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[8]D. Campbell and S.-J. Chang, Phys. Rev. D 4, 1151 (1971); 4, 3658 (1971).

[9]The over-all energy-momentum-conserving δ function is absorbed in the definition of g_n.

[10]A somewhat restrictive form of (2.11) appears in Ref. 6.

[11]This is a weaker hypothesis than that proposed in Ref. 1 and 2 if the total cross section is not a constant at high energy.

[12]This idea appears in Ref. 4 and also in K. Wilson, Cornell Report No. CLNS-131 (unpublished).

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[16]See, for example, K. Huang, *Statistical Mechanics* (Wiley, New York, 1963), Chap. 14.

[17]N. Nakanishi, Phys. Rev. 135, B1430 (1964).

[18]The term "multiperipheral processes" is used here to denote those processes in which most of the produced particles are in the pionization region.

[19]This point has been emphasized by K. Wilson, Ref. 12.

[20]This observation has been made by Predazzi and Veneziano in Ref. 5.

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[22]Z. Koba, H. B. Nielsen, and P. Olesen, Nucl. Phys. B (to be published).

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Asymptotic Behavior of the Electromagnetic Form Factor in Quantum Electrodynamics: A Functional Approach*

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We give a simple derivation of the large-momentum-transfer behavior of the electromagnetic vertex in quantum electrodynamics by means of functional methods. Jackiw's result $\exp[-(e^2/16\pi^2)\ln^2(q^2/\mu^2)]$ is obtained by using an eikonal approximation for the electron propagator in an external field.

INTRODUCTION AND DISCUSSION

It has been known for some time that the relevant graphs contributing to the high-energy behavior of scattering amplitudes are the ladders and crossed ladders.[1,2] It is again this same set of graphs which apparently determines the asymptotic behavior of the vertex function in quantum electrodynamics,[3] and which has been used by Appelquist and Primack to study hadronic form factors in several model field theories.[4] Abarbanel and Itzykson[5] were the first to realize that the relativistic eikonal expansion for the scattering amplitude in quantum field theory could easily be derived by functional methods.

In the present paper, we will show how functional techniques can be used to determine the asymptotic behavior of the on-mass-shell electromagnetic vertex, determined by Jackiw[3] to be

$$\exp\left[-\frac{e^2}{16\pi^2}\ln^2(q^2/\mu^2)\right] \tag{1}$$

and previously studied by others.[6] It is hoped that this alternative derivation, besides illustrating the elegance and economy of the functional approach, will prove useful in analyzing more complicated processes.[7]

Our starting point is the equation[8]

$$\tilde{S}^B(x,x') = \exp\left(\frac{i}{2}\frac{1}{i}\frac{\delta}{\delta B^\mu}\cdot D^{\mu\nu}\cdot\frac{1}{i}\frac{\delta}{\delta B^\nu}\right)S_0^B(x,x'), \tag{2}$$

where $\tilde{S}^B(x,x')$ is the exact electron propagator without vacuum-polarization graphs, $S_0^B(x,x')$ is the electron propagator in an external field $B_\mu(x)$ with the quantum field interaction switched off, $D_{\mu\nu}(zz')$ is the zeroth-order photon propagator in an arbitrary gauge (which we will take for simplicity to be the Feynman gauge), and where we are using the functional notation

$$\frac{i}{2}\frac{1}{i}\frac{\delta}{\delta B^\mu}\cdot D^{\mu\nu}\cdot\frac{1}{i}\frac{\delta}{\delta B^\nu} \equiv \frac{i}{2}\int dz dz'\frac{1}{i}\frac{\delta}{\delta B^\mu(z)}D^{\mu\nu}(zz')\frac{1}{i}\frac{\delta}{\delta B^\nu(z')}. \tag{3}$$

The correctness of the statement following Eq. (2) will become clear to the reader upon examination of a few low-order graphs. However, since we will need the generalization of (2) to the case in which an external current is present in order to calculate the vertex function, and since in any case it is instructive to see how one obtains (2) from the exact expression for the propagator, we give in Sec. I a derivation based on path integrals.

The removal of the vacuum-polarization graphs to all orders simplified the high-energy behavior of quantum electrodynamics a great deal, as has been emphasized by Baker and Johnson,[9] and thus one holds the hope that at least for the leading terms, Eq. (2) can be used to give an *explicit* solution to all orders in perturbation theory. We show in Sec. III that this is indeed the case.

Section II is devoted to the calculation of the propagator in an external field. A closed-form expression for this propagator has been obtained before.[2,5] In particular, the authors of Ref. 2 have given an iterative solution for the propagator which in the eikonal limit, and by suitable averaging over the field coordinates, can be recombined into closed form. Their technique yields an "eikonal factor" of the form

$$\frac{e^{\chi}-1}{\chi}, \tag{4}$$

where χ is a functional of the potential $B_\mu(x)$ to be defined in Sec. III. If we use their propagator to calculate the vertex via Eq. (35), we find that the asymptotic behavior of the vertex comes out wrong. If, however, we directly differentiate the iterative solution for S_0^B, and only then go to the eikonal limit, we find that the averaging procedure of Lévy and Sucher is bypassed, and one obtains instead the eikonal factor $\exp(\chi)$ which in turn gives the correct large-momentum-transfer dependence for the vertex. This form of the eikonal factor could also be obtained for the propagator by summing over the field coordinates rather than averaging and has been advocated by other authors.[10] We refer the reader to the second paper of Ref. 2 for a discussion of this point.

I. PATH-INTEGRAL FORMULATION FOR ELECTRON PROPAGATOR

In this section we will give a derivation of Eq. (2) based on Feynman's path-integral approach. We will not be concerned with any of the subtleties involved in defining these integrals, and only use them as a formal device to handle the combinatorics of the perturbation expansion. There are many works on path integrals to which the reader is referred for details.[11]

The Lagrange function for quantum electrodynamics in the presence of external sources $J_\mu(x)$, $B_\mu(x)$, $\eta(x)$, and $\bar\eta(x)$ is given by

$$L(x)=L_0(\psi)+L_0(A)+j^\mu A_\mu+J^\mu A_\mu+j^\mu B_\mu+\bar\eta\psi+\bar\psi\eta\,, \tag{5}$$

where

$$\begin{aligned} &L_0(\psi)=-\bar\psi\left(\gamma^\mu\frac{1}{i}\partial_\mu+m\right)\psi\,,\quad L_0(A)=-\tfrac14 F^{\mu\nu}F_{\mu\nu}\,,\\ &F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,\quad j_\mu=e\bar\psi\cdot\gamma_\mu\psi\,. \end{aligned} \tag{6}$$

The vacuum-vacuum amplitude in the presence of the sources can be written as

$$\begin{aligned}\langle 0+\infty\,|\,0-\infty\rangle^{JB\eta\bar\eta}&\equiv Z^{JB\eta\bar\eta}\\ &=\lambda\int dA d\psi d\bar\psi\exp[i(-\bar\psi\cdot S_0^{-1}\cdot\psi-\tfrac12 A^\mu\cdot D^{-1}{}_{\mu\nu}\cdot A^\nu+j^\mu\cdot A_\mu+J^\mu\cdot A_\mu+j^\mu\cdot B_\mu+\bar\eta\cdot\psi+\bar\psi\cdot\eta)]\,,\end{aligned} \tag{7}$$

where S_0^{-1} is the Dirac operator, and we are using the functional notation

$$\begin{aligned}&-\bar\psi\cdot S_0^{-1}\cdot\psi\equiv-\int d^4x\,\bar\psi(x)\left(\gamma^\mu\frac{1}{i}\partial_\mu+m\right)\psi(x)\,,\\ &j^\mu\cdot A_\mu\equiv\int d^4x\, j^\mu(x)A_\mu(x)\,,\quad \text{etc.}\end{aligned} \tag{8}$$

λ is a constant determined by the boundary condition

$$\lambda^{-1}=\int dAd\psi d\bar\psi\exp[i(-\bar\psi S_0^{-1}\psi-\tfrac12 A\cdot D^{-1}\cdot A+j\cdot A)]\,. \tag{9}$$

From $Z^{JB\eta\bar\eta}$ one can obtain all the Green's functions of the theory by functional differentiation. It will be sufficient for our purposes to restrict our attention to the two-point function

$$\langle 0|T\psi(x)\bar{\psi}(x')|0\rangle = \frac{1}{i}\frac{\delta}{\delta\eta(x')}\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}Z^{JB\eta\bar{\eta}}$$

$$=\lambda\int dAd\psi d\bar{\psi}\,\psi(x)\bar{\psi}(x')\exp[i(-\bar{\psi}S_0{}^{-1}\psi-\tfrac{1}{2}AD^{-1}A+jA+jB+JA+\bar{\eta}\psi+\bar{\psi}\eta)]\,, \tag{10}$$

and for the remainder of this work we set the fermion sources η and $\bar{\eta}$ to zero. Using the operator identity

$$e^{a^m}f(b)e^{-a^m}=f(b+ma^{m-1})\,, \tag{11}$$

where a and b are any two operators such that $[a,b]=1$, and f is an arbitrary function, we can rewrite Z^{JB} as

$$Z^{JB}=\lambda\exp\left(\frac{1}{i}\frac{\delta}{\delta J}\cdot\frac{\delta}{\delta B}\right)\int dAd\psi d\bar{\psi}\exp[i(-\bar{\psi}S_0{}^{-1}\psi-\tfrac{1}{2}AD^{-1}A+JA+jB)]\,. \tag{12}$$

Equation (12) expresses the well-known fact that the generating functional for the interacting theory can be written in terms of the classical actions of the fields in the presence of external sources. The electron propagator S^{JB} is defined by

$$S^{JB}(x,x')=i(Z^{JB})^{-1}\langle 0|T\psi(x)\bar{\psi}(x')|0\rangle \tag{13}$$

which becomes upon substitution of (10)

$$S^{JB}(xx')=\frac{\exp\left(\frac{1}{i}\frac{\delta}{\delta J}\cdot\frac{\delta}{\delta B}\right)\int dAd\psi d\bar{\psi}\,i\psi(x)\bar{\psi}(x')\exp[i(-\bar{\psi}S_0{}^{-1}\psi-\frac{1}{2}AD^{-1}A+JA+jB)]}{\exp\left(\frac{1}{i}\frac{\delta}{\delta J}\cdot\frac{\delta}{\delta B}\right)\int dAd\psi d\bar{\psi}\exp[i(-\bar{\psi}S_0{}^{-1}\psi-\frac{1}{2}AD^{-1}A+JA+jB)]}\,. \tag{14}$$

The integral over the electromagnetic potential A_μ can easily be done by a field translation,[11] and (14) becomes

$$S^{JB}(xx')=\frac{\exp\left(\frac{1}{i}\frac{\delta}{\delta J}\cdot\frac{\delta}{\delta B}\right)\exp\left(\frac{i}{2}J\cdot D\cdot J\right)\int d\psi d\bar{\psi}\,i\psi(x)\bar{\psi}(x')\exp[i(-\bar{\psi}S_0{}^{-1}\psi+jB)]}{\exp\left(\frac{1}{i}\frac{\delta}{\delta J}\cdot\frac{\delta}{\delta B}\right)\exp\left(\frac{1}{i}J\cdot D\cdot J\right)\int d\psi d\bar{\psi}\exp[i(-\bar{\psi}S_0{}^{-1}\psi+jB)]}\,, \tag{15}$$

and where, according to our convention,

$$(i/2)J\cdot D\cdot J=(i/2)\int dzdz'J^\mu(z)D_{\mu\nu}(zz')J^\nu(z')\,. \tag{16}$$

The remaining path integrals are now recognized as those appropriate for the electron propagator in an external field; thus writing

$$S_0^B=\frac{\int d\psi d\bar{\psi}\,i\psi(x)\bar{\psi}(x')\exp[i(-\bar{\psi}S_0{}^{-1}\psi+jB)]}{\int d\psi d\bar{\psi}\exp[i(-\bar{\psi}S_0{}^{-1}\psi+jB)]} \tag{17}$$

and

$$Z_0^B=\frac{\int d\psi d\bar{\psi}\exp[i(-\bar{\psi}S_0{}^{-1}\psi+jB)]}{\int d\psi d\bar{\psi}\exp[i(-\bar{\psi}S_0{}^{-1}\psi)]}\,, \tag{18}$$

Eq. (15) takes the form

$$S^{JB}(xx')=\frac{\exp\left(\frac{1}{i}\frac{\delta}{\delta J}\cdot\frac{\delta}{\delta B}\right)\exp\left[\frac{i}{2}(J\cdot D\cdot J)\right](Z_0^BS_0^B)}{\exp\left(\frac{1}{i}\frac{\delta}{\delta J}\cdot\frac{\delta}{\delta B}\right)\exp\left[\left(\frac{i}{2}J\cdot D\cdot J\right)\right](Z_0^B)}\,. \tag{19}$$

Finally, using again the operator relation of Eq. (11), we transform (19) into its final form:

$$S^{JB}(xx') = \frac{\exp\left\{\frac{i}{2}\left[\left(J+\frac{1}{i}\frac{\delta}{\delta B}\right)\cdot D\cdot\left(J+\frac{1}{i}\frac{\delta}{\delta B}\right)\right]\right\}(Z_0^B S_0^B)}{\exp\left\{\frac{i}{2}\left[\left(J+\frac{1}{i}\frac{\delta}{\delta B}\right)\cdot D\cdot\left(J+\frac{1}{i}\frac{\delta}{\delta B}\right)\right]\right\}Z_0^B}\,. \tag{20}$$

Equation (20) for S^{JB} has a very simple diagrammatic interpretation. S_0^B contains only graphs of the form shown in Fig. 1, while Z_0^B supplies us with the vacuum loops shown in Fig. 2. The differential operator

$$\exp\left\{\frac{i}{2}\left(\frac{1}{i}\frac{\delta}{\delta B}\cdot D\cdot\frac{1}{i}\frac{\delta}{\delta B}\right)\right\}$$

(we are setting $J=0$ for the moment) then joins the photon lines in all possible ways with the appropriate weights. The numerator in Eq. (20) gives us then the correct perturbation expansion for the electron propagator plus the disconnected vacuum loops which are in turn canceled by the graphs in the denominator.

Equation (20) is therefore equivalent to the series expansion of the propagator, and thus correct independently of the method used to derive it. It gives us, however, new insight into the structure of the perturbation series, and provides us with a means of separating the contribution due to the polarization of the vacuum to all orders in the charge. For, if in Eq. (20) we simply neglect the factor Z_0^B in both numerator and denominator, the resulting approximation

$$\exp\left\{\frac{i}{2}\left[\left(J+\frac{1}{i}\frac{\delta}{\delta B}\right)\cdot D\cdot\left(J+\frac{1}{i}\frac{\delta}{\delta B}\right)\right]\right\}S_0^B \tag{21}$$

contains all the diagrams except the vacuum loops. Setting $J=0$ obtains the result quoted in the Introduction, Eq. (2).

II. PROPAGATOR IN AN EXTERNAL FIELD

We will limit ourselves in this section to writing down the iterative solution for the propagator in an external field in order to establish our notation.

S_0^B satisfies the equation

$$S_0^B(x,x') = S_0(xx') + \int S_0(xx'')e\gamma^\mu B_\mu(x'')S_0^B(x''x')dx'', \tag{22}$$

where

$$S_0(xx') = \int (dp)\,\frac{m-\gamma\cdot p}{p^2+m^2-i\epsilon}\,e^{ip(x-x')}, \tag{23}$$

$$(dp) = d^4p/(2\pi)^4 .$$

It will prove convenient to solve instead for the "T matrix" defined by the equation

$$S_0^B = S_0 + S_0 T^B S_0 \tag{24}$$

and which satisfies $(e\gamma^\mu B_\mu \equiv V)$

$$T^B = V + VS_0T . \tag{25}$$

Taking the partial Fourier transform

$$T^B(px') = \int dx\, e^{-ip(x-x')}T^B(x,x'), \tag{26}$$

Eq. (25) becomes

$$T^B(p,x') = V(x') + \int (dk)\,V(k)S_0(p-k)T^B(p-k,x')e^{ik\cdot x'} . \tag{27}$$

An iterative solution of (27) can be immediately written down:

$$T^B(p,x') = \sum_{n=0}^{\infty}\int (dk_1)\cdots(dk_n)V(k_1)S_0(p-k_1)\cdots \times V(k_n)S_0(p-k_1-\cdots-k_n)V(x') \times e^{i(k_1+\cdots+k_n)\cdot x'} . \tag{28}$$

Finally, taking the transform with respect to both coordinates,

$$T^B(p,p') = \int dx' e^{-i(p-p')\cdot x'}T^B(p,x'), \tag{29}$$

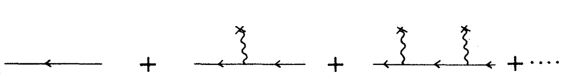

FIG. 1. Diagrams contributing to S_0^B.

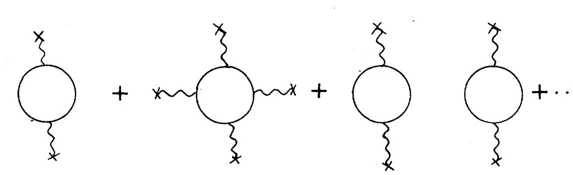

FIG. 2. Expansion of Z_0^B.

we write

$$T^B(p,p') = \sum_{n=1}^{\infty} \int (dk_1)\cdots(dk_n)\, V(k_1)S_0(p-k_1)\cdots S_0(p-k_1-\cdots-k_{n-1})\, V(k_n)(2\pi)^4\delta^4(q-k_1-\cdots-k_n), \tag{30}$$

where $q=p-p'$. One could now go to the eikonal limit in (30) to obtain a generalization to spin $\frac{1}{2}$ of the Lévy-Sucher equation, which for the on-mass-shell case reads

$$T(pp') = e\gamma^\mu \int dz\, e^{-iq\cdot z} B_\mu(z)\, \frac{e^{\chi(B)}-1}{\chi(B)}, \tag{31}$$

where

$$\chi(z;p,p';B) = e\int (dk) e^{ik\cdot z} B_\mu(k)\left(\frac{2p^\mu}{k^2-2p\cdot k} + \frac{2p'^\mu}{k^2+2p'\cdot k}\right). \tag{32}$$

We will refer again to (31) in the next section, and the reader is referred to the work of Lévy and Sucher for the details of its derivation.

III. VERTEX FUNCTION

We finally come to the central theme of this paper, the calculation of the vertex function. The vertex function of quantum electrodynamics is defined by[12]

$$\Gamma^\mu(xx';z) = -\left.\frac{\delta S^{J-1}(xx')}{\delta e a_\mu(z)}\right|_{J=0}, \tag{33}$$

where

$$a_\mu(z) = \frac{\langle 0|A_\mu(z)|0\rangle}{\langle 0|0\rangle}.$$

Γ^μ is easily related to the improper vertex

$$\langle 0|\, T\psi(x)\bar{\psi}(x')A^\mu(z)|0\rangle = \frac{1}{i}\,\frac{\delta}{\delta J_\mu(z)}\langle 0|\, T\psi(x)\bar{\psi}(x')|0\rangle \tag{34}$$

by a simple application of the chain rule:

$$eS\Gamma^\nu S D_{\nu\mu} = \left.\frac{\delta S^{JB}}{\delta J^\mu}\right|_{J=B=0} = -\langle 0|T\psi\bar{\psi}A_\mu|0\rangle, \tag{35}$$

where the function $D_{\nu\mu} \equiv \delta a_\nu/\delta J^\mu$ is the full unrenormalized photon propagator. In the large-momentum-transfer limit, we will assume that this propagator is asymptotic to [in Feynman gauge] $\delta_{\mu\nu}/(q^2-i\epsilon)$. This is of course consistent with neglecting vacuum loops and is suggested by canonical field theory.

The right-hand side of Eq. (35) can now be calculated, neglecting vacuum polarization, by making use of (21); thus

$$\begin{aligned}\left.\frac{\delta \tilde{S}(xx')}{\delta J_\mu(0)}\right|_{J=0} &= \frac{\delta}{\delta J_\mu(0)} \exp\left[\frac{i}{2}\left(J+\frac{1}{i}\frac{\delta}{\delta B}\right)\cdot D\cdot\left(J+\frac{1}{i}\frac{\delta}{\delta B}\right)\right] S_0^B(xx')\Big|_{J=0} \\ &= \exp\left(\frac{i}{2}\,\frac{1}{i}\frac{\delta}{\delta B}\cdot D\cdot\frac{1}{i}\frac{\delta}{\delta B}\right)\int dz\, D^{\mu\nu}(0,z)\,\frac{\delta}{\delta B^\nu(z)} S_0^B(xx').\end{aligned} \tag{36}$$

Now using the iterative solution for S_0^B, Eq. (30), and the fact that

$$\delta B^\mu(k)/\delta B^\nu(z) = e^{-ikz}\delta^\mu_\nu,$$

we have

$$\begin{aligned}\frac{\delta S_0^B(p,p')}{\delta B^\nu(z)} &= S_0(p)\,\frac{\delta T^B(p,p')}{\delta B^\nu(z)}\, S_0(p') \\ &= S_0(p)\sum_{n=1}^{\infty} e^n \int (dk_1)\cdots(dk_n)\sum_{l=1}^{n} \gamma\cdot B(k_1)S_0(p-k_1)\cdots\gamma\cdot B(k_{l-1})S_0(p-k_1-\cdots-k_{l-1}) \\ &\quad\times \gamma_\nu e^{-ik_l\cdot z}\, S_0(p-k_1-\cdots-k_l)\gamma\cdot B(k_{l+1})\cdots S_0(p-k_1-\cdots-k_{n-1})\gamma\cdot B(k_n)\, S_0(p')(2\pi)^4\delta^4(q-\Sigma k)\end{aligned}$$

$$=S_0(p)\sum_{n=1}^{\infty}e^n\sum_{l=1}^{n}\int(dk_1)\cdots(dk_{l-1})(dk_{l+1})\cdots(dk_n)$$

$$\times\gamma\cdot B(k_1)S_0(p-k_1)\cdots\gamma\cdot B(k_{l-1})S_0(p-k_1-\cdots-k_{l-1})\gamma_\nu e^{-iqz+i\sum_{i\neq l}k_iz}$$

$$\times S_0(p'+k_{l+1}\cdots+k_n)\gamma\cdot B(k_{l+1})\cdots S_0(p'+k_n)\gamma\cdot B(k_n)S_0(p')\,. \tag{37}$$

We will now pass to the eikonal limit making the replacement

$$\gamma_\mu S_0(p-k_1-\cdots-k_i)=\frac{2p_\mu}{p^2+m^2+a_1+\cdots+a_i}\,,$$
$$S_0(p'+k_1+\cdots+k_j)\gamma_\mu=\frac{2p'_\mu}{p'^2+m^2+b_1+\cdots+b_j}\,, \tag{38}$$

where $a_i=k_i^2-2p\cdot k_i$ and $b_i=k_i^2+2p'\cdot k_i$. It is not the purpose of this paper to enter into the assumptions which go into (38), and the reader is referred to the literature[3,4,13] for a discussion of this point. Using (38), Eq. (37) can be cast into the form

$$\frac{\delta S_0^B(p,p')}{\delta B^\nu(z)}=S_0(p)\sum_{n=1}^{\infty}e^n\sum_{l=1}^{n}\int(dk_1)\cdots(dk_{l-1})(dk_{l+1})\cdots(dk_n)\frac{2p\cdot B(k_1)e^{+ik_1z}}{\delta+a_1}\;\frac{2p\cdot B(k_2)e^{ik_2z}}{\delta+a_1+a_2}\cdots\frac{2p\cdot B(k_{l-1})e^{ik_{l-1}z}}{\delta+a_1+\cdots+a_{l-1}}$$

$$\times\gamma_\nu e^{-iqz}\frac{2p'\cdot B(k_{l+1})e^{ik_{l+1}z}}{\delta'+b_{l+1}\cdots+b_n}\cdots\frac{2p'\cdot B(k_n)e^{ik_nz}}{\delta'+b_n}S_0(p')\,, \tag{39}$$

where $q=p-p'$, $\delta=p^2+m^2$, and $\delta'=p'^2+m^2$. Since we are calculating the on-mass-shell vertex we will set $\delta=\delta'=0$, and the off-mass-shell case will be treated in an appendix. We next symmetrize among the $(l-1)$ variables $k_1\cdots k_{l-1}$ and the $(n-l)$ variables $k_{l+1}\cdots k_n$ separately. Then using the relation[14]

$$\sum_{\pi(i)}\frac{1}{a'_1}\;\frac{1}{a'_1+a'_2}\cdots\frac{1}{a'_1+a'_2+\cdots+a'_{l-1}}=\frac{1}{a_1a_2\cdots a_{l-1}}\,, \tag{40}$$

where $\sum_{\pi(i)}$ is a sum over all permutations of the $(l-1)$ indices, $a'_i=a_{\pi(i)}$, and the corresponding equation for the b's, we recast (39) into the form

$$\left.\frac{\delta T^B(p,p')}{\delta B^\nu(z)}\right|_{p^2=p'^2=-m^2}=e\gamma_\nu e^{-iqz}\sum_{n=1}^{\infty}\sum_{l=1}^{n}\frac{1}{(l-1)!(n-l)!}\left(e\int(dk)\,\frac{2p\cdot B(k)e^{ikz}}{a(k)}\right)^{l-1}\left(e\int(dk)\,\frac{2p'\cdot B(k)e^{ikz}}{b(k)}\right)^{n-l}$$

$$=e\gamma_\nu e^{-iqz}\exp[\chi(z;p,p';B)]\,, \tag{41}$$

where $\chi(z;p,p';B)$ is the functional defined by Eq. (32).

The important point is that χ is a linear functional of the field $B_\mu(x)$:

$$\chi(z;p,p';B)=\int dz'L^\mu(z,z';p,p')B_\mu(z')\,; \tag{42}$$

the explicit form of L^μ can be easily deduced from (32) and is given in Appendix A; as a result,

$$\exp\left(\frac{i}{2}\,\frac{1}{i}\,\frac{\delta}{\delta B}\,D\,\frac{1}{i}\,\frac{\delta}{\delta B}\right)\exp[\chi(B)]=\exp\left[\chi\left(B-iD\,\frac{\delta}{\delta B}\right)\right]$$

$$=\exp\left(L\cdot B-iL\cdot D\cdot\frac{\delta}{\delta B}\right)$$

$$=\exp(L\cdot B)\exp\left(-iL\cdot D\cdot\frac{\delta}{\delta B}\right)\exp\left\{\frac{i}{2}\left[L\cdot B,\,L\cdot D\cdot\frac{\delta}{\delta B}\right]\right\}, \tag{43}$$

where we have used the operator identity

$$e^{a+b}=e^ae^be^{-[a,b]/2},\qquad [a,b]=c\text{ number} \tag{44}$$

which applies since

$$\left[B_\mu(z),\,\frac{S}{\delta B^\nu(z')}\right]=-\delta_{\mu\nu}\delta(z-z')\,. \tag{45}$$

Substituting (45) into (43) we obtain, after setting $B_\mu = 0$,

$$\exp\left[\frac{i}{2}\left(\frac{1}{i}\frac{\delta}{\delta B}D\frac{1}{i}\frac{\delta}{\delta B}\right)\right]\exp[\chi(B)]\Big|_{B=0} = \exp[\phi(z)]\,, \tag{46}$$

where

$$\phi(z) = -\frac{i}{2}\int dz'dz''L^\mu(zz')D_{\mu\nu}(z'z'')L^\nu(zz'')\,. \tag{47}$$

As we show in Appendix B, ϕ is in fact independent of z, and can be written, Eq. (B1),

$$\phi = \phi^{(1)}(q^2) + \phi^{(2)}(p) + \phi^{(3)}(p')\,.$$

Let us now rewrite the vertex function using (37) and (41) as

$$eS(p)\Gamma^\nu(p,p')S(p')D_{\nu\mu}(q) = eS_0(p)\gamma^\nu S_0(p')\int dz\, D_{\mu\nu}(0z)e^{-iqz}\exp\left[\frac{i}{2}\left(\frac{1}{i}\frac{\delta}{\delta B}D\frac{\delta}{\delta B}\right)\right]\exp[\chi(B)]\,, \tag{48}$$

which becomes, upon using Eqs. (46) and (47),

$$S(p)\Gamma_\mu(p,p')S(p') = S_0(p)e^{\phi^{(2)}(p)}\gamma_\mu S_0(p')e^{\phi^{(3)}(p')}\,e^{\phi^{(1)}(q^2)}\,. \tag{49}$$

As is shown in Appendix C, in the eikonal approximation

$$S(p) = S_0(p)e^{\phi^{(2)}(p)}\,,$$
$$S(p') = S_0(p')e^{\phi^{(3)}(p')}\,. \tag{50}$$

Therefore, (49) yields the final result

$$\Gamma_\mu^{\mathrm{eik}}(p,p') = \gamma_\mu \exp\left[-\frac{e^2}{16\pi^2}\ln^2(q^2/\mu^2)\right], \tag{51}$$

where Eq. (B5) has been used for the leading behavior of $\phi^{(1)}(q^2)$.

To bring this section to a conclusion, let us briefly discuss the effect of using the eikonal approximation to the propagator, Eq. (31), to calculate the vertex instead of proceeding as described above. Instead of Eq. (41), we obtain

$$\frac{\delta T^B(p,p')}{\delta B_\nu(z)} = e\gamma^\alpha\int_0^1 d\lambda\int dz' e^{-iqz'}\frac{\delta}{\delta B_\nu(z)}\{B_\alpha(z')\exp[\lambda\cdot\chi(z';B)]\}\,, \tag{52}$$

where we have used the identity

$$\frac{e^\chi - 1}{\chi} = \int_0^1 e^{\lambda\chi}d\lambda\,. \tag{53}$$

Since

$$\frac{\delta}{\delta B_\nu(z)}\{B_\alpha(z')\exp[\lambda\chi(z';B)]\} = [\delta_\alpha^\nu\delta(z-z') + \lambda B_\alpha(z')L^\nu(z'z)]\exp[\lambda\chi(z';B)]\,, \tag{54}$$

we can write Eq. (36) in the form

$$\frac{\delta\tilde{S}(p,p')}{\delta J^\mu(0)} = S_0(p)e\gamma^\alpha S_0(p')\int_0^1 d\lambda\int dz dz' e^{-iqz'}D_{\mu\nu}(0z)\exp\left(\frac{i}{2}\frac{1}{i}\frac{\delta}{\delta B}\cdot D\cdot\frac{1}{i}\frac{\delta}{\delta B}\right)$$
$$\times\exp[\lambda\chi(z';B)][\delta_\alpha^\nu\delta(z-z') + \lambda B_\alpha(z')L^\nu(z'z)]\,. \tag{55}$$

Now, using Eq. (46), we can rewrite (54) as

$$\frac{\delta\tilde{S}(pp')}{\delta J^\mu(0)} = S_0(p)e\gamma^\alpha S_0(p')\int_0^1 d\lambda\int dz dz' e^{-iqz'}D_{\mu\nu}(0z)\exp(\lambda^2\phi)\exp\left(-i\lambda L\cdot D\cdot\frac{\delta}{\delta B}\right)[\delta_\alpha^\nu\delta(z-z') + \lambda B_\alpha(z')L^\nu(z'z)]$$
$$= S_0(p)e\gamma^\alpha S_0(p')\int_0^1 d\lambda\int dz dz' e^{-iqz'}D_{\mu\nu}(0z)\exp(\lambda^2\phi^2)$$
$$\times\left[\delta_\alpha^\nu\delta(z-z') - i\lambda^2 L^\nu(z'z)\int dz'' L^\beta(z'z'')D_{\beta\alpha}(z''z')\right]. \tag{56}$$

It is now straightforward, albeit tedious, to show that, using the definition of L^μ given in Appendix A, one can write

$$\frac{\delta\tilde{S}(p,p')}{\delta J^\mu(0)} = S_0(p)e\gamma^\alpha S_0(p')[I_{1\alpha\mu}+I_{2\alpha\mu}], \tag{57}$$

where

$$I_{1\alpha\mu}=\frac{\delta_{\alpha\mu}}{q^2-i\epsilon}\int_0^1 d\lambda\, e^{\lambda^2\phi} \tag{58}$$

and

$$I_{2\alpha\mu}=-\frac{ie^2}{q^2-i\epsilon}\left[\frac{2p_\mu}{a(q)}+\frac{2p'_\mu}{b(q)}\right]\int\frac{(dk)}{k^2-i\epsilon}\left[\frac{2p_\alpha}{a(k)}+\frac{2p'_\alpha}{b(k)}\right]\int_0^1\lambda^2 d\lambda\, e^{\lambda^2\phi}. \tag{59}$$

Since

$$\int_0^1 d\lambda\, e^{\lambda^2\phi}\sim[-\phi]^{-1/2}$$

and

$$\int_0^1 \lambda^2 d\lambda\, e^{\lambda^2\phi}\sim[-\phi]^{-3/2}, \tag{60}$$

it is obviously $I_{1\alpha\mu}$ which contributes to the leading term in the vertex. However, by comparison with the graphical analysis, order by order, of Refs. 3 and 4, we are led to the conclusion that the asymptotic behavior predicted by Eq. (57), as a result of using the eikonal factor (4), is incorrect. Exponentiation of the second-order graph takes place only if we use the iterative solution of Eq. (30).

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APPENDIX A

The generalization to the off-shell case is straightforward, and follows from an application of the identity[15]

$$\sum_\pi \frac{1}{\delta+a'_1}\,\frac{1}{\delta+a'_2}\cdots\frac{1}{\delta+a'_1+\cdots+a'_n}=i\delta\int_0^\infty d\beta\, e^{-i\beta\delta}\pi_i\,\frac{1-e^{-i\beta a_i}}{a_i}. \tag{A1}$$

Thus, as before

$$\frac{\delta T(p,p')}{\delta B_\nu(z)}=e\gamma^\nu e^{-iqz}i\delta i\delta'\int_0^\infty d\beta d\beta' e^{-i\beta\delta-i\beta'\delta'}\exp\chi, \tag{A2}$$

where now

$$\chi(z;p,p';\beta,\beta';B)=e\int(dk)e^{ikz}B_\mu(k)\left[\frac{2p^\mu(1-e^{-i\beta a})}{a}+\frac{2p'^\mu(1-e^{-i\beta' b})}{b}\right], \tag{A3}$$

where

$$a=k^2-2p\cdot k,\qquad b=k^2+2p'\cdot k. \tag{A4}$$

Comparing (A3) with (42) we see that

$$L^\mu(zz';pp';\beta\beta')=e\int(dk)e^{ik(z-z')}\left[\frac{2p^\mu(1-e^{-i\beta a})}{a}+\frac{2p'^\mu(1-e^{-i\beta' b})}{b}\right]. \tag{A5}$$

For the on-mass-shell case $\delta=\delta'=0$, which implies

$$\frac{\delta T(p,p')}{\delta B_\nu(z)}=e\gamma^\nu e^{-iqz}\lim_{\delta,\delta'\to 0}\int_0^\infty d\beta d\beta'\left(\frac{\partial}{\partial\beta}\,\frac{\partial}{\partial\beta'}\,e^{-i\beta\delta-i\beta'\delta'}\right)\exp\chi, \tag{A6}$$

and since

$$L^{\mu}(\beta=\beta'=\infty)=e\int(dk)e^{ik(z-z')}\left(\frac{2p^{\mu}}{a}+\frac{2p'^{\mu}}{b}\right) \tag{A7}$$

and

$$L^{\mu}(\beta=\beta'=0)=0\,, \tag{A8}$$

it follows that

$$\frac{\delta T(p,p')}{\delta B_{\nu}(z)}=e\gamma^{\nu}e^{-iqz}\exp[\chi(z;p,p';\beta=\beta'=\infty;B)]\,, \tag{A9}$$

which is the result used in Sec. III.

APPENDIX B

Using Eq. (A7), the function ϕ can be written as

$$\begin{aligned}\phi\equiv-\frac{i}{2}L\cdot D\cdot L&=-\frac{i}{2}\int\frac{(dk)}{k^2+\mu^2-i\epsilon}\int dz'dz''L^{\mu}(zz')L_{\mu}(zz'')e^{ik(z'-z'')}\\&=-\frac{ie^2}{2}\int\frac{(dk)}{k^2+\mu^2-i\epsilon}\left[\frac{2p^{\mu}}{a(k)}+\frac{2p'^{\mu}}{b(k)}\right]\left[\frac{2p_{\mu}}{a(-k)}+\frac{2p'_{\mu}}{b(-k)}\right]\\&\equiv\phi^{(1)}(q^2)+\phi^{(2)}(p)+\phi^{(3)}(p')\,,\end{aligned} \tag{B1}$$

where

$$\phi^{(1)}(q^2)=2ie^2q^2\int\frac{(dk)}{k^2+\mu^2-i\epsilon}\,\frac{1}{a(k)b(-k)}\,, \tag{B2}$$

$$\phi^{(2)}(p)=-2ie^2p^2\int\frac{(dk)}{k^2+\mu^2-i\epsilon}\,\frac{1}{a(k)a(-k)}\,, \tag{B3}$$

$$\phi^{(3)}(p')=-2ie^2p'^2\int\frac{(dk)}{k^2+\mu^2-i\epsilon}\,\frac{1}{b(k)b(-k)}\,. \tag{B4}$$

Equation (B2) contains the relevant q^2 dependence, and its leading term can easily be obtained by parametric techniques[16]:

$$\phi^{(1)}(q^2)\sim-\frac{e^2}{16\pi^2}\ln^2(q^2/\mu^2)+O(\ln(q^2/\mu^2))\,. \tag{B5}$$

APPENDIX C

Here we endeavor to demonstrate that in the eikonal limit, the unrenormalized electron propagator becomes

$$S^{\text{eik}}(p)=S_0(p)\exp\left(-2ie^2p^2\int\frac{(dk)}{k^2+\mu^2-i\epsilon}\,\frac{1}{k^2-2p\cdot k-i\epsilon}\,\frac{1}{k^2+2p\cdot k-i\epsilon}\right). \tag{C1}$$

We begin by rewriting Eq. (2) for the case of zero external field:

$$\tilde{S}^{B=0}(x,x')\equiv\tilde{S}(x-x')=\exp\left[\frac{i}{2}\left(\frac{1}{i}\,\frac{\delta}{\delta B}\cdot D\cdot\frac{1}{i}\,\frac{\delta}{\delta B}\right)\right]S_0^B(x,x')\,, \tag{C2}$$

which when Fourier transformed gives

$$\begin{aligned}\tilde{S}(p)&=\int d^4x e^{-ip(x-x')}\tilde{S}(x-x')\\&=\exp\left[\frac{i}{2}\left(\frac{1}{i}\,\frac{\delta}{\delta B}\cdot D\cdot\frac{1}{i}\,\frac{\delta}{\delta B}\right)\right]S_0^B(p,x')\,.\end{aligned} \tag{C3}$$

In the same fashion in which we obtained (30), we can construct an iterative solution for $S_0^B(p,x')$; thus

$$S_0^B(p,x')=S_0(p)\left[1+\sum_{n=1}^{\infty}\int(dk_1)\cdots(dk_n)V(k_1)S_0(p-k_1)\cdots V(k_n)S_0(p-k_1-\cdots-k_n)e^{i(k_1+\cdots+k_n)\cdot x'}\right], \tag{C4}$$

which becomes upon making the replacement of Eq. (38), and using the combinatorial formula (40),

$$S_0^{B\,\mathrm{eik}}(p,x')=S_0(p)\exp\left(e\int\frac{(dk)2p^\mu B_\mu(k)e^{ikx'}}{a(k)}\right). \tag{C5}$$

It is now straightforward, and completely analogous to the argument presented in Sec. III, to show that operating with

$$\exp\left[\frac{i}{2}\left(\frac{1}{i}\,\frac{\delta}{\delta B}\cdot D\cdot\frac{1}{i}\,\frac{\delta}{\delta B}\right)\right]$$

on (C5) we obtain (C1).

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