Fig. 2(a) has to vanish. For variety, we chose to emphasize the converse here. The correspondence between the couplings of the Pomeranchukon and those of the f and f' Regge poles [see R. Carlitz, M. B. Green, and A. Zee, Phys. Rev. D 4, 3439 (1971)] is maintained by Fig. 2(b) even if we assume that the f and f' poles in Fig. 2(b) have very small residues.

[26]For a discussion of Regge-Regge cuts in phenomenology, see H. Harari, Phys. Rev. Letters 26, 1079 (1971).

[27]This property is shared by Reggeon-multiple Pomeranchukon cuts ($R\times P\times P$, etc.) obtained by adding several handles.

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Spectrum of Physical States in a Dual-Resonance Model

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The dual-resonance model entails an exponentially growing resonance spectrum for the hadrons. It is well known that not all the states which achieve the factorization are physical ones; there are spurious and ghost states among them. We give a general formula for counting the ghost states at any given mass value and also an asymptotic expression for the ghost spectrum. The Virasoro model, in which the leading trajectory intercept is unity, is not considered here. A method for constructing the subspace of real states, which is orthogonal to the space of spurious states, is also given. After removing the ghost states from the real states, the remaining ones are taken as physical states, which then constitute the resonance spectrum. Implications of this resonance spectrum in the statistical approach are briefly discussed.

I. INTRODUCTION

After the n-point dual-resonance amplitudes were shown to be factorizable in terms of harmonic-oscillator states,[1,2] there was brought to light the unexpectedly rich level structure of resonances making up the amplitudes. It has been shown[1-3] that the multiplicity of independent states at a certain mass value increases exponentially with the mass. However, it is immediately apparent from the method employed that many of the states included in the factorization cannot be interpreted as physical or resonance states since they have negative norms (ghost states), and would give rise to negative-pole residues. Moreover, there are spurious states[4] due to linear dependence which do not couple to the external-particle states at all. In order to reach some understanding of the spectrum of the physical resonance states implied by the dual-resonance amplitudes, it is necessary to be able to enumerate precisely the number of ghost and spurious states. For the special case in which the leading trajectory has intercept $\alpha_0 = 1$, Brower[5] has shown recently that the ghost states can be completely eliminated. In this paper we shall only consider cases with $\alpha_0 \neq 1$. We obtain a general formula through a generating function to count the ghost states at any mass value. We also work out the asymptotic expression for the ghost spectrum. The result is actually quite surprising. We find that nearly all of the factorized states are spurious and ghost states. However, the number of the remaining physical states, like the number of the total factorized states, still grows exponentially with increasing mass.

The exponential rate of growth of resonance degeneracy, interpreted as the growth of the density of states in a statistical ensemble of strongly interacting system, can lead to far-reaching consequences. In fact, in 1965 Hagedorn[6] obtained precisely an exponentially increasing density of states within a statistical-bootstrap model by requiring that the hadronic resonance spectrum be identified with the density of states of the ensemble. An ultimate temperature of hadron matter in thermal equilibrium is predicted leading to interesting cosmological speculations.[7-9] If this identification could indeed by justified, the level structure of the dual-resonance model would provide added support to these results. Some properties of hadron matter at high temperatures and high densities have been derived on this basis.[10]

What we have attempted to do here is to obtain for each propagator pole position a count of the number of "real" states (states orthogonal to the spurious states) that will contribute effectively to the pole residues. Further, we try to distinguish among the real states those which contribute positive-pole residues to the propagator from those which contribute negative-pole residues. The real states giving rise to positive-pole residues are called the physical states. The spectrum of physical states are given in Eq. (46) and in a slightly different form by Eq. (48).

II. FACTORIZATION

The factorization of the dual-resonance amplitude for n spinless particles has been proposed by several authors.[1,2] Most of the early works deal with trajectories which are degenerate with each other and hence can all be specified by a single intercept α_0 and a slope α'. Subsequently, the same method of factorization has been extended to n-point amplitudes involving parallel trajectories but with unequal intercepts, and several interesting features have been pointed out.[11-13]

For simplicity we shall carry out our analysis for the degenerate-trajectories case as an illustration of the general method of approach. The analysis can be easily generalized to the unequal-intercepts case. As is well known for the degenerate-trajectories case factorization can be achieved by introducing harmonic oscillators of tensor dimension, $D=4$, whereas in the general unequal-intercepts case harmonic oscillators of higher tensor dimension are needed and the momentum 4-vectors are similarly extended to a higher dimension. We shall postpone the discussion of the choice of D till later (see Sec. IV), although it has direct bearing on the numerics of the level degeneracy of the model.

As we have mentioned before, we are only interested in the cases where $\alpha_0 \neq 1$. The Virasoro[14] model will not be considered here.

We shall follow mainly the work (and notations) of Ref. 3.[15] These authors have shown that the process described in Fig. 1 can be written as

$$A(p_0 \cdots p_r; q_0 \cdots q_s) = \langle p | D(R, \pi) | q \rangle = \sum_\lambda \langle p | \lambda \rangle \langle \lambda | D(R, \pi) | \lambda \rangle \langle \lambda | q \rangle , \tag{1}$$

where $D(R, \pi)$ is the propagator

$$D(R, \pi) = \int_0^1 dx\, x^{R-\alpha(\pi^2)-1}(1-x)^{\alpha_0 - 1} , \tag{2}$$

with

$$R = \sum_{n=1}^{\infty} n a_n^\dagger a_n , \tag{3}$$

and the states $|q\rangle$ are the "tree" states of $(s+1)$ external scalar particles

$$|q\rangle = V(q_s) D(\pi_{s-1}) V(q_{s-1}) \cdots V(q_1) |0\rangle , \tag{4}$$

with

$$V(q) = \exp\left(i\sqrt{2\alpha'}\, q \sum_{n=1}^{\infty} a_n^\dagger / \sqrt{n}\right) \times \exp\left(i\sqrt{2\alpha'}\, q \sum_{n=1}^{\infty} a_n / \sqrt{n}\right) . \tag{5}$$

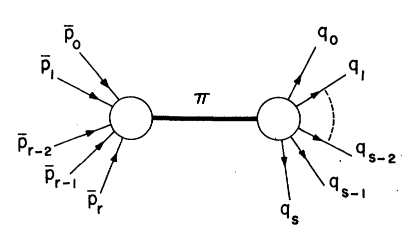

FIG. 1. Momentum labels for a scattering amplitude involving $(r+s+2)$ scalar particles.

$\alpha(\pi^2)$ denotes the linear Regge trajectory, $\alpha(\pi^2) = \alpha_0 - \alpha'\pi^2$, on which the intermediate states lie, and $a_{n,\mu}$ and $a^\dagger_{n,\mu}$ (the μ indices are suppressed in the above equations) are multidimensional tensor operators satisfying harmonic-oscillator commutation relations:

$$[a_{n,\mu}, a^\dagger_{m,\nu}] = \delta_{nm} g_{\mu\nu}, \quad [a_{n,\mu}, a_{m,\nu}] = [a^\dagger_{n,\mu}, a^\dagger_{n,\nu}] = 0, \tag{6}$$

where $g_{00} = -1$, $g_{ii} = +1$, $i = 1, 2, \ldots, D-1$, and the dimension D is still to be specified. The states $|\lambda\rangle$ are a complete set of eigenstates of the operator R:

$$|\lambda\rangle = \prod_{n,\mu} [(a^\dagger_{n,\mu})^{\lambda_{n,\mu}}/(\lambda_{n,\mu}!)^{1/2}]\,|0\rangle, \tag{7}$$

with

$$R|\lambda\rangle = \sum_{n,\mu} n\lambda_{n,\mu}|\lambda\rangle. \tag{8}$$

At a given mass position, say m_N, with

$$N = \alpha_0 + \alpha' m_N{}^2, \tag{9}$$

the "on-shell" harmonic oscillator states are those eigenstates of R with eigenvalues equal to $0, 1, 2, \ldots, N$ (N an integer), since the propagator $D(-\pi^2 = m_N{}^2)$ develops poles at these values. That is to say, the $\lambda_{n,\mu}$ of (8) become integers and satisfy the condition

$$\sum_{n,\mu} n\lambda_{n,\mu} = r, \quad \text{with} \quad r = 0, 1, 2, \ldots, N. \tag{10}$$

The number of independent ways that (10) can be satisfied is given by the partition $P_D(r)$.[16] The manifold of "on-shell" states at position N has therefore $\sum_{r=0}^{N} P_D(r)$ linearly independent members. However, due to the fact that $g_{00} = -1$, many of the eigenstates of R have negative norms, and further it has been shown by several authors[4] that many are spurious states, i.e., states which do not couple to the tree states. None of these can be categorized as physical states.

III. THE REAL STATES

We shall first discuss the problem concerning the spurious states in the factorization. As we shall see there are many more spurious states than there are ghost states. Let us introduce a "Ward" operator, defined by[4,17]

$$W(p) = L_0(p) - L_-(p), \tag{11}$$

where

$$L_0(p) = -\alpha' p^2 - R$$

and

$$L_-(p) = -i(2\alpha')^{1/2} p \cdot a_1 - \sum_{n=1}^{\infty} [n(n+1)]^{1/2} a^\dagger_n a_{n+1}. \tag{12}$$

L_0, L_- together with $L_+ = (L_-)^\dagger$, where the superscript † denotes Hermitian adjoint, define a simple algebra SO(2, 1) satisfying the commutation relations

$$[L_0, L_\pm] = \mp L_\pm, \quad [L_+, L_-] = 2L_0. \tag{13}$$

Since the "on-shell" oscillator states are eigenstates of R,

$$R|\lambda\rangle = r|\lambda\rangle. \tag{14}$$

By the commutation relations (13), $L_\pm$ can be identified as raising and lowering operators:

$$RL_\pm|\lambda\rangle = (r \pm 1)L_\pm|\lambda\rangle. \tag{15}$$

$W(p)$ has the property that it annihilates an arbitrary tree state of an arbitrary number of scalar particles:

$$W(p)|p\rangle = 0. \tag{16}$$

This can be shown by commuting $W(p)$ past the V and D operators in (4) until it acts on $|0\rangle$ which it annihilates, using the relations

$$W(p)V(k) = V(k)[W(p-k) + \alpha_0]$$

and

$$W(p)D(R,p) = D(R+1,p)W(p) - \alpha_0 D(R,p). \tag{17}$$

Consequently,

$$\langle\lambda|D(R+1,p)W(p)|p\rangle = \langle\lambda|[W(p)+\alpha_0]D(R,p)|p\rangle = 0, \tag{18}$$

where $\langle\lambda|$ is an arbitrary oscillator state. Thus all states

$$|S\rangle = [W^\dagger(p) + \alpha_0]|\lambda\rangle \tag{19}$$

will not contribute to the summation over intermediate states in (1), and they will be called spurious states. So far no other Ward operators[18] with this property have been found, and for the time being we shall concentrate on spurious states defined by (19).

At each position N corresponding to the resonance mass m_N by (9), we shall be mainly interested in the states which are needed to effect factorization but are at the same time orthogonal to the manifold of spurious states. We shall call these the real states. The question of ghost states will be discussed later.

The real states $|\Psi\rangle$ are defined by[19]

$$\langle S|\Psi\rangle = 0, \tag{20}$$

or by the condition[20]

$$[W(p) + \alpha_0]|\Psi\rangle = 0, \tag{21}$$

and, in addition, $|\Psi\rangle$ should not be zero-norm states. The on-shell real states can be explicitly constructed. At position N, $|\Psi\rangle$ will be formed from linear combinations of the $|\lambda\rangle$ states of (7). Let us write

$$|\Psi\rangle = \sum_{r=0}^{N} \sum_{i_r} a_{r,i_r} |r, i_r\rangle , \tag{22}$$

where $|r, i_r\rangle$ denote the $|\lambda\rangle$ states, with

$$R|r, i_r\rangle = r|r, i_r\rangle , \tag{23}$$

and each i_r labels one of the $P_D(r)$ components in the r level. From (21) we have

$$[L_0(p) + \alpha_0]|\Psi\rangle = L_-(p)|\Psi\rangle . \tag{24}$$

Using the fact that

$$(L_0 + \alpha_0)|r, i_r\rangle = (N - r)|r, i_r\rangle , \tag{25}$$

and L_- is a lowering operator, (24) implies that not all of the coefficients a_{N,i_N} of $|\Psi\rangle$ can be zero. With

$$\begin{aligned} L_-^2|\Psi\rangle &= L_-(L_0 + \alpha_0)|\Psi\rangle \\ &= (L_0 L_- - L_- + \alpha_0 L_-)|\Psi\rangle \\ &= [(L_0 + \alpha_0)^2 - (L_0 + \alpha_0)]|\Psi\rangle , \end{aligned} \tag{26}$$

we have in general,

$$L_-^n|\Psi\rangle = Q_n(L_0 + \alpha_0)|\Psi\rangle , \tag{27}$$

where $Q_n(x)$ is a polynomial of order n, and is given by

$$Q_n(x) = \prod_{m=0}^{n-1} (x - m) . \tag{28}$$

In this manner, a real state $|\Psi\rangle$ can be generated from any one of the $P_D(N)$ oscillator states of the $r = N$ level. Take one $|N, i_N\rangle$ state given by (7) at a time and choose $a_{N,i_N} = 1$; then,

$$|\Psi_i\rangle = \sum_{r=0}^{N} a_r (L_-)^r |N, i_N\rangle , \tag{29}$$

where

$$a_0 = 1, \quad a_r = \frac{1}{r!} .$$

Also, (29) can be rewritten as

$$|\Psi_i\rangle = e^{+L_-}|N, i_N\rangle . \tag{30}$$

The real states given by (29) or (30) are not orthogonal to each other. For example,

$$\begin{aligned} \langle\Psi_{i'}|\Psi_i\rangle &= \sum_{r,r'} a_r a_{r'} \langle N, i'_N|(L_+)^{r'}(L_-)^r|N, i_N\rangle \\ &= \sum_r a_r^2 \langle N, i'_N|(L_+)^r(L_-)^r|N, i_N\rangle \end{aligned} \tag{31}$$

do not vanish, even though

$$\langle N, i'_N|N, i_N\rangle = 0 . \tag{32}$$

In order to obtain a set of orthogonal real states, each $|\Psi\rangle$ is formed from a proper mixture of $|N, i_N\rangle$ states, and to facilitate counting, which we would have to do eventually, the mixing problem has to be carried out systematically. We shall demonstrate below a particular way of achieving this. Let us define

$$|N, i_N\rangle_m = \sum_{n=0}^{N} b_n (L_+ L_-)^n |N, i_N\rangle , \tag{33}$$

where $b_0 = 1$; we claim that the states defined by

$$|\overline{\Psi}_i\rangle = e^{+L_-}|N, i_N\rangle_m \tag{34}$$

can be made to be orthogonal to any other $|\overline{\Psi}_{i'}\rangle$ by a proper choice of the b_n. The reason is that in the product $\langle\overline{\Psi}_{i'}|\overline{\Psi}_i\rangle$ there will be operators like $(L_+L_+L_- \cdots L_+L_-L_-)$ sandwiched between $\langle N, i'_N|\cdots|N, i_N\rangle$. By means of the commutation relations (13) and the condition (24) they can be reduced to a form with all the L_- to the right and all the L_+ to the left. For example, the residue of the propagator $D(R, \pi)$ can be written as follows:

$$\text{Res}\{\langle\overline{\Psi}_{i'}|D(R, \pi)|\overline{\Psi}_i\rangle\} = \text{Res}\left\{\langle N, i'_N|D(R, \pi)|N, i_N\rangle + \sum_{m=1}^{3N} c_m \langle N, i'_N|(L_+)^m D(R - m, \pi)(L_-)^m|N, i_N\rangle\right\} , \tag{35}$$

where

$$\text{Res} = \lim_{(\alpha_0 - \alpha'\pi^2) \to N} (N - \alpha_0 + \alpha'\pi^2) \tag{36}$$

and c_m are some functions of α_0, b_n, and a_r. In (35) we also make use of the relations

$$\begin{aligned} D(R, \pi)L_+ &= L_+ D(R - 1, \pi) , \\ L_- D(R, \pi) &= D(R - 1, \pi)L_- . \end{aligned} \tag{37}$$

Since

$$(L_-)^m|N, i_N\rangle = 0 , \text{ whenever } m > N , \tag{38}$$

the sum in (35) stops effectively at $m = N$. Now, we shall take the b_n to be such that all $c_m = 0$ for $m = 1, 2, \ldots$. This can in principle be done since there are N adjustable b_n, and hence

$$\begin{aligned} \text{Res}\{\langle\overline{\Psi}_{i'}|D(R, \pi)|\overline{\Psi}_i\rangle\} &= \text{Res}\{\langle N, i'_N|D(R, \pi)|N, i_N\rangle\} \\ &\sim \pm\delta_{i'_N, i_N} . \end{aligned} \tag{39}$$

Thus, $|\overline{\Psi}_i\rangle$ have basically the orthogonality properties of $|N, i_N\rangle$ as constructed in (7). Since there are $P_D(N)$ members of $|N, i_N\rangle$ states, there are, therefore, also $P_D(N)$ members of real states at the N level.

IV. THE PHYSICAL STATES

The real states $|\overline{\Psi}\rangle$ constructed in (34) are orthogonal to the manifold of spurious states, and according to (39), none of them would cause the propagator to vanish although some of them may make the propagator take on the wrong sign. The latter correspond to states giving rise to negative pole residues in (39). These are real states generated from oscillator states $|N, i_N\rangle$ having odd numbers of creation operators $a^\dagger_{n,0}$ with Lorentz indices $\mu=0$ in (7). We shall first give an enumeration of the number of ghost oscillator states, which possess odd numbers of the zeroth-component Lorentz indices, for each level r (i.e., the level with eigenvalue of R equal to r), since this number will be useful later.

In the same way that the on-shell oscillator states at each level r is found to be given by $P_D(r)$, the number of ghost oscillator states $G_D(r)$ at the same level can easily be shown to be

$$G_D(r)=\tfrac{1}{2}[P_D(r)-\overline{P}_D(r)], \qquad (40)$$

where

$$\sum_{r=0}^{\infty} P_D(r)x^r=\left[\prod_k (1-x^k)\right]^{-D}, \qquad (41)$$

$$\sum_{r=0}^{\infty} \overline{P}_D(r)x^r=\left[\prod_k (1-x^k)\right]^{-D+1}\left[\prod_j (1+x^j)\right]^{-1}. \qquad (42)$$

The expression for $\overline{P}_D$ differs from that for P_D by altering the sign in one of the sequences of infinite products. Some properties of $\overline{P}_D$ are given in the Appendix. In $\overline{P}_D$, therefore, every ghost component is canceled by a positive-norm component. Hence, (40) gives the number of ghost components. The remaining positive-norm components are just

$$\tfrac{1}{2}[P_D(r)+\overline{P}_D(r)]. \qquad (43)$$

Asymptotically, we have the following approximate expressions for $P_D(r)$ and $\overline{P}_D(r)$, valid for r large:

$$P_D(r)\to\sqrt{2}\,(D/24)^{(D+1)/4}(r)^{-(D+3)/4}\times\exp[2\pi(Dr/6)^{1/2}], \qquad (44)$$

$$\overline{P}_D(r)\to[(D-3/2)/24]^{D/4}(r)^{-(D+2)/4}\times\exp\{2\pi[(D-3/2)r/6]^{1/2}\} \quad (D\geq 2). \qquad (45)$$

Since $\overline{P}_D(r)$ in Eq. (45) has a smaller exponential than $P_D(r)$, it can be ignored for large r, and $G_D(r)\cong(1/2)P_D(r)$. Hence, the elimination of all ghost states from the manifold of on-shell states will reduce the total number of such states by at most a factor of two but would not change the asymptotic form of the level degeneracy, which is the dominant feature of the model.

According to (39), the allowed states generated from the ghost oscillator states would give the wrong sign to the pole residues of the propagator, and therefore would not be acceptable as physical states. However, an arbitrary removal of these states may destroy the Lorentz invariance of the amplitude. We do not yet have a completely satisfactory way to overcome the ghost problem.

A possible way to rid the amplitude of ghosts and at the same time not destroy Lorentz invariance can be accomplished as follows. First, stay in the rest frame and decompose the Lorentz indices of the allowed states in terms of spin states. This decomposition can be achieved without the ghost states (i.e., the ghost states will belong to separate irreducible representations). Thus, the explicit particle contents of the factorization are exhibited. Then by allowing every spin-j state to transform according to the $(j, 0)$ or $(0,j)$ representations of the Lorentz group, full Lorentz invariance of the amplitude is restored. In other words, the original allowed states transforming under the direct products $(\frac{1}{2},\frac{1}{2})\times(\frac{1}{2},\frac{1}{2})\times\cdots\times(\frac{1}{2},\frac{1}{2})$, which have reductions into representations of the (m, n) type, are replaced entirely by states transforming under representations of the $(j, 0)$ and $(0,j)$ types. In terms of the latter the ghost states may be removed without causing obvious injury to the factorized amplitude.

Due to the high degree of degeneracy at every resonance position, the spin quantum numbers alone are insufficient to distinguish the different particle states. Hence, the Lorentz quantum numbers (m, n) are also needed to classify the states, and these additional quantum numbers will not be available if the $(j, 0)$ and $(0,j)$ representations are used. However, since the Lorentz quantum numbers are still insufficient to classify all states anyway and other quantum numbers are needed to aid in the classification, this argument should not be used to reject the $(j, 0)$ scheme. The higher symmetry which gives rise to the new quantum numbers must be so broken that the particles are distinguishable by their differences in mass.

Therefore with the justifications given above, we take now the maximal set of physical states to be those real states which give the right sign to the propagator residues. The number of physical states at level N corresponding to resonance mass m_N is given by [see (39) and (43)]

$$\rho^{(N)}_{\text{phy}}=\tfrac{1}{2}[P_D(D)+\overline{P}_D(N)], \qquad (46)$$

with $N=\alpha_0+\alpha' m_N{}^2$. This presumably is the number of states needed to effect a factorization of the dual amplitude without redundant components. This would also be the spectrum of resonances predicted by the model, and in the framework of statistical bootstrap, ρ_{phy} is related to the density of states of an ensemble of hadrons.

The final question relates to the tensor dimension D. In one of the pioneer works on the factorization problem, Nambu[1] has already suggested that in order to account for the fact that not all of the boson trajectories have identical intercepts, the tensor dimension D should be at least five. Several authors[11, 13] have subsequently given detailed investigations into the choice of dimension D within the framework of hadron dynamics. They find that if the dual-resonance amplitude is required to be factorizable in every multiperipheral configuration, a tensor dimension $D=5$ is in general insufficient to accommodate all arbitrary Regge trajectories (i.e., trajectories of the same slope but different intercepts). Instead, there are mass relations among the mesons. Since most of these relations seem to be satisfied, there appears to be no need to go beyond $D=5$.

The choice of D is quite important in view of its implication on the statistical ensemble, since it appears in the exponential of (44) which is directly related to ρ_{phy}. In the statistical-bootstrap framework[6-10] it essentially determines the ultimate temperature T_0 of hadron matter in thermal equilibrium, $T_0=(1/2\pi k)(6/D\alpha')^{1/2}$, where k is the Boltzmann constant. Taking $D=5$ and $\alpha'=1.0$ BeV^{-2}, the ultimate temperature is $T_0=2\times10^{12}$ °K, which is very close to that suggested by Hagedorn.[6] ($T_0=174$ MeV in units where $k=1$.)

Instead of $\rho_{\text{phy}}(N)$ of Eq. (46) it is common to write $\rho_{\text{phy}}(m)$; these are related to each other by

$$dm\rho_{\text{phy}}(m)=dN\rho_{\text{phy}}(N)\,, \tag{47}$$

with $N=M^2$. For $D=5$, $\rho_{\text{phy}}(m)$ takes on the following asymptotic form:

$$\rho_{\text{phy}}(m)\xrightarrow[m\text{ large}]{}(2)^{-1/2}(\tfrac{5}{24})^{5/4}m^{-3}\exp(m/T_0)\,. \tag{48}$$

This spectrum is similar to but not identical to the one derived by Hagedorn,[6] who employs a statistical bootstrap model to fix the power of m in Eq. (48) at $-\frac{5}{2}$ instead of -3 as we have here. However, in a recent study Frautschi[21] employing a somewhat different statistical-bootstrap model favors the situation where the power of m in Eq. (48) is less than $-\frac{5}{2}$ instead of being identically equal to it as in Hagedorn's case. Also, if $m<-\frac{7}{2}$, it has been shown by Huang and Weinberg[9] and by Frautschi[21] that even *local* thermal equilibrium cannot be attained. $\rho_{\text{phy}}(m)$ given by (48) is basically the so-called "Case II" in Ref. 10.

From all of these considerations it seems that the choice of $D=5$ is not without merit, and apparently receives a great deal of theoretical justification.

Before we end our discussion we should perhaps mention the results of a recent paper by Gliozzi *et al.*[22] who describe the "allowed states" $|x, r\rangle$ of the factorized amplitudes. These states, unlike the real states, satisfy a more restrictive condition that

$$L_-|x, r\rangle=(R-r)|x, r\rangle=0\,, \tag{49}$$

which comes from the fact that very often a full dual-resonance amplitude is obtained by summing over all noncyclic permutations of the external lines and therefore the amplitude possesses a higher symmetry: invariance under an arbitrary permutation of the external lines.

The manifold of allowed states is smaller than that of the real states. At the resonance position N the number of allowed states is given by $[P_D(N)-P_D(N-1)]$. Hence, if the spectrum of physical states were based on the allowed states the new density of states $\rho'_{\text{phy}}(m)$ will be down by a factor of m^{-1} compared with $\rho_{\text{phy}}(m)$ of (48),

$$\rho'_{\text{phy}}(m)\xrightarrow[m\text{ large}]{}\frac{d}{dN}\rho_{\text{phy}}(m)\sim m^{-4}\exp(m/T_0)\,. \tag{50}$$

$\rho'_{\text{phy}}(m)$ given by (50) corresponds to the "Case III" of Ref. 10.

It is, however, not clear at the moment whether or not the higher symmetry injected by condition (49) is realistic and whether all dual amplitudes should be invariant under permutations of the external lines. We shall leave the answer to this question open for the time being.

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TABLE I. The numerical values of $q(n)$, $P_D(n)$, and $\bar{P}_D(n)$.

n	$q(n)$	$P_4(n)$	$P_5(n)$	$\bar{P}_5(n)$
0	1	1	1	1
1	1	4	5	3
2	1	14	20	10
3	2	40	65	25
4	2	105	190	62
5	3	252	506	136
6	4	574	1265	293
7	5	1240	2990	590
8	6	2580	6765	1165
9	8	5180	14 725	2205
10	10	10 108	31 027	4097

TABLE II. Generators for harmonic-oscillator states.

Levels	States generated by (μ, ν, σ = 0, 1, 2, 3, 4)	ρ_+ = No. of positive-norm states (D = 5)	ρ_- = No. of negative-norm states (D = 5)	ρ_+/ρ_-
$r=1$	$a^\dagger_{1,\mu}$	4	1	4
$r=2$	$a^\dagger_{2,\mu}$, $a^\dagger_{1,\mu}a^\dagger_{1,\nu}/\sqrt{2}$	4 + 11 = 15	1 + 4 = 5	3
$r=3$	$a^\dagger_{3,\mu}$, $a^\dagger_{2,\mu}a^\dagger_{1,\nu}$, $a^\dagger_{1,\mu}a^\dagger_{1,\nu}a^\dagger_{1,\sigma}/\sqrt{3}$	4 + 17 + 24 = 45	1 + 8 + 11 = 20	2.25
$r=4$	$a^\dagger_{4,\mu}$, $a^\dagger_{3,\mu}a^\dagger_{1,\nu}$, $a^\dagger_{2,\mu}a^\dagger_{2,\nu}/\sqrt{2}$, $a^\dagger_{2,\mu}a^\dagger_{1,\sigma}/\sqrt{2}$, $a^\dagger_{1,\mu}a^\dagger_{1,\nu}a^\dagger_{1,\sigma}a^\dagger_{1,\eta}/\sqrt{4}$	4 + 17 + 11 + 48 + 47 = 126	1 + 8 + 4 + 27 + 24 = 64	1.97

APPENDIX

In the Appendix of Ref. 10, the recursion relation for $P_D(n)$ is given as

$$P_D(n) = (D/n) \sum_{m=0}^{n-1} P_D(m)\sigma(n-m),$$

where the divisor function $\sigma(m)$ can be computed by the recursion relation[23]

$$\sigma(m) = \sum_{1 \le (3k^2 \pm k)/2 \le m} (-1)^k \sigma'\left(m - \frac{3k^2 \pm k}{2}\right),$$

with

$$\sigma'(m-x) = \sigma(m-x) \quad \text{if } m \ne x,$$

$$\sigma'(m-x) = m \quad \text{if } m \ne x.$$

We give here the recursion relation for the computation of $\overline{P}_D(n)$:

$$n\overline{P}_D(n) = \sum_{m=0}^{n-1} \overline{P}_D(m)[(D-1)\sigma(n-m) - \lambda(n-m)],$$

where

$$\lambda(m) = \sigma(m), \quad m \text{ odd}$$

$$\lambda(m) = \sigma(m) - 2\sigma(m/2), \quad m \text{ even}.$$

Since the divisor functions satisfy the factorization property $\sigma(ab) = \sigma(a)\sigma(b)$ if a and b are relatively prime, $\lambda(m)$ can be expressed as $\lambda(m) = \sigma(d_m)$, where d_m is the largest odd divisor of m. Alternatively,

$$\sum_{m=0}^{n} q(m)\overline{P}_D(n-m) = P_{D-1}(n),$$

where $q(m)$ is the number of ways to write m as distinct sums of integers without regard to order[23], i.e.,

$$\sum_{m=0}^{\infty} q(m)x^m = \prod_{k=1}^{\infty} (1 + x^k).$$

Some numerical values of $q(n)$, $P_D(n)$, and $\overline{P}_D(n)$ are given in Table I. Equation (40) can be verified by comparing the numbers given in Table I with some explicit evaluations of the number of positive-norm and negative-norm states given in Table II. Notice that the ratio ρ_+/ρ_- approaches 2 rapidly as r increases.

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[16] See Eq. (41).

[17] A generalized form of the Ward operator applicable to the unequal-Regge-intercept case is given in Ref. 13.

[18] We are considering explicitly the case where $\alpha_0 \neq 1$; for the special case $\alpha_0 = 1$ there exist more than one Ward operator of this kind.

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High-Energy Scattering in ϕ^3 Theory and the Breakdown of Eikonal Approximation. II*

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In this paper we study, in ϕ^3 theory, the high-energy amplitudes for the exchange of two or more ladders. We obtain these amplitudes in closed forms to all orders of the coupling constant. The conclusions are as follows: (i) All of the scattering amplitudes satisfy the impact-factor representation. (ii) Except for the leading term (in the coupling constant) the scattering amplitude of two-ladder exchange is not in the form dictated by the eikonal approximation. (iii) The scattering amplitudes for the exchange of N ladders, $N \geq 3$, are never in the form dictated by the eikonal approximation – even for the leading term. (When the coupling constant approaches zero, all of them are of the order of s^{-3} instead of s^{-2N+1}, as dictated by the eikonal approximation.) Thus the eikonal approximation is not valid in ϕ^3 theory. The amplitude for the exchange of n scalar mesons, $n > 4$, is also given. Contrary to the popular notion, it is not of the order of s^{-n+1} when $n > 4$. Summing over such amplitudes does not lead to the exponentiation form commonly conceded in the past.

I. INTRODUCTION

In the preceding paper,[1] henceforth referred to as I, we discussed the high-energy behavior of (i) the one-ladder amplitude and (ii) the amplitude for the exchange of a ladder plus a scalar particle. The method we used to treat these problems is the one we had developed in treating high-energy amplitudes of quantum electrodynamics. This powerful method enabled us to obtain the asymptotic form of the above-mentioned amplitudes, *without* any assumption on the magnitude of the coupling constant. We were then able to verify the validity of the impact picture as well as the breakdown of the eikonal approximation in ϕ^3 theory.

In this paper, we shall continue our study of ϕ^3 theory by treating the amplitude for the exchange of two or more ladders. As before, our purpose is to extract general physical principles which are independent of perturbation. We shall show in various examples that, just as in I, the impact picture is correct while the eikonal approximation fails.

Before we go into the details of the calculations,

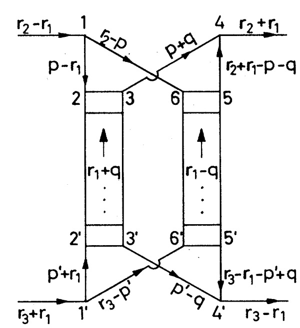

FIG. 1. A diagram of two-ladder exchange. The s channel is from left to right and the t channel is from bottom to top. A ladder with dots represents the sum of ladder diagrams over all rungs.