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Higher-Order Corrections to Leptonic Processes and the Renormalization of Weinberg's Theory of Weak Interactions in the Unitary Gauge*

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Higher-order corrections to the leptonic processes such as muon decay and neutrino-lepton scattering are investigated in order to study the problems associated with the renormalization of the Weinberg theory of weak interactions in the unitary gauge. It is shown that there are definite ways to carry out the renormalization programs and the residual divergences (quadratic as well as logarithmic) cancel out systematically. As a result we find that the higher-order corrections to these processes and the neutrino charge radius are all finite. As a by-product, we find that the logarithmic residual divergences in the elastic ν-ν scattering also cancel out.

I. INTRODUCTION

Some time ago Weinberg[1] proposed a unified theory of weak and electromagnetic interactions and suggested that it might be renormalizable. Recently, several studies[2,3] have indicated that various models of the same general class as Weinberg's actually are renormalizable in the R gauge (renormalizable gauge). However, it is not at all obvious that the theory is renormalizable in the unitary gauge. This is because if one tries to calculate the higher-order S matrix in this gauge, one encounters the following problems: (i) Since in general the S-matrix elements are complicated functions of the two unrenormalized coupling constants, there seems to be no clear-cut way to carry out the renormalization programs. (ii) The divergences arising from the weak and electromagnetic interactions are usually mixed up. (iii) There are divergences which cannot be absorbed into the renormalization constants. These problems manifest themselves especially in the higher-order corrections to the leptonic processes such as muon decay and neutrino-lepton scattering. It is the purpose of this paper to show how to deal with these problems by studying these leptonic processes.

In the case of nondiagonal leptonic processes such as muon decay, things become a little simpler because problem (ii) above does not arise. We find that the conventional renormalization program can be carried out and all the residual divergences cancel out. We refer to the residual divergence as the divergences which still remain even after the renormalization. Thus it yields a finite result of higher-order weak and electromagnetic corrections to the muon decay in Weinberg's theory.

In the case of diagonal leptonic process such as the neutrino-electron scattering, all the problems mentioned above are present. To see how to deal with them, we note that one can unambiguously define the renormalized neutrino–Z-boson coupling constant in the process of ν-ν elastic scattering discussed by Weinberg,[4] where it was shown that the quadratic residual divergences canceled out (as will be shown later, so do the logarithmic residual divergences), while in the process of muon decay the renormalized lepton–W-boson coupling constant is properly defined. We observe that once they are defined in these processes one has to stick to these definitions in other physical processes as well, such as neutrino-lepton scattering, if the theory is actually renormalizable.

Guided by the above observation, we find that the S matrix for the fourth-order elastic neutrino-lepton scattering in the unitary gauge can be separated into three distinct parts. The first part is associated with the neutrino charge radius. The second part is similar to that in ν-ν scattering while the third part is similar to that in μ decay. The conventional renormalization programs can then be carried out separately in these three parts. It is shown that after renormalization the residual divergences cancel out in each of these three parts. As a result, we find that the S matrix for the higher-order neutrino-lepton scattering and the neutrino charge radius are all finite.

The paper is organized as follows. In Sec. II, we first discuss the higher-order corrections to the muon decay and show that the conventional renormalization program can be carried out and the residual divergences cancel out. In Sec. III, we discuss the higher-order neutrino-lepton scattering where for convenience we take the lepton to be the electron and the neutrino to be ν_e. It is shown that one can renormalize the ν-Z coupling constant and lepton-W coupling constant in the same way as discussed in the ν-ν elastic scattering and μ decay, respectively. In Sec. IV, some relevant points are discussed.

II. MUON DECAY

In this section we shall discuss the second-order weak and electromagnetic corrections to muon decay in Weinberg's unified theory of weak and electromagnetic interactions.[1,4]

A. Notation and Kinematics

We consider the decay process

$$\mu \to e + \bar{\nu}_e + \nu_\mu . \tag{2.1}$$

The momenta of the muon and electron are denoted by p and p', respectively, while we use k_1 and k_2 to denote the momenta of $\bar{\nu}_e$ and ν_μ, respectively. In Weinberg's theory of weak and electromagnetic interactions of leptons, there are charged intermediate vector bosons W, neutral intermediate vector bosons Z, and massive neutral scalar mesons ϕ as well as photons A. There are two fundamental coupling constants g and g' and one spontaneous symmetry-breaking parameter λ. The electric charge e and Fermi coupling constant G are related to these three constants as follows:

$$e = \frac{gg'}{(g^2+g'^2)^{1/2}} , \quad \frac{G}{\sqrt{2}} = \frac{1}{2\lambda^2} , \tag{2.2}$$

while the vector-boson masses (to zeroth order in the fine-structure constant) are

$$m_W = \frac{\lambda g}{2} , \quad m_Z = \frac{\lambda (g^2+g'^2)^{1/2}}{2} . \tag{2.3}$$

To the lowest order in g, the decay matrix element M_0 for (2.1) is given by the contribution of the diagram in Fig. 1,

$$M_0 = i\frac{(-g^{\alpha\beta} + q^\alpha q^\beta / m_W{}^2)}{q^2 - m_W{}^2} M_{\alpha\beta} , \tag{2.4}$$

where q is the momentum of the charged W boson and $M_{\alpha\beta}$ is defined by

$$M_{\alpha\beta} = \bar{u}(k_2)\gamma_\alpha(1+\gamma_5)u(p) \times \bar{u}(p')\gamma_\beta(1+\gamma_5)\, v(k_1)\left(\frac{g}{2\sqrt{2}}\right)^2 . \tag{2.5}$$

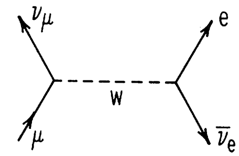

FIG. 1. The lowest-order diagram for μ decay.

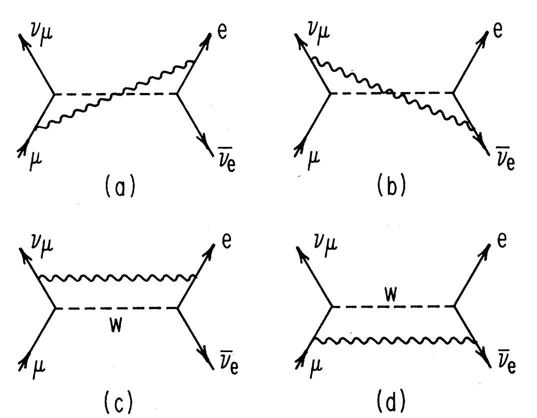

FIG. 2. Higher-order skeleton diagrams. The wavy line in (a) means either Z, A, or ϕ, while in (b), (c), and (d) it stands only for Z.

From the conservation of momentum, we have

$$q = p - k_2 = p' + k_1 . \tag{2.6}$$

The next-order corrections to the decay matrix element M_0 are given by the skeleton diagrams in Fig. 2 and all possible self-energy and vertex insertions to the diagrams in Fig. 1. All these diagrams are divergent. We are going to show that some of these divergences can be absorbed into the renormalization constants, while those divergences which cannot be lumped into the constants cancel each other out.

B. Contributions of the Skeleton Diagrams

Since the neutral Z boson can couple to neutral and charged leptons, there are four skeleton diagrams due to the exchange of a Z between leptons as is given in Fig. 2. However since photons and massive neutral scalar mesons interact only with charged leptons, there are two more skeleton diagrams of the type given in Fig. 2(a), due to exchange of a photon or a ϕ between two charged leptons. Let us first consider the diagram in Fig. 2(a) due to the exchange of a Z between the muon and the electron, and denote its amplitude by B_Z which reads

$$\begin{aligned} B_Z = \frac{(g/2\sqrt{2})^2}{16}(g^2+g'^2)\int \frac{d^4k}{(2\pi)^4}\, \frac{-g^{\mu\nu}+k^\mu k^\nu/m_Z{}^2}{k^2-m_Z{}^2}\, \frac{-g^{\alpha\beta}+(q-k)^\alpha(q-k)^\beta/m_W{}^2}{(q-k)^2-m_W{}^2} \\ \times \bar{u}(k_2)O_\alpha \frac{\not{p}+m-\not{k}}{(p-k)^2-m^2}\gamma_\mu(a+\gamma_5)u(p) \\ \times \bar{u}(p')\gamma_\nu(a+\gamma_5)\frac{\not{p}'+m'-\not{k}}{(p'-k)^2-m'^2}O_\beta v(k_1)\,, \end{aligned} \tag{2.7}$$

where m and m' are the masses of the muon and electron, respectively,

$$O_\alpha = \gamma_\alpha(1+\gamma_5)\,, \tag{2.8}$$

and

$$a = 1 - \frac{4g'^2}{g^2+g'^2}\ . \tag{2.9}$$

Introducing the Feynman parameters α_1, α_2, α_3, and α_4 to combine the terms in the denominator and making a translation of the dummy variable k, we obtain

$$\begin{aligned} B_Z = \left(\frac{g}{2\sqrt{2}}\right)^2 \frac{3!}{16}(g^2+g'^2)\int \frac{d^4k\,d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4\,\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{(2\pi)^4(k^2-C)^4} \\ \times\left[-g^{\mu\nu}+\frac{(k+d)^\mu(k+d)^\nu}{m_Z{}^2}\right]\left[-g^{\alpha\beta}+\frac{(k+h_1)^\alpha(k+h_1)^\beta}{m_W{}^2}\right] \\ \times \bar{u}(k_2)O_\alpha(-\not{h}_2+m-\not{k})\gamma_\mu(a+\gamma_5)u(p)\bar{u}(p')\gamma_\nu(a+\gamma_5)(-\not{h}_3+m'-\not{k})O_\beta v(k_1)\,, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} d &\equiv \alpha_2 q + \alpha_3 p + \alpha_4 p', \\ h_1 &\equiv d - q\,, \\ h_2 &\equiv d - p\,, \\ h_3 &\equiv d - p', \end{aligned} \tag{2.11}$$

and

$$C \equiv \alpha_1 m_Z{}^2 - \alpha_2(q^2 - m_W{}^2) + d^2\,.$$

After some algebra, we find that B_Z becomes

$$\begin{aligned} B_Z = \frac{3!}{16}(g^2+g'^2)\int \frac{d^4k\,d\alpha_1\cdots d\alpha_4\,\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{(2\pi)^4(k^2-C)^4} \\ \times\left\{\frac{-(a+1)^2}{4m_Z{}^2m_W{}^2}k^6g^{\alpha\beta} - (a+1)^2\left(\frac{1}{m_Z{}^2}+\frac{1}{m_W{}^2}\right)k^4g^{\alpha\beta}\right. \end{aligned}$$

$$+\frac{(a+1)^2k^4}{m_Z{}^2m_W{}^2}[\tfrac{1}{6}h_2\cdot h_3+\tfrac{1}{4}(h_2{}^2+h_3{}^2)]g^{\alpha\beta}$$

$$+\frac{(a+1)^2k^4}{m_Z{}^2m_W{}^2}[\tfrac{1}{6}(h_2^\alpha h_3^\beta+h_2^\beta h_3^\alpha)+\tfrac{1}{3}d^\alpha d^\beta-\tfrac{1}{2}(q^\alpha d^\beta+q^\beta d^\alpha)-\tfrac{1}{2}d^\alpha p^\beta-\tfrac{1}{2}p'^\alpha d^\beta]$$

$$+\frac{(a^2-1)k^4}{m_Z{}^2m_W{}^2}[\tfrac{1}{2}(h_2^\alpha q^\beta+h_2^\beta q^\alpha)-\tfrac{3}{2}(q^\alpha d^\beta+q^\beta d^\alpha)+\tfrac{1}{2}(q^\alpha p^\beta+q^\beta p'^\alpha)-h_1^\alpha q^\beta-h_1^\beta q^\alpha]$$

$$\left.+\frac{(a-1)^2k^4}{m_Z{}^2m_W{}^2}q^\alpha q^\beta\right\}M_{\alpha\beta}+\text{convergent terms}, \tag{2.12}$$

where $M_{\alpha\beta}$ is defined in (2.5). Let us denote the ultraviolet divergent terms in B_Z to be $\overline{B}_Z$. Upon carrying out the integration in (2.12), we get

$$\overline{B}_Z=\frac{1}{16}(g^2+g'^2)\left[\frac{(a+1)^2}{4m_Z{}^2m_W{}^2}g^{\alpha\beta}M_{\alpha\beta}\right]\left(-\frac{i}{16\pi^2}\Lambda^2\right)$$

$$+\frac{1}{16}(g^2+g'^2)\left\{\frac{(a+1)^2}{4m_Z{}^2m_W{}^2}[-3(m_Z{}^2+m_W{}^2)-\tfrac{1}{3}q^2]g^{\alpha\beta}M_{\alpha\beta}+\frac{(a+1)^2}{m_Z{}^2m_W{}^2}\frac{q^\alpha q^\beta}{3}M_{\alpha\beta}-\frac{2(a-1)}{m_Z{}^2m_W{}^2}q^\alpha q^\beta M_{\alpha\beta}\right.$$

$$\left.+\frac{2(a+1)}{4m_Z{}^2m_W{}^2}(m^2+m'^2)g^{\alpha\beta}M_{\alpha\beta}\right\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right), \tag{2.13}$$

where Λ is the cutoff of the integral. The first term in $\overline{B}_Z$ is quadratically divergent, while the rest of the term is logarithmically divergent.

Similarly one can get the ultraviolet-divergent parts for the diagrams in Figs. 2(b), 2(c), and 2(d), which are denoted respectively by $\overline{B}_Z^b$, $\overline{B}_Z^c$, and $\overline{B}_Z^d$. Since the calculations are similar, straightforward and a little tedious, we do not bother to write them down but simply state the results:

$$\overline{B}_Z^b=\frac{1}{16}(g^2+g'^2)\left[\frac{4}{4m_Z{}^2m_W{}^2}g^{\alpha\beta}M_{\alpha\beta}\right]\left(\frac{-i}{16\pi^2}\Lambda^2\right)$$

$$+\frac{1}{16}(g^2+g'^2)\left\{\frac{4}{4m_Z{}^2m_W{}^2}[-3(m_Z{}^2+m_W{}^2)-\tfrac{1}{3}q^2]g^{\alpha\beta}M_{\alpha\beta}+\frac{4}{4m_Z{}^2m_W{}^2}\frac{q^\alpha q^\beta}{3}M_{\alpha\beta}\right\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right), \tag{2.14}$$

$$\overline{B}_Z^c=\frac{1}{16}(g^2+g'^2)\left[\frac{2(a+1)}{4m_Z{}^2m_W{}^2}g^{\alpha\beta}M_{\alpha\beta}\right]\left(\frac{-i}{16\pi^2}\Lambda^2\right)$$

$$+\frac{1}{16}(g^2+g'^2)\left\{\frac{2(a+1)}{4m_Z{}^2m_W{}^2}[-3(m_Z{}^2+m_W{}^2)-\tfrac{1}{3}q^2]g^{\alpha\beta}M_{\alpha\beta}+\frac{2(a+1)}{m_Z{}^2m_W{}^2}\frac{q^\alpha q^\beta}{3}M_{\alpha\beta}\right.$$

$$\left.+\frac{1}{m_Z{}^2m_W{}^2}(m'^2g^{\alpha\beta}-2q^\alpha q^\beta)M_{\alpha\beta}\right\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right), \tag{2.15}$$

and

$$\overline{B}_Z^d=\frac{1}{16}(g^2+g'^2)\left[\frac{2(a+1)}{4m_Z{}^2m_W{}^2}g^{\alpha\beta}M_{\alpha\beta}\right]\left(\frac{-i}{16\pi^2}\Lambda^2\right)$$

$$+\frac{1}{16}(g^2+g'^2)\left\{\frac{2(a+1)}{4m_Z{}^2m_W{}^2}[-3(m_Z{}^2+m_W{}^2)-\tfrac{1}{3}q^2]g^{\alpha\beta}M_{\alpha\beta}+\frac{2(a+1)}{m_Z{}^2m_W{}^2}\frac{q^\alpha q^\beta}{3}M_{\alpha\beta}\right.$$

$$\left.+\frac{1}{m_Z{}^2m_W{}^2}[m^2g^{\alpha\beta}-2q^\alpha q^\beta]M_{\alpha\beta}\right\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \tag{2.16}$$

Summing (2.13)–(2.16) and making use of (2.9) yields

$$\overline{B}_Z^{\text{tot}}\equiv\overline{B}_Z+\overline{B}_Z^b+\overline{B}_Z^c+\overline{B}_Z^d$$

$$=\frac{1}{4m_Z{}^2m_W{}^2}\frac{g^4}{g^2+g'^2}g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right)$$

$$+\left\{\frac{1}{4m_Z{}^2m_W{}^2}\frac{g^4}{g^2+g'^2}[-3(m_Z{}^2+m_W{}^2)-\tfrac{1}{3}q^2]g^{\alpha\beta}M_{\alpha\beta}+\frac{1}{m_Z{}^2m_W{}^2}\frac{g^4}{g^2+g'^2}\frac{q^\alpha q^\beta}{3}M_{\alpha\beta}\right.$$

$$+\frac{1}{8m_Z{}^2m_W{}^2}\frac{g^2(g^2+g'^2)}{g^2+g'^2}(m^2+m'^2)g^{\alpha\beta}M_{\alpha\beta}+\frac{1}{4m_Z{}^2m_W{}^2}\frac{g'^4-g^4}{g^2+g'^2}q^\alpha q^\beta M_{\alpha\beta}\Big\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \tag{2.17}$$

Now let us proceed to calculate the contributions of the skeleton diagrams due to the exchange of a photon or a ϕ between the electron and muon, denoted respectively by B_A and B_ϕ. The explicit expression for B_A and B_ϕ can be readily written down as

$$B_A=\left(\frac{g}{2\sqrt{2}}\right)^2e^2\int\frac{d^4k}{(2\pi)^4}\frac{(-g^{\mu\nu})}{k^2}\frac{-g^{\alpha\beta}+(q-k)^\alpha(q-k)^\beta/m_W{}^2}{(q-k)^2-m_W{}^2}$$
$$\times\bar{u}(k_2)O_\alpha\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\gamma_\mu u(p)\bar{u}(p')\gamma_\nu\frac{\not{p}'-\not{k}+m'}{(p'-k)^2-m'^2}O_\beta v(k_1), \tag{2.18}$$

$$B_\phi=\left(\frac{g}{2\sqrt{2}}\right)^2\frac{mm'}{\lambda^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{-g^{\alpha\beta}+(q-k)^\alpha(q-k)^\beta/m_W{}^2}{(q-k)^2-m_W{}^2}$$
$$\times\bar{u}(k_2)O_\alpha\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}u(p)\bar{u}(p')\frac{\not{p}'-\not{k}+m'}{(p'-k)^2-m'^2}O_\beta v(k_1). \tag{2.19}$$

Both of these integrals are logarithmically divergent. Let us denote $\bar{B}_A$ and $\bar{B}_\phi$ to be the divergent parts for B_A and B_ϕ, respectively. They can be easily gotten by calculating the k^4 terms in the numerators of the integrands. Thus we have

$$\bar{B}_A=-\frac{e^2}{m_W{}^2}g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right), \tag{2.20}$$

$$\bar{B}_\phi=\frac{mm'}{\lambda^2}\bar{u}(k_2)(1-\gamma_5)u(p)\bar{u}(p')(1-\gamma_5)v(k_1)\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)$$
$$=\frac{g^2}{4m_W{}^2}q^\alpha q^\beta M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right), \tag{2.21}$$

where use is made of (2.3) and (2.6).

These divergences certainly cannot be absorbed into any renormalization constants. However we are going to see that they will be canceled out by the divergences arising from the self-energy and vertex corrections.

C. Wave-Function Renormalization and Self-Energy Corrections

Since the couplings of W, Z, A, and ϕ to the leptons do not depend upon the momenta of the coupled particles, all the divergences arising from the self-energy insertions on the external leptons are taken care of by the wave-function renormalization constants just like that in pure quantum electrodynamics. So are the divergences in the self-energy of W due to the lepton loop. However, it is quite different for the self-energy of W due to Z, A, or ϕ. In this case only part of the divergences can be absorbed into the wave-function renormalization constants. To see this, let us first look at the self-energy part $I_{\alpha\delta}$ of W due to Z as given in Figs. 3(a) and 3(b). The diagram in Fig. 3(b), however, need not be considered here because it is taken care of by the mass renormalization. So we get

$$I_{\alpha\delta}=\frac{g^4}{g^2+g'^2}\int\frac{d^4k}{(2\pi)^4}\frac{-g^{\mu\nu}+(q-k)^\mu(q-k)^\nu/m_Z{}^2}{(q-k)^2-m_Z{}^2}\frac{-g^{\beta\gamma}+k^\beta k^\gamma/m_W{}^2}{k^2-m_W{}^2}$$
$$\times[(k+q)_\mu g_{\alpha\beta}-(2k-q)_\alpha g_{\mu\beta}-(2q-k)_\beta g_{\mu\alpha}][(k+q)_\nu g_{\delta\gamma}-(2k-q)_\delta g_{\nu\gamma}-(2q-k)_\gamma g_{\nu\delta}]. \tag{2.22}$$

Introducing the Feynman parameters α_1 and α_2 to combine the terms in the denominator of the integrand and making a change of variable of integration by replacing k by $k+\alpha_2 q$, we get

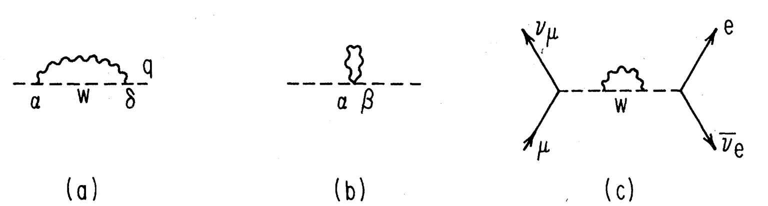

FIG. 3. The diagrams of the self-energy of the W boson and the self-energy correction to μ decay. The wavy line means either Z, A, or ϕ.

$$I_{\alpha\delta} = \frac{g^4}{g^2+g'^2}\int \frac{d^4k\, d\alpha_1 d\alpha_2 \delta(\alpha_1+\alpha_2-1)}{(2\pi)^4 (k^2-C_0)^2}$$

$$\times \left\{ \frac{1}{m_W^2}\left[-\tfrac{3}{4}k^4 g_{\alpha\delta} - \tfrac{1}{4}k^2(\alpha_2^2-2\alpha_2)(11\mathcal{Q}_{\alpha\delta}+3q_\alpha q_\delta) - \tfrac{1}{4}k^2(5\mathcal{Q}_{\alpha\delta}+3q_\alpha q_\delta) - (\alpha_2^4-4\alpha_2^3+4\alpha_2^2)q^2\mathcal{Q}_{\alpha\delta}\right]\right.$$

$$+\frac{1}{m_Z^2}\left[-\tfrac{3}{4}k^4 g_{\alpha\delta} - \tfrac{1}{4}k^2\alpha_2^2(11\mathcal{Q}_{\alpha\delta}+3q_\alpha q_\delta) + \tfrac{3}{2}k^2\mathcal{Q}_{\alpha\delta} - (\alpha_2^2-1)^2 q^2\mathcal{Q}_{\alpha\delta}\right]$$

$$\left. + \frac{k^2}{4m_Z^2 m_W^2} q^2\mathcal{Q}_{\alpha\delta} + \tfrac{9}{2}k^2 g_{\alpha\delta} + 5\mathcal{Q}_{\alpha\delta} + 3q_\alpha q_\delta + 2(\alpha_2^2-\alpha_2)(\mathcal{Q}_{\alpha\delta}+6q_\alpha q_\delta)\right\} + \text{convergent terms,} \tag{2.23}$$

where

$$C_0 = \alpha_1 m_W^2 + \alpha_2 m_Z^2 + \alpha_2 \tag{2.24}$$

and

$$\mathcal{Q}_{\alpha\delta} = q^2 g_{\alpha\delta} - q_\alpha q_\delta\,. \tag{2.25}$$

After carrying out the integrations and performing the mass renormalization, we arrive at

$$I_{\alpha\delta} = \frac{g^4}{g^2+g'^2}\left[\frac{(q^4-m_W^4)g_{\alpha\delta}-q^2q_\alpha q_\delta}{4m_Z^2 m_W^2}\right]\left(\frac{-i}{16\pi^2}\Lambda^2\right)$$

$$+\frac{g^4}{g^2+g'^2}\left\{\frac{(m_Z^2+m_W^2)}{4m_Z^2m_W^2}\left[(q^4-m_W^4)g_{\alpha\delta}-q^2q_\alpha q_\delta\right] - \frac{\left[(q^6-m_W^6)g_{\alpha\delta}-q^4q_\alpha q_\delta\right]}{12m_Z^2m_W^2}\right.$$

$$+\frac{1}{4}\left(\frac{m_Z^2}{m_W^2}+\frac{m_W^2}{m_Z^2}-2\right)q_\alpha q_\delta - \frac{5}{6}\left(\frac{1}{m_W^2}+\frac{1}{m_Z^2}\right)\left[(q^4-m_W^4)g_{\alpha\delta}-q^2q_\alpha q_\delta\right]$$

$$+\frac{1}{m_W^2}\left[\tfrac{9}{24}(m_W^2+m_Z^2)+\tfrac{3}{24}m_W^2+\tfrac{25}{24}m_Z^2\right]\left[(q^2-m_W^2)g_{\alpha\delta}-q_\alpha q_\delta\right]$$

$$\left.+\frac{1}{m_Z^2}\left[\tfrac{9}{24}(m_W^2+m_Z^2)+\tfrac{25}{24}m_W^2+\tfrac{3}{24}m_Z^2\right]\left[(q^2-m_W^2)g_{\alpha\delta}-q_\alpha q_\delta\right]\right\} + \text{convergent terms.} \tag{2.26}$$

Because of the self-energy insertion, the W-boson propagator becomes $D^{\beta\gamma}(q^2)$, which reads

$$D^{\beta\gamma}(q^2) = \frac{-g^{\beta\gamma}+q^\beta q^\gamma/m_W^2}{q^2-m_W^2} + \frac{-g^{\beta\alpha}+q^\beta q^\alpha/m_W^2}{q^2-m_W^2} I_{\alpha\delta} \frac{-g^{\gamma\delta}+q^\gamma q^\delta/m_W^2}{q^2-m_W^2}\,. \tag{2.27}$$

We define the wave-function renormalization constant $Z_{3,W}$ for the W boson conventionally as

$$Z_{3,W}\left[-g^{\beta\gamma}+\frac{q^\beta q^\gamma}{m_W^2}\right] = \lim_{q^2\to m_W^2}(q^2-m_W^2)D^{\beta\gamma}(q^2)\,. \tag{2.28}$$

Denoting $Z_{3,W}^Z$ to be the contribution to $Z_{3,W}$ due to Z, we find from (2.26) to (2.28) that

$$Z_{3,W}^Z - 1 = \frac{-g^4}{g^2+g'^2}\left[\frac{2m_W^2}{4m_Z^2m_W^2}\right]\left(\frac{-i}{16\pi^2}\Lambda^2\right)$$

$$-\frac{g^4}{g^2+g'^2}\left\{\frac{2m_W^2}{4m_Z^2m_W^2}(m_Z^2+m_W^2) - \frac{3m_W^4}{12m_Z^2m_W^2} - \frac{5}{6}\left(\frac{1}{m_W^2}+\frac{1}{m_Z^2}\right)(2m_W^2)\right.$$

$$\left.+\frac{1}{m_W^2}\left[\tfrac{9}{24}(m_W^2+m_Z^2)+\tfrac{3}{24}m_W^2+\tfrac{25}{24}m_Z^2\right]+\frac{1}{m_Z^2}\left[\tfrac{9}{24}(m_W^2+m_Z^2)+\tfrac{25}{24}m_W^2+\tfrac{3}{24}m_Z^2\right]\right\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)\,. \tag{2.29}$$

The divergent terms in (2.26) are momentum-dependent. It is clear that if q^2 is not on the mass shell, the divergence in (2.26) then cannot be entirely taken care of by the wave-function renormalization. As a matter of fact, with (2.26) and (2.29), the divergent part H_Z of the contribution of the diagram in Fig. 3(c) can be readily written down as

$$H_Z = (Z_{3,W}^Z - 1)M_0 + \overline{H}_Z\,, \tag{2.30}$$

where $\overline{H}_Z$ is the divergent term which cannot be absorbed into the wave-function renormalization constant,

$$\overline{H}_Z = \frac{g^4}{g^2+g'^2}\left[\frac{g^{\alpha\beta}M_{\alpha\beta}}{4m_Z{}^2m_W{}^2}\right]\left(\frac{-i}{16\pi^2}\Lambda^2\right) + \frac{g^4}{g^2+g'^2}\left\{-\frac{7}{12}\left(\frac{1}{m_Z{}^2}+\frac{1}{m_W{}^2}\right)g^{\alpha\beta} - \frac{1}{12m_Z{}^2m_W{}^2}[(q^2+2m_W{}^2)g^{\alpha\beta} - q^\alpha q^\beta] + \frac{1}{4}\left(\frac{m_Z{}^2}{m_W{}^2}+\frac{m_W{}^2}{m_Z{}^2}\right)\frac{q^\alpha q^\beta}{m_W{}^4} - \frac{1}{2}\frac{q^\alpha q^\beta}{m_W{}^4}\right\} M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \tag{2.31}$$

Similarly we can find the divergent contributions $\overline{H}_A$ and $\overline{H}_\phi$ of the self-energy corrections to the muon decay due to A or ϕ, respectively, which cannot be taken care of by the wave-function renormalization,

$$\overline{H}_A = e^2\left[-\frac{5}{6}\frac{g^{\alpha\beta}}{m_W{}^2} - \frac{1}{2}\frac{q^\alpha q^\beta}{m_W{}^4}\right]M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right) \tag{2.32}$$

and

$$\overline{H}_\phi = \frac{g^2}{4m_W{}^4}q^\alpha q^\beta M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \tag{2.33}$$

D. Vertex Renormalization and Vertex Corrections

Since the couplings of W or Z to the leptons do not depend on the momenta of the coupled particles, all the divergences in the vertex corrections in Figs. 4(a) to 4(d) can be entirely absorbed into the vertex renormalization constants just like those in pure quantum electrodynamics. However, things become quite different for the vertex correction in Fig. 4(e), because the W-Z coupling is momentum-dependent. Let us denote Γ_β to be the contribution of the diagram in Fig. 4(e). Its explicit expression reads

$$\Gamma_\beta(q^2) = -\frac{g^2}{4}\int\frac{d^4k}{(2\pi)^4}\,\frac{-g^{\mu\nu}+k^\mu k^\nu/m_Z{}^2}{k^2-m_Z{}^2}\,\frac{-g^{\gamma\alpha}+(q-k)^\gamma(q-k)^\alpha/m_W{}^2}{(q-k)^2-m_W{}^2}\,O_\gamma\,\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\,\gamma_\nu(a+\gamma_5) \times[(q-k+q)_\mu g_{\alpha\beta} - (2q-2k-q)_\beta g_{\mu\alpha} - (2q-q+k)_\alpha g_{\mu\beta}], \tag{2.34}$$

where the lepton momenta are taken to be on the mass shell. Applying the same techniques that we used in calculating the divergent part of an integral in the last two sections, we find that

$$\begin{aligned}\Gamma_\beta = &-\frac{g^2}{4}\left[\frac{-(a+1)}{4m_Z{}^2m_W{}^2}O^\alpha\mathcal{Q}_{\alpha\beta} - \frac{3}{4}\left(\frac{1}{m_Z{}^2}+\frac{1}{m_W{}^2}\right)O_\beta\right]\left(\frac{-i}{16\pi^2}\Lambda^2\right)\\ &-\frac{g^2}{16m_Z{}^2m_W{}^2}\{(a+1)[-(m_Z{}^2+m_W{}^2)+\tfrac{1}{3}q^2]-2m^2\}O^\alpha\mathcal{Q}_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)\\ &-\frac{g^2}{16m_W{}^2}[2(a+1)]\{[-\tfrac{3}{2}(m_Z{}^2+m_W{}^2)-2m^2+\tfrac{11}{6}q^2]O_\beta - \tfrac{7}{3}mq_\beta(1-\gamma_5)\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)\\ &-\frac{g^2}{16m_Z{}^2}[2(a+1)]\{[-\tfrac{3}{2}(m_Z{}^2+m_W{}^2)+\tfrac{11}{6}q^2]O_\beta - \tfrac{4}{3}mq_\beta(1-\gamma_5)\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)\\ &-\frac{g^2}{8}\left(\frac{1}{m_W{}^2}-\frac{1}{m_Z{}^2}\right)[m^2O_\beta+2mq_\beta(1-\gamma_5)]\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right).\end{aligned} \tag{2.35}$$

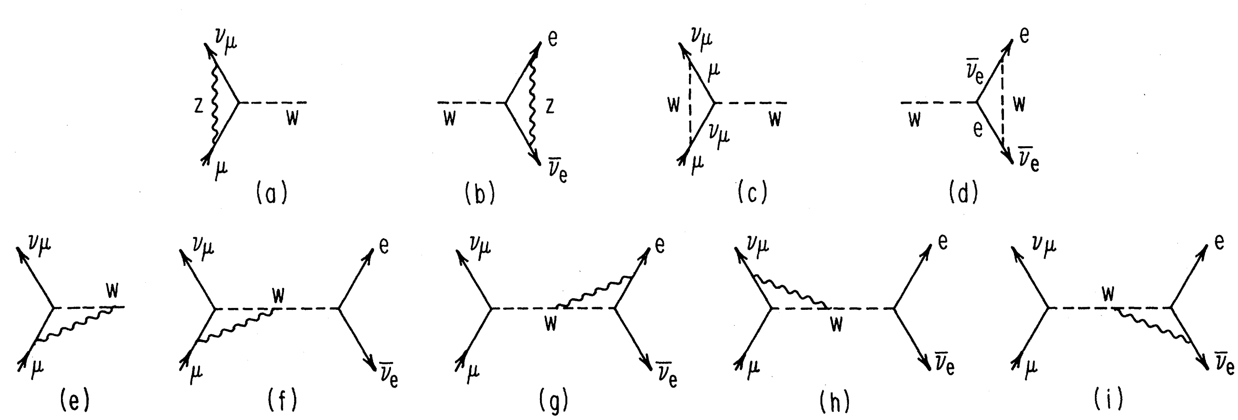

FIG. 4. The diagrams of the vertices and vertex corrections to μ decay. The wavy line in (e)–(g) means either Z, A, or ϕ, while in the rest of the diagrams it stands only for Z.

As usual, we define the vertex renormalization constant inverse Z_1^{-1} as one plus the coefficient of O_β in $\bar{u}(k_1)\Gamma_\beta(q^2)u(p)$ with $q^2=m_W^2$; i.e.,

$$Z_1^{-1}-1=-\tfrac{1}{4}g^2\left[\frac{-(a+1)}{4m_Z^2m_W^2}m_W^2-\frac{3}{4}\left(\frac{1}{m_Z^2}+\frac{1}{m_W^2}\right)\right]\left(\frac{-i}{16\pi^2}\Lambda^2\right)$$
$$-\frac{g^2m_W^2}{16m_Z^2m_W^2}\left\{(a+1)[-(m_Z^2+m_W^2)+\tfrac{1}{3}m_W^2]-2m^2\right\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)$$
$$-\frac{2g^2(a+1)}{16m_W^2}\left[-\tfrac{3}{2}(m_Z^2+m_W^2)-2m^2+\tfrac{11}{6}m_W^2\right]\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)$$
$$-\frac{2g^2(a+1)}{16m_Z^2}\left[-\tfrac{3}{2}(m_Z^2+m_W^2)+\tfrac{11}{6}m_W^2\right]\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)-\frac{m^2g^2}{8}\left(\frac{1}{m_W^2}-\frac{1}{m_Z^2}\right)\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \qquad (2.36)$$

With (2.35) and (2.36), we find the divergent part V_Z^f of the vertex corrections to the μ-decay matrix element due to the Z boson as given in Fig. 4(f),

$$V_Z^f=(Z_1^{-1}-1)M_0+\bar{V}_Z^f, \qquad (2.37)$$

where $\bar{V}_Z^f$ represent those divergences which cannot be taken care of by the vertex renormalization, i.e.,

$$\bar{V}_Z^f=-\frac{g^2(a+1)}{4m_Z^2m_W^2}\frac{g^{\alpha\beta}M_{\alpha\beta}}{4}\left(\frac{-i}{16\pi^2}\Lambda^2\right)-\tfrac{1}{4}g^2\left\{(a+1)[(m_Z^2+m_W^2)g^{\alpha\beta}-\tfrac{1}{3}(q^2+m_W^2)g^{\alpha\beta}+\tfrac{1}{3}q^\alpha q^\beta]\right.$$
$$+\frac{2m^2}{4m_Z^2m_W^2}g^{\alpha\beta}+\frac{(a+1)}{4m_Z^2}\left(-\tfrac{11}{3}g^{\alpha\beta}+\frac{q^\alpha q^\beta}{m_W^2}\right)+\frac{(a+1)}{4m_W^2}\left[-\tfrac{11}{3}g^{\alpha\beta}-\frac{q^\alpha q^\beta}{m_W^2}\right]$$
$$\left.+\frac{4}{4m_W^2}\left(\frac{1}{m_W^2}-\frac{1}{m_Z^2}\right)q^\alpha q^\beta\right\}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \qquad (2.38)$$

Similarly we find the divergent contributions $\bar{V}_Z^g$, $\bar{V}_Z^h$, and $\bar{V}_Z^i$, which cannot be absorbed into the vertex renormalization constants, for the diagrams in Figs. 4(g), 4(h), and 4(i), respectively, to be

$$\bar{V}_Z^g=\bar{V}_Z^f+\frac{g^2}{8m_Z^2m_W^2}(m^2-m'^2)\,g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right) \qquad (2.39)$$

and

$$\bar{V}_Z^h=-\frac{2g^2}{16m_Z^2m_W^2}g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right)-\frac{g^2}{4}\left\{\frac{2}{4m_Z^2m_W^2}[(m_Z^2+m_W^2)g^{\alpha\beta}-\tfrac{1}{3}(q^2+m_W^2)g^{\alpha\beta}+\tfrac{1}{3}q^\alpha q^\beta]\right.$$
$$\left.-\frac{2}{4m_Z^2}\left(-\tfrac{11}{3}g^{\alpha\beta}+\frac{q^\alpha q^\beta}{m_W^2}\right)-\frac{2}{4m_W^2}\left(-\tfrac{11}{3}g^{\alpha\beta}-\frac{q^\alpha q^\beta}{m_W^2}\right)\right\}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)$$
$$(2.40)$$

and

$$\bar{V}_Z^i=\bar{V}_Z^h. \qquad (2.41)$$

Summing up (2.38) to (2.41) and making use of (2.9), we obtain the total divergent part $\bar{V}_Z$ of the vertex corrections to the μ-decay matrix element due to the Z boson which cannot be taken care of by the vertex renormalization,

$$\bar{V}_Z=\bar{V}_Z^f+\bar{V}_Z^g+\bar{V}_Z^h+\bar{V}_Z^i$$
$$=-\frac{g^4}{g^2+g'^2}\frac{2}{4m_Z^2m_W^2}g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right)$$
$$-\frac{g^4}{g^2+g'^2}\left\{\frac{2}{4m_Z^2m_W^2}[(m_Z^2+m_W^2)g^{\alpha\beta}-\tfrac{1}{3}(q^2+m_W^2)g^{\alpha\beta}+\tfrac{1}{3}q^\alpha q^\beta]\right.$$
$$\left.+\frac{2}{4m_W^2}\left(-\tfrac{11}{3}g^{\alpha\beta}+\frac{q^\alpha q^\beta}{m_W^2}\right)+\frac{2}{4m_Z^2}\left(-\tfrac{11}{3}g^{\alpha\beta}-\frac{q^\alpha q^\beta}{m_W^2}\right)\right\}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)$$
$$-g^2\left\{\frac{1}{8m_Z^2m_W^2}(m^2+m'^2)g^{\alpha\beta}-\frac{2q^\alpha q^\beta}{4m_Z^2m_W^2}+\frac{2q^\alpha q^\beta}{4m_W^4}\right\}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \qquad (2.42)$$

Similarly we find the total divergent parts $\overline{V}_A$ and $\overline{V}_\phi$ of the vertex corrections to the μ-decay matrix element due to A and ϕ, respectively, to be

$$\overline{V}_A = \frac{-2e^2}{4m_W^2}\left(-\tfrac{11}{3}g^{\alpha\beta} - \frac{q^\alpha q^\beta}{m_W^2}\right)M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right), \tag{2.43}$$

$$\overline{V}_\phi = \frac{-2g^2}{4m_W^4}\, q^\alpha q^\beta M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \tag{2.44}$$

E. Cancellation of Divergences

We now group together those divergences which cannot be lumped into the renormalization constants and show that they cancel out. First let us consider the divergent higher-order corrections due to the Z boson. With (2.3), we note that terms like m_Z^2/m_W^6 in the self-energy correction $\overline{H}_Z$ in (2.31) can be written as

$$\frac{m_Z^2}{m_W^6} = \frac{1}{m_Z^2 m_W^2}\left(\frac{m_Z^2}{m_W^2}\right)^2 = \frac{1}{m_Z^2 m_W^2}\,\frac{(g^2+g'^2)^2}{g^4}\,. \tag{2.45}$$

With (2.45), the sum of the divergences $\overline{B}_Z^{\rm tot}$ in (2.17) and $\overline{H}_Z$ in (2.31) becomes

$$\begin{aligned}\overline{B}_Z^{\rm tot} + \overline{H}_Z = {} & \frac{2}{4m_Z^2 m_W^2}\,\frac{g^4}{g^2+g'^2}\, g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right) \\ & + \frac{g^4}{g^2+g'^2}\left\{-\frac{16}{12}\,\frac{m_Z^2+m_W^2}{m_Z^2 m_W^2} - \frac{2}{12 m_Z^2 m_W^2}(q^2+m_W^2)g^{\alpha\beta} - \frac{q^\alpha q^\beta}{2m_W^4} + \frac{2}{3}\,\frac{q^\alpha q^\beta}{m_Z^2 m_W^2}\right\}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right) \\ & + \frac{1}{8m_Z^2 m_W^2}\,\frac{g^2(g^2+g'^2)}{g^2+g'^2}\,(m^2+m'^2)g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right) \\ & + \frac{2}{4m_Z^2 m_W^2}\,\frac{g'^2(g^2+g'^2)}{g^2+g'^2}\, q^\alpha q^\beta M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right).\end{aligned} \tag{2.46}$$

Using (2.3), (2.42) can be rewritten as

$$\begin{aligned}\overline{V}_Z = {} & -\frac{2}{4m_Z^2 m_W^2}\,\frac{g^4}{g^2+g'^2}\, g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right) \\ & - \left\{\frac{2}{4m_Z^2 m_W^2}\,\frac{g^4}{g^2+g'^2}\left[-\tfrac{8}{3}(m_Z^2+m_W^2)g^{\alpha\beta} - \tfrac{1}{3}(q^2+m_W^2)g^{\alpha\beta} + \tfrac{4}{3}q^\alpha q^\beta\right] - \frac{2q^\alpha q^\beta}{4m_W^4}\,\frac{g^4}{g^2+g'^2}\right\}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right) \\ & - \left\{\frac{1}{8m_Z^2 m_W^2}(m^2+m'^2)\frac{g^2(g^2+g'^2)}{g^2+g'^2}\, g^{\alpha\beta} + \frac{2}{4m_Z^2 m_W^2}\,\frac{g'^2(g^2+g'^2)}{g^2+g'^2}\, q^\alpha q^\beta\right\}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right).\end{aligned} \tag{2.47}$$

Comparing (2.46) to (2.47), we get

$$\overline{B}_Z^{\rm tot} + \overline{H}_Z = -\overline{V}_Z$$

or

$$\overline{B}_Z^{\rm tot} + \overline{H}_Z + \overline{V}_Z = 0. \tag{2.48}$$

Summing up (2.20), (2.32), and (2.43) yields

$$\begin{aligned}\overline{B}_A + \overline{H}_A + \overline{V}_A &= e^2\left(-\frac{5}{6m_W^2}g^{\alpha\beta} - \frac{g^{\alpha\beta}}{m_W^2} - \frac{q^\alpha q^\beta}{2m_W^4} + \frac{11}{6m_W^2}g^{\alpha\beta} + \frac{q^\alpha q^\beta}{2m_W^2}\right)M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right) \\ &= 0.\end{aligned} \tag{2.49}$$

Also summing up (2.21), (2.33), and (2.44), we find

$$\begin{aligned}\overline{B}_\phi + \overline{H}_\phi + \overline{V}_\phi &= g^2\left(\frac{q^\alpha q^\beta}{4m_W^2} + \frac{q^\alpha q^\beta}{4m_W^2} - \frac{2q^\alpha q^\beta}{4m_W^2}\right)M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right) \\ &= 0.\end{aligned} \tag{2.50}$$

So we see from (2.48) to (2.50) that the divergences arising from the skeleton diagrams and self-energy corrections just cancel out those from the vertex corrections.

III. ELASTIC NEUTRINO-ELECTRON SCATTERING

A. Scattering Amplitude

We consider the process

$$\nu_e + e \to \nu_e + e\,.$$

The momenta of the neutrinos in the initial and final state are denoted by k_1 and k_2, respectively, while p and p' denote the corresponding momenta for the electrons in the initial and final state. To the lowest order in the coupling constants, the S-matrix in the unitary gauge is given by the sum of diagrams in Fig. 5. We note that to this lowest order the ν-ν scattering amplitude is given by the Z-exchange diagram similar to Fig. 5(a) and the μ-decay amplitude is given by the W-exchange diagram similar to Fig. 5(b). To fourth order in the coupling constants, the scattering amplitude is given by the sum of diagrams in Figs. 6–10 and a number of other diagrams not drawn here (self-energy and vertex insertions to Fig. 5) which do not generate residual divergences.[5] It is clear that contributions of the diagrams in Figs. 6–10 can be divided into three distinct groups. The first group is given by the contributions of diagrams in Fig. 6 and the part of the contributions of the diagrams in Figs. 7(a)–7(c) associated with the charge radius of the neutrino.

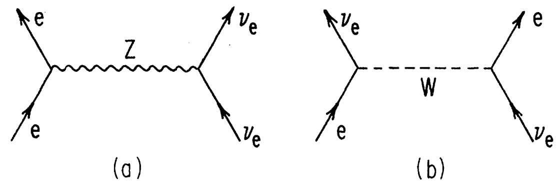

FIG. 5. The lowest-order diagrams for neutrino-electron scattering.

The second group is given by the contributions of the diagrams in Figs. 7(a)–7(c) excluding the part associated with the charge radius of the neutrino and of the diagrams in Fig. 8. They are similar to those in the elastic ν-ν scattering. The third group includes all diagrams in Figs. 9 and 10. They are almost identical to those in μ decay. We are going to show in the next three subsections that the renormalization programs can be carried out separately in these three groups and that the residual divergences all cancel out in each of them.

B. Neutrino Charge Radius

The mean-square charge radius of the neutrino is usually defined to be proportional to the first derivative of the neutrino electromagnetic form factor at zero momentum transfer. At first glance, one tends to think that diagrams in Figs. 7(a)–7(c) have nothing to do with it because it seems only weak interactions are involved. This is not at all so. To see this, let us first calculate the contribution V_Z of the diagram in Fig. 7(c):

$$\begin{aligned} V_Z &= \frac{g^2}{4(g^2+g'^2)^{1/2}}\,\bar{u}(k_2)\Gamma_\mu u(k_1)\frac{-g^{\mu\nu}+q^\mu q^\nu/m_Z^{\,2}}{q^2-m_Z^{\,2}}\,\bar{u}(p')\gamma_\nu\left(1+\gamma_5-\frac{4g'^2}{g^2+g'^2}\right)(g^2+g'^2)^{1/2}u(p) \\ &= V_Z^{\rm em}+V_Z^W, \end{aligned} \tag{3.1}$$

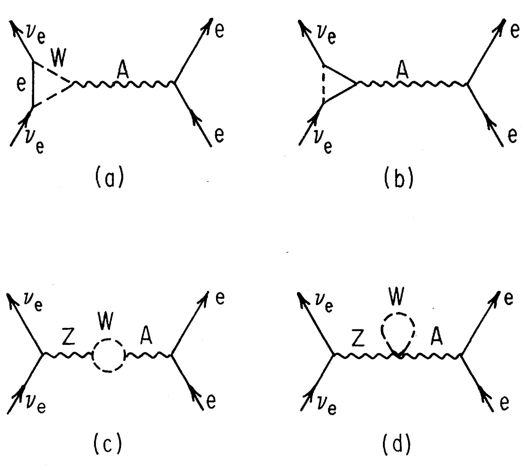

FIG. 6. Diagrams with one virtual photon exchange which contribute to the charge radius of the neutrino.

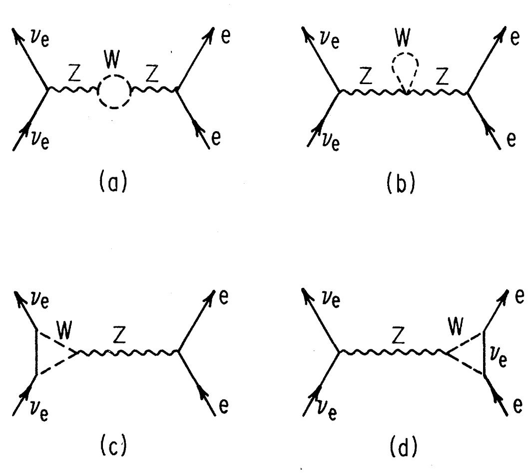

FIG. 7. Self-energy and vertex insertions to the diagram in Fig. 5(a).

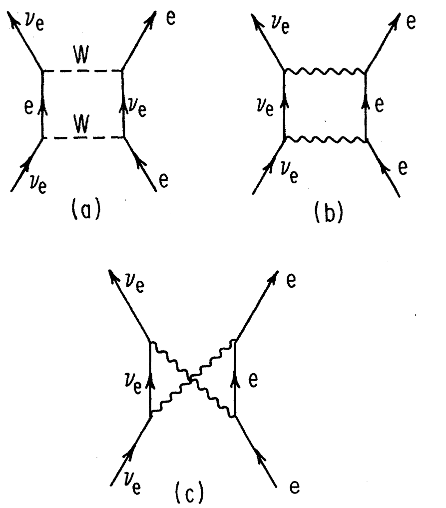

FIG. 8. Skeleton diagrams with two W or two Z exchanges.

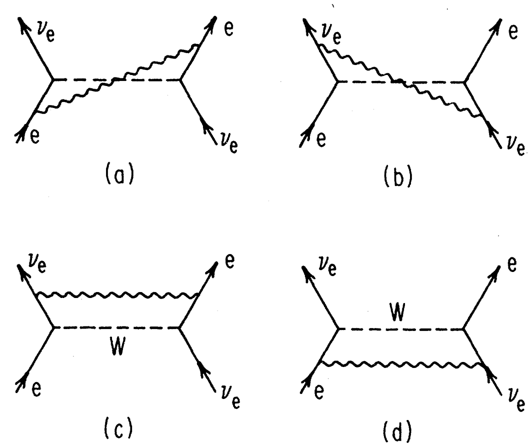

FIG. 9. Skeleton diagrams with one W exchange and one Z, A, or ϕ exchange. The wavy line in (a) represents either Z, A, or ϕ, while in the rest of the diagrams it stands for Z only.

where

$$q = p - p', \tag{3.2}$$

$$V_Z^{\text{em}} = -e^2 \bar{u}(k_2)\Gamma_\mu u(k_1) \frac{-g^{\mu\nu} + q^\mu q^\nu / m_Z^{\,2}}{q^2 - m_Z^{\,2}} \bar{u}(p')\gamma_\nu u(p), \tag{3.3}$$

$$V_Z^W = \tfrac{1}{4} g^2 \bar{u}(k_2)\Gamma_\mu u(k_1) \frac{-g^{\mu\nu} + q^\mu q^\nu / m_Z^{\,2}}{q^2 - m_Z^{\,2}} \bar{u}(p')\gamma_\nu (1+\gamma_5) u(p), \tag{3.4}$$

and

$$\Gamma_\mu = \left(\frac{g}{2\sqrt{2}}\right)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-g^{\alpha\delta} + k^\alpha k^\delta / m_W^{\,2}}{k^2 - m_W^{\,2}} \frac{-g^{\beta\gamma} + (k-q)^\beta (k-q)^\gamma / m_W^{\,2}}{(k-q)^2 - m_W^{\,2}} V_{\mu\alpha\beta} O_\delta \frac{\not{k}_2 + \not{k} + m_e}{(k+k_2)^2 - m_e^{\,2}} O_\gamma, \tag{3.5}$$

with

$$V_{\mu\alpha\beta} \equiv k_\mu g_{\alpha\beta} - k_\beta g_{\mu\alpha} + (k-q)_\mu g_{\alpha\beta} - (k-q)_\alpha g_{\mu\beta} + q_\alpha g_{\mu\beta} - q_\beta g_{\mu\alpha} \tag{3.6}$$

and

$$O_\delta \equiv \gamma_\delta (1+\gamma_5). \tag{3.7}$$

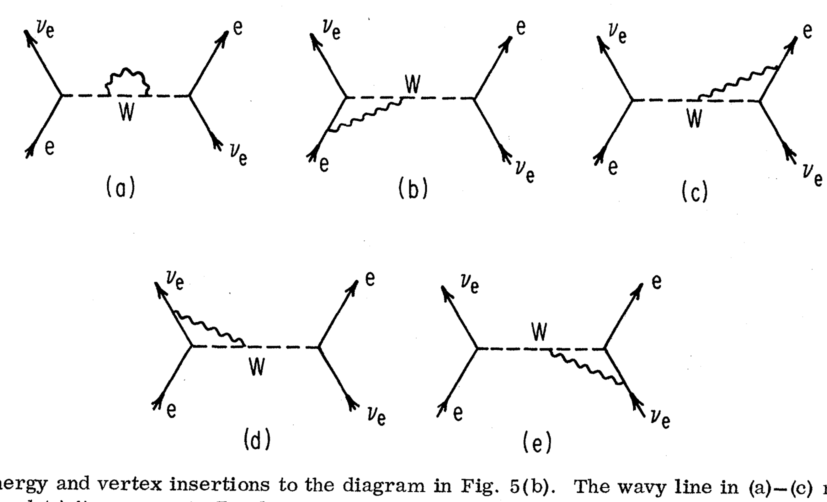

FIG. 10. Self-energy and vertex insertions to the diagram in Fig. 5(b). The wavy line in (a)–(c) means either Z, A, or ϕ, while in (d) and (e) it represents Z only.

Of course, in (3.3) we have used the relation between the electric charge e and the coupling constant g and g', i.e.,

$$e=\frac{gg'}{(g^2+g'^2)^{1/2}}\ . \tag{3.8}$$

From (3.3), it is then clear that the part V_Z^{em} in V_Z does contribute to the charge radius of the neutrino. We are going to show that the charge radius of the neutrino is finite. To see this, let us first calculate the divergent part in Γ_μ. Introducing the Feynman parameters α_1, α_2, and α_3 to combine the terms in the numerator and changing the variables of integration, one finds that

$$\begin{aligned}\Gamma_\mu=&\left(\frac{g}{2\sqrt2}\right)^2\int\frac{d^4k}{(2\pi)^4}\,\frac{2\,!\,\delta(\alpha_1+\alpha_2+\alpha_3-1)\,d\alpha_1 d\alpha_2 d\alpha_3}{(k^2-C)^3}\left(-g^{\alpha\delta}+\frac{(k+d)^\alpha(k+d)^\delta}{m_W^{\ 2}}\right)\left(-g^{\beta\gamma}+\frac{(k+h)^\beta(k+h)^\gamma}{m_W^{\ 2}}\right)\\&\times[(k+d)_\mu g_{\alpha\beta}-(k+d)_\beta g_{\mu\alpha}+(k+h)_\mu g_{\alpha\beta}-(k+h)_\alpha g_{\mu\beta}+q_\alpha g_{\mu\beta}-q_\beta g_{\mu\alpha}]O_\delta(\not k+\not k_2+\not d)O_\gamma\,,\end{aligned} \tag{3.9}$$

where

$$d=\alpha_2 q-\alpha_3 k_2\,, \tag{3.10}$$

$$h=d-q\,, \tag{3.11}$$

$$C=(\alpha_1+\alpha_2)m_W^{\ 2}-\alpha_2 q^2+\alpha_3 m_e^{\ 2}+d^2\,. \tag{3.12}$$

After some algebra, we obtain

$$\begin{aligned}\Gamma_\mu=&4\left(\frac{g}{2\sqrt2}\right)^2\int\frac{d^4k\,d\alpha_1\,d\alpha_2\,d\alpha_3\,\delta(\alpha_1+\alpha_2+\alpha_3-1)}{(2\pi)^4\,(k^2-C)^3}\\&\times\left\{\left[\frac{k^4}{4m_W^{\ 4}}+\frac{k^2}{4m_W^{\ 4}}(d^2+2k_2\cdot d)\right]q^2+\frac{2}{m_W^{\ 2}}[-\tfrac34k^4+k^2(-\tfrac{11}{4}d^2+3q\cdot d-\tfrac52k_2\cdot d+q\cdot k_2]-3k^2\right\}O_\mu\\&+\text{convergent terms.}\end{aligned}$$

Introducing the cutoff Λ and performing the integrations we find the divergent part $\overline{\Gamma}_\mu$ in Γ_μ to be

$$\begin{aligned}\overline{\Gamma}_\mu=&\tfrac18 g^2\left(\frac{2}{4m_W^{\ 4}}q^2O_\mu+\frac{3}{m_W^{\ 2}}O_\mu\right)\left(\frac{-i}{16\pi^2}\Lambda^2\right)\\&+\tfrac18g^2\left\{\frac{2}{4m_W^{\ 4}}[(2m_W^{\ 2}+m_e^{\ 2})q^2-\tfrac13q^4]-\frac{4}{4m_W^{\ 2}}[\tfrac{11}{3}q^2-3(m_W^{\ 2}+m_e^{\ 2})]-6\right\}O_\mu\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right).\end{aligned} \tag{3.13}$$

Let us define the vertex renormalization constant by setting q^2 in (3.13) to be on the Z-meson mass shell so that we have

$$\begin{aligned}Z_1^{\ -1}-1=&\tfrac18g^2\left(\frac{2m_Z^{\ 2}}{4m_W^{\ 4}}+\frac{3}{m_W^{\ 2}}\right)\left(\frac{-i}{16\pi^2}\Lambda^2\right)\\&+\tfrac18g^2\left\{\frac{2}{4m_W^{\ 4}}[(2m_W^{\ 2}+m_e^{\ 2})m_Z^{\ 2}-\tfrac13m_Z^{\ 4}]-\frac{1}{m_W^{\ 2}}[\tfrac{11}{3}m_Z^{\ 2}-3(m_W^{\ 2}+m_e^{\ 2})]-6\right\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right).\end{aligned} \tag{3.14}$$

With (3.14), (3.13) can be rewritten as

$$\overline{\Gamma}_\mu=(Z_1^{\ -1}-1)O_\mu+D_RO_\mu\,, \tag{3.15}$$

where D_R is the residual divergence given by

$$D_R=\tfrac18g^2\frac{2}{4m_W^{\ 4}}(q^2-m_Z^{\ 2})\left(\frac{-i}{16\pi^2}\Lambda^2\right)+\tfrac18g^2\left\{\frac{2}{4m_W^{\ 4}}[(2m_W^{\ 2}+m_e^{\ 2})(q^2-m_Z^{\ 2})-\tfrac13(q^4-m_Z^{\ 4})]-\frac{(q^2-m_Z^{\ 2})}{3m_W^{\ 2}}\right\}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \tag{3.16}$$

With (3.3), (3.15), and (3.16), the residual divergence $\overline{V}_Z^{\text{em}}$ in V_Z^{em} then reads

$$\overline{V}_Z^{\text{em}}=\frac{2e^2}{4m_W^{\ 4}}g^{\mu\nu}N_{\mu\nu}\left(\frac{-i}{16\pi^2}\Lambda^2\right)+\left\{\frac{2}{4m_W^{\ 4}}[(2m_W^{\ 2}+m_e^{\ 2})-\tfrac13(q^2+m_Z^{\ 2})]-\frac{1}{3m_W^{\ 2}}\right\}e^2g^{\mu\nu}N_{\mu\nu}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right), \tag{3.17}$$

where $N_{\mu\nu}$ is defined to be

$$N_{\mu\nu} \equiv \tfrac{1}{8} g^2 \bar{u}(k_2) O_\mu u(k_1) \bar{u}(p') \gamma_\nu u(p) . \tag{3.18}$$

Similarly we find the contribution V_A of the diagram in Fig. 6(a) to be

$$V_A = e^2 \bar{u}(k_2) \Gamma_\mu u(k_1) \frac{-g^{\mu\nu}}{q^2} \bar{u}(p') \gamma_\nu u(p) . \tag{3.19}$$

With (3.13) and (3.19), the residual divergence $\overline{V}_A$ in V_A is readily found to be

$$\overline{V}_A = \frac{-2e^2}{4m_W^4} g^{\mu\nu} N_{\mu\nu} \left(\frac{-i}{16\pi^2} \Lambda^2 \right) - \left\{ \frac{2}{4m_W^4} [(2m_W^2 + m_e^2) - \tfrac{1}{3} q^2] - \frac{1}{3m_W^2} \right\} e^2 g^{\mu\nu} N_{\mu\nu} \left(\frac{-i}{16\pi^2} \ln \Lambda^2 \right) . \tag{3.20}$$

Summing up (3.17) and (3.20) yields

$$\overline{V}_Z^{\text{em}} + \overline{V}_A = -\frac{2m_Z^2 e^2}{12 m_W^4} g^{\mu\nu} N_{\mu\nu} \left(\frac{-i}{16\pi^2} \ln \Lambda^2 \right) . \tag{3.21}$$

Now let us go on to calculate the contribution H_A of the sum of diagrams in Figs. 6(c) and 6(d). Its explicit expression can be readily written down as

$$H_A = \tfrac{1}{4} g^2 e^2 \bar{u}(k_2) O_\alpha u(k_1) \frac{-g^{\alpha\mu} + q^\alpha q^\mu / m_Z^2}{q^2 - m_Z^2} \Pi_{\mu\nu} \frac{-g^{\nu\beta}}{q^2} \bar{u}(p') \gamma_\nu u(p) , \tag{3.22}$$

where $\Pi_{\mu\nu}$ is the self-energy insertion of the W loop. Its divergent part $\overline{\Pi}_{\mu\nu}$ is calculated in the Appendix and given by the expression in (A7),

$$\overline{\Pi}_{\mu\nu} = \frac{1}{4m_W^4} q^2 \mathcal{Q}_{\mu\nu} \left(\frac{-i}{16\pi^2} \Lambda^2 \right) + \left[\frac{2}{4m_W^2} q^2 \mathcal{Q}_{\mu\nu} - \frac{1}{12 m_W^4} q^4 \mathcal{Q}_{\mu\nu} + \frac{2}{m_W^2} (\tfrac{1}{4} m_W^2 q_\mu q_\nu - \tfrac{5}{6} q^2 \mathcal{Q}_{\mu\nu}) + \tfrac{25}{4} \mathcal{Q}_{\mu\nu} - \tfrac{1}{2} q_\mu q_\nu \right] \left(\frac{-i}{16\pi^2} \ln \Lambda^2 \right) . \tag{3.23}$$

Using (3.22) and (3.23), one finds that after renormalization the residual divergence $\overline{H}_A$ in H_A reads

$$\overline{H}_A = \frac{2e^2}{4m_W^4} g^{\alpha\beta} N_{\alpha\beta} \left(\frac{-i}{16\pi^2} \Lambda^2 \right) - \left[\frac{2}{12 m_W^4} (q^2 + m_Z^2) + \frac{7}{3m_W^2} \right] e^2 g^{\alpha\beta} N_{\alpha\beta} \left(\frac{-i}{16\pi^2} \ln \Lambda^2 \right) . \tag{3.24}$$

Similarly the contribution H_Z of the sum of diagrams in Figs. 7(a) and 7(b) can be written down as

$$\begin{aligned} H_Z &= \tfrac{1}{16} g^2 \bar{u}(k_2) O_\alpha u(k_1) \frac{-g^{\alpha\mu} + q^\alpha q^\mu / m_Z^2}{q^2 - m_Z^2} \Pi_{\mu\nu} \frac{-g^{\nu\beta} + q^\nu q^\beta / m_Z^2}{q^2 - m_Z^2} \bar{u}(p') \gamma_\beta \left(1 + \gamma_5 - \frac{4g'^2}{g^2 + g'^2} \right) u(p) \\ &= H_Z^{\text{em}} + H_Z^W , \end{aligned} \tag{3.25}$$

where

$$H_Z^{\text{em}} = -\tfrac{1}{4} e^2 g^2 \bar{u}(k_2) O_\alpha u(k_1) \frac{-g^{\alpha\mu} + q^\alpha q^\mu / m_Z^2}{q^2 - m_Z^2} \Pi_{\mu\nu} \frac{-g^{\nu\beta} + q^\nu q^\beta / m_Z^2}{q^2 - m_Z^2} \bar{u}(p') \gamma_\beta u(p) , \tag{3.26}$$

$$H_Z^W = \tfrac{1}{16} g^4 \bar{u}(k_2) O_\alpha u(k_1) \frac{-g^{\alpha\mu} + q^\alpha q^\mu / m_Z^2}{q^2 - m_Z^2} \Pi_{\mu\nu} \frac{-g^{\nu\beta} + q^\nu q^\beta / m_Z^2}{q^2 - m_Z^2} \bar{u}(p') O_\beta u(p) . \tag{3.27}$$

Thus we see immediately that there is a part of H_Z, namely H_Z^{em}, which is associated with the neutrino charge radius. Using (3.23) and (3.26), we find that after wave-function renormalization the residual divergence $\overline{H}_Z^{\text{em}}$ in H_Z^{em} reads

$$\overline{H}_Z^{\text{em}} = -\frac{2e^2}{4m_W^4} g^{\alpha\beta} N_{\alpha\beta} \left(\frac{-i}{16\pi^2} \Lambda^2 \right) + \left[\frac{2}{12 m_W^4} (q^2 + 2m_Z^2) + \frac{7}{3m_W^2} \right] e^2 g^{\alpha\beta} N_{\alpha\beta} \left(\frac{-i}{16\pi^2} \ln \Lambda^2 \right) . \tag{3.28}$$

Summing up (3.24) and (3.28) yields

$$\overline{H}_A + \overline{H}_Z^{\text{em}} = \frac{2m_Z^2}{12 m_W^4} e^2 g^{\alpha\beta} N_{\alpha\beta} \left(\frac{-i}{16\pi^2} \ln \Lambda^2 \right) .$$

Since α and β are dummy summation variables, the above equation can be rewritten as

$$\overline{H}_A + \overline{H}_Z^{\text{em}} = \frac{2m_Z^2}{12 m_W^4} e^2 g^{\mu\nu} N_{\mu\nu} \left(\frac{-i}{16\pi^2} \ln \Lambda^2 \right) . \tag{3.29}$$

The total divergence associated with the charge radius of the neutrino is just the sum of $\overline{V}_A$, $\overline{V}_Z^{\text{em}}$, $\overline{H}_A$, and

$\overline{H}_Z^{\rm em}$. From (3.21) and (3.29), we find that

$$\overline{V}_A + \overline{V}_Z^{\rm em} + \overline{H}_A + \overline{H}_Z^{\rm em} = \left(\frac{-2m_Z^2}{12m_W^4} + \frac{2m_Z^2}{12m_W^4}\right)e^2 g^{\mu\nu} N_{\mu\nu}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)$$
$$= 0 . \tag{3.30}$$

Thus we see that all divergences associated with the charge radius of the neutrino cancel out. So in Weinberg's theory of weak interaction, the neutrino charge radius is *finite* .

C. Renormalization of the ν-Z Coupling Constant

Now let us go on to calculate the contributions H_Z^W and V_Z^W of the diagrams in Figs. 7(a)–7(c) put aside in the previous section, which belong to the second group. From (3.4), (3.15), and (3.16), the residual divergence $\overline{V}_Z^W$ in V_Z^W can be readily written down as

$$\overline{V}_Z^W = -\frac{2g^2}{16m_W^4} g^{\alpha\beta} M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right) + \tfrac{1}{4}g^2\left[\frac{2}{12m_W^4}(q^2 + m_Z^2) - \frac{2m_e^2}{4m_W^4} + \frac{8}{3m_W^2}\right] g^{\alpha\beta} M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right), \tag{3.31}$$

where $M_{\alpha\beta}$ is defined to be

$$M_{\alpha\beta} \equiv \tfrac{1}{8}g^2 \overline{u}(k_2) O_\alpha u(k_1)\overline{u}(p') O_\beta u(p) . \tag{3.32}$$

The residual divergence $\overline{V}'$ of the contribution of the diagram in Fig. 3(d) can be similarly calculated and is found to be equal to $\overline{V}_Z^W$, i.e.,

$$\overline{V}' = \overline{V}_Z^W . \tag{3.33}$$

Using (3.23) and (3.27), we find that after performing the wave-function renormalization the residual divergence $\overline{H}_Z^W$ in H_Z^W reads

$$\overline{H}_Z^W = \frac{g^2}{8m_W^4} g^{\alpha\beta} M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right) - \tfrac{1}{2}g^2\left[\frac{1}{12m_W^4}(q^2 + 2m_Z^2) + \frac{7}{6m_W^2}\right] g^{\alpha\beta} M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right) . \tag{3.34}$$

Next let us calculate the contributions of the skeleton diagrams in Figs. 8(a), 8(b), and 8(c), denoted respectively by B_a, B_b, and B_c . The explicit expression of B_a is readily written down as

$$B_a = \left(\frac{g}{2\sqrt{2}}\right)^4 \int \frac{d^4k}{(2\pi)^4} \frac{-g^{\alpha\beta} + k^\alpha k^\beta/m_W^2}{k^2 - m_W^2} \frac{-g^{\mu\nu} + (k-q)^\mu (k-q)^\nu}{(k-q)^2 - m_W^2}$$
$$\times \overline{u}(k_2) O_\nu \frac{\not{k} + \not{k}_1 + m_e}{(k+k_1)^2 - m_e^2} O_\beta u(k_1)\overline{u}(p') O_\mu \frac{\not{k} - \not{p}}{(k-p)^2} O_\alpha u(p) . \tag{3.35}$$

Introducing the Feynman parameters α_1, α_2, α_3, and α_4 to combine the terms in the denominator and changing the variable of integration yields

$$B_a = 3!\left(\frac{g}{2\sqrt{2}}\right)^4 \int \frac{d^4k\, d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 \delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{(2\pi)^4 (k^2 - C_0)^4}\left[-g^{\alpha\beta} + \frac{(k+\overline{d})^\alpha (k+\overline{d})^\beta}{m_W^2}\right]\left[-g^{\mu\nu} + \frac{(k+\overline{h})^\mu (k+\overline{h})^\nu}{m_W^2}\right]$$
$$\times \overline{u}(k_2) O_\nu(\not{k} + \not{\overline{d}} + \not{k}_1) O_\beta u(k_1)\overline{u}(p') O_\mu(\not{k} + \not{\overline{d}} - \not{p}) O_\alpha u(p), \tag{3.36}$$

where

$$\overline{d} = \alpha_3 p + \alpha_2 q - \alpha_4 k_1 , \tag{3.37}$$

$$\overline{h} = \overline{d} - q , \tag{3.38}$$

$$C_0 = (\alpha_1 + \alpha_2)m_W^2 - \alpha_2 q^2 - \alpha_3 m_e^2 + \alpha_4 m_e^2 + \overline{d}^2 . \tag{3.39}$$

After some algebra, we get

$$B_a = 3!\left(\frac{g}{2\sqrt{2}}\right)^2 4\int \frac{d^4k\, d\alpha_1 \cdots d\alpha_4 \delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{(2\pi)^4 (k^2 - C_0)^4}$$
$$\times\left\{\frac{1}{4m_W^4}k^6 + \left[\tfrac{1}{6}(\overline{d}-p)\cdot(\overline{d}+k_1) + \tfrac{1}{2}\overline{d}^2 - \tfrac{1}{2}p\cdot\overline{d} + \tfrac{1}{2}m_e^2 + \tfrac{1}{2}k_1\cdot\overline{d}\right]\frac{k^4}{m_W^4} - \frac{2}{m_W^2}k^4\right\} g^{\alpha\beta} M_{\alpha\beta}$$
$$+ \text{convergent terms.} \tag{3.40}$$

Denote $\overline{B}_a$ to be the divergent part of B_a. From (3.40), one immediately finds that after introducing the cutoff Λ and performing the integration, $\overline{B}_a$ reads

$$\overline{B}_a=\frac{g^2}{8m_W^4}g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right)+\left(\frac{2}{8m_W^4}m_e^2-\frac{3}{4m_W^2}-\frac{q^2}{24m_W^4}\right)g^2g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \tag{3.41}$$

Summing up (3.34) and (3.41) yields

$$\overline{H}_Z^W+\overline{B}_a=2\left\{\frac{g^2}{8m_W^4}g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right)-\tfrac{1}{4}g^2\left[\frac{2}{12m_W^4}(q^2+m_Z^2)-\frac{2m_e^2}{4m_W^4}+\frac{8}{3m_W^2}\right]g^{\alpha\beta}M_{\alpha\beta}\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)\right\}. \tag{3.42}$$

Using (3.31) and (3.33), the above equation becomes

$$\overline{H}_Z^W+\overline{B}_a=-\overline{V}_Z^W-\overline{V}' \tag{3.43}$$

or

$$\overline{H}_Z^W+\overline{B}_a+\overline{V}_Z^W+\overline{V}'=0. \tag{3.44}$$

So, we see that the sum of the divergences arising in the skeleton diagram with two-W exchange and in the self-energy corrections just cancel out those in the vertex corrections.

Next we are going to show that the divergence in B_b just cancel out those in B_c. The explicit expression of B_b reads

$$B_b=(\tfrac{1}{4})^4(g^2+g'^2)^2\int\frac{d^4k}{(2\pi)^4}\frac{-g^{\alpha\beta}+k^\alpha k^\beta/m_Z^2}{k^2-m_Z^2}\frac{-g^{\mu\nu}+(k-q)^\mu(k-q)^\nu/m_Z^2}{(k-q)^2-m_Z^2}$$
$$\times\overline{u}(p')\gamma_\mu(a+\gamma_5)\frac{\not p-\not k+m_e}{(p-k)^2-m_e^2}\gamma_\alpha(a+\gamma_5)u(p)\overline{u}(k_2)O_\nu\frac{\not k+\not k_1}{(k+k_1)^2}O_\beta u(k_1). \tag{3.45}$$

Introducing the Feynman parameters α_1, α_2, α_3, and α_4 to combine the terms in the denominator and changing the variable of integration yields

$$B_b=3!(\tfrac{1}{4})^4(g^2+g'^2)^2\int\frac{d^4k\,d\alpha_1\cdots d\alpha_4\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{(2\pi)^4(k^2-C_1)^4}\left[-g^{\alpha\beta}+\frac{(k+\overline{d})^\alpha(k+\overline{d})^\beta}{m_Z^2}\right]\left[-g^{\mu\nu}+\frac{(k+\overline{h})^\mu(k+\overline{h})^\nu}{m_Z^2}\right]$$
$$\times\overline{u}(p')\gamma_\mu(a+\gamma_5)(\not p-\not{\overline{d}}+\not k+m_e)\gamma_\alpha(a+\gamma_5)u(p)\,\overline{u}(k_2)O_\nu(\not k+\not k_1+\not{\overline{d}})O_\beta u(k_1), \tag{3.46}$$

where

$$C_1=(\alpha_1+\alpha_2)m_Z^2-\alpha_2q^2+\overline{d}^2 \tag{3.47}$$

while $\overline{d}$ and $\overline{h}$ are given by (3.37) and (3.38), respectively. After some algebra, one finds

$$B_b=3!\int\frac{d^4k\,d\alpha_1\cdots d\alpha_4\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{(2\pi)^4(k^2-C_1)^4}$$
$$\times\left\{\left[\frac{-k^6}{4m_Z^4}-\frac{k^4}{6m_W^4}(\overline{d}-p)\cdot(\overline{d}+k_1)-\frac{k^4}{2m_W^4}(\overline{d}^2-p\cdot\overline{d}+m_e^2+k_1\cdot\overline{d})+\frac{2k^4}{m_W^2}\right]g^{\alpha\beta}M^a_{\alpha\beta}+\frac{k^4m_e^2}{m_Z^4}g^{\alpha\beta}\overline{M}^a_{\alpha\beta}\right\}$$
$$+\text{convergent terms}, \tag{3.48}$$

where

$$M^a_{\alpha\beta}=(\tfrac{1}{4})^4(g^2+g'^2)^2\overline{u}(k_2)O_\alpha u(k_1)\overline{u}(p')\gamma_\beta(a^2+1+2a\gamma_5)u(p), \tag{3.49}$$

$$\overline{M}^a_{\alpha\beta}=(\tfrac{1}{4})^4(g^2+g'^2)^2(a^2-1)\overline{u}(k_2)\gamma_\alpha u(k_1)\overline{u}(p')\gamma_\beta u(p). \tag{3.50}$$

Performing the integration in (3.48), we obtain the divergent part $\overline{B}_b$ in B_b,

$$\overline{B}_b=-\frac{1}{4m_Z^4}g^{\alpha\beta}M^a_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right)-\left[\left(\frac{m_e^2}{4m_Z^4}-\frac{3}{4m_Z^2}-\frac{q^2}{24m_W^4}\right)g^{\alpha\beta}M^a_{\alpha\beta}-\frac{m_e^2}{m_Z^4}g^{\alpha\beta}\overline{M}^a_{\alpha\beta}\right]\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \tag{3.51}$$

From the diagram in Fig. 8(c) we find that B_c reads

$$B_c=(\tfrac{1}{4})^4(g^2+g'^2)^2\int\frac{d^4k}{(2\pi)^4}\frac{-g^{\alpha\nu}+k^\alpha k^\nu/m_Z^2}{k^2-m_Z^2}\frac{-g^{\mu\beta}+(k-q)^\mu(k-q)^\beta/m_Z^2}{(k-q)^2-m_Z^2}$$
$$\times\overline{u}(p')\gamma_\mu(a+\gamma_5)\frac{\not p-\not k+m_e}{(p-k)^2-m_e^2}\gamma_\alpha(a+\gamma_5)u(p)\overline{u}(k_2)O_\nu\frac{\not k_2-\not k}{(k-k_2)^2}O_\beta u(k_1). \tag{3.52}$$

Using steps similar to those leading from (3.45) to (3.46), one finds that

$$B_c = 3!\,(\tfrac{1}{4})^4(g^2+g'^2)^2\int \frac{d^4k\,d\alpha_1\cdots d\alpha_4\,\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{(2\pi)^4\,(k^2-C_2)^4}\left[-g^{\alpha\nu}+\frac{(k+d_2)^\alpha(k+d_2)^\nu}{m_Z{}^2}\right]\left[-g^{\mu\beta}+\frac{(k+h_2)^\mu(k+h_2)^\beta}{m_Z{}^2}\right]$$
$$\times\,\bar u(p')\gamma_\mu(a+\gamma_5)(\not p-\not d_2-\not k+m_e)\gamma_\alpha(a+\gamma_5)u(p)\bar u(k_2)O_\nu(\not k_2-\not d_2-\not k)O_\beta u(k_1)\,, \tag{3.53}$$

where

$$d_2=\alpha_2 q+\alpha_3 p+\alpha_4 k_2\,, \tag{3.54}$$

$$h_2=d_2-q\,, \tag{3.55}$$

$$C_2=(\alpha_1+\alpha_2)m_Z{}^2-\alpha_2 q^2+d_2{}^2\,. \tag{3.56}$$

After some algebra we get

$$B_c = 3!\int\frac{d^4k\,d\alpha_1\cdots d\alpha_4\,\delta(\alpha_1+\alpha_2+\alpha_3+\alpha_4-1)}{(2\pi)^4\,(k^2-C_2)^4}$$
$$\times\left\{\left[\frac{k^6}{4m_Z{}^4}+\left(\frac{1}{6m_Z{}^4}(p-d_2)\cdot(k_2-d_2)+\frac{1}{2m_Z{}^4}(d_2{}^2-p\cdot d_2+m_e{}^2-k_2\cdot d_2)-\frac{2}{m_Z{}^2}\right)k^4\right]g^{\alpha\beta}M^a_{\alpha\beta}\right.$$
$$\left.-\frac{k^4m_e{}^2}{m_Z{}^4}g^{\alpha\beta}\overline{M}{}^a_{\alpha\beta}\right\}+\text{convergent terms.} \tag{3.57}$$

Performing the integration in (3.57), one finds that the divergent part $\overline{B}_c$ in B_c reads

$$\overline{B}_c=\frac{1}{4m_Z{}^4}g^{\alpha\beta}M^a_{\alpha\beta}\left(\frac{-i}{16\pi^2}\Lambda^2\right)+\left[\left(\frac{m_e{}^2}{4m_Z{}^4}-\frac{3}{4m_Z{}^2}-\frac{q^2}{24m_W{}^4}\right)g^{\alpha\beta}M^a_{\alpha\beta}-\frac{m_e{}^2}{m_Z{}^4}g^{\alpha\beta}\overline{M}{}^a_{\alpha\beta}\right]\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right). \tag{3.58}$$

Comparing (3.51) to (3.58), we immediately see the cancellation of the divergences in $\overline{B}_b$ and $\overline{B}_c$, i.e.,

$$\overline{B}_b=-\overline{B}_c$$

or

$$\overline{B}_b+\overline{B}_c=0\,. \tag{3.59}$$

Thus we have shown that residual divergences in this second group all cancel out. Those divergences already contained in the renormalization constants in this group are used to renormalize the ν-Z coupling constant $\sqrt{g^2+g'^2}$ just like what happens in the case of ν-ν elastic scattering.

We note that the cancellation of the residual divergence in this group is independent of the value of the mass of the external particle. Since the diagrams in the higher-order elastic ν-ν scattering are similar to what are discussed here, we find as a by-product that the logarithmic residual divergences in the elastic ν-ν scattering all cancel out. Weinberg[4] has only shown that the quadratic one cancels out.

D. Renormalization of Lepton-W Coupling Constant

Now we come to the third group, the sum of the diagrams in Figs. 9 and 10. Comparing these diagrams to those appearing in higher-order μ decay one finds immediately that the contributions of these diagrams can be easily obtained by simply replacing the muon mass in what is found in Sec. II by the electron mass. It is found that the residual divergences in higher-order μ decay all cancel out, independent of the value of muon mass. So we see that after renormalization the contributions of the sum of diagrams in this group are finite. The renormalization of the lepton-W coupling constant g can be carried out in exactly the same way as was done in the case of muon decay.

IV. DISCUSSION

Summing up what we have obtained in the last two sections, we find that there are definite ways to carry out the renormalization programs and the residual divergences cancel out systematically. As a result, the S-matrix for the neutrino-lepton scattering calculated in the unitary gauge and the neutrino charge radius are all finite. To actually calculate the finite part, one has to employ some kind of regularization scheme which is gauge invariant. Perhaps the regularization scheme recently discussed by 't Hooft and Veltman[6] is suitable for this purpose.

From the calculation associated with the neutrino charge radius, one sees that weak and electromagnetic interactions help out in solving each other's problems of divergences. This certainly is one of the most satisfying features of the theory.

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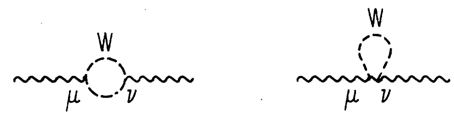

(a) (b)

FIG. 11. Diagrams of the self-energy insertion of a W loop.

APPENDIX

In this appendix we calculate the divergent part in the contributions of the diagrams in Fig. 11. Denote as $\Pi^{(a)}_{\mu\nu}$ and $\Pi^{(b)}_{\mu\nu}$ the contributions of Figs. 11(a) and 11(b), respectively, which read

$$\Pi^{(a)}_{\mu\nu}=\int\frac{d^4k}{(2\pi)^4}\,\frac{-g^{\alpha\delta}+k^\alpha k^\delta/m_W{}^2}{k^2-m_W{}^2}\,\frac{-g^{\beta\gamma}+(q-k)^\beta(q-k)^\gamma/m_W{}^2}{(q-k)^2-m_W{}^2}\,V_{\mu\alpha\beta}V_{\nu\delta\gamma}\,, \tag{A1}$$

where

$$V_{\mu\alpha\beta}\equiv k_\mu g_{\alpha\beta}-k_\beta g_{\mu\alpha}+(k-q)_\mu g_{\alpha\beta}-(k-q)_\alpha g_{\mu\beta}+q_\alpha g_{\mu\beta}-q_\beta g_{\mu\alpha} \tag{A2}$$

and

$$\Pi^{(b)}_{\mu\nu}=-\int\frac{d^4k}{(2\pi)^4}\,\frac{-g^{\alpha\beta}+k^\alpha k^\beta/m_W{}^2}{k^2-m_W{}^2}\,(2g_{\mu\nu}g_{\alpha\beta}-g_{\mu\alpha}g_{\nu\beta}-g_{\nu\alpha}g_{\mu\beta})\,. \tag{A3}$$

Introducing the Feynman parameters σ and λ to combine the denominators in (A1) and changing k to $k+\lambda q$, one finds that after some algebra $\Pi^{(a)}_{\mu\nu}$ becomes

$$\begin{aligned}\Pi^{(a)}_{\mu\nu}=\int&\frac{d^4k\,d\lambda\,d\sigma\,\delta(\lambda+\sigma-1)}{(2\pi)^4\,(k^2-\overline{C})^2}\\&\times\Big\{\frac{1}{m_W{}^2}\big[-\tfrac{3}{4}k^4g_{\mu\nu}-\tfrac{1}{4}k^2(\lambda^2-2\lambda)(11\mathcal{Q}_{\mu\nu}+3q_\mu q_\nu)-\tfrac{1}{4}k^2(5\mathcal{Q}_{\mu\nu}+3q_\mu q_\nu)-(\lambda^2-\lambda)^2q^2\mathcal{Q}_{\mu\nu}\big]\\&\quad+\frac{k^2}{4m_W{}^4}\,q^2\mathcal{Q}_{\mu\nu}+\frac{1}{m_W{}^2}\big[-\tfrac{3}{4}k^4g_{\mu\nu}-\tfrac{1}{4}k^2\lambda^2(11\mathcal{Q}_{\mu\nu}+3q_\mu q_\nu)+\tfrac{1}{4}k^2\cdot 6\mathcal{Q}_{\mu\nu}-(\lambda^2-1)^2q^2\mathcal{Q}_{\mu\nu}\big]\\&\quad+\tfrac{18}{4}k^2g_{\mu\nu}+(5\mathcal{Q}_{\mu\nu}+3q_\mu q_\nu)+2(\lambda^2-\lambda)(\mathcal{Q}_{\mu\nu}+6q_\mu q_\nu)\Big\}+\text{convergent terms,}\end{aligned} \tag{A4}$$

where

$$\mathcal{Q}_{\mu\nu}\equiv q^2g_{\mu\nu}-q_\mu q_\nu \tag{A5}$$

and

$$\overline{C}=m_W{}^2+(\lambda^2-\lambda)q^2\,. \tag{A6}$$

The divergent parts $\overline{\Pi}^{(a)}_{\mu\nu}$ in $\Pi^{(a)}_{\mu\nu}$ in $\overline{\Pi}^{(b)}_{\mu\nu}$ in $\Pi^{(b)}_{\mu\nu}$ can be easily obtained by introducing the cutoff Λ and performing the integrations. We find that the sum of $\overline{\Pi}^{(a)}_{\mu\nu}$ and $\overline{\Pi}^{(b)}_{\mu\nu}$ denoted by $\overline{\Pi}_{\mu\nu}$ is

$$\begin{aligned}\overline{\Pi}_{\mu\nu}&=\overline{\Pi}^{(a)}_{\mu\nu}+\overline{\Pi}^{(b)}_{\mu\nu}\\&=\frac{1}{4m_W{}^4}\,q^2\mathcal{Q}_{\mu\nu}\left(\frac{-i}{16\pi^2}\Lambda^2\right)+\left[\frac{q^2\mathcal{Q}_{\mu\nu}}{4m_W{}^4}(2m_W{}^2-\tfrac{1}{3}q^2)+\frac{2}{m_W{}^2}(\tfrac{1}{4}m_W{}^2q_\mu q_\nu-\tfrac{5}{6}q^2\mathcal{Q}_{\mu\nu})-\tfrac{1}{2}q_\mu q_\nu+\tfrac{25}{4}\mathcal{Q}_{\mu\nu}\right]\left(\frac{-i}{16\pi^2}\ln\Lambda^2\right)\,.\end{aligned} \tag{A7}$$

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[5] The diagram in Fig. 6(b) does not generate residual divergences either. It is drawn here because it is one of the lowest order diagrams associated with the charge radius of the neutrino. The vertex renormalization constant for this diagram just cancels out the corresponding one for Fig. 6(a) to maintain the neutrality of the neutrino. See J. Bernstein and T. D. Lee, Phys. Rev. Letters 11, 512 (1963).

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