Equal-Time and Null-Plane Commutators of Conserved Currents on the Light Cone

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The usual manifestly conserved expansions of current densities on the light cone do not lead to the correct commutation relations at equal times or on a lightlike plane. Therefore a piece must be added which is compatible with current conservation only for a specific light-cone singularity, namely, $(z^2 - i\epsilon z_0)^{-1}$, which obeys $\Box^{(z)}(z^2 - i\epsilon z_0)^{-1} = 0$. A close relation exists between such a nonmanifestly conserved addition to the operator expansion and the well-known phenomenon of a noncausal part in the invariant function of the matrix element between one-particle states.

I. INTRODUCTION

The connection between the deep-inelastic limit of electron-proton scattering and the properties of a product of current densities at relative lightlike distances is now a well-established theoretical fact[1] – at least for the proton-proton matrix element relevant for this process. Actually not only for this case of a cross section for an inclusive reaction with one virtual photon, but also for matrix elements with two photons, the light cone becomes important,[2] if the limit[3] $q^2 \to \infty$, $\nu \to \infty$ with $q^2/2\nu$ = fixed (A-limit) is performed. As usual we denote by q the momentum transfer via the virtual photon and let $\nu = (P \cdot q)$, where for deep-inelastic electron-proton scattering, P coincides with the four-momentum of the proton. In more general cases P can be chosen to be any timelike linear combination of the hadronic momenta relevant to the problem.[4] Then the light cone is tested in the matrix element, integrated with respect to the angles of the virtual photon in the system where $\vec{P} = 0$.

It has been found that the light cone also determines matrix elements of semihadronic processes in which some internal integration is dominated by large values of the virtual four-momentum. Examples are the general semileptonic process in the renormalizable box model of weak interactions[5] and also electromagnetic[4] and weak[6] corrections to hadronic reactions.

In order to correlate all these applications, the essential ingredient is a universal light-cone expansion of a product of current densities including both electromagnetic currents and conserved current densities related to internal symmetries,[7] valid for T products as well. Only if it is possible to assume such a general operator expression can the light-cone technique become a powerful tool for a general phenomenological description of the scaling behavior of amplitudes at high energies.

Such an expansion can be obtained in two ways. It was proposed that the expansion might be abstracted from certain models, e.g., the free-quark model,[8] the quark-gluon model in a conventional approach[9] or after quantization on a lightlike plane.[10] Another approach is based on a generalization of the scaling behavior of operator products at small distances to lightlike distances.[11] The singularity of such products on the light cone is determined by the dimension of the operators appearing in the expansion. At least for the current densities in deep-inelastic electron-proton scattering, the dimensions seem to be the "naive" physical ones which gives hope that this remains true for charged currents as well. A general light-cone expansion can then be written down simply from the requirements of current conservation, causality, and C and T conservation[4,11] if certain assumptions are made about the type of operators to be included in the expansion, e.g., objects behaving like the Dirac scalar, pseudoscalar, vector, axial-vector, and tensor densities made up of quark fields.[4] This expansion is often written in a manifestly conserved way,[4,11] that is, the differential operators

$$\mathcal{P}_{\mu\nu\alpha\beta} = g_{\mu\nu}\partial_\alpha^{(y)}\partial_\beta^{(x)} - g_{\nu\alpha}\partial_\mu^{(y)}\partial_\beta^{(x)} - g_{\mu\beta}\partial_\alpha^{(y)}\partial_\nu^{(x)} + g_{\mu\beta}g_{\nu\alpha}(\partial^{(x)}\partial^{(y)}), \tag{1.1}$$

$$\mathcal{P}^{(\pm)}_{\mu\nu\gamma} = \epsilon_{\nu\rho\alpha\gamma}\partial^{(y)\rho}\,\mathcal{P}_\mu{}^\alpha \pm \epsilon_{\mu\rho\alpha\gamma}\partial^{\rho(x)}\mathcal{P}^\alpha{}_\nu$$

in the expansion

$$j^a_\mu(x) j^b_\nu(y) \underset{z^2\sim 0}{=} M^{ab}_{\mu\nu} = \mathcal{P}_{\mu\nu\alpha\beta}R^{\alpha\beta,ab} + \mathcal{P}^{(\pm)}_{\mu\nu\gamma}R^{\gamma,ab} \tag{1.2}$$

guarantee the vanishing of $\partial_\mu^{(x)}$ or $\partial_\nu^{(y)}$ applied to (1.2), irrespective of the quite general bilocal operators and singularities in $z = x - y$ contained in R, e.g.,

$$R^{\alpha\beta}=[z^\gamma j_{(1)\gamma}(z;y)g^{\alpha\beta}+(z^\alpha j^\beta_{(2)}\pm z^\beta j^\alpha_{(2)})](z^2-i\epsilon z_0)^{-1}+z^\alpha z^\beta z^\gamma j_{(3)\gamma}(z^2-i\epsilon z_0)^{-2}+(j^{\alpha\beta}_{(4)}+g^{\alpha\beta}j_\gamma{}^\gamma)\ln(-z^2+i\epsilon z_0)\,. \tag{1.3}$$

We have not indicated other possible contributions from the unit operator, the axial vector, scalar, etc.[4] and other bilocal operators containing more derivatives ($j_{\alpha\beta}, j_{\alpha\beta\gamma}, \ldots$) if the latter may be absorbed into redefined terms of the type retained in (1.3). Terms with $\ln z^2$ are kept only where the differentiations in the full $M_{\mu\nu}$ exempt the leading light-cone singularities of logarithms.

An expression like (1.2) is to be confronted with a typical result from, e.g., the free-quark model,[8,9,12]

$$j_\mu(x)j_\nu(y) \underset{z^2\sim 0}{=} c[s_{\mu\nu\alpha\beta}V^\alpha(x|y)+\epsilon_{\mu\nu\alpha\beta}A^\alpha(x|y)]\times\partial_\beta(z^2-i\epsilon z_0)^{-1}\,, \tag{1.4}$$

$$s_{\mu\nu\alpha\beta}=g_{\mu\alpha}g_{\nu\beta}+g_{\nu\alpha}g_{\mu\beta}-g_{\mu\nu}g_{\alpha\beta}\,,$$

where V^α and A^α are special bilocal operators of the type

$$V_\alpha(x|y)=\bar\psi(x)\gamma_\alpha\psi(y)+(x\leftrightarrow y) \tag{1.5}$$

or

$$\tilde V_\alpha(x|y)=i[\bar\psi(x)\gamma_\alpha\psi(y)-(x\leftrightarrow y)]\,. \tag{1.6}$$

The first one appears with f coupling, the second with d coupling of the internal symmetry, say SU(3). For the similar axial-vector operator the connection is just inverted. Current conservation is not manifest in (1.4); rather, it follows from $\Box(z^2-i\epsilon z_0)^{-1}=0$ and the equations of motion for free massless quarks.

The purpose of this paper is to show that – contrary to remarks in the literature[9] – it is *not* possible in general to write a not manifestly conserved (N.M.C.) expression like (1.4) in the manifestly conserved (M.C.) form (1.2). In Sec. II this is exhibited by taking a special limit, the equal-time commutator (E.T.C.), of (1.2). The whole usefulness of such an expansion is based on light-cone singularities being absent from the bilocal operators. Under this assumption, the E.T.C. of $\mu=\nu=0$ in (1.2) always vanishes by locality, whereas the supposedly equivalent (1.4) leads, of course, to the usual current-algebra result. Thus a N.M.C. term must obviously be added to (1.2). The problem is actually closely related to a well-known phenomenon for one-particle intermediate states in a one-particle matrix element of two current densities carrying internal symmetries[13] as seen in Sec. III.

II. EQUAL-TIME AND NULL-PLANE COMMUTATOR OF GENERAL EXPANSIONS

For $z_0\to 0$ only terms of $O(z^{-3})$ contribute to the E.T.C. in the short-distance ($z\sim 0$) limit.[11] In order to recover the current-algebra result, it must be sufficient to retain the local limit $j_\alpha(0;y)\equiv j_\alpha(y)$ of the bilocal operators in (1.3) containing the vector current density. Then also $\partial^{(x)}\sim -\partial^{(y)}\sim\partial^{(z)}=\partial$ so that the term with $\mathcal{P}_{\mu\nu\gamma}$ has no E.T.C. in this limit.

$$P_{\mu\nu\alpha\beta}[z^\alpha j^\beta(y)+z^\beta j^\alpha(y)]z^{-2}$$

is easily seen to be proportional to

$$P_{\mu\nu}[z^\alpha j_\alpha(y)]z^{-2}\,,$$

where

$$P_{\mu\nu\alpha\beta}=g_{\mu\nu}\partial_\alpha\partial_\beta-g_{\mu\alpha}\partial_\nu\partial_\beta-g_{\nu\beta}\partial_\mu\partial_\alpha+g_{\mu\alpha}g_{\nu\beta}\Box\,,$$
$$P_{\mu\nu}=P_{\mu\nu\alpha}{}^\alpha\,. \tag{2.1}$$

Because the other terms in (1.3) vanish altogether to this order in z we consider $P_{\mu\nu}(z^\alpha j_\alpha)z^{-2}$ only. Because this is the only term of $O(z^{-3})$ to be constructed from a vector current operator with physical dimensions, also more complicated expansions than (1.2), (1.3) must reduce to this term.

A necessary condition for a non-Abelian internal symmetry is a nonvanishing ($y=0$)

$$\left(\int j_0^a(x)d^3x, j_0^b(0)\right)\propto f_{abc}\int d^3x(\vec\partial)^2x_\alpha j^{\alpha,c}(0)[(x^2-i\epsilon x_0)^{-1}-\text{c.c.}]\,. \tag{2.2}$$

However, for any finite or vanishing x_0 the right-hand side of (2.2) is a surface integral for $|\vec x|\to\infty$ which can never be made different from zero – unless $j_\alpha(x;0)$ developsa singularity at $x\to 0$. Such bilocal operators have been excluded.

On the other hand, from dimensional arguments in the scale-invariant limit, the only N.M.C. expression to $O(z^{-3})$ is

$$K^{(0)ab}_{\mu\nu}=\frac{f_{abc}}{4\pi^2}[j^c_\nu(y)\partial_\mu+j^c_\mu(y)\partial_\nu-g_{\mu\nu}j^c_\alpha\partial^\alpha](z^2-i\epsilon z_0)^{-1}, \tag{2.3}$$

which coincides, of course, with the $z \sim 0$ limit of (1.4) and leads to

$$\left(\int j_0^a(x)d^3x, j_0^b(0)\right) = i f_{abc} j_0^c(0) \tag{2.4}$$

if the constant of proportionality of $K_{\mu\nu}^{(0)}$ is chosen as in (2.3). $K_{\mu\nu}^{(0)}$ is conserved only up to terms $O(z^{-2})$. Current conservation can be used to determine successively the higher orders in z. The appropriate combination of $j_{\beta\alpha} = \partial_\alpha^{(y)} j_\beta(y)$ and factors z^α for the next order becomes for a conserved current $(j_\alpha{}^\alpha = 0)$

$$K_{\mu\nu}^{(1)ab} = \frac{f_{abc}}{8\pi^2}\{j_{\mu\nu}^c - j_{\nu\mu}^c + \tfrac{1}{2} z^\beta[(j_{\nu\beta}^c - j_{\beta\nu}^c)\partial_\mu + (j_{\mu\beta}^c - j_{\beta\mu}^c)\partial_\nu]\}(z^2 - i\epsilon z_0)^{-1} \tag{2.5}$$

plus, of course, other combinations of $O(z^{-2})$ which are not related to $K_{\mu\nu}^{(0)}$ and can be written in M.C. form:

$$P_{\mu\nu} z^\alpha z^\beta j_{\alpha\beta} z^{-2}, \quad P_{\mu\nu\alpha\beta}(j^{\alpha\beta} + j^{\beta\alpha}) \ln z^2 .$$

One verifies easily that a nonconserved (local) current density on the right-hand side of (1.3) cannot create a N.M.C. $K_{\mu\nu}$. The same is true if a current density like $\tilde{V}_\alpha$ with C parity $= +1$ [cf. (1.6)] is used. Such a bilocal operator does not have a local limit $[\tilde{V}_\alpha(y|y) \equiv 0]$ and again all the terms of $O(z^{-2})$ are in M.C. form. From symmetry under charge conjugation such a bilocal operator is easily seen to occur in the expansion only for d coupling.[4] This is precisely the reason why the application to the electromagnetic currents in deep-inelastic electron-proton scattering[8,9] requires a M.C. term alone.

It is instructive to consider also the limit on a lightlike plane $z^+ = 0$, in the lightlike variables[10]

$$z^\pm = (z^0 \pm z^3)/\sqrt{2} ,$$

$$\vec{z}_\perp = (z^1, z^2) .$$

The null-plane commutator (N.P.C.) reads in a free-quark model, but also for a model with a gluon interaction,[10]

$$[j^{+a}(x), j^{+b}(y)]_{z^+=0} = i f_{abc} j^{+c}(y)\delta(z^-)\delta^2(\vec{z}_\perp) + \partial_-^{(x)}\partial_-^{(y)} S^{ab}(x,y)\epsilon(z^-)\delta^2(\vec{z}_\perp), \tag{2.6}$$

where the Schwinger term S cannot be determined in such a model. It must be at least a nonzero c number.[10] The limit $z^+ = 0$ of the commutator of our M.C. expansion (1.3)

$$[j^+(x), j^+(y)]_{z^+=0} = i\pi^2\{\partial_-^{(x)}\partial_-^{(y)} z^- j_{(1)}^+ + \mathcal{P}^{++\alpha\beta}[(z_\alpha j_\beta^{(2)} + z_\beta j_\alpha^{(2)}) + z_\alpha z_\beta j^\gamma \partial_\gamma]\}\epsilon(z^-)\delta^2(\vec{z}_\perp) \tag{2.7}$$

is obtained using

$$\delta(\vec{z}_\perp{}^2) = \tfrac{1}{2}\pi\delta^2(\vec{z}_\perp),$$

$$\epsilon(z^0)\theta(z^2)|_{z^+=0} = 0 .$$

The explicit form of the terms with $j_{(2)}$ and $j_{(3)}$ does not yield any further insight. The term with $j_{(1)}$ is just of the form of the simplest light-cone q-number Schwinger term. If we had included

$$g^{\alpha\beta}(z^2 - i\epsilon z_0)^{-2}$$

times the unit operator in (1.3) a c-number Schwinger term of precisely the same form would have emerged. $j_{(2)}$ and $j_{(3)}$ produce more complicated operator Schwinger terms in (2.7). It is well known that the absence of all operator Schwinger terms is sufficient to explain the absence of σ_L in deep-inelastic electron-proton scattering. Again it is, however, not possible to reproduce the typical (local) commutator terms in the right-hand side of (2.6) starting from the M.C. (1.2). On the other hand, $K_{\mu\nu}^{(0)}$ reproduces exactly the latter, whereas $K_{\mu\nu}^{(1)}$ and higher orders are seen to vanish for $z^+ \to 0$.

The conclusion of this section is that no manifestly conserved operator expansion with nonsingular bilocal operators reproduces the E.T.C. or N.P.C. for conserved currents. Only the not manifestly conserved addition fulfills this task. Clearly no projection technique, adding less singular expressions on the light cone, connects $K_{\mu\nu}$ and a M.C. expansion.[14] Nevertheless, as it will become clear from the example below, the explicit forms (2.3), (2.5) of $K_{\mu\nu}$ must not be taken seriously. $K_{\mu\nu}$ cannot be determined from dimensional arguments alone.

III. RELATION TO NONCAUSAL CONTRIBUTIONS IN A ONE-PARTICLE MATRIX ELEMENT

The presence of a N.M.C. $K_{\mu\nu}$ must be reflected in certain parts of a matrix element. It has been known for some time for a simple case, the one-particle matrix element, that the one-particle intermediate state has peculiar properties, if current densities of an internal symmetry are considered.[13] The discussion of the Fourier transform of the commutator ($\nu = pq$)

$$W^{ab}_{\mu\nu} = \int e^{iqx} d^4x \langle p | [j^a_\mu(x), j^b_\nu(0)] | p \rangle$$
$$= (q_\mu q_\nu - q^2 g_{\mu\nu}) V^{ab}_1(\nu, q^2)$$
$$+ [(p_\mu q_\nu + p_\nu q_\mu)\nu - p_\mu p_\nu q^2 - g_{\mu\nu}\nu^2] V^{ab}_2(\nu, q^2) \quad (3.1)$$

is especially simple for a fermion with four-momentum p, averaged over the spin, because the E.T.C. can be picked out simply by the operation $\int dq_0/(2\pi)$. The coefficients of the polynomial in $\vec{q}$ obtained in this way are the matrix element for the E.T.C. – the term which remains as $\vec{q} \to 0$ – plus all the matrix elements of possible operator-Schwinger terms of arbitrarily high order. The N.P.C. is produced by $\int dq^-/(2\pi)$ and $\lim_{q^+ \to 0}$. We shall see below that in fact this sequence is the only correct one.

On the light cone a M.C. ansatz like (1.1) yields V^{ab}_i as matrix elements of the bilocal operators contained in (1.3),

$$V_1 = \int e^{iqx} d^4x [\epsilon(x_0)\delta(x^2)\overline{V}_1(px) + \epsilon(x_0)\theta(x^2)\overline{V}_2(px)],$$
$$V_2 = \int e^{iqx} d^4x [\epsilon(x_0)\theta(x^2)\overline{V}_3(px)], \quad (3.2)$$

where we have kept only the leading singularities near $x^2 \sim 0$. The matrix elements $\langle p | j_{\alpha(1)} | p \rangle$ and $\langle p | j_{\alpha(2)} | p \rangle$ contribute to $\overline{V}_1$ and $\overline{V}_2$, respectively. $\langle p | j_{\alpha(2)} | p \rangle$ is important for $\overline{V}_3$, whereas the remaining bilocal operators yield contributions to all $\overline{V}_i$.[1,4] For general indices of the internal symmetry, $j_{\alpha(i)}$ are appropriate combinations of bilocal operators with $c = \pm 1$ (d or f coupling).

For W^{ab}_{00} the factors in front of V_1 and V_2 are independent of q_0,

$$W_{00} = \vec{q}^2 V_1 + [p_0^2 \vec{q}^2 - (\vec{p}\vec{q})^2] V_2 . \quad (3.3)$$

Therefore, applying $\int dq_0/(2\pi)$ in the special system $\vec{p} = 0$, the square brackets in (3.2) are simply taken at $x_0 = 0$ for the E.T.C. yielding a vanishing result for W_{00} as $\vec{q} \to 0$, as long as $\overline{V}_i(0)$ is finite. This is, however, precisely the implication of the ansatz (1.2) with bilocal operators $j(x; 0)$ which are regular at $x \to 0$.

In order to obtain a nonvanishing E.T.C., V_1 and/or V_2 must develop a (kinematical) singularity. As noticed by Meyer and Suura[13] the only possibility is (α is an arbitrary constant)

$$V_2 \sim \frac{\delta(q^2 + \alpha\nu)}{q^2} . \quad (3.4)$$

The one-particle intermediate state for pointlike particles in (3.1) is just of this type with $\alpha = \pm 2$.

Actually the semiconnected three-particle state must be added to recover the full E.T.C. for the case of the quark fields[15]:

$$V^{ab}_{2,\,\text{n.c.}} = -\frac{\pi i f_{abc}}{q^2} [\epsilon(p_0 + q_0)\delta(q^2 + 2\nu) + \epsilon(p_0 - q_0)\delta(q^2 - 2\nu)] . \quad (3.5)$$

This expression is noncausal (n.c.) because the Fourier transform contains a piece proportional to

$$[(p \cdot x)^2 - p^2 x^2]^{-1/2}$$

which is nonvanishing outside the light cone. This does not prevent the full $W_{\mu\nu}$ from being causal.

Calculating $\int dq_0/(2\pi)$ *before* $\vec{q} \to 0$ the factor

$$[p_0^2 \vec{q}^2 - (\vec{p} \cdot \vec{q})^2]$$

in (3.3) is canceled and the result is different from zero in the limit $\vec{q} \to 0$. Clearly for general hadrons (3.5) is part of the full one-particle and semiconnected three-particle intermediate states.

Let us compare this with the matrix element of (2.3):

$$\int e^{iqx} d^4x \langle p | K^{ab}_{\mu\nu} - (K^{ab}_{\mu\nu})^\dagger | p \rangle = \pi i f_{abc} \lambda_c (p_\nu q_\mu + p_\mu q_\nu - g_{\mu\nu}\nu)\epsilon(q_0)\delta(q^2) . \quad (3.6)$$

$K^{(1)}_{\mu\nu}$ and the higher orders are proportional to

$$\langle p | \partial_{\alpha_1} \partial_{\alpha_1} j_\beta | p \rangle \equiv 0 .$$

Using $q^2 \delta(q^2) = 0$, (3.6) can be written as a contribution to V_2,

$$V^{ab}_{2,K} = \pi i f_{abc} \lambda_c \epsilon(q_0)\delta(q^2)/\nu , \quad (3.7)$$

which is again of the form (3.4). Now the Fourier transform of (1.7) is proportional to

$$\theta(x^2)[(p \cdot x)^2 - p^2 x^2]^{-1/2}$$

and hence causal. For $x^2 \sim 0$ the factor $|p \cdot x|^{-1}$ cannot be related to a nonsingular bilocal operator or $\overline{V}_i$ in (3.2) at $x \to 0$. Despite their different causality properties, (3.5) and (3.7) must be basically related: $K_{\mu\nu}$ is – as a result of scaling – also essentially the one-quark plus (semiconnected) three-quark intermediate state for a theory with (pointlike) free-quarks. Neglecting the quark mass in $K_{\mu\nu}$, the operator expansion corresponds, however, to $p \to 0$ in the matrix element. This leads in fact from (3.5) to (3.7). In reality, therefore, $\epsilon(p_0 \pm q_0)$, which is responsible for the noncausality of (3.5), never completely degenerates to the causal expression (3.7). Even more serious is the fact that the transition from (3.5) to (3.7) assumes also that $\lim_{p \to 0} \nu = 0$, whereas for the only interesting case, namely, the A limit, $\nu \to \infty$. This is the reason why the correct singularities at $q^2=\pm 2\nu$ degenerate into the line $q^2=0$ in the q^2-ν plane. Such a singularity is unacceptable for the matrix element under consideration.

The consideration of the N.P.C. parallels this case closely, so that it has not been reproduced here.

IV. CONCLUSION

The study of commutators at equal times and on a lightlike plane tells us that any manifestly conserved expansion of current densities on the light cone is incomplete. We have constructed a nonmanifestly conserved addition $K_{\mu\nu}$ to $O(z^{-3})$ and $O(z^{-2})$ using dimensional arguments alone. Current conservation can be used in principle to calculate $K_{\mu\nu}$ to all orders, i.e., on the whole light cone. We have not done this, however, because the discussion of the one-particle matrix element has already revealed the unsatisfactory nature of a universal $K_{\mu\nu}$. We suggest therefore that for general applications one should write

$$j_\mu^a(x) j_\nu^b(y) = M_{\mu\nu}^{ab} + K_{\mu\nu}^{ab}, \quad (4.1)$$

where $M_{\mu\nu}^{ab}$ is a manifestly conserved expression involving bilocal operators and explicit singularities on the light cone like (1.3) or more generally, as given in Eqs. (3.6) to (3.8) of Ref. 4. $K_{\mu\nu}^{ab}$ cannot be determined in general. It is an operator containing all the information about those masses, intermediate states, etc. in an arbitrary matrix element, which for free-quark currents (with the external particles also replaced by appropriate free-quark wave functions, but with correct masses) are necessary to lead to correct equal-time commutators. In every practical case

$$\langle\beta|K_{\mu\nu}|\alpha\rangle$$

will be automatically incorporated in the matrix element

$$\langle\beta|\ |\alpha\rangle$$

if certain simple intermediate states are not neglected. Any such matrix element remains, of course, undetermined to the extent that a manifestly conserved expression may be added to $K_{\mu\nu}$ – or a kinematically less singular part to the Fourier-transform of the matrix element. We note a few consequences of the additional $K_{\mu\nu}$. $K_{\mu\nu}^{ab}$ is proportional to f_{abc} – at least the singular part necessary for E.T.C.'s. Therefore $M_{\mu\nu}$ alone suffices for a description of situations with a symmetric combination of SU(3) indices. This is, e.g., the case for the discussion of electromagnetic or weak corrections to hadronic processes.[4,6] The connection between an expansion in the free-quark model and a general light-cone expansion[8] used for electromagnetic currents in deep-inelastic electron proton scattering, exists only insofar as the singularities on the light cone are of the same degree, which is essentially a consequence of scale invariance for the operators involved. The model-dependent results are always related to $K_{\mu\nu}$ rather than $M_{\mu\nu}$. This criticism does not apply when free field[8] or interacting field[9,10] models are used only to determine the E.T.C. or N.P.C. The many sum rules obtained in this way[16] are based on a specific

$$\langle p|K_{\mu\nu}|p\rangle$$

which is reasonable if restricted to equal times or to a lightlike plane. However, our example for $V_{2,K}$ and $V_{2,\text{n.c.}}$ in deep lepton-nucleon scattering shows that the problem of having to interchange the infinite-momentum limit,

$$\lim_{p_0\to\infty}\int d\nu = \int d\nu \lim_{p_0\to\infty},$$

for the derivation with E.T.C.'s, persists for the related $\lim_{q^+\to 0}\int d\nu$ required for the derivation with a N.P.C.[16] The reason is that, e.g., the "truncated" one-plus three-particle intermediate states are proportional to

$$\int e^{iqx}d^4x\langle p|K^{++}(x,0)-[K^{++}(x,0)]^\dagger|p\rangle$$
$$\propto p^+q^+\delta(2q^+q^- - \vec{q}_\perp{}^2)\epsilon(q^++q^-)$$

which yield the finite result only for

$$\lim_{q^+\to 0}\int\frac{dq^-}{(2\pi)} \sim \lim_{q^+\to 0}\int\frac{d\nu}{(2\pi p^+)}$$

and not if the $\lim_{q^+\to 0}$ is taken first.

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Duality Constraints on Inclusive Reactions. II. The Role of the Harari-Freund Conjecture

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We present a scheme for the generalization of the two-component theory of Harari and Freund to multiparticle amplitudes and, in particular, to inclusive reactions. Our scheme is based on duality and borrows the terminology of dual perturbation theory but we do not restrict our generalizations to any specific model. We obtain a criterion for the absence of secondary (nonscaling) Regge contributions in inclusive reactions. We show how inclusive reactions may approach their limiting values from below, in contrast to total cross sections which always have positive secondary contributions. The scheme suggests a novel interpretation of diffraction dissociation which we discuss in a separate paper.

I. INTRODUCTION

Phenomenological duality[1] has had striking success in accounting for certain features of hadronic interactions, especially for two-body processes and, via the optical theorem, for total cross sections. The predictions of duality (supplemented by the absence of exotic resonances) include (1) exchange degeneracy among Regge trajectories and Reggeon residues[2]; (2) certain quark-model results such as the "magic" ϕ-ω mixing angle[3]; (3) the *fall* of the total cross section, $\sigma(a+b\to\text{anything})$, to its asymptotic value when the channel (ab) is a nonexotic channel, i.e., the positivity of secondary Regge-pole contributions; (4) the approximate energy independence of the total cross section, $\sigma(a+b\to\text{anything})$, at surprisingly low energies when (ab) is an exotic channel, i.e., the absence of secondary Regge-pole contributions when (ab) is exotic.