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Spectrum-Generating Algebra and No-Ghost Theorem for the Dual Model*

Richard C. Brower
Laboratory for Nuclear Science and Department of Physics,
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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The dual model is generally factorized using Lorentz oscillators a_n^μ with ghost (or negative-norm) states arising from the indefinite metric ($[a_n^0, a_n^{0\dagger}] = -1$). Here *all* ghost states are proven to decouple for unit Regge intercept ($\alpha_0 = 1$) as a consequence of the Virasoro gauges (L_n). By reformulating vertices in light-cone variables and exploiting the local commutators (for Q^μ, P^μ) on the Koba-Nielson circle, the spectrum-generating algebra ($A_n^i, A_n^{(+)}$) is found that commutes with all the gauges L_n. All physical states are explicitly constructed. The no-ghost theorem follows from the remarkable isomorphism of the transverse generators A_n^i ($i = 1, 2$) of Del Giudice, Di Vecchia, and Fubini to the original oscillators $\sqrt{n}\, a_n^i$, $[A_n^i, A_m^j] = n\delta_{ij}\delta_{n+m,0}$, and the isomorphism (up to c numbers) of the longitudinal generators $A_n^{(+)}$ with the conformal group generators L_l, $[A_n^{(+)}, A_m^{(+)}] = (n-m)A_{n+m}^{(+)} + 2n^3\delta_{n+m,0}$. Increasing the number of spatial oscillators (a_n^i, $i = 1, \ldots, D-1$), one observes a critical dimension $D = 26$. For $D > 26$ ghosts appear, for $D < 26$ there are no ghosts, and $A_1^{(+)}$ gives the null states postulated by Brower and Thorn. But for $D = 26$, all $A_n^{(+)}$ correspond to null states, so that the second-order Pomeranchukon is precisely a Regge pole ($\alpha_P = \frac{1}{2}\alpha' s + 2$) as proposed by Lovelace.

I. INTRODUCTION

Solutions to finite-energy sum rules are often plagued by resonances with complex couplings ($g^2 < 0$), called ghost states.[1] Indeed this is a typical difficulty of Lorentz-covariant theories with spin. For example, suppose we modify the nonrelativistic harmonic oscillator

$$\begin{aligned} 2mE &= \vec{p}^{\,2} + m^2\omega^2\vec{q}^{\,2} \\ &= 2m\omega\vec{a}^\dagger \cdot \vec{a} + \text{const} \end{aligned} \tag{1.1}$$

by replacing $\vec{a}^\dagger \cdot \vec{a}$ with $a^{\mu\dagger} g_{\mu\nu} a^\nu$ and $m^2 + 2mE$ with $s = (m+E)^2$ to construct a covariant theory. Since this gives a series of integer-spaced zero-width trajectories linear in s, the Lorentz oscillator might be appropriate for approximating hadronic resonances. However, the negative metric of the time-component oscillator introduces ghost states, which as in quantum electrodynamics should be decoupled by a subsidiary (or gauge) condition.

The dual model is a particular covariant harmonic oscillator with an infinite[2] set of modes ($n = 1, 2 \ldots$)

$$[a_n^\mu, a_m^{\nu\dagger}] = \delta_{n,m} g^{\mu\nu}, \quad g = (-,+,+,+), \tag{1.2}$$

with resonances of masses, $-p^2 = m^2 + \sum_n n a_n^\dagger \cdot a_n$. Over two years ago, Virasoro[3] observed that the dual model for unit Regge intercept ($\alpha_0 = 1$) possesses an infinite set of gauges L_l and boldly conjectured that these decoupled all the negative-metric states. *Here we prove Virasoro's no-ghost conjecture.*

Moreover, the demonstration[4-7] follows from a remarkably simple algebraic structure. The generators for the physical spectrum are naturally expressed in light-cone variables [$\vec{P} = (P^1, P^2)$, $P_\pm = (P^0 \pm P^3)/\sqrt{2}$] as averages (denoted by $\langle \cdots \rangle$; see Sec. II) over the "densities" $P^\mu(\theta)$ and $Q^\mu(\theta)$ for momentum and position:

$$\begin{aligned} \vec{A}_n &= \langle \vec{P} e^{inQ_-} \rangle, \\ A_n^{(-)} &= \langle P_- e^{inQ_-} \rangle = 0, \\ A_n^{(+)} &= \langle :P_+ e^{inQ_-}: \rangle + \cdots. \end{aligned} \tag{1.3}$$

The difficult mathematical problem (see Sec. III) is the need to add corrections to $A_n^{(+)}$ to restore the conformal gauge invariance ($[L_l, A_n] = 0$) destroyed by normal ordering.

However, ignoring infinities, the three nonzero components of $A_n^\mu = \langle P^\mu e^{inQ_-} \rangle$ close algebraically (see Sec. IV) lacking only the c numbers of the rigorous derivation, and the algebra obeys the isomorphism[6,7] $\vec{A}_n \to \sqrt{n}\,\vec{a}_n$, $A_n^{(+)} \to L_n$ up to c numbers.

The no-ghost theorem follows essentially because the isomorphism does not require the use of the time components in L_n. Hence, we can project our states into a positive definite (unitary) Hilbert space. The vital questions of normal ordering, infinities, and linear independence aside, our demonstration is a surprisingly simple solution to an infinite gauge algebra.

There are two major goals that this construction may serve. First it should lead to a new formulation of the dual theory on the physical states, in light-cone variables. Also, extending the proof to other theories[8] with the same conformal invariance should allow one to explore rigorously the options available for a more realistic spectrum.

Alternately, the results may have a qualitative significance that can be abstracted. After all, we have here a theory for multiparticle amplitudes with positive-norm resonances that (1) is covariant, (2) has Regge behavior, and (3) is crossing-symmetric (i.e., dual). This provides a new theoretical laboratory in which to study their combined consequences. Indeed, as we note at several points, a sort of parton (or multiperipheral) picture seems to emerge.

The transverse oscillators $\vec{A}_n$ obey nonrelativistic commutators, and the longitudinal modes $A_n^{(+)}$ are consistent with positivity and the notorious angular conditions. The $\vec{A}_n$ algebra can be understood as implementing the transverse momentum cutoff, and work is proceeding to understand the $A_n^{(+)}$ algebra in terms of the consequences of covariance and positivity.

Perhaps we can see how the mathematical "partons" are mapped onto the resonance states or at least how to introduce resonances in a more systematic fashion reflecting partially the above constraints.

II. BASIC OPERATORS

Here we review those algebraic properties of the model essential to our subsequent construction. We hope that the presentation is complete enough so that the uninitiated can supply the derivations. Moreover, the intuitive content is emphasized to motivate the mathematics.

A. N-Particle Amplitude

We wish to study the pole structure (or spectrum) in the dual amplitude,

$$B_N(p_1, \ldots, p_N) = \int \frac{dz_1 \cdots dz_N}{d^3\omega} \prod_{i=1}^{N} |z_{i+1} - z_i|^{\alpha_0 - 1} \prod_{i\neq j} |z_i - z_j|^{p_i \cdot p_j} \quad , \tag{2.1}$$

which is the direct generalization[9] of the beta function to N external particles of mass m and leading trajectory intercept of $\alpha_0 = -m^2$ and slope $\alpha' = 1$. Factorization is performed on the oscillator basis

$$|\{\lambda\}, p\rangle = \prod_n (a_n^{\mu\dagger})^{\lambda_n} (\lambda_n!)^{-1/2} |0, p\rangle , \tag{2.2}$$

where $|0, p\rangle$ is the lowest-mass state with momentum p.

It is convenient to replace the a_n's with the Fourier expansion[10]

$$Q^\mu(z) = q_0^\mu - i p_0^\mu \ln z + \sum_{n=1}^{\infty} \frac{a_n^\mu z^{-n} + a_n^{\mu\dagger} z^n}{\sqrt{n}} , \tag{2.3}$$

where $p_0^\mu/\sqrt{2}$ is the momentum operator ($p_0^\mu|\{\lambda\}, p\rangle = \sqrt{2}\, p^\mu|\{\lambda\}, p\rangle$) and $\sqrt{2}\, q_0^\mu$ is the conjugate position operator ($[q_0^\mu, p_0^\nu] = i g^{\mu\nu}$). In terms of the complex Koba-Nielsen variable

$$z = \exp(\tau + i\theta) , \tag{2.4}$$

we can replace the fundamental oscillator commutators (1.2) by the local (in θ) equal-τ commutation relation

$$[Q^\mu(\theta, \tau), P^\nu(\theta', \tau)] = 2\pi i g^{\mu\nu} \delta(\theta - \theta') , \tag{2.5}$$

where the conjugate "momentum" operator is

$$P^\mu(\theta, \tau) = i z \frac{d}{dz} Q^\mu(z) , \tag{2.6}$$

or for $\alpha_0 = 1$, $P^\mu = m \partial_\tau Q^\mu(\theta, \tau)$.

This local structure is not only suggestive intuitively,[11] but also provides the essential mathematical ingredient for our construction. One may regard $Q^\mu(\theta)$ and $P^\mu(\theta)$ as position and momentum for mathematical "partons" labeled by θ (infinite in number, $-\pi \leq \theta \leq \pi$) as they move in their proper time τ. The central theorem of Fubini and Veneziano[10] reexpresses the amplitude B_N for $\alpha_0 = 1$ as the vacuum expectation value of the τ-ordered product,[12]

$$B_N = \int_{-\infty}^{\infty} \frac{d\tau_1 \cdots d\tau_N}{d^3\omega} \left\langle 0 \left| T\left[\prod_{j=1}^{N} V(p_j, z_j) \right] \right| 0 \right\rangle , \tag{2.7}$$

with the vertices $[z_j = \exp(\tau_j),\ \theta_j = 0]$

$$V(p, z) = :\exp[-i\sqrt{2}\, p^\mu Q_\mu(z)]: . \tag{2.8}$$

By setting $\alpha_0 = 1$ we have dropped the only factor,

$$\prod_i |z_{i+1} - z_i|^{\alpha_0 - 1} ,$$

dependent on ordering the external particles $(p_1, \ldots, p_N)$. So by integrating over all orderings of the τ_i's the fully symmetrized amplitude is achieved. Of course, *all* the properties of the operators can be understood as a direct consequence of the equivalence[13] between Eq. (2.1) and Eq. (2.7).

B. On-Shell Physical States $|\psi, N\rangle$

By introducing the dilatation operator L_0 ($[L_0, Q] = z dQ/dz$), from the vertex condition

$$V(z, p) = z^{L_0} V(1, p) z^{-L_0} , \tag{2.9}$$

it is easy to discover the poles in B_N. For the term with $z_i \leq z_{i+1}$, setting $(z_1, z_{N-1}, z_N) = (0, 1, \infty)$ and introducing $x_i = z_2 \cdots z_i$ and evaluating the integral

$$\int_0^1 dx\, x^{-L_0} = (1 - L_0)^{-1} ,$$

we obtain

$$B_N = \left\langle 0, p_N \middle| V(1, p_{N-1}) \frac{1}{L_0 - 1} V(1, p_{N-2}) \cdots \frac{1}{L_0 - 1} V(1, p_2) \middle| 0, p_1 \right\rangle . \tag{2.10}$$

The states $|\psi, N\rangle$ contributing to a pole at $p^2 = 1 - N$ satisfy the (Klein-Gordon) equation

$$(L_0 + m^2)|\psi, N\rangle = 0, \quad m^2 = -1 \tag{2.11}$$

where $L_0 = \frac{1}{2} p_0^2 + \sum n a_n^\dagger \cdot a_n$.

Under the general conformal generator,

$$[L_l, Q(z)] = z^{+l+1} \frac{d}{dz} Q(z) , \tag{2.12}$$

the vertex for $\alpha_0 = 1$ (or $p^2 = 1$) is a conformal vector,[14]

$$[L_l, V(z, p)] = z \frac{d}{dz} [z^{+l} V(z, p)] . \tag{2.13}$$

As a consequence[10] there are (spurious) states $\langle\{\lambda\}, p| L_l$ at each pole ($p^2 = 1 - N$) that decouple, and all the physical states[15] (orthogonal to these) must obey the subsidiary conditions

$$L_l |\psi, N\rangle = 0 \text{ for } l > 0 . \tag{2.14}$$

A bizarre feature of introducing a four-vector (parton) position operator $Q^\mu(\theta)$ is the unconstrained (noncausal) time $Q^0(\theta)$. Classically, a four-velocity $\dot{x}^\mu(\tau)$ satisfies $\dot{x}^2 = -1$, so that τ is the (Lorentz scalar) proper time. Curiously our gauges may be viewed as constraining $Q^0(\theta, \tau)$ so that $\sqrt{2\alpha'}\, \tau$ *is* the proper time,[16] on the physical subspace. The constraint

$$\langle \psi', N' | : \partial_\tau Q^\mu \partial_\tau Q_\mu : | \psi, N\rangle = -2\alpha' \langle \psi', N' | \psi, N\rangle \tag{2.15}$$

follows from (2.11), (2.14), and the expansion $:\dot{Q}^2: = -2 \sum_l z^{-l} L_l$. It is attractive to suppose that the ghost states (decaying backward in time) violate this causality constraint.

Finally, we notice that all the basic operators are expressible as averages over θ (i.e., sums over partons). For example, the gauges are

$$\begin{aligned} L_l &= \tfrac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{+il\theta} :P^2: \\ &= \frac{1}{4\pi i} \oint \frac{dz}{z} z^l :P^2: . \end{aligned} \tag{2.16}$$

Also as realized by Del Giudice, Di Vecchia, and Fubini (DDF), the *on*-shell vertex transition operator ($p^2 = 1$) is

$$\frac{1}{2\pi i} \oint \frac{dz}{z} V(z, p) \tag{2.17}$$

as one can readily verify by taking the matrix elements ($p_i^2 = 1 - N_i$, $N_i = \sum_k k \lambda_k^{(i)}$),

$$\begin{aligned} &\left\langle \{\lambda^{(1)}\}, p_1 \middle| \oint \frac{dz}{2\pi i z} z^{L_0} V(1, p) z^{-L_0} \middle| \{\lambda^{(2)}\}, p_2 \right\rangle \\ &\qquad = \langle \{\lambda^{(1)}\}, p_1 | V(1, p) | \{\lambda^{(2)}\}, p_2 \rangle , \end{aligned} \tag{2.18}$$

using (2.11) and Cauchy's theorem. We will often use the shorthand notation[17]

$$\langle V \rangle \equiv \frac{1}{2\pi i} \oint \frac{dz}{z} V(z, p) , \tag{2.19}$$

since these integrals are used so frequently. Also note that the gauge condition (2.13) for the on-shell vertex $\langle V \rangle$ becomes

$$[L_l, \langle V \rangle] = 0 \tag{2.20}$$

for all l.

III. GENERATING THE PHYSICAL STATES

Here we shall explicitly construct all physical states $|\psi, N\rangle$. The physical states on shell $[p^2 = 1 - N,\ (L_0 - 1)|\psi, N\rangle = 0]$ form a linear subspace of the states

$$(a_1^{0\dagger})^{\lambda_1}\cdots(a_m^{0\dagger})^{\lambda_m}(\vec{a}_1^{\dagger})^{\mu_1}\cdots(\vec{a}_n^{\dagger})^{\mu_n}|0, p\rangle, \tag{3.1}$$

with $\sum i\lambda_i = m,\ \sum i\mu_i = n,\ m+n = N$ that satisfy the condition[15]

$$L_l|\psi, N\rangle = 0\,. \tag{3.2}$$

The first step taken by Brower and Thorn[5] was to count the number of (linearly independent) solutions to Eq. (3.2). In this section, we shall construct the correct number of solutions, which are proven in Sec. IV to be independent.

A. Counting

Using the Virasoro algebra $(D = 4)$

$$[L_n, L_m] = (n-m)L_{n+m} + \tfrac{1}{12}D(n^3 - n)\delta_{n+m,0}\,, \tag{3.3}$$

we demonstrated[5] that the physical states $|\Psi, N\rangle$ have the same degeneracy, spin, and parity as those states generated by the spatial oscillators $(\vec{a}_n^{\dagger})$. [In subsequent discussions, we shall want to vary the number $(D-1)$ of spatial dimensions.]

Specifically, a D-component oscillator a_n^{μ} $(\mu = 0, \ldots, D-1)$ generates $T^D(N)$ states at $p^2 = 1 - N$, where $T^D(N)$ is the coefficient of x^N in

$$\left[\prod_{n=1}^{\infty}(1 - x^n)\right]^{-D}.$$

In Ref. 5, we constructed a basis for the spurious states,

$$|\{\lambda\}, m, \mu\rangle = (L_m^{\dagger})^{\lambda_m}\cdots(L_1^{\dagger})^{\lambda_1}|\mu, N-m\rangle\,,$$

where $\sum i\lambda_i = m$ and $|\mu, N-m\rangle$ are orthogonal[18] to the spurious states with $\sum i\lambda_i = N - m$. Comparison with (3.1) shows that there are $S(N)$ spurious states

$$\begin{aligned} S(N) &= \sum_{m=1}^{N} T^1(m)T^{D-1}(N-m) \\ &= T^D(N) - T^{D-1}(N)\,, \end{aligned} \tag{3.4}$$

or there are $T^{D-1}(N)$ physical states.

To generate the physical states we might try to replace the four- (D) component vectors (a_n^{μ}) by three- $(D-1)$ component objects (A_n).

B. Transverse Algebra

As a consequence, the beautiful idea[6] of Del Giudice, Di Vecchia, and Fubini (DDF) to use the transition operators A_n^i for the vector zero-mass particle (at $\alpha = 1$) appears natural, at least in retrospect. This vertex operator takes a physical state $|\Psi_1, N_1\rangle$ into another physical state $|\Psi_2, N_2\rangle$ by absorbing a "photon" of momentum k, and polarization $i = 1, 2$ (see Fig. 1)

$$A^i(k') = \frac{1}{2\pi i}\oint P^i(z)e^{-i\sqrt{2}\,k'\cdot Q}\,. \tag{3.5}$$

The procedure is the same as for constructing $\langle V(p)\rangle$ except that it is convenient to pick a standard vector k since the mass-shell condition $p_i^2 = 1 - N_i$, for $p_2 = p_1 + k'$, can be satisfied by scaling $k' = (N_2 - N_1)k = nk$ with $2k\cdot p_1 = 2k\cdot p_2 = -1$ (or $\sqrt{2}\,k\cdot p_0 = -1$).[19]

By taking the vector k along the z axis $k = (1, 0, 0, 1)/\sqrt{2}$ we can convert to light-cone coordinates $\vec{P} = (P^1, P^2)$, $P_{\pm} = (P^0 \pm P^3)/\sqrt{2}$ and

$$\vec{A}_n \equiv \langle \vec{P}e^{inQ_-}\rangle\,. \tag{3.6}$$

Since there is no normal ordering, one can easily check conformal gauge invariance

$$[L_l, \vec{A}_n] = 0\,. \tag{3.7}$$

For arbitrary orientation of the "photons," the vertices acting on the ground state $[\prod_i \vec{A}^{\dagger}(k_i')|0, p\rangle]$ probably generate all physical states. However, their norms are extremely difficult to calculate, because of the difficult (nonlocal) commutator

$$[Q^{\mu}(\theta, \tau), Q^{\nu}(\theta', \tau)] = 2\pi i g^{\mu\nu}\epsilon(\theta' - \theta)\,. \tag{3.8}$$

Subsequently,[7] Goddard and I realized that since

$$[k_1'\cdot Q(\theta_1), k_2'\cdot Q(\theta_2)] \sim k_1'\cdot k_2' = 0$$

for collinear "photons" $(k_1' = nk,\ k_2' = mk)$, the commutator of A_n^i and A_m^j could be easily calculated from the local commutators for Q and P. Indeed reminiscent of a field theory in light-cone variables, the transverse subalgebra is precisely that of the (nonrelativistic) oscillators $\sqrt{n}\,a_n^i$, $\sqrt{n}\,a_n^{i\dagger}$,

$$\begin{aligned} &[A_n^i, A_m^j] = n\delta_{ij}\delta_{n+m,0}\,, \\ &(A_n^i)^{\dagger} = A_{-n}^i,\quad A_n^i|0, p\rangle = 0,\quad n > 0\,. \end{aligned} \tag{3.9}$$

One more series of operators $A_n^{(+)}$ purely in the longitudinal (P_+) and scalar (P_-) subspace is needed to generate all the physical states.

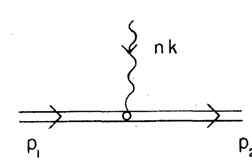

FIG. 1. On-shell vertex for $|\Psi_1, N_1\rangle + |\gamma, nk\rangle \rightarrow |\Psi_2, N_2\rangle$. The mass-shell constraints $(p_i^2 = 1 - N_i,\ k^2 = 0)$ with energy conservation $p_2 = p_1 + nk$ imply $n = N_2 - N_1$ and $2k\cdot p_1 = 2k\cdot p_2 = -1$.

C. Longitudinal Generators

On grounds of covariance, Ref. 7 suggests you consider the vector

$$A_n^\mu = \tfrac{1}{2}\langle : [P^\mu(z) e^{inQ_-} + e^{inQ_-} P^\mu(z)] : \rangle , \qquad (3.10)$$

which is in the light-cone gauge ($A_n^{(-)} = \delta_{n,0}$). Since the scalar part is zero ($n \neq 0$), the only new component is the longitudinal part

$$A_n^L = \tfrac{1}{2}\langle : \{P_+, e^{inQ_-}\} : \rangle . \qquad (3.11)$$

Now the normal ordering is required to obtain a finite operator, and this alone destroys gauge invariance,[7]

$$[L_l, A_n^L] = -\tfrac{1}{2} n \langle l^2 z^l e^{inQ_-} \rangle . \qquad (3.12)$$

However, the nonzero commutator is entirely in the scalar ($k\cdot a_n$) modes, so we suggest adding a scalar correction term $F_n(k\cdot a)$ to make $A_n^{(+)} = A_n^L + \tfrac{1}{2} n F_n$ gauge invariants

$$[L_l, A_n^{(+)}] = 0 . \qquad (3.13)$$

To construct F_n, again the strategy is to exploit the local commutators in light-cone variables.

Using the expansion

$$\begin{aligned} P^2(z') &= -P_+P_- - P_-P_+ + \vec{P}^2 \\ &= 2\sum_l (z')^{-l} L_l , \end{aligned} \qquad (3.14)$$

the constraint on F_n takes the local form[20]

$$[P_+(\theta'), F_n] = \left(\frac{inQ_-'' - (nQ_-')^2}{Q_-'} \right) e^{inQ_-} , \qquad (3.15)$$

where $Q_-' = P_-(\theta'), Q_-'' = \partial_{\theta'} P_-(\theta')$. Representing F_n as a general integral of the functional $\mathfrak{F}_n$,

$$F_n = \int \frac{d\theta}{2\pi} \mathfrak{F}_n(Q_-, Q_-', Q_-'', \dots) , \qquad (3.16)$$

and using

$$[P_+(\theta'), \mathfrak{F}_n] = \sum_r [P_+(\theta'), Q_-^{(r)}(\theta)] \frac{\partial}{\partial Q_-^{(r)}} \mathfrak{F}_n(Q_-^{(r)})$$

with the local commutators, the equation reduces to a set of differential equations. The solution gives (the entropy)

$$F_n = -n \int \frac{d\theta}{2\pi} P_- \ln(P_-) e^{inQ_-} \qquad (3.17)$$

or integrating by parts $F_n = \langle \dot{P}_-(P_-)^{-1} e^{inQ_-} \rangle$. A direct check on the solution follows immediately from

$$[L_l, \dot{P}_-(P_-)^{-1}] = z \frac{d}{dz}[z^l \dot{P}_-(P_-)^{-1}] + l^2 z^l . \qquad (3.18)$$

The integral for F_n is well defined since $\sqrt{2}\, k\cdot p_0 = -1$ allows $\ln(P_-)$ or $(P_-)^{-1}$ to be expanded in a series of $k\cdot a_n$, $k\cdot a_n^\dagger$ – indeed a finite series when F_n acts on a state.

A basis for the physical states ($|\{\lambda_r^i\}, N\rangle$) at $(p+Nk)^2 = 1 - N$ is

$$|\{\lambda_r^i\}, N\rangle = \left[\prod_{i=1}^{D-2} \prod_{r=1}^{N} (A_{-r}^i)^{\lambda_r^i} \right] \times (A_{-N}^{(+)})^{\lambda_N^0} \cdots (A_{-1}^{(+)})^{\lambda_1^0} |0, p\rangle , \qquad (3.19)$$

where $\sum_{i,r} i\lambda_r^i = N$. The states are generated by *ordered* products from the lowest state $|0, p\rangle$ at at $p^2 = 1$. As we shall show in Sec. IV, they are independent and the null states (for $D < 26$) have $\lambda_1^0 \neq 0$.

IV. CONSEQUENCES OF THE GENERATING ALGEBRA

A. Generating Algebra

With a little care with the normal ordering the generating algebra is calculated:

$$\begin{aligned} [A_n^{(+)}, A_m^{(+)}] &= (n-m) A_{n+m}^{(+)} + 2n^3 \delta_{n+m,0} , \\ [A_n^i, A_m^j] &= n\delta_{ij}\delta_{n+m,0} , \\ [A_n^i, A_m^{(+)}] &= nA_{n+m}^i . \end{aligned} \qquad (4.1)$$

Remarkably, only the c number in the commutator for

$$A_n^{(+)} = \tfrac{1}{2}\langle : \{P_+, e^{inQ_-}\} : + n\dot{P}_-(P_-)^{-1} e^{inQ_-} \rangle$$

is affected by the normal ordering[7] and the F_n correction term. In this sense, the results are extremely simple.

The term $\langle (e^{iQ_-})^n P_+ \rangle$ has the form of a conformal generator

$$[L_n, Q^\mu] = i(e^{i\theta})^n \partial_\theta Q^\mu(\theta) ,$$

with $Q_- \leftrightarrow \theta, P_+ \leftrightarrow \partial_\theta$. This is the root of the conformal algebra for the $A_n^{(+)}$, except that F_n must obey

$$[F_n, A_m^{(+)}] = nF_{n+m} + n^3 \delta_{n+m,0} \qquad (4.2)$$

and normal ordering only doubles the c number. In closing the algebra, the collinearity is essential on merely kinematical grounds. As is well known to current algebra experts, $[A(k_1), A(k_2)]$ is proportional to $A(k_1+k_2)$ with k_1+k_2 lightlike only if k_1 and k_2 are collinear.

By the replacement

$$A_n^{(+)} \to \bar{A}_n^{(+)} = A_n^{(+)} - \tfrac{1}{2} \sum_{l=-\infty}^{\infty} : \bar{\mathrm{A}}_{n+l} \cdot \bar{\mathrm{A}}_{-l} : , \qquad (4.3)$$

the algebra is diagonalized:

$$[\bar{A}_n^{(+)}, A_m^j] = 0 \qquad (4.4)$$

and only the c number for $\bar{A}_n^{(+)}$ is changed to $2m^3 - (D-2)m^3/12$.

B. Linear Independence and $p \to \infty$ Limit

To prove that we have constructed *all* the $T^{D-1}(N)$ physical states $|\Psi, N\rangle$, we must show they are linearly independent. This is most easily done by introducing the auxiliary operator[7] Φ_n (or F_n),

$$\Phi_n = \frac{1}{2\pi i}\oint \frac{dz}{z} e^{inQ_-} \tag{4.5}$$

and proving that A_n^i, $A_n^{(+)}$, and Φ_n generate all independent states (physical *and* spurious). Since they close algebraically,

$$[\Phi_n, A_m^j] = 0,$$
$$[A_n^{(+)}, \Phi_m] = m\Phi_{n+m}, \tag{4.6}$$

the $T^D(N)$ *ordered* products,

$$\prod_s (\Phi_{-s})^{\lambda_s} \prod_{i,s} (A_{-r}^i)^{\lambda_r^i} (A_{-N}^{(+)})^{\lambda_N^0} \cdots (A_{-1}^{(+)})^{\lambda_1^0} |0,p\rangle ,$$

should give all the on-shell states.

We consider the boost along the z axis ($Q_- \to \eta Q_-$, $P_+ \to \eta^{-1}P_+$, $\vec{P} \to \vec{P}$) on the generators. We have

$$\eta A_{-n}^{(+)} \to i\left(\frac{n}{2}\right)^{1/2} (a_n^{3\dagger} + a_n^{0\dagger}),$$
$$A_{-n}^j \to i\sqrt{n}\, a_n^{j\dagger}, \quad \eta^{-1}\Phi_{-n} \to i\left(\frac{n}{2}\right)^{1/2} (a_n^{3\dagger} - a_n^{0\dagger}), \tag{4.7}$$

to first order in $\eta = e^{-y}$ [Boost = $\exp(iyK_3)$]. Hence we can expand any state

$$\prod_n (a_n^{\mu\dagger})^{\lambda_n} |0,p\rangle$$

in terms of products of A_{-n} and Φ_{-n}. The procedure is to replace each $a_n^\dagger$ by the corresponding A_{-n} (or Φ_{-n}), then expand to find the $O(\eta)$ term, and repeat substitution to identify the $O(\eta^2)$ and so on. Hence (A, Φ) generate all $T^D(N)$ states, and there are no linear dependences. So the subset of $T^{D-1}(N)$ *ordered* products of A_n have no linear dependences.

C. No-Ghost Theorem

The inner product between physical states $|\Psi, N\rangle = \prod_n (A_{-n})^{\lambda_n}|0,p\rangle$ or the metric tensor is calculated entirely from the algebra, by moving the A_n right (or $A_n^\dagger$ left) and the eigenvalue equation

$$A_0^{(+)}|0,p\rangle = |0,p\rangle . \tag{4.8}$$

Consequently the calculation can be done with an isomorphic representation.

The isomorphism (suggested by the $p \to \infty$ limit) is

$$A_n^i \to \sqrt{n}\, a_n^i, \quad A_{-n}^i \to \sqrt{n}\, a_n^{i\dagger},$$
$$A_n^{(+)} \to \mathfrak{L}_n, \quad |0,p\rangle \to |0\rangle , \tag{4.9}$$

where $\mathfrak{L}_n$ is constructed from 24 spatial (positive-metric) oscillators and $i = 1, \ldots, D-2 \leq 24$. The algebra for $\mathfrak{L}_n$ requires only spatial oscillators because the eigenvalue relation is $\mathfrak{L}_0|0\rangle = 0$. Consequently, we can choose to drop the linear term $(p\cdot a_n)$ for $\mathfrak{L}_n(p^i = 0)$ and use spatial oscillators for the quadratic terms.

As a consequence of this representation of the metric tensor in a positive-definite space, all eigenvalues (and norms) are either positive or zero. The zero eigenvalues are due to linear dependences in the a_n^i, $\mathfrak{L}_n$ basis and null states in the original A_n basis.[21]

The null states postulated by Brower and Thorn are given by any state with $A_{-1}^{(+)}$ adjacent to $|0,p\rangle$, since $L_1^\dagger \prod (A_{-n})^{\lambda_n}|0,p\rangle = \prod (A_{-n})^{\lambda_n} L_1^\dagger |0,p\rangle$ and

$$L_1^\dagger e^{i\sqrt{2}\,k\cdot q_0}|0,p\rangle = A_{-1}^{(+)}|0,p\rangle \to \mathfrak{L}_{-1}|0\rangle = 0 . \tag{4.10}$$

There are no other null states for $D < 25$, because the other states in the isomorphic representation $(a_n^i, \mathfrak{L}_1)$ are easily shown to be linearly independent. The argument is too close to that given in the appendix of Ref. 5 to necessitate repetition.

The bizarre feature noted by Thorn was that a ghost state appeared if the spatial dimensions were increased to $D-1 > 25$. This is now understood as a consequence of the c numbers involved. For $D = 26$ there are new null states due to the dependence of $\mathfrak{L}_n$ on a_n^i in accord with new null states found on the second daughter.[5]

V. CONCLUSIONS

The models with $D > 4$ are interesting,[10] since one can assign to each particle an extended momentum $k_i = (k_i^0, k_i^1, k_i^2, k_i^3, C_i^1, \ldots, C_i^{D-4})$, where the C's are new conserved quantum numbers and the trajectory in the ij channel $(k_i + \cdots + k_j)^2$ has intercept

$$\alpha_{ij}(0) = 1 - \sum_{l=1}^{D-4} (C_i^l + C_{i+1}^l + \cdots + C_j^l)^2 . \tag{5.1}$$

Only the channel with zero ("vacuum") quantum numbers has $\alpha_0 = 1$. Hence there is a large class of unequal intercept models which are free of ghosts.

For $D \leq 25$, the physical states $|\{\lambda_r^i\}, N\rangle$ [defined in Eq. (3.19)] have positive norm, except the null states with powers of $A_{-1}^{(+)}$ ($\lambda_1^0 \neq 0$). At a pole the null states also decouple[15] so there are actually $T^{D-1}(N) - T^{D-1}(N-1)$ states.[5] For $D = 26$ all states with $\lambda_k^0 \neq 0$ are dependent (in the $\overline{A}_{-n}^{(+)}$ basis these *are* the null states), so there are $T^{24}(N)$ states contributing to a pole.[22]

As pointed out by Lovelace[23] for dimensionality $D = 26$, the new singularity in the second-order diagram can be a factorizable pole, if one assumes the removal of *two* dimensions of states by the gauges. Here, we see that the new null states (not new gauges) give precisely the required effect to make the Lovelace Pomeranchukon into a pole $[\alpha_P(s) = \frac{1}{2}\alpha' s + 2]$.[24]

It is interesting that there is a maximum density[25] of states $\rho(m)$ consistent with a positive-norm space, and that the loop theory is most elegant at saturation ($D=26$). Above $D=26$, renormalization is impossible.

These techniques are being extended to the Neveu-Schwarz model[8] where the leading trajectory has no tachyon and saturation occurs at $D=10$. The $D=5$ model of Halpern and Thorn[26] should have no ghosts, no tachyons, and no zero masses if the trajectory at $\alpha_0=1$ is given positive signature (like the Pomeranchukon).[27] Any example of a zero-width resonance-saturation scheme with positive masses and norms is an interesting achievement.

Work is now proceeding to reformulate dual models entirely on the physical states (in the light-cone gauge). Although one may lose manifest covariance,[28] the new formulation may help us to understand the role (if any) of a parton concept in the dual theory. Here is the first opportunity to study the interplay between covariance and positivity for a resonance spectrum consistent with the multi-Regge limits.

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[1]S. Fubini, in *Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1967,* edited by A. Perlmutter and B. Kurşunoğlu (Freeman, San Francisco, Calif., 1967).

[2]S. Fubini and G. Veneziano, Nuovo Cimento 64A, 811 (1969); K. Bardakci and S. Mandelstam, Phys. Rev. 184, 1640 (1969); S. Fubini, D. Gordon, and G. Veneziano, Phys. Letters 29B, 679 (1969).

[3]M. A. Virasoro, Phys. Rev. D 1, 2933 (1970).

[4]References 5, 6, and 7 contain *essential* ingredients leading to the present proof, as should be abundantly clear in Sec. III.

[5]R. C. Brower and C. B. Thorn, Nucl. Phys. B31, 163 (1971).

[6]E. Del Giudice, P. Di Vecchia, and S. Fubini (denoted DDF), Ann. Phys. (N.Y.) 70, 378 (1972).

[7]R. C. Brower and P. Goddard, Nucl. Phys. B40, 437 (1972).

[8]A forthcoming paper with K. Friedman extends the present results to the model of A. Neveu and J. H. Schwarz, Nucl. Phys. B31, 86 (1971). For a review of conformally invariant models see J.-L. Gervais and B. Sakita, paper presented at International Symposium on Functional Methods in Field Theory and Statistics, P. N. Lebedev Physical Institute, 1971 (unpublished).

[9]We use the variables of Z. Koba and H. B. Nielsen, Nucl. Phys. B12, 517 (1969), that exhibit directly the projective invariance, $z_i \to (\alpha z_i+\beta)/(\gamma z_i+\delta)$ of the integrand. The invariant differential volume,

$$d^3\omega = dz_a dz_b dz_c/[(z_a-z_b)(z_b-z_c)(z_c-z_a)],$$

allows any three z_i's to be set to constants.

[10]S. Fubini and G. Veneziano, Nuovo Cimento 67, 29 (1970); Ann. Phys. (N.Y.) 63, 12 (1971). The notation follows closely these papers and Ref. 6 (DDF).

[11]The intuitive picture is similar to that suggested by L. Susskind, Phys. Rev. D 1, 1182 (1970), and Y. Nambu, University of Chicago report, 1969 (unpublished); although for complex z, they utilize the field $X^\mu(\theta,\tau) = \frac{1}{2}Q^\mu(z)+\frac{1}{2}Q^\mu(z^*)$. Because $Q^\mu(z)$ is not the general solution to $(\partial_\theta{}^2+\partial_\tau{}^2)Q=0$, the commutator $[Q^\mu(\theta,\tau),Q^\nu(\theta',\tau)] = 2\pi i g^{\mu\nu}\epsilon(\theta'-\theta)$ is not zero. Our analysis is easily extended to the Shapiro-Virasoro model [J. Shapiro, Phys. Letters 34B, 79 (1971)], which uses the full set of modes $[a_n z^n$ and $b_n^\dagger(z^*)^n]$, but our restricted solution yields a more reasonable spectrum.

[12]This is not a second-quantized theory. The "locality" and apparent field-theoretical properties reside in the auxiliary *two*-dimensional space (θ,τ), with no simple connection to physical space-time (x_i^μ for particle i) (see Gervais and Sakita, Ref. 8).

[13]To establish this equivalence, one simply uses the commutator

$$[\hat{Q}^{\mu\dagger}(z_1),\hat{Q}^\nu(1/z_2)] = g^{\mu\nu}\ln(1-z_1/z_2)$$

for the positive $[\hat{Q}^\dagger(z)]$ and negative $[\hat{Q}(1/z)]$ powers of z, to move the $a_n^\dagger$'s left, and energy conservation on the factor $\exp[i\sqrt{2}\,(\sum_j p_j)\cdot q_0]=1$.

[14]Because of the normal ordering, the vertex $V(z,p) = :\exp[i\sqrt{2}\,p\cdot Q(z)]:$ is a conformal spin-p^2 object,

$$[L_l, V(z,p)] = z^{+l}\left(z\frac{d}{dz}+lp^2\right)V(z,p)$$

as demonstrated in Ref. 10.

[15]E. Del Giudice and P. Di Vecchia, Nuovo Cimento 70A, 579 (1970).

[16]Essentially the observation of L. N. Chang and F. Mansouri, Phys. Rev. D 5, 2535 (1972). As one can see from the Hermiticity condition $Q^\dagger(\theta,\tau)=Q(\theta,-\tau)$, τ has been Wick rotated; hence the sign in Eq. (2.15) is wrong reflecting the noncausality of the tachyon.

[19]The condition $\sqrt{2}\,k\cdot P_0=-1$ removes the logarithmic branch point from the integrand.

[20]Actually $2\sum(z')^{-l}L_l = :P^2(z'):$ but this normal ordering can be dropped inside the commutator of Eq. (3.15).

[21]F. P. Gantmacher, *Matrix Theory* (Chelsea, New York, 1959), Vol. 1.

[22]Of course given the basis for the physical states at $p^2=1-N$ $(|\{\lambda_r^i\},N\rangle)$ and the metric tensor ($M_{\lambda\lambda'}$ excluding null states), the physical projection operator is

$$\mathcal{P}_N = \sum_{\lambda\lambda'} |\{\lambda\},N\rangle (M^{-1})_{\lambda\lambda'} \langle\{\lambda'\},N| .$$

In the $(A_n, \overline{A}_n^{(+)})$ basis M is diagonal except in a one-dimensional (longitudinal) space. Curiously, for $D=26$, the A_n^i of DDF are sufficient, and they give a *completely* diagonalized metric tensor. The loop correction factor

$(1-x)\prod_n(1-x^n)$ in Ref. 5, becomes $[\prod_n(1-x^n)]^2$ for $D=26$.

[23]C. Lovelace, Phys. Letters 34B, 500 (1971).

[24]Independently, C. Thorn has come to the same conclusions regarding the special properties of the $D=26$ model (private communications).

[25]The asymptotic density $\rho(m)\to Am^{-B}e^{\beta_0 m}$ given by K. Huang and S. Weinberg, Phys. Rev. Letters 25, 895 (1970), in the ghost-free model has $B=\frac{1}{2}(D+2)$, $\beta_0=2\pi[(D-1)/6]^{1/2}$ for $D<26$ and $B=\frac{1}{2}(D-1)$, $\beta_0=2\pi[(D-2)/6]^{1/2}$ for $D=26$.

[26]M. B. Halpern and C. B. Thorn, Phys. Letters 35B, 441 (1971).

[27]Individual low-point functions of phenomenological interest may have no ghost even with $\alpha_0<1$. DDF (Ref. 6) prove the beta function has no ghosts for $\alpha_0=1$ by factorizing with spatial oscillators in the frame $k_2=(0,0,0,\sqrt{-m^2})$. Actually for $m^2\le 0$, the proof is also valid since $(1-x)^{\alpha_0-1}$ has a positive power series. Almost the same argument proves the Lovelace-Shapiro four-pion amplitude has no ghosts for $m_\pi{}^2=0$.

[28]To establish covariance, one should enlarge the generating algebra (A_n) to include the Lorentz generators

$$J^{\mu\nu}=\langle P^\mu Q^\nu - Q^\mu P^\nu\rangle\,,$$

where

$$U(\Lambda)=\exp[i\alpha^{\mu\nu}(\Lambda)J_{\mu\nu}]\,,$$

$$U(\Lambda)V(z,p)U^\dagger(\Lambda)=V(z,\Lambda p)\,.$$

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Off-Shell Extension of the Partial-Wave Transition Amplitude*

Bengt R. Karlsson†
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
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The Marchenko approach to the inverse problem of scattering theory is transformed into a procedure for the calculation of the (half-) off-shell partial-wave transition amplitude from the on-shell amplitude and certain bound-state parameters.

I. INTRODUCTION

In many theories for multiparticle systems, like the Faddeev equations for three-particle systems, the input information is not the explicit interparticle potentials but rather the two-particle transition amplitudes $t(\vec{p}\,',\vec{p};E+i\epsilon)$. However, two-particle scattering experiments provide direct information only on the on-the-energy-shell part of these amplitudes, corresponding to $|\vec{p}\,'|=|\vec{p}|=k$, where $k^2/2\mu=E$ is the energy; while in the multiparticle theories the scattering amplitude is in general also needed for off-shell values of the momenta, and for negative energies.

The methods to extract information on the off-shell parts of t from on-shell parts in one way or another exploit the assumption that t corresponds to a unitary S matrix or, more precisely, that the solutions to the Schrödinger equation corresponding to different energies form a complete set. The best-known consequence of this so-called unitarity condition on t is of course that the on-shell transition amplitude itself in every partial wave can be parametrized in terms of a *real* function of energy, the phase shift $\delta_l(k)$,

$$\mathrm{Im}\,t_l(k,k;E+i\epsilon)=-\frac{1}{\pi\mu k}\sin\delta_l(k)e^{i\delta_l(k)}\,. \qquad (1.1)$$

Other well-known consequences,[1] also applying to partial-wave amplitudes, are that the amplitude can be expressed in closed form in terms of half-off-shell amplitudes,[2-4] e.g., for the imaginary part of t_l,

$$\mathrm{Im}\,t_l(p',p;E+i\epsilon)=-\pi\mu k t_l(p',k;E+i\epsilon)t_l^*(k,p;E+i\epsilon) \qquad (1.2)$$

and, moreover, that only the modulus of the half-off-shell amplitude depends on the off-shell momentum, leaving it with the same phase factor as the corresponding on-shell amplitude,

$$t_l(p,k;E+i\epsilon)=f_l(p,k)t_l(k,k;E+i\epsilon)\,, \qquad (1.3)$$

where the half-off-shell factor $f_l(p,k)$ is real. Thus, in every partial wave, the completely off-shell amplitude can be parametrized in terms of the two real functions $\delta_l(k)$ and $f_l(p,k)$.

The remaining general restrictions on t due to the unitarity condition[3] are less transparent, since they are expressed in the form of an integral relation which is quadratic in the half-off-shell amplitude. However, in the important special case when the transition amplitude corresponds to an interaction potential which is *diagonal* in configuration space (a condition that excludes, for instance, separable interactions) the situation is considerably simplified. From the solutions to the inverse problem of scattering theory,[5] it is known that in