{3}}, \underline{3})$, is thus characterized by a single independent amplitude and hence by a unique f/d. This, of course, must be the f/d of $SU(6)_W$, and we have verified that such is indeed the case.

On the other hand, for $L \geq 3$, we cannot justify the use of coplanar symmetry at present, and hence, have no way of proving Eqs. (8b) or (44) except by direct reference to the absence of graphs such as Fig. 1(c), as mentioned above. When $L \leq 2$, a fortunate circumstance allows one to prove Eq. (10). Since the decay of a quark-spin-$\frac{1}{2}$ resonance with $L_z = 2$ to $\frac{1}{2}^+ 0^-$ is forbidden (remember the z axis is the decay axis, so the final J_z must be $\pm\frac{1}{2}$), decays such as $N(\frac{5}{2}^+; 1688) \to N\pi$ involve initial $L_z \leq 1$. As usual, one can treat the $L_z = 0$ substate via $SU(6)_W$. The states with $L_z = \pm 1$ can be written

$$|L=2, L_z=\pm 1\rangle = \frac{1}{\sqrt{2}}[\,|L=1, L_z=\pm 1\rangle \otimes |L=1, L_z=0\rangle + |L=1, L_z=0\rangle \otimes |L=1, L_z=\pm 1\rangle\,], \tag{45}$$

or, symbolically, recalling Eqs. (14)–(16), as

$$|L=2, L_z=\pm 1\rangle = \tfrac{1}{2}[(\mp\hat{n}_x - i\hat{n}_y)\hat{n}_z + \hat{n}_z(\mp\hat{n}_x - i\hat{n}_y)]. \tag{46}$$

Hence these states are linear superpositions of ones involving only one direction besides the z direction. We then expect coplanar symmetry to be applicable. Since only the transition $[(\underline{6}, \underline{3})$ or $(\underline{3}, \underline{6})] \to [(\underline{6}, \underline{3})$ or $(\underline{3}, \underline{6})] \otimes [(\underline{3}, \underline{\bar{3}})$ or $(\underline{\bar{3}}, \underline{3})]$ is involved, a unique f/d [that of Eq. (10)] results.

There are predictions of f/d ratios made by $SU(6)_W$ and the $^3P_0$ model but *not* by coplanar symmetry: notably, that *all* $\underline{70}$, $L = 1 \to \underline{56} \otimes \underline{35}$ decays should have

$$f/d = \tfrac{5}{3} \text{ for } \underline{8}(S_q = \tfrac{1}{2}) \to \underline{8} \otimes \underline{8}, \tag{47}$$

$$f/d = -\tfrac{1}{3} \text{ for } \underline{8}(S_q = \tfrac{3}{2}) \to \underline{8} \otimes \underline{8}. \tag{48}$$

The reason unique f/d values do not result in coplanar symmetry is that – except for the states with $J^P = \frac{5}{2}^-$, mentioned above – all the $J_z = \pm\frac{1}{2}$ states are mixtures of $(\underline{8}, \underline{1})$, $(\underline{6}, \underline{3})$, $(\underline{\bar{3}}, \underline{3}) + (W_y \to -W_y)$. There are thus, in principle, *three* independent $|L_z| = 1$ amplitudes describing decays of such resonances: $\mathfrak{M}^{8,\,63}_{1,\,3\bar{3}}$, $\mathfrak{M}^{6,\,33}_{3,\,6\bar{3}}$, and $\mathfrak{M}^{\bar{3},\,33}_{3,\,6\bar{3}}$. (See Appendix C for notation.) There is another $|L_z| = 1$ amplitude describing decays to $\underline{10} \otimes \underline{8}$, namely $\mathfrak{M}^{6,\,10\bar{3}}_{3,\,13}$. This means *four* amplitudes describe $|L_z| = 1$ decays and one [the $SU(6)_W$-invariant one] describes $L_z = 0$ decays of the $\underline{70}$, $L = 1$ into $\underline{56} \otimes \underline{35}$. This total of *five* amplitudes makes for trouble in comparison with experiment.[38]

Experimentally there does seem to be some evidence for the existence of a $J^P = \frac{3}{2}^-$ baryon octet, containing the $N(1520)$, with f/d close to the value of $\frac{5}{3}$ that would be predicted by $SU(6)_W$, the $^3P_0$ model, or the relativistic quark model *for a pure* $S_q = \frac{1}{2}$ *state*.[27–29] Speaking optimistically, this may mean one or more of these more specialized schemes is correct. Speaking pessimistically, one is disturbed by the lack of clear separation of the $\Sigma(1660)$ states, *two* of which seem to be $\frac{3}{2}^-$, the arbitrariness of mixing of the Λ state with the unitary singlet, and the uncertain status of the

$\Xi(1820)$ as a $\frac{3}{2}^-$ resonance.

It is amusing that the prediction $f/d = -\frac{1}{3}$ for the $\frac{5}{2}^-$ octet, which is stronger than the other $SU(6)_W$ predictions of f/d ratios as just mentioned, is also the most reliable such prediction in various studies based on duality.[39]

At present, coplanar symmetry does not seem to say very much regarding photoproduction. It forbids

$$\gamma p \rightarrow N^{*+}(1670, \tfrac{5}{2}^-; \lambda = \pm \tfrac{1}{2}) \tag{49}$$

but does not seem to say anything regarding excitation of the $\lambda = \pm \frac{3}{2}$ state. Other selection rules of the relativistic quark model[3] do not seem to follow readily. This suggests that – especially for photoproduction – the relativistic quark model may embody some very desirable dynamical features which our weak symmetry arguments are unable to provide.

We have steered clear of $SU(6)_W$ (Ref. 40) and coplanar $U(3) \otimes U(3)$ (Ref. 41) for $2 \rightarrow 2$ reactions, in view of their more questionable applicability in such cases.[42] Nonetheless, if one applied coplanar symmetry *only to* $K^{\pm}N$ *and* πN *reactions*, one might expect relations of roughly the same quality as the Johnson-Treiman relations. We should note, moreover, that the apparent violation of $SU(6)_W$ in comparing (6) with experiment is probably as bad as that of the Johnson-Treiman relations. Thus, without a more sophisticated understanding of "kinematic" symmetry-breaking effects, $SU(6)_W$ for $L = 0$ decays and consequently coplanar symmetry for $L = 1$ decays may be no more accurate than ±50%, and their violation *may indeed* spring from the same unknown mechanism as the violation of the Johnson-Treiman relations. The violation of (6) enhances the contribution of the Δ pole over the N pole in $\pi N \rightarrow \pi N$. This turns out, by duality, to push the D/F ratio of the crossed-channel nonflip coupling of the tensor and vector trajectories to octet baryons from 0 to a negative value.[43] This is indeed the direction required by experimental data.[5,43,44] It is notable that some saturation schemes for superconvergence relations[45] indeed violate $SU(6)_W$ in places where ΔL_z *must* be zero, e.g., in relations such as (7).

The physical circumstances under which the various symmetries presented here are likely to be valid are summarized in Fig. 2. We believe the results of coplanar symmetry to be as valid as the particle classification itself. The most likely source of deviation from such relations as Eq. (6) and Eqs. (28)–(34) would thus be mixing of multiplets: e.g., $\underline{56}$, $L = 0$ mixing with $\underline{56}$, $L = 2$.[46] At present no evidence has appeared for spin-orbit effects,[2,3,7] so that even the $^3P_0$ model may be a good approximation to decays. In decays of 2^+, 1^+, and 0^+ mesons to 1^-0^- or 0^-0^-, and in baryonic decays to $\frac{1}{2}^+0^-$ and $\frac{3}{2}^+0^-$, the $^3P_0$ model involves two independent partial-wave amplitudes.[2,7] These in turn are related to one another in more restrictive symmetries: apparently incorrectly in $SU(6)_W$ and correctly in the relativistic quark model. This model thus seems to contain a couple of nontrivial dynamical statements. We have shown that some but by no means all of its relations follow from symmetry arguments alone.

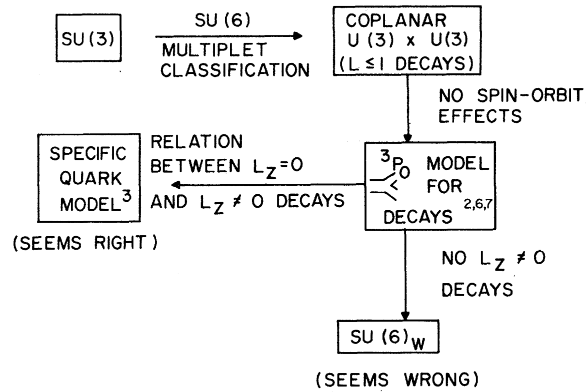

FIG. 2. Hierarchy of symmetries expressed by Eq. (1). The labels on the arrows denote sets of assumptions that lead to successively more restrictive symmetries.

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APPENDIX A: CLASSIFICATION OF HADRONS IN MULTIPLETS OF $[U(3) \otimes U(3)]_{\beta\sigma y}$

1. Quarks[47,48]

The projection operators

$$\textstyle\sum_y^{\pm} \equiv \frac{1}{2}(1 \pm \beta\sigma_y) \tag{A1}$$

have the following effect on quarks and antiquarks:

$$\textstyle\sum_y^{\pm} \left\{ \begin{matrix} q \\ \bar{q} \end{matrix} \right\} (S_y = \pm \tfrac{1}{2}) = + \left\{ \begin{matrix} q \\ \bar{q} \end{matrix} \right\} (S_y = \pm \tfrac{1}{2}), \tag{A2}$$

$$\textstyle\sum_y^{\mp} \left\{ \begin{matrix} q \\ \bar{q} \end{matrix} \right\} (S_y = \pm \tfrac{1}{2}) = 0. \tag{A3}$$

Then

$$q, S_y = \tfrac{1}{2} \in (\underline{3}, \underline{1})_y\,,$$
$$\overline{q}, S_y = \tfrac{1}{2} \in (\underline{\overline{3}}, \underline{1})_y\,,$$
$$q, S_y = -\tfrac{1}{2} \in (\underline{1}, \underline{3})_y\,, \qquad \text{(A4)}$$
$$\overline{q}, S_y = -\tfrac{1}{2} \in (\underline{1}, \underline{\overline{3}})_y\,.$$

Specifically, we shall denote (say) a u quark[47] with $S_y = \frac{1}{2}$ by the symbol $(u, -)$, where the dash indicates a unitary singlet.

2. Mesons

The states of $q\overline{q}$ are ${}^3S_1(S_y = 1, 0, -1) \equiv V$ and ${}^1S_0 \equiv P$. In terms of quark spin, one may write

$$V(S_y = 1) = q(S_y = \tfrac{1}{2})\overline{q}(S_y = \tfrac{1}{2})\,,$$
$$V(S_y = 0) = [\,q(S_y = \tfrac{1}{2})\overline{q}(S_y = -\tfrac{1}{2}) + q(S_y = -\tfrac{1}{2})\overline{q}(S_y = \tfrac{1}{2})]/\sqrt{2}\,, \qquad \text{(A5)}$$
$$V(S_y = -1) = q(S_y = -\tfrac{1}{2})\overline{q}(S_y = -\tfrac{1}{2})\,,$$

and

$$P = [\,q(S_y = \tfrac{1}{2})\overline{q}(S_y = -\tfrac{1}{2}) - q(S_y = -\tfrac{1}{2})\overline{q}(S_y = \tfrac{1}{2})]/\sqrt{2}\,. \qquad \text{(A6)}$$

These then belong to the following multiplets of $[\mathrm{U}(3)\otimes\mathrm{U}(3)]_{\beta\sigma_y}$:

$$V(S_y = 1) \in (\underline{8}, \underline{1})_y \oplus (\underline{1}, \underline{1})_y\,,$$
$$V(S_y = 0) \in [\,(\underline{3}, \underline{\overline{3}})_y + (\underline{\overline{3}}, \underline{3})_y\,]/\sqrt{2}\,, \qquad \text{(A7)}$$
$$V(S_y = -1) \in (\underline{1}, \underline{8})_y \oplus (\underline{1}, \underline{1})_y\,,$$

and

$$P \in [\,(\underline{3}, \underline{\overline{3}})_y - (\underline{\overline{3}}, \underline{3})_y\,]/\sqrt{2}\,. \qquad \text{(A8)}$$

The total multiplicity of V and P is that of $(\underline{8}, \underline{1}) \oplus (\underline{1}, \underline{1}) \oplus (\underline{3}, \underline{\overline{3}}) \oplus (\underline{\overline{3}}, \underline{3}) + (\underline{1}, \underline{8}) + (\underline{1}, \underline{1})$, i.e., 36. The corresponding $q\overline{q}$ multiplets in $\mathrm{SU}(6)_W$ are an $\mathrm{SU}(6)\underline{1}$, containing the $S_z = 0$ vector-meson unitary singlet and a $\underline{35}$, containing all the remaining particles.

The specific members of the multiplets (A7) and (A8) are easily written down. We shall adopt notation whereby an octet member is labeled by the corresponding pseudoscalar meson: e.g.,

$$\rho^+(S_y = 1) = (\pi^+, -)_y\,, \qquad \text{(A9)}$$
$$\rho^+(S_y = 0) = [\,(u, \overline{d})_y + (\overline{d}, u)_y\,]/\sqrt{2}\,, \qquad \text{(A10)}$$
$$\rho^+(S_y = -1) = (-, \pi^+)_y\,, \qquad \text{(A11)}$$

while

$$\pi^+(S_y = 0) = [\,(u, \overline{d})_y - (\overline{d}, u)_y\,]/\sqrt{2}\,. \qquad \text{(A12)}$$

The multiplet assignments for octet-singlet mixtures are straightforward except in the case of $S_y = \pm 1$, for which

$$\omega(S_y = 1) = (\tfrac{1}{3})^{1/2}(\eta, -)_y + (\tfrac{2}{3})^{1/2}(-, -)_y\,, \qquad \text{(A13)}$$
$$\phi(S_y = 1) = (\tfrac{2}{3})^{1/2}(\eta, -)_y - (\tfrac{1}{3})^{1/2}(-, -)_y\,. \qquad \text{(A14)}$$

3. qq states

In building the baryons it will be helpful to define members of the $\underline{21}$ and $\underline{15}$ of SU(6), which are formed by respective symmetric and antisymmetric combinations of $\underline{6}\otimes\underline{6}$. Under $\mathrm{SU}(3)\otimes\mathrm{SU}(2)$, these multiplets reduce as follows:

$$\underline{21} = (\underline{6}, \underline{3}) \oplus (\underline{\overline{3}}, \underline{1})\,, \qquad \text{(A15)}$$
$$\underline{15} = (\underline{6}, \underline{1}) \oplus (\underline{\overline{3}}, \underline{3})\,. \qquad \text{(A16)}$$

The SU(3) representations of nonzero triality indicated on the right-hand side occur often enough in our calculations that we find it helpful to define them in Fig. 3.

The reduction of the multiplets under $[\mathrm{U}(3)\otimes\mathrm{U}(3)]_{\beta\sigma_y}$ is as follows:

$$(qq)_{\underline{21}}, S_y = 1 \in (\underline{6}, \underline{1})_y\,, \qquad \text{(A17)}$$
$$(qq)_{\underline{21}}, S_y = 0 \in (\underline{3}, \underline{3})_{y,\underline{21}}\,, \qquad \text{(A18)}$$
$$(qq)_{\underline{21}}, S_y = -1 \in (\underline{1}, \underline{6})_y\,, \qquad \text{(A19)}$$

while

$$(qq)_{\underline{15}}, S_y = 1 \in (\underline{\overline{3}}, \underline{1})_y\,, \qquad \text{(A20)}$$
$$(qq)_{\underline{15}}, S_y = 0 \in (\underline{3}, \underline{3})_{y,\underline{15}}\,, \qquad \text{(A21)}$$
$$(qq)_{\underline{15}}, S_y = -1 \in (\underline{1}, \underline{\overline{3}})_y\,. \qquad \text{(A22)}$$

Assignments are unambiguous except for the $S_y = 0$ states. Here one has the following examples (see Fig. 3 for the state labels):

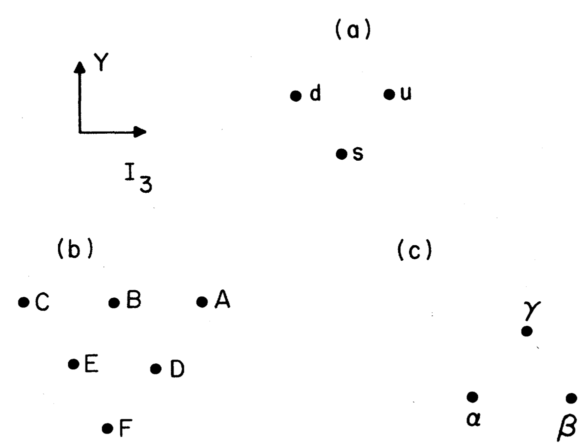

FIG. 3. (a) Quark states; (b) qq states belonging to $\underline{6}$ of SU(3); (c) qq states belonging to $\underline{3}$ of SU(3).

$$A(S=1, S_y=0)(\in \underline{21}) = (p,p)_{y,\underline{21}}, \tag{A23}$$

$$A(S=0, S_y=0)(\in \underline{15}) = (p,p)_{y,\underline{15}}, \tag{A24}$$

$$\gamma(S=1, S_y=0)(\in \underline{15}) = [(p,n)_{y,\underline{15}} - (n,p)_{y,\underline{15}}]/\sqrt{2}, \tag{A25}$$

$$\gamma(S=0, S_y=0)(\in \underline{21}) = [(p,n)_{y,\underline{21}} - (n,p)_{y,\underline{21}}]/\sqrt{2}, \tag{A26}$$

to show the ambiguity that arises if the origin of the $(\underline{3},\underline{3})$ representation is not labeled. Such a (purely formal) labeling allows one to keep track of the symmetry or antisymmetry of the SU(6) part of the quark wave function. One could imagine this to be important in a "realistic" quark calculation for which such symmetry or antisymmetry would determine permutation properties of the *spatial* part of the wave function.

4. Baryons as qqq states

The qqq baryons encompass the SU(6) representations $\underline{56} \oplus \underline{70} = \underline{21} \otimes \underline{6}$ and $\underline{70} \oplus \underline{20} = \underline{15} \otimes \underline{6}$. Since under $[\mathrm{U}(3) \otimes \mathrm{U}(3)]_{\beta\sigma_y}$ one has

$$\underline{21} = (\underline{6},\underline{1})_y \oplus (\underline{3},\underline{3})_{y,\underline{21}} \oplus (\underline{1},\underline{6})_y, \tag{A27}$$

$$\underline{15} = (\underline{\bar{3}},\underline{1})_y \oplus (\underline{3},\underline{3})_{y,\underline{15}} \oplus (\underline{1},\underline{\bar{3}})_y, \tag{A28}$$

$$\underline{6} = (\underline{3},\underline{1})_y \oplus (\underline{1},\underline{3})_y, \tag{A29}$$

the following decompositions can be extracted:

$$\underline{56} = (\underline{10},\underline{1})_y \oplus (\underline{6},\underline{3})_{y,\underline{56}} \oplus (\underline{3},\underline{6})_{y,\underline{56}} \oplus (\underline{1},\underline{10})_y, \tag{A30}$$

$$\underline{70} = (\underline{8},\underline{1})_y \oplus (\underline{6},\underline{3})_{y,\underline{70}} \oplus (\underline{\bar{3}},\underline{3})_{y,\underline{70}} \oplus (\underline{3},\underline{\bar{3}})_{y,\underline{70}} \oplus (\underline{3},\underline{6})_{y,\underline{70}} \oplus (\underline{1},\underline{8})_y, \tag{A31}$$

$$\underline{20} = (\underline{1},\underline{1})_y \oplus (\underline{\bar{3}},\underline{3})_{y,\underline{20}} \oplus (\underline{3},\underline{\bar{3}})_{y,\underline{20}} \oplus (\underline{1},\underline{1})_y. \tag{A32}$$

Some ambiguities arise in identifying multiplets of definite quark spin. For this reason we shall be more explicit in our construction.

The $S_y = \pm\frac{3}{2}$ states are clearly identified in each of Eqs. (A30)–(A32) since the $S=\frac{3}{2}$ members of the $\underline{56}$, $\underline{70}$, $\underline{20}$ are, respectively, $\underline{10}$, $\underline{8}$, and $\underline{1}$ of SU(3). The $S_y = \pm\frac{1}{2}$ states, however, pose more of a problem.

Let us take $\{\Delta^{++}, S_y=\frac{1}{2}\} = (A,u)$. Then acting with I_- we have

$$\{\Delta^+, S_y=\tfrac{1}{2}\} = (\tfrac{2}{3})^{1/2}(B,u) + (\tfrac{1}{3})^{1/2}(A,d).$$

The orthogonal state is a proton:

$$\{p, S_y=\tfrac{1}{2}\} = (\tfrac{2}{3})^{1/2}(A,d) - (\tfrac{1}{3})^{1/2}(B,u).$$

Then acting with I_- we construct the neutron:

$$\{n, S_y=\tfrac{1}{2}\} = (\tfrac{1}{3})^{1/2}(B,d) - (\tfrac{2}{3})^{1/2}(C,u).$$

Note, however, that the members of the $(\underline{6},\underline{3})$ representation of $[\mathrm{U}(3)\otimes \mathrm{U}(3)]_{\beta\sigma_y}$ we have just constructed have nothing to label them as having come from the $\underline{56}$ of $\mathrm{SU}(6)_W$. A similar $(\underline{6},\underline{3})$ comes from the $\underline{70}$. For *this* $(\underline{6},\underline{3})$, however, the decimet has $S=\frac{1}{2}$ and the octet is an (*a priori*) arbitrary mixture of $S=\frac{1}{2}$ and $S=\frac{3}{2}$.

Our practical problem is this: Given an assignment of quark spin S to a member of the $\underline{70}$ (no firm evidence for the $\underline{20}$ exists), what does this say about the $[\mathrm{U}(3)\otimes \mathrm{U}(3)]_{\beta\sigma_y}$ assignment? For example, suppose we take the $D_{13}(1520)$ $\frac{3}{2}^-$ πN resonance to have $S_q=\frac{1}{2}$. Does the $S_y=\frac{1}{2}$ state of this particle then fit into $(\underline{6},\underline{3})_{\underline{70}}$ into $(\underline{\bar{3}},\underline{3})_{\underline{70}}$, or into some mixture?

At present we suspect the answer to this question is nontrivial and would involve assumptions which are more model-dependent than those we choose to make.[49] For the time being, therefore, we can only write

$$\{\underline{70}:(\underline{8},\underline{4}); S_y=\tfrac{1}{2}\} \in \cos\theta\,(\underline{6},\underline{3})_{y,\underline{70}} + \sin\theta\,(\underline{\bar{3}},\underline{3})_{y,\underline{70}}, \tag{A33}$$

$$\{\underline{70}:(\underline{8},\underline{2}); S_y=\tfrac{1}{2}\} \in -\sin\theta\,(\underline{6},\underline{3})_{y,\underline{70}} + \cos\theta\,(\underline{\bar{3}},\underline{3})_{y,\underline{70}}. \tag{A34}$$

APPENDIX B: TRANSFORMATION BETWEEN STATES OF DEFINITE S_z AND THOSE OF DEFINITE S_y, S_x

The symbols $|Sm\rangle_{x,y,z}$ will denote spin eigenfunctions when $x, y,$ and z are the respective quantization axes. For definiteness in phases we shall build spin eigenfunctions from those with $S=\frac{1}{2}$. Define

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle_y \equiv R_x(-\tfrac{1}{2}\pi)\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_z, \tag{B1}$$

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle_x \equiv R_y(\tfrac{1}{2}\pi)\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_z, \tag{B2}$$

where $R_i(\theta)$ is a rotation by θ about the i axis. In terms of Pauli spinors this means that with

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle_z = \begin{bmatrix}1\\0\end{bmatrix} \tag{B3}$$

one has

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle_y = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} = \frac{1}{\sqrt{2}}[\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_z + i\,|\tfrac{1}{2}-\tfrac{1}{2}\rangle_z], \tag{B4}$$

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle_x = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}[\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_z + |\tfrac{1}{2}-\tfrac{1}{2}\rangle_z]. \tag{B5}$$

We shall then define

$$|\tfrac{1}{2}-\tfrac{1}{2}\rangle_{x,y} \equiv S_{x,y}^-\,|\tfrac{1}{2}\tfrac{1}{2}\rangle_{x,y}, \tag{B6}$$

where

$$S_y^- = S_z - iS_x = S_z - \tfrac{1}{2}i(S_z^+ + S_z^-) \tag{B7}$$

and

$$S_x^- = S_y - iS_z$$
$$= \tfrac{1}{2}i(S_z^- - S_z^+) - iS_z \ . \tag{B8}$$

Applying these to (B4) and (B5), one has

$$|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_y = \frac{1}{\sqrt{2}}\left[\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_z - i\,|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_z\right] \tag{B9}$$

and

$$|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_x = \frac{i}{\sqrt{2}}\left[-|\tfrac{1}{2}\,\tfrac{1}{2}\rangle_z + |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle_z\right] . \tag{B10}$$

The states with spin 1 are defined by setting

$$|11\rangle_{x,y,z} \equiv |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_{x,y,z} \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_{x,y,z} \tag{B11}$$

and then applying $S^-_{x,y,z}$ successively. Similarly we define

$$|\tfrac{3}{2}\,\tfrac{3}{2}\rangle_{x,y,z} \equiv |11\rangle_{x,y,z} \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle_{x,y,z} \ . \tag{B12}$$

In proceeding from (B11) or (B12) we assume all phases are positive:

$$S^-_{x,y,z}\,|Sm\rangle_{x,y,z} = +\left[(S+m)(S-m+1)\right]^{1/2} \times |S\,m-1\rangle_{x,y,z} \ . \tag{B13}$$

We shall also define the spin-0 states in terms of spin $\tfrac{1}{2}$:

$$|00\rangle_{x,y,z} = \frac{1}{\sqrt{2}}\left[\,|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right]_{x,y,z} . \tag{B14}$$

Expressions for $|Sm\rangle_z$ in terms of $|Sm'\rangle_{x,y}$ are given in Tables IV and V for $S=0$, $\tfrac{1}{2}$, 1, and $\tfrac{3}{2}$. These tables are in the form of matrices (essentially Wigner D functions, which we have thus calculated for specific cases):

$$|Sm\rangle_z = M^{(S;zy)}_{m\,m'}\,|Sm'\rangle_y \tag{B15}$$
$$= M^{(S;zx)}_{m\,m'}\,|Sm'\rangle_x \ . \tag{B16}$$

These transformations are unitary, so that

$$|Sm'\rangle_y = M^{*(S;zy)}_{m\,m'}\,|Sm\rangle_z \ , \tag{B17}$$

$$|Sm'\rangle_x = M^{*(S;zx)}_{m\,m'}\,|Sm\rangle_z \ . \tag{B18}$$

APPENDIX C: CONSTRAINTS DUE TO ANGULAR MOMENTUM CONSERVATION

Consider an amplitude for the decay $(\alpha_1, \alpha_2) \to (\beta_1, \beta_2) \otimes (\gamma_1, \gamma_2)$. We shall label such an amplitude by $\mathfrak{M}^{\alpha_1;\,\beta_1\gamma_1}_{\alpha_2;\,\beta_2\gamma_2}$ (x or y), depending on whether

TABLE IV. Expressions for $|Sm\rangle_z$ in terms of $|Sm'\rangle_y$ for $S=0$, $\tfrac{1}{2}$, 1, and $\tfrac{3}{2}$. Matrices $M^{(S;zy)}_{mm'}$.

$S=0$ $|00\rangle_z = i\,|00\rangle_y$

$S=\tfrac{1}{2}$

$m \backslash m'$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$
$\tfrac{1}{2}$	$(\tfrac{1}{2})^{1/2}$	$(\tfrac{1}{2})^{1/2}$
$-\tfrac{1}{2}$	$-i(\tfrac{1}{2})^{1/2}$	$i(\tfrac{1}{2})^{1/2}$

$S=1$

$m \backslash m'$	1	0	-1
1	$\tfrac{1}{2}$	$(\tfrac{1}{2})^{1/2}$	$\tfrac{1}{2}$
0	$-i(\tfrac{1}{2})^{1/2}$	0	$i(\tfrac{1}{2})^{1/2}$
-1	$-\tfrac{1}{2}$	$(\tfrac{1}{2})^{1/2}$	$-\tfrac{1}{2}$

$S=\tfrac{3}{2}$

$m \backslash m'$	$\tfrac{3}{2}$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$	$-\tfrac{3}{2}$
$\tfrac{3}{2}$	$(\tfrac{1}{8})^{1/2}$	$(\tfrac{3}{8})^{1/2}$	$(\tfrac{3}{8})^{1/2}$	$(\tfrac{1}{8})^{1/2}$
$\tfrac{1}{2}$	$-i(\tfrac{3}{8})^{1/2}$	$-i(\tfrac{1}{8})^{1/2}$	$i(\tfrac{1}{8})^{1/2}$	$i(\tfrac{3}{8})^{1/2}$
$-\tfrac{1}{2}$	$-(\tfrac{3}{8})^{1/2}$	$(\tfrac{1}{8})^{1/2}$	$(\tfrac{1}{8})^{1/2}$	$-(\tfrac{3}{8})^{1/2}$
$-\tfrac{3}{2}$	$i(\tfrac{1}{8})^{1/2}$	$-i(\tfrac{3}{8})^{1/2}$	$i(\tfrac{3}{8})^{1/2}$	$-i(\tfrac{1}{8})^{1/2}$

TABLE V. Expressions for $|Sm\rangle_z$ in terms of $|Sm'\rangle_x$ for $S=0$, $\tfrac{1}{2}$, 1, and $\tfrac{3}{2}$. Matrices $M^{(S;zx)}_{mm'}$.

$S=0$ $|00\rangle_z = -i\,|00\rangle_x$

$S=\tfrac{1}{2}$

$m \backslash m'$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$
$\tfrac{1}{2}$	$(\tfrac{1}{2})^{1/2}$	$i(\tfrac{1}{2})^{1/2}$
$-\tfrac{1}{2}$	$(\tfrac{1}{2})^{1/2}$	$-i(\tfrac{1}{2})^{1/2}$

$S=1$

$m \backslash m'$	1	0	1
1	$\tfrac{1}{2}$	$i(\tfrac{1}{2})^{1/2}$	$-\tfrac{1}{2}$
0	$(\tfrac{1}{2})^{1/2}$	0	$(\tfrac{1}{2})^{1/2}$
-1	$\tfrac{1}{2}$	$-i(\tfrac{1}{2})^{1/2}$	$-\tfrac{1}{2}$

$S=\tfrac{3}{2}$

$m \backslash m'$	$\tfrac{3}{2}$	$\tfrac{1}{2}$	$-\tfrac{1}{2}$	$-\tfrac{3}{2}$
$\tfrac{3}{2}$	$(\tfrac{1}{8})^{1/2}$	$i(\tfrac{3}{8})^{1/2}$	$-(\tfrac{3}{8})^{1/2}$	$-i(\tfrac{1}{8})^{1/2}$
$\tfrac{1}{2}$	$(\tfrac{3}{8})^{1/2}$	$i(\tfrac{1}{8})^{1/2}$	$(\tfrac{1}{8})^{1/2}$	$i(\tfrac{3}{8})^{1/2}$
$-\tfrac{1}{2}$	$(\tfrac{3}{8})^{1/2}$	$-i(\tfrac{1}{8})^{1/2}$	$(\tfrac{1}{8})^{1/2}$	$-i(\tfrac{3}{8})^{1/2}$
$-\tfrac{3}{2}$	$(\tfrac{1}{8})^{1/2}$	$-i(\tfrac{3}{8})^{1/2}$	$-(\tfrac{3}{8})^{1/2}$	$i(\tfrac{1}{8})^{1/2}$

the $U(3)\otimes U(3)$ generated by λ_i and $\lambda_i\beta\sigma_x$ or $\lambda_i\beta\sigma_y$ is being considered.

We are treating the decays "K" → "K"π. In order to do this most simply we shall consider a particular charge state, "K^+" → "K^0"π^+, and then multiply by $-(\frac{3}{2})^{1/2}$, the inverse of the isospin Clebsch-Gordan coefficient, to obtain an isoscalar factor. The isoscalar factors for other decays may then be gleaned from Table II.

In the decay "K^+" → "K^0"π^+ from $L_z \neq 0$ substates the following amplitudes contribute: $\mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(x,y)$, $\mathfrak{M}^{1,3\bar{3}}_{8,\bar{3}3}(x,y)$, $\mathfrak{M}^{\bar{3},8\bar{3}}_{3,13}(x,y)$, and $\mathfrak{M}^{3,13}_{\bar{3},8\bar{3}}(x,y)$. By considering enough examples, we shall show that in fact only two of these eight are independent.

Consider a 2^+ meson with $J_z=\pm1$ (recall we are taking the z axis to be the direction of the first final particle, in this case the "K^0"). Then it is in the state

$$\begin{aligned}|J=2,J_z=\pm1\rangle &= \frac{1}{\sqrt{2}}\,|S=1,S_z=0\rangle|L=1,L_z=\pm1\rangle \\ &\quad+\frac{1}{\sqrt{2}}\,|S=1,S_z=\pm1\rangle|L=1,L_z=0\rangle \\ &=\pm\frac{i}{2\sqrt{2}}\,[\,|11\rangle_y-|1-1\rangle_y]\,|\vec{L}=\hat{n}_x\rangle \\ &\quad-\frac{i}{2\sqrt{2}}\,[\,|11\rangle_x+|1-1\rangle_x]\,|\vec{L}=\hat{n}_y\rangle \\ &\quad+\frac{1}{\sqrt{2}}\,|1\pm1\rangle_z\,|\vec{L}=\hat{n}_z\rangle\,, \end{aligned} \tag{C1}$$

where the $|Sm\rangle_{x,y,z}$ on the right-hand side refer to quark spin.

We have used Tables IV and V to quantize the quark spin along the appropriate axes for each $\vec{L}$ substate. Similarly a 1^- meson with $J_z=1$ can be represented as

$$\begin{aligned}|J=1,J_z=\pm1\rangle &= \tfrac{1}{2}|11\rangle_y+\frac{1}{\sqrt{2}}\,|10\rangle_y+\tfrac{1}{2}|1\,-1\rangle_y \\ &=\tfrac{1}{2}|11\rangle_x+\frac{i}{\sqrt{2}}\,|10\rangle_x-\tfrac{1}{2}|1\,-1\rangle_x \end{aligned} \tag{C2}$$

and a 0^- meson as

$$|00\rangle_z=i\,|00\rangle_y=-i\,|00\rangle_x\,, \tag{C3}$$

again using Tables IV and V.

Equations (C1)–(C3) may be translated into $[U(3)\otimes U(3)]_{\beta\sigma_{x,y}}$ representations as follows, using Appendix A:

$$|\text{“}K^+\text{”},J=2,J_z=\pm1,\vec{L}=\hat{n}_x\rangle =\pm\frac{i}{2\sqrt{2}}\,[\,(K^+,-)_y-(-,K^+)_y\,]\,, \tag{C4}$$

$$|\text{“}K^+\text{”},J=2,J_z=\pm1,\vec{L}=\hat{n}_y\rangle =-\frac{i}{2\sqrt{2}}\,[\,(K^+,-)_x+(-,K^+)_x\,]\,, \tag{C5}$$

$$\begin{aligned}|\text{“}K^0\text{”},J=1,J_z=1\rangle &=\tfrac{1}{2}(K^0,-)_y+\tfrac{1}{2}(d,\bar{s})_y+\tfrac{1}{2}(\bar{s},d)_y+\tfrac{1}{2}(-,K^0)_y &\text{(C6)}\\ &=\tfrac{1}{2}(K^0,-)_x+\tfrac{1}{2}i\,(d,\bar{s})_x+\tfrac{1}{2}i\,(\bar{s},d)_x-\tfrac{1}{2}(-,K^0)_x\,, &\text{(C7)}\end{aligned}$$

$$|\pi^+\rangle=\frac{i}{\sqrt{2}}\,(u,\bar{d})_y-\frac{i}{\sqrt{2}}\,(\bar{d},u)_y \tag{C8}$$

$$=-\frac{i}{\sqrt{2}}\,(u,\bar{d})_x+\frac{i}{\sqrt{2}}\,(\bar{d},u)_x\,, \tag{C9}$$

$$|K^0\rangle=\frac{i}{\sqrt{2}}\,(d,\bar{s})_y-\frac{i}{\sqrt{2}}\,(\bar{s},d)_y \tag{C10}$$

$$=-\frac{i}{\sqrt{2}}\,(d,\bar{s})_x+\frac{i}{\sqrt{2}}\,(\bar{s},d)_x\,. \tag{C11}$$

In calculating the overlap of such states we need the following Clebsch-Gordan coefficients of SU(3):

$$\langle d\bar{d}\,|\,-\rangle=1/\sqrt{3}\,, \tag{C12}$$

$$\langle \bar{s}u\,|\,K^+\rangle=1\,, \tag{C13}$$

and elsewhere we will also need

$$\langle K^0\bar{d}\,|\,\bar{s}\rangle=1/\sqrt{3}\,. \tag{C14}$$

To obtain constraints we simply calculate some decays forbidden by J^P conservation and demand that they be forbidden. For example,

$$\begin{aligned}&\mathfrak{M}[\text{“}K^+\text{”}(J=2,J_z=\pm1)\to K^0\pi^+] \\ &\quad=0 \\ &\quad=\frac{i}{4\sqrt{6}}\,\{\pm[\,\mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(y)-\mathfrak{M}^{1,3\bar{3}}_{8,\bar{3}3}(y)\,] \\ &\qquad-[\,\mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(x)+\mathfrak{M}^{1,3\bar{3}}_{8,\bar{3}3}(x)\,]\,\}\,, \end{aligned} \tag{C15}$$

which requires

$$\begin{aligned}&\mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(y)=\mathfrak{M}^{1,3\bar{3}}_{8,\bar{3}3}(y) \\ &\text{and} \\ &\mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(x)=-\mathfrak{M}^{1,3\bar{3}}_{8,\bar{3}3}(x)\,. \end{aligned} \tag{C16}$$

Moreover with

$$\begin{aligned}&\mathfrak{M}[\text{“}K^+\text{”}(J=2,J_z=-1)\to K^{*0}(J_z=1)\pi^+] \\ &\quad=0 \\ &\quad=\frac{1}{8\sqrt{3}}\,\{-[\,\mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(y)+\mathfrak{M}^{1,3\bar{3}}_{8,\bar{3}3}(y)\,] \\ &\qquad-i\,[\,\mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(x)-\mathfrak{M}^{8,3\bar{3}}_{1,\bar{3}3}(x)\,]\} \end{aligned} \tag{C17}$$

we now have [using (C16)]

$$\mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(x) = i\,\mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(y)\,. \tag{C18}$$

Hence, with (C16), this relates four amplitudes to one another. Similar relations may be shown to hold among four other amplitudes one of which is $\mathfrak{M}^{\bar{3},8\bar{3}}_{3,13}(y)$. We shall define

$$a^{(8)} \equiv \mathfrak{M}^{8,\bar{3}3}_{1,3\bar{3}}(y)\,,$$
$$a^{(3)} \equiv \mathfrak{M}^{\bar{3},8\bar{3}}_{3,13}(y)\,. \tag{C19}$$

All the $L_z \neq 0$ decays of "K" into "K"π are then described in terms of these two amplitudes, as noted in Table II.

An alternative discussion, which helps to define the group "coplanar U(3)⊗U(3)" more clearly, may be of some use here. We regard this group as an accidental symmetry of specific states and not as a dynamical symmetry. These states, as mentioned in the Introduction, are those with "linear polarization" states of $\vec{L}$ lying in a plane, and may be constructed as superpositions of states of definite J and J_z. For example, referring to Eq. (C1), the combination

$$[\,|J=2, J_z=1\rangle - |J=2, J_z=-1\rangle\,]/\sqrt{2}$$
$$= \tfrac{1}{2} i [\,|11\rangle_y - |1\,{-1}\rangle_y\,]\,|\,\vec{L}=\hat{n}_x\rangle$$
$$+ \tfrac{1}{2} [\,|11\rangle_z - |1\,{-1}\rangle_z\,]\,|\,\vec{L}=\hat{n}_z\rangle \tag{C20}$$

is such a state, with $\vec{L}$ lying in the x-z plane. The group $[\mathrm{U}(3)\otimes \mathrm{U}(3)]_{\beta\sigma_y}$ is then applicable to decays of this state. We may form similar states for $\vec{L}$ in the y-z plane and calculate their decays. For a suitable choice of relations between reduced matrix elements for the two schemes, we will find that decays which violate conservation of J_z are forbidden. Hence the two calculation schemes are not independent, and it suffices to use either. In practice, this amounts to taking the $\vec{L}=\hat{n}_x$ projection of a state with definite J, J_z: for example, from Eq. (C1),

$$\langle\,\vec{L}=\hat{n}_x\,|\,J=2, J_z=1\rangle = \frac{i}{2\sqrt{2}}\,[\,|\,11\rangle_y - |1\,{-1}\rangle_y\,]\,. \tag{C21}$$

This is, in fact, how the calculations in the present paper were first performed.

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