Coupled Form Factors in Two-Channel Problems*

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The problem of coupled (S-wave) form factors is formulated in the two-body two-channel spinless case. Iterative and exact numerical solutions of coupled integral equations are presented with model parametrizations of the T matrix. The problem of diagonalization is discussed and the difficulties are highlighted. A method of solution for the weak coupling to first order in T_{12} is presented.

I. INTRODUCTION

The complete solution of the problem of form factors necessarily involves a set of coupled integral equations in the form factors. A great deal of literature exists on the study of the form factors in the elastic unitarity limit which involves essentially a one-channel problem. The problem of the inclusion of inelasticity *but still using only the elastic intermediate states* in the spectral function for the form factor is also a standard one and a good deal of published literature exists in this field.[1] The problem of including the coupling of form factors which will necessarily occur if inelasticity is allowed, however, has not been studied in a satisfactory way to the best of our knowledge. The literature in this field is scanty and somewhat incomplete.[2,3] In this paper we have studied the problem of inclusion of inelasticity and also that of coupling of the form factors in the context of a coupled two-channel problem. We assume that in the first (elastic) channel we have two scalar particles of mass m each and that in the second (inelastic) channel we have two scalar particles of mass M ($>m$) each. We also restrict our discussion to that of scalar form factors. There are two such form factors which we call $J_1(s)$ and $J_2(s)$, $\sqrt{s}$ being the total center-of-mass energy.

Most of the discussion in the literature on form factors relies on the use of the lowest-mass states for the saturation of unitarity relations. In many cases, like the form factors entering the K_{l3} problem, this amounts to using the elastic unitarity. This corresponds to the use of elastic amplitudes in the big blob of the unitarity diagram of Fig. 1. The solution is then the familiar Omnes-Muskhelishvili representation[1,4]

$$J_1(s)=J_1(0)\exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{\delta(s')ds'}{s'(s'-s)}\right)\ ,\qquad (1.1)$$

where δ is the scattering phase shift.

The next refinement one may introduce is to keep only channel-1 intermediate states in Fig. 1 but introduce the scattering amplitude T_{11} in the big blob of Fig. 1 which takes into account the absorption. The phase $\delta(s)$ is then replaced by a phase $\phi(s)$,[1]

$$\tan\phi(s)=\frac{\eta(s)\sin 2\delta(s)}{1+\eta(s)\cos 2\delta(s)}\ ,\qquad (1.2)$$

where

$$T_{11}(s)=\frac{\eta e^{2i\delta(s)}-1}{2ik}\qquad (1.3)$$

with k the magnitude of the center-of-mass three-momentum. The solution is then given by Eq. (1.1) with a replacement $\delta(s)\to\phi(s)$.

The use of full unitarity forces one to include the unitarity diagrams of Fig. 2 which involve the form factor $J_2(s)$ with the off-diagonal T-matrix element, T_{12}, connecting it to channel 1. The use of full unitarity thus necessarily yields a set of coupled equations in the form factors. It is this problem that we address ourselves to in this paper.

In Sec. II we have set up the basic formalism that leads to an iteration scheme. This iteration procedure, though in principle possible, is numerically rather involved. In Sec. III the formalism developed in Sec. II is used to calculate the form factor in the vicinity of $s=0$ in the first iteration. In Sec. IV we present a calculation of coupled form factors using a scattering-length description for the scattering amplitudes. Various approximations are discussed for cases with and without a resonance in the elastic region. In Sec. V we discuss the problem of diagonalization highlighting the difficulties one encounters. We also solve the problem for weak interchannel couplings in a particular model.

II. FORMALISM

The two form factors are defined via

$$\langle \vec{k}, \vec{l}, i|j(0)|0\rangle=\frac{1}{(4\omega_k\omega_l)^{1/2}}J_i^*(s),\qquad (2.1)$$

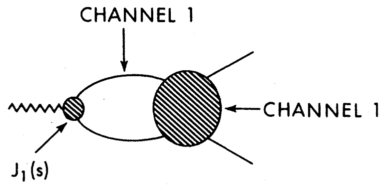

FIG. 1. Elastic (first-channel) diagram. Small blob: form-factor vertex; big blob: T matrix (scattering vertex).

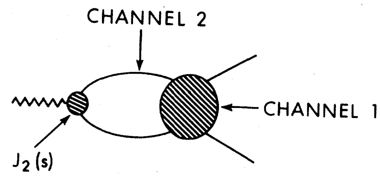

FIG. 2. Inelastic (second-channel) diagram.

where $j(0)$ is a local scalar operator and ω_k and ω_1 are the on-shell energies of the two particles. The channel label is $i\,(=1, 2)$. The 2×2 T matrix has elements T_{ij} $(i, j=1, 2)$. The normalization of our T matrix is such that for a purely elastic scattering,

$$T=k^{-1}e^{i\delta}\sin\delta\,. \tag{2.2}$$

The imaginary part of the form factors can be worked out through standard procedures.[1] The imaginary part of $J_1(s)$ arising from the unitarity diagram of Fig. 1 is

$$\sigma_{11}(s)\doteq \mathrm{Re}[k_1T_{11}J_1^*(s)]\theta(k_1)\,, \tag{2.3}$$

where

$$4k_1^2=s-4m^2\,.$$

The imaginary part of $J_1(s)$ coming from the diagram of Fig. 2 is

$$\sigma_{12}(s)=\mathrm{Re}[k_2T_{12}J_2^*(s)]\theta(k_2)\,, \tag{2.4}$$

where

$$4k_2^2=s-4M^2\,.$$

In a completely analogous way the imaginary part of $J_2(s)$ has contributions

$$\sigma_{21}(s)=\mathrm{Re}[k_1T_{21}(s)J_1^*(s)]\theta(k_1) \tag{2.5}$$

and

$$\sigma_{22}(s)=\mathrm{Re}[k_2T_{22}(s)J_2^*(s)]\theta(k_2)\,. \tag{2.6}$$

Define a matrix $\underline{f}$,

$$\underline{f}=\begin{pmatrix} k_1T_{11}\theta(k_1) & k_2T_{12}\theta(k_2)\\ k_1T_{21}\theta(k_1) & k_2T_{22}\theta(k_2)\end{pmatrix}\,. \tag{2.7}$$

Then the imaginary parts of $J_1(s)$ and $J_2(s)$ can be written as

$$\mathrm{Im}J_1(s)=\mathrm{Re}f_{11}\,\mathrm{Re}J_1+\mathrm{Im}f_{11}\,\mathrm{Im}J_1 +\mathrm{Re}f_{12}\,\mathrm{Re}J_2+\mathrm{Im}f_{12}\,\mathrm{Im}J_2 \tag{2.8}$$

and

$$\mathrm{Im}J_2(s)=\mathrm{Re}f_{21}\,\mathrm{Re}J_1+\mathrm{Im}f_{21}\,\mathrm{Im}J_1 +\mathrm{Re}f_{22}\,\mathrm{Re}J_2+\mathrm{Im}f_{22}\,\mathrm{Im}J_2\,. \tag{2.9}$$

These two equations can be combined in a matrix equation,

$$\mathrm{Im}\underline{J}=\mathrm{Re}\underline{f}\;\mathrm{Re}\underline{J}+\mathrm{Im}\underline{f}\;\mathrm{Im}\underline{J}\,, \tag{2.10}$$

with a solution

$$\mathrm{Im}\underline{J}=(1-\mathrm{Im}\underline{f})^{-1}\,\mathrm{Re}\underline{f}\;\mathrm{Re}\underline{J}\equiv \underline{F}\,\mathrm{Re}\underline{J}\,. \tag{2.11}$$

In particular,

$$\mathrm{Im}J_1(s)=H(s)\,\mathrm{Re}J_1(s)+h(s)\,\mathrm{Re}J_2(s)\,, \tag{2.12}$$

where

$$H(s)=\frac{\mathrm{Re}f_{11}(1-\mathrm{Im}f_{22})+\mathrm{Re}f_{21}\,\mathrm{Im}f_{12}}{X} \tag{2.13}$$

and

$$h(s)=\frac{\mathrm{Re}f_{12}(1-\mathrm{Im}f_{22})+\mathrm{Re}f_{22}\,\mathrm{Im}f_{12}}{X}\,, \tag{2.14}$$

with

$$X=(1-\mathrm{Im}f_{11})(1-\mathrm{Im}f_{22})-\mathrm{Im}f_{21}\,\mathrm{Im}f_{12}\,. \tag{2.15}$$

The once-subtracted form of the dispersion relation for $J_1(s)$ is

$$\begin{aligned} J_1(s)&=J_1(0)+\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{\mathrm{Im}J_1(s')ds'}{s'(s'-s-i\epsilon)}\\ &=J_1(0)+\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{H(s')\,\mathrm{Re}J_1(s')}{s'(s'-s-i\epsilon)}ds'\\ &\quad+\frac{s}{\pi}\int_{4M^2}^{\infty}\frac{h(s')\,\mathrm{Re}J_2(s')}{s'(s'-s-i\epsilon)}ds'\,. \end{aligned} \tag{2.16}$$

Let us write

$$f(s)=\frac{s}{\pi}\int_{4M^2}^{\infty}\frac{h(s')\,\mathrm{Re}J_2(s')}{s'(s'-s-i\epsilon)}ds'\,. \tag{2.17}$$

We have shown in the Appendix that Eq. (2.16) has a solution (all symbols are defined in the Appendix)

$$J_1(s_+)=f(s)+\left(J_1(0)+\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{e^{-\rho(s')}\sin\phi_1(s')\,\mathrm{Re}f(s')}{s'(s'-s-i\epsilon)}ds'\right)\exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{\phi_1(s')ds'}{s'(s'-s-i\epsilon)}\right)\,. \tag{2.18}$$

If we had a complete knowledge of $J_2(s)$ and the T matrix, then Eq. (2.18) would determine $J_1(s)$. Unfortunately, $J_2(s)$ is not known but is given by an equation analogous to Eq. (2.18) and in turn demands a knowledge of $J_1(s)$. In principle, we can thus set up an iteration scheme as follows: One solves the decoupled problem for $J_2(s)$ by approximating Eq. (2.11) as

$$\operatorname{Im} J_2^{(0)} \approx F_{22}^{(0)} \operatorname{Re} J_2^{(0)}(s), \tag{2.19}$$

with

$$F_{22}^{(0)} = \frac{\operatorname{Re} f_{22}}{1 - \operatorname{Im} f_{22}} .$$

The decoupled $J_2^{(0)}(s)$ is then given by the Omnes-Muskhelishvili function with phase $\phi_2^{(0)} = \tan^{-1} F_{22}^{(0)}(s)$. In this approximation we include the interchannel coupling in the T matrix (essentially inelasticity). With $J_2^{(0)}(s)$ we can next determine the lowest-order approximation to $f(s)$ via Eq. (2.17) and finally get the approximate $J_1(s)$ through Eq. (2.18). This $J_1(s)$ can in turn be inserted in the formula [analogous to Eq. (2.18)] for $J_2(s)$ to yield a better approximation to $J_2(s)$ and restart the whole cycle of iteration again. This procedure is in practice very complex as one deals with multiple integrals (all principal values) in Eq. (2.18). Fortunately for many problems the range of energy over which the form factor is required is limited enough that a linear approximation to the form factor is considered an adequate approximation (an example is the form factor in the K_{l3} problem). The parameter of interest in this case is the slope of $J_1(s)$ at $s=0$. From Eq. (2.18) we get

$$J_1{}'(0) = \frac{J_1(0)}{\pi} \int_{4m^2}^{\infty} \frac{\phi_1(s)\,ds}{s^2} + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{e^{-\rho(s)} \sin\phi_1(s) \operatorname{Re} f(s)}{s^2}\, ds + \frac{1}{\pi} \int_{4M^2}^{\infty} \frac{h(s) \operatorname{Re} J_2(s)}{s^2}\, ds . \tag{2.20}$$

To order f_{12} the formulas given by Eqs. (2.13) and (2.14) take the form

$$H(s) = \tan\phi_1(s) \approx \frac{\operatorname{Re} f_{11}}{1 - \operatorname{Im} f_{11}} , \tag{2.21}$$

$$h(s) \approx \frac{\operatorname{Re} f_{12} + \tan\phi_2^{(0)} \operatorname{Im} f_{12}}{1 - \operatorname{Im} f_{11}} , \tag{2.22}$$

with

$$\tan\phi_2^{(0)}(s) = \frac{\operatorname{Re} f_{22}}{1 - \operatorname{Im} f_{22}} . \tag{2.23}$$

We then see that the first term on the right-hand side of Eq. (2.20) would be the only term present if only Fig. 1 were to be considered in writing the unitarity condition. It therefore includes the effects of inelasticity in the T matrix. The next two terms arise due to the coupling of the form factors, i.e., from the inhomogeneous term in the unitarity equation [Eq. (2.12)] which is diagrammatically represented by Fig. 2. In the next two sections we present model calculations where the formalism developed in this section is applied.

The method outlined in the foregoing would fail if $\underline{F}$ of Eq. (2.11) does not exist. This would happen if X of Eq. (2.15) were to be identically zero for a range of energies.

We should add here a comment on the nonuniqueness of these solutions. It is well known that the solution of the one-channel Omnes-Muskhelishvili equation is only determined up to a holomorphic function.[1] This ambiguity, of course, remains in the coupled-channel case. It can be overcome by specifying the asymptotic behavior of the form factors at infinity. This can easily be done for the perturbation solutions by choosing as the zeroth-order approximation the Omnes solution [of the uncoupled form factors $J_1^{(0)}(s)$ and $J_2^{(0)}(s)$] which has the fastest possible decay at infinity compatible with the phase shifts. This then makes the iterative solution unique. A thorough discussion of the nonuniqueness and asymptotic behavior of the solution of singular integral equations of the Omnes-Muskhelishvili type and the connection with the rescattering parameters (phase shifts) has been given by Resnick.[5]

III. A MODEL CALCULATION

We assume the following parametrization for the S matrix[6]:

$$S_{11} = e^{2i\delta_1}, \quad s < 4M^2 \tag{3.1}$$

$$\underline{S} = \begin{pmatrix} \eta e^{2i\delta_1} & i\nu \\ i\nu & \eta e^{2i\delta_2} \end{pmatrix}, \quad s > 4M^2 \tag{3.2}$$

where

$$\nu = (1-\eta^2)^{1/2} e^{i(\delta_1+\delta_2)} .$$

The T matrix is obtained from Eq. (3.2) through

$$S_{ij} = 2ik_i{}^{1/2} T_{ij} k_j{}^{1/2} + \delta_{ij} , \tag{3.3}$$

$$\underline{T} = \frac{1}{2i} \begin{pmatrix} \dfrac{\eta e^{2i\delta_1} - 1}{k_1} & \dfrac{i\nu}{(k_1k_2)^{1/2}} \\ \dfrac{i\nu}{(k_1k_2)^{1/2}} & \dfrac{\eta e^{2i\delta_2} - 1}{k_2} \end{pmatrix} . \tag{3.4}$$

The $\underline{f}$ matrix is then [Eq. (2.7)]

$$\underline{f} = \frac{1}{2i} \begin{pmatrix} \eta e^{2i\delta_1} - 1 & (k_2/k_1)^{1/2} i\nu \\ (k_1/k_2)^{1/2} i\nu & \eta e^{2i\delta_2} - 1 \end{pmatrix} . \tag{3.5}$$

In this model we find (to order f_{12}) from Eqs. (2.21)–(2.23)

$$\phi_1(s)=\delta_1(s),\quad 4m^2<s<4M^2$$

$$=\tan^{-1}\frac{\eta(s)\sin 2\delta_1(s)}{1+\eta(s)\cos 2\delta_1(s)},\quad s>4M^2 \qquad (3.6)$$

$$\phi_2(s)=\tan^{-1}\frac{\eta(s)\sin 2\delta_2(s)}{1+\eta(s)\cos 2\delta_2(s)},\quad s>4M^2 \qquad (3.7)$$

$$h(s)=\frac{\mathrm{Re} f_{12}+\tan\phi_2(s)\,\mathrm{Im} f_{12}}{1-\mathrm{Im} f_{11}}, \qquad (3.8)$$

$$\mathrm{Re} f_{12}=\left(\frac{k_2}{k_1}\right)^{1/2}\tfrac{1}{2}(1-\eta^2)^{1/2}\cos(\delta_1+\delta_2),\quad s>4M^2 \qquad (3.9)$$

$$\mathrm{Im} f_{12}=\left(\frac{k_2}{k_1}\right)^{1/2}\tfrac{1}{2}(1-\eta^2)^{1/2}\sin(\delta_1+\delta_2),\quad s>4M^2\,. \qquad (3.10)$$

The parametrizations we used were

$$\delta_1(s)=\tan^{-1}\left(\frac{a_1 m(s-4m^2)^{1/2}}{s_1-s}\right),\quad s>4m^2 \qquad (3.11)$$

$$\delta_2(s)=\tan^{-1}\left(\frac{a_2 M(s-4M^2)^{1/2}}{s_2+s}\right),\quad s>4M^2 \qquad (3.12)$$

$$\eta(s)=1-40\frac{m^2}{s}\,\frac{s-4M^2}{s-4m^2},\quad s>4M^2 \qquad (3.13)$$

with

$m=0.14$ GeV, $M=0.5$ GeV,

$a_1=10$, $a_2=0.5$,

$s_1=1.5\ \mathrm{GeV}^2$, $s_2=1.0\ \mathrm{GeV}^2$.

With this set we get the following parametrizations for $J_1(s)$ near $s=0$.

Elastic approximation ($\eta=1$):

$$J_1(s)=1+1.85s\,. \qquad (3.14)$$

[We normalize $J_1(0)=1$, s to be expressed in GeV^2.]

Inelastic approximation *but no coupling to* $J_2(s)$; i.e., the first term on the right-hand side of Eq. (2.20):

$$J_1(s)=1+1.20s. \qquad (3.15)$$

Inelastic approximation *and coupling to* $J_2(s)$; i.e., use of Eq. (2.20) up to order f_{12} with approximations (2.21) and (2.22):

$$J_1(s)=1+1.14s. \qquad (3.16)$$

The importance of these results is that by switching on the inelasticity, yet using only the elastic intermediate states in the unitarity relation for the form factors one can decrease the slope over that for the purely elastic case [Eqs. (3.15) and (3.14)]. By coupling the form factor of the second channel, i.e., using the second-channel intermediate states in the unitarity relation for the form factors one can decrease the slope yet further [Eq. (3.16)]. This is only a model calculation. There will certainly be models where the trend is reversed, i.e., the slope is made to increase rather than decrease. The results of this section are relevant to the problem of K_{l3} scalar form factors.[7] One word of caution: In this model, X^{-1} has a pole at $s=1.488$, which seems to make the third integral in Eq. (2.20) not defined. However, one easily verifies that this pole is canceled by a zero in the numerator. Using the (uncoupled) Omnes solution for $J_2(s)$ on the right-hand side of Eq. (2.20), one finds

$$J_1{}'(0)=\frac{J_1(0)}{\pi}\int_{4m^2}^{\infty}\frac{\phi_1(s)ds}{s^2}+\frac{J_2(0)}{\pi}\int_{4m^2}^{\infty}\frac{e^{-\rho_1(s)}\sin\phi_1(s)}{s}\,ds\,\frac{P}{\pi}\int_{4M^2}^{\infty}\frac{h(s')e^{\rho_2(s')}\cos\phi_2(s')}{s'(s'-s)}ds'$$
$$+\frac{J_2(0)}{\pi}\int_{4M^2}^{\infty}\frac{e^{\rho_2(s)}h(s)\cos\phi_2(s)}{s^2}ds\,,$$

with

$$\rho_1(s)=\frac{s}{\pi}P\int_{4m^2}^{\infty}\frac{\phi_1(s')ds'}{s'(s'-s)}\,,$$

and similarly for $\rho_2(s)$.

It can now be shown that

$$h(s)\cos\phi_2(s)=\left(\frac{k_2}{k_1}\right)^{1/2}\left(\frac{1-\eta}{1+\eta}\right)^{1/2}\frac{\cos\psi_1}{\cos\delta_1}\,,$$

with

$$\frac{\cos\psi_1}{\cos\delta_1}=\left(\frac{1+\tan^2\delta_1}{1+[(1-\eta)/(1+\eta)]^2\tan^2\delta_1}\right)^{1/2}\,.$$

This quantity is certainly finite in the whole range of integration, showing a cancellation of the (spurious) pole in X^{-1}.

IV. A MODEL: SCATTERING-LENGTH APPROXIMATION

In this section we calculate the coupled two-channel form factors exactly and in various approximations using the scattering-length description for the coupled T matrix as described by Ross and Shaw.[8] For the f matrix of Sec. II we have

$$f_{11}=k_1(M_{22}-ik_2)D^{-1},$$
$$f_{22}=k_2(M_{11}-ik_1)D^{-1}, \tag{4.1}$$
$$f_{12}=-k_2M_{12}D^{-1}=\frac{k_2}{k_1}f_{21},$$

where

$$D=(M_{11}-ik_1)(M_{22}-ik_2)-M_{12}{}^2,$$
$$M_{11}=qk_1(Bc-A),$$
$$M_{12}{}^2=M_{21}{}^2=q^2k_1B(1+c^2), \tag{4.2}$$
$$M_{22}=cq,$$
$$q=-(Ac+B)^{-1}$$

with $k_1{}^2=k_2{}^2+k_0{}^2$. This is a scattering-length approximation around the inelastic threshold $k_0{}^2$. To have a zero elastic phase shift at the elastic threshold as well, we impose the condition

$$k_0=-cq. \tag{4.3}$$

We can decouple the problem by choosing $B=0$ and $c=0$ which makes $f_{11}=T_{11}=0$ and leaves the one-channel scattering-length approximation for f_{22}.

For this model we can solve the coupled integral equations for the form factors in the case where there is no resonance in the elastic region. With Eq. (2.11) for $\operatorname{Im}J_i(s)$, we get two coupled (once-subtracted) integral equations for $\operatorname{Re}J_1(s)$:

$$\begin{aligned}\operatorname{Re}J_1(s)&=\operatorname{Re}J_1(0)+\frac{s}{\pi}P\int_{4m^2}^{\infty}\frac{ds'}{s'(s'-s)}\operatorname{Im}J_1(s)\\&=\operatorname{Re}J_1(0)+\frac{s}{\pi}P\int_{4M^2}^{\infty}\frac{ds'}{s'(s'-s)}\frac{(1-\operatorname{Im}f_{22})\operatorname{Re}f_{12}+\operatorname{Re}f_{22}\operatorname{Im}f_{12}}{X}\operatorname{Re}J_2(s')\\&\quad+\frac{s}{\pi}P\int_{4m^2}^{4M^2}\frac{ds'}{s'(s'-s)}\frac{\operatorname{Re}f_{11}}{1-\operatorname{Im}f_{11}}\operatorname{Re}J_1(s')\\&\quad+\frac{s}{\pi}P\int_{4M^2}^{\infty}\frac{ds'}{s'(s'-s)}\frac{(1-\operatorname{Im}f_{22})\operatorname{Re}f_{11}+\operatorname{Re}f_{21}\operatorname{Im}f_{12}}{X}\operatorname{Re}J_1(s'),\end{aligned} \tag{4.4}$$

$$\begin{aligned}\operatorname{Re}J_2(s)&=\operatorname{Re}J_2(0)+\frac{s}{\pi}P\int_{4m^2}^{4M^2}\frac{ds'}{s'(s'-s)}\left(\operatorname{Re}f_{21}+\frac{\operatorname{Im}f_{21}\operatorname{Re}f_{11}}{\mathrm{I}-\operatorname{Im}f_{11}}\right)\operatorname{Re}J_1(s')\\&\quad+\frac{s}{\pi}P\int_{4M^2}^{\infty}\frac{ds'}{s'(s'-s)}\frac{(1-\operatorname{Im}f_{11})\operatorname{Re}f_{21}+\operatorname{Re}f_{11}\operatorname{Im}f_{21}}{X}\operatorname{Re}J_1(s')\\&\quad+\frac{s}{\pi}P\int_{4M^2}^{\infty}\frac{ds'}{s'(s'-s)}\frac{(1-\operatorname{Im}f_{11})\operatorname{Re}f_{22}+\operatorname{Re}f_{12}\operatorname{Im}f_{21}}{X}\operatorname{Re}J_2(s'),\end{aligned} \tag{4.5}$$

where

$$X=(1-\operatorname{Im}f_{11})(1-\operatorname{Im}f_{22})-\operatorname{Im}f_{12}\operatorname{Im}f_{21}.$$

All the f_{ij} are of course to be taken at s'. From the integrands we can easily read off $\operatorname{Im}J_1(s)$ in the various energy regions. After transforming the infinite integrals to a finite range we solve this set of integral equations by a simple matrix inversion technique. The poles in the principal-value integrals are handled by subtracting the integrand at the pole position and differentiating numerically. Unfortunately, this technique only works as long as no resonances are present. It seems that in this latter case one of the form factors goes rapidly to zero as $s\to\infty$ so that the once-subtracted integral equations allow at least two solutions differing by a polynomial of first order in s. This nonuniqueness makes our matrix singular, and the matrix inversion technique breaks down.

Figures 3 and 4 show two numerical examples of coupled form factors for weak and strong couplings when no resonances are present. We have chosen a fairly large value for the scattering length A in order to exhibit clearly the threshold effect which, of course, already exists in the one-channel scattering-length approximation when no bound-state poles are present.[9] To study the influence of the interchannel coupling we can simply increase c. Instead, in Fig. 4, we have moved the thresholds closer together to achieve the same effect. We notice that $\mathrm{Im}\,J_2(s)$ is nonzero starting at the elastic threshold, as it ought to be. In a purely elastic one-channel case $\mathrm{Im}\,J_2(s)$ would be nonzero only above the inelastic threshold. Moreover, $\mathrm{Re}\,J_2(s)$ shows a remarkable threshold enhancement at the elastic threshold, as this channel opens and begins to contribute to the imaginary part of $J_2(s)$.

To get an idea about the coupled form factors in the presence of an elastic resonance we have developed a perturbative solution for $J_1(s)$. In zero order we include the coupling between the two

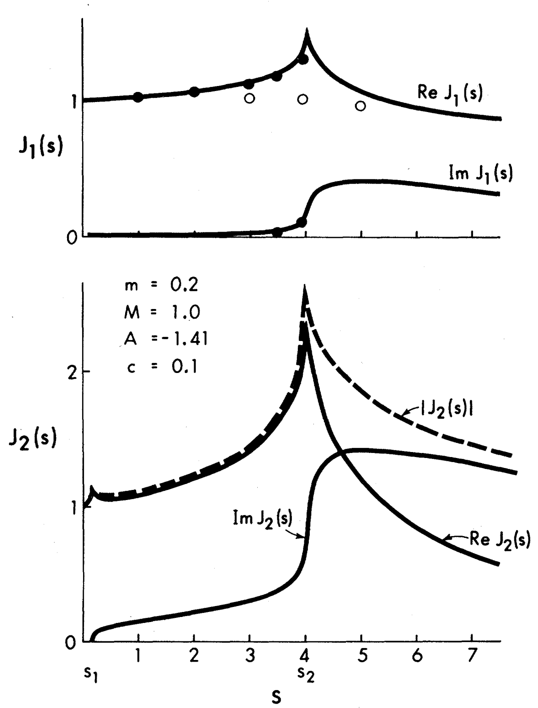

FIG. 3. Form factors in the scattering-length model. Open circles: $J_1^{(0)}(s)$; dots: $J^{(1)}(s)$ (see Sec. IV for details). $s_1 = 4m^2$, $s_2 = 4M^2$. $B = c$ has been chosen. (m, M in GeV; A in GeV^{-1}; s in GeV^2.)

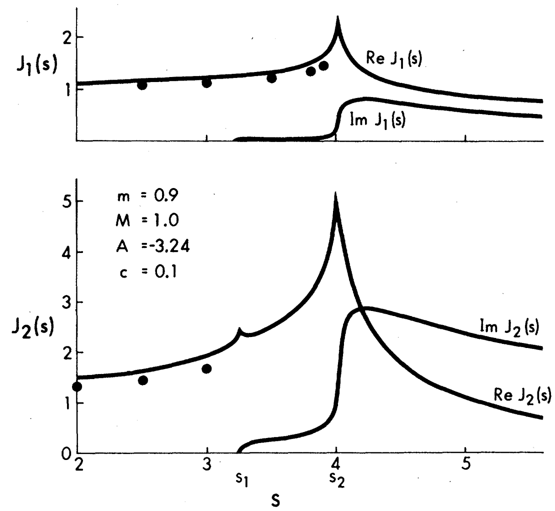

FIG. 4. See Fig. 3.

channels in the calculation of the T matrices only but assume that the form factors are still uncoupled by putting [Eqs. (2.8) and (2.9)]

$$\mathrm{Im}\,J_i^{(0)} = \mathrm{Re}\,f_{ii}\,\mathrm{Re}\,J_i^{(0)} + \mathrm{Im}\,f_{ii}\,\mathrm{Im}\,J_i^{(0)}. \qquad (4.6)$$

Starting from $J_1^{(0)}(s)$ we get a perturbative expansion in the coupling between the two channels by taking for $n = 0, 1, 2, \ldots$

$$\mathrm{Im}\,J_i^{(n+1)}(s) = \mathrm{Im}\,J_i^{(n)}(s) + \mathrm{Re}\,f_{ij}\,\mathrm{Re}\,J_j^{(n)} + \mathrm{Im}\,f_{ij}\,\mathrm{Im}\,J_j^{(n)} \qquad (4.7)$$

to be inserted into the integral equations for $\mathrm{Re}\,J_i(s)$.

We have compared a first-order calculation of this type with the exact solution in the case without any resonances (Figs. 3 and 4). We found that for

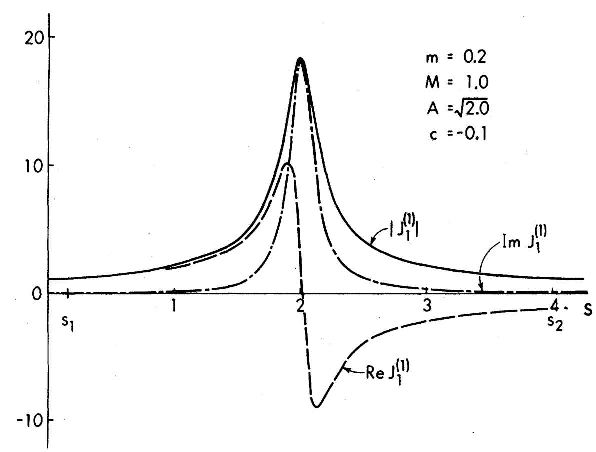

FIG. 5. First-order approximation $J_1^{(1)}(s)$ in the presence of a resonance in the elastic region. $B = c$ has been chosen. (m, M in GeV; A in GeV^{-1}; s in GeV^2.)

weak couplings the T coupling alone $J_i^{(0)}(s)$ (open circles in Fig. 3) gives too small an effect, whereas $J_1^{(1)}(s)$ (dots) gives excellent agreement with the exact result. For the same (weak) interchannel coupling we next calculated $J_1^{(1)}(s)$ for the case of a resonance in the elastic region (Fig. 5). This resonance in the elastic region of the first channel is in fact the bound state of the uncoupled second channel which is reflected via channel coupling as a resonance in the first channel.[10] The form factor shows the typical resonance behavior. In this case $J_1^{(0)}(s)$ is a much better approximation due to the fact that the coupled T matrices, of course, already contain all the information about the resonance, whereas the inelastic region simply adds a smooth background without much effect on the resonance structure. We should mention that the peak in $|J_1^{(1)}(s)|$ is slightly shifted back towards the original position of the bound-state pole in the uncoupled channel problem as compared with the peak in $|J_1^{(0)}(s)|$. Both peaks are, of course, at higher energies than the corresponding bound-state pole in the uncoupled second channel.

V. DIAGONALIZATION AND MORE MODELS

Written in the matrix form the unsubtracted form of the form factors [see Eq. (2.11)] is

$$\underline{J}(s)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\,\frac{\underline{F}(s')\,\mathrm{Re}\,\underline{J}(s')}{s'-s-i\epsilon}\,. \tag{5.1}$$

Let $\underline{R}(s)$ be a matrix which diagonalizes $\underline{F}(s)$,

$$\underline{R}(s)\,\underline{F}(s)\,\underline{R}^{-1}(s)=\underline{M}(s), \tag{5.2}$$

with eigenvalues $M_{11}(s)$ and $M_{22}(s)$. Then Eq. (5.1) yields

$$\underline{R}(s)\underline{J}(s)=\frac{1}{\pi}\int ds'\,\frac{\underline{K}(s,s')\underline{M}(s')\underline{R}(s')\,\mathrm{Re}\,\underline{J}(s')}{s'-s-i\epsilon}, \tag{5.3}$$

where we have defined[2]

$$\underline{K}(s,s')=\underline{R}(s)\underline{R}^{-1}(s')\,. \tag{5.4}$$

In general the matrix $\underline{K}(s,s')$ carries extra kinematical singularities and this feature causes some of the difficulties one faces in solving the problem through diagonalization. In a model, later in this section, we have explicitly demonstrated the appearance of these kinematical singularities. If the diagonalizing matrix $\underline{R}(s)$ were to be energy independent, then, of course, these kinematical singularities would not appear. In the following we have studied a particular model where such a situation occurs.

A. Degenerate Thresholds

Let us take a model with degenerate thresholds which saturates the two-channel unitarity,[11]

$$S_{11}=\frac{Z+ibk}{Z-iak},\quad Z=s_R-s$$
$$S_{12}=S_{21}=\frac{i(a^2-b^2)^{1/2}k}{Z-iak}, \tag{5.5}$$
$$S_{22}=\frac{Z-ibk}{Z-iak}\,.$$

The T matrix is

$$T_{11}=\frac{a+b}{2(Z-iak)},$$
$$T_{12}=T_{21}=\frac{(a^2-b^2)^{1/2}}{2(Z-iak)}, \tag{5.6}$$
$$T_{22}=\frac{a-b}{2(Z-iak)}\,.$$

The f matrix [Eq. (2.7)] is

$$\underline{f}=\frac{k}{2(Z-iak)}\begin{pmatrix}a+b & (a^2-b^2)^{1/2}\\ (a^2-b^2)^{1/2} & a-b\end{pmatrix}. \tag{5.7}$$

The eigenvalues of F [Eq. (2.11)] are ak/Z and 0. The constant matrix

$$\underline{R}=\frac{1}{\sqrt{2a}}\begin{pmatrix}(a+b)^{1/2} & (a-b)^{1/2}\\ -(a-b)^{1/2} & (a+b)^{1/2}\end{pmatrix} \tag{5.8}$$

serves to diagonalize $\underline{F}$. Defining a matrix $\underline{G}$ as

$$\underline{G}(s)=\underline{R}(s)\underline{J}(s), \tag{5.9}$$

one gets

$$G_1(s)=\left(\frac{a+b}{2a}\right)^{1/2}J_1(s)+\left(\frac{a-b}{2a}\right)^{1/2}J_2(s),$$
$$G_2(s)=-\left(\frac{a-b}{2a}\right)^{1/2}J_1(s)+\left(\frac{a+b}{2a}\right)^{1/2}J_2(s). \tag{5.10}$$

In this model $\underline{K}(s,s')$ is energy-independent and indeed equal to unity. As the eigenvalues of $\underline{F}$ are $M_{11}=ak/Z\neq 0$ and $M_{22}=0$, the integral equations for $G_i(s)$ are [Eq. (5.3)]

$$G_1(s)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{M_{11}(s')\,\mathrm{Re}\,G_1(s')}{s'-s-i\epsilon} \tag{5.11}$$

and

$$G_2(s)=0. \tag{5.12}$$

The solution of Eq. (5.11) is the standard Omnes-Muskhelishvili function[4]

$$G_1(s)=G_1(0)\exp\left(\frac{s}{\pi}\int\frac{\phi(s')ds}{s'(s'-s-i\epsilon)}\right) \tag{5.13}$$

with

$$\phi(s)=\tan^{-1}\left(\frac{ak}{Z}\right). \tag{5.14}$$

The solutions for $J_1(s)$ and $J_2(s)$ on inversion of Eq. (5.10) are

$$J_1(s)=J_1(0)\exp\left(\frac{s}{\pi}\int\frac{\phi(s')ds'}{s'(s'-s-i\epsilon)}\right), \tag{5.15}$$

$$J_2(s)=\left(\frac{a-b}{a+b}\right)^{1/2}J_1(s). \tag{5.16}$$

B. Nondegenerate Thresholds

With the S matrix defined in Eq. (5.5) and using

$$S_{ij}=2ik_i^{1/2}T_{ij}k_j^{1/2}+\delta_{ij}, \tag{5.17}$$

one gets for $s>4M^2$ (i.e., above the inelastic threshold)

$$T_{11}=\frac{a+b}{2(Z-iak_1)},$$
$$T_{12}=T_{21}=\frac{k_1^{1/2}(a^2-b^2)^{1/2}k_2^{-1/2}}{2(Z-iak_1)}, \tag{5.18}$$
$$T_{22}=\frac{k_1}{k_2}\,\frac{a-b}{2(Z-iak_1)},$$

with $4k_1^2=s-4m^2$ and $4k_2^2=s-4M^2$. The eigenvalues of the matrix $\underline{F}$ above the inelastic threshold ($s>4M^2$) are

$$M_{11}=\frac{ak_1}{Z},\qquad M_{22}=0. \tag{5.19}$$

The matrix that diagonalizes $\underline{F}$ above the inelastic threshold is

$$\underline{R}(s)=\frac{1}{(2a)^{1/2}}\begin{pmatrix}(a+b)^{1/2} & (k_2/k_1)^{1/2}(a-b)^{1/2}\\ -(k_1/k_2)^{1/2}(a-b)^{1/2} & (a+b)^{1/2}\end{pmatrix}. \tag{5.20}$$

Note that for degenerate thresholds $\underline{R}(s)$ reduces to that of Eq. (5.8). The form of $\underline{R}(s)$ clearly displays the kinematical branch points that appear because of the square roots. For both s and $s'>4M^2$ we have then

$$\underline{K}(s,s')=\frac{1}{2a}\begin{pmatrix}a+b+\left(\frac{k_2}{k_1}\right)^{1/2}\left(\frac{k_2'}{k_1'}\right)^{-1/2}(a-b) & \left[\left(\frac{k_2}{k_1}\right)^{1/2}-\left(\frac{k_2'}{k_1'}\right)^{1/2}\right](a^2-b^2)^{1/2}\\ \left[\left(\frac{k_2'}{k_1'}\right)^{-1/2}-\left(\frac{k_2}{k_1}\right)^{-1/2}\right](a^2-b^2)^{1/2} & a+b+\left(\frac{k_2}{k_1}\right)^{-1/2}\left(\frac{k_2'}{k_1'}\right)^{1/2}(a-b)\end{pmatrix}. \tag{5.21}$$

The kinematic branch points are displayed in Eq. (5.21). For energies below the inelastic threshold, $4m^2<s<4M^2$, we have from Eq. (2.7)

$$\underline{f}=\begin{pmatrix}k_1T_{11} & 0\\ k_1T_{21} & 0\end{pmatrix} \tag{5.22}$$

and Eq. (2.11) gives

$$\underline{F}=\frac{1}{1-\mathrm{Im}f_{11}}\begin{pmatrix}\mathrm{Re}f_{11} & 0\\ B(s) & 0\end{pmatrix}, \tag{5.23}$$

where

$$B(s)=(1-\mathrm{Im}f_{11})\,\mathrm{Re}f_{21}+\mathrm{Re}f_{11}\,\mathrm{Im}f_{21} \tag{5.24}$$

and the diagonalizing matrix is ($4m^2<s<4M^2$)

$$\underline{R}(s)=\begin{pmatrix}1 & 0\\ -B/\mathrm{Re}f_{11} & 1\end{pmatrix}. \tag{5.25}$$

The eigenvalues of $\underline{F}$ are in this case

$$M_{11}=\frac{\mathrm{Re}f_{11}}{1-\mathrm{Im}f_{11}} \tag{5.26}$$

and $M_{22}=0$. For both s and $s'<4M^2$ one can also evaluate $\underline{K}(s,s')$,

$$\underline{K}(s,s')=\begin{pmatrix}1 & 0\\ \dfrac{B(s')}{\mathrm{Re}f_{11}(s')}-\dfrac{B(s)}{\mathrm{Re}f_{11}(s)} & 1\end{pmatrix}. \tag{5.27}$$

$\underline{K}(s,s')$ for other regions of s and s' can be constructed out of Eq. (5.20) and Eq. (5.25). Defining

$$\underline{G}(s)=\underline{R}(s)\underline{J}(s) \tag{5.28}$$

and noting that $M_{11}\neq 0$ while $M_{22}=0$, we get the following equations:

$$G_1(s)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{K_{11}(s,s')M_{11}(s')\,\mathrm{Re}\,G_1(s')}{s'-s-i\epsilon}, \tag{5.29}$$

$$G_2(s)=\frac{1}{\pi}\int_{4m^2}^{\infty}ds'\frac{K_{21}(s,s')M_{11}(s')\,\mathrm{Re}\,G_1(s')}{s'-s-i\epsilon}, \tag{5.30}$$

where the integrand is defined on the upper lip of the cut. The solution of Eq. (5.29) cannot proceed in the manner given in the Appendix because of the extra kinematic branch points in $K_{11}(s,s')$. A complete solution of this problem is therefore difficult. However, it is clear from Eqs. (5.18) and (5.20) that for small inelasticities the diagonal elements of $\underline{R}$ are proportional to T_{11} and the off-diagonal elements to T_{12}. This implies that

$$\text{for } s,s'<4M^2,\quad K_{11}(s,s')=1 \text{ (exact)};$$
$$\text{for } s,s'>4M^2,\quad K_{11}(s,s')=\frac{a+b}{2a}+O(T_{12}^2). \tag{5.31}$$

If the coupling between the channels is weak (small inelasticity) and we wish to study the effect of coupling only to first order in T_{12}, then the element K_{11} is energy-independent and there are no kinematic branch points. The problem is then solvable in the manner shown in the Appendix. If we write

$$K_{11}(s, s') = K_{11}^{(0)} + O(T_{12}^2), \tag{5.32}$$

then we get the standard Omnes-Muskhelishvili solution

$$G_1(s) \approx G_1(0) \exp \frac{s}{\pi} \int_{4m^2}^{\infty} \frac{\xi(s') ds'}{s'(s' - s - i\epsilon)}, \tag{5.33}$$

where

$$\xi(s) = \tan^{-1}[K_{11}^{(0)} M_{11}(s)]. \tag{5.34}$$

In this particular model $G_2(s) \sim O(T_{12})$. Then solving Eq. (5.28) for $J_1(s)$ one gets to order $T_{12}{}^2$

$$J_1(s) \approx [\underline{R}^{-1}(s)\underline{G}(s)]_1 \tag{5.35}$$

or

$$J_1(s) = G_1(s), \quad s < 4M^2$$
$$= (\underline{R}^{-1})_{11}(s) G_1(s), \quad s > 4M^2. \tag{5.36}$$

In this model the phase function $\xi(s)$ is

$$\xi(s) = \tan^{-1} \frac{ak_1}{Z}, \quad s < 4M^2$$
$$= \tan^{-1} \frac{a+b}{2a} \frac{ak_1}{Z}, \quad s > 4M^2. \tag{5.37}$$

This phase is to be compared with the phase $\phi(s)$ which enters the same problem if we were to ignore the second-channel form factor, i.e., drop the contribution of Fig. 2 from the unitarity relation,

$$\psi(s) = \tan^{-1}\left(\frac{ak_1}{Z}\right), \quad s < 4M^2$$
$$= \tan^{-1}\left(\frac{Zk_1(a+b)}{2Z^2 + ak_1{}^2(a-b)}\right), \quad s > 4M^2. \tag{5.38}$$

This model does not lend itself to easy numerical work since the inelasticity is switched on suddenly. For $s < 4M^2$, $a = b$, but for $s > 4M^2$, $a \neq b$. We tried a model calculation with

$$m = 0.14 \text{ GeV}, \quad M = 0.5 \text{ GeV},$$
$$a = 0.3 \text{ GeV}, \quad b = 0.27 \text{ GeV}, \tag{5.39}$$
$$s_R = 0.5 \text{ GeV}^2.$$

The use of Eq. (5.37) resulted in a form factor, $|J_1(s)|$, which lies below that which results from the use of Eq. (5.38) for energies less than s_R and above for energies larger than s_R. The quantitative difference was too small to plot. The reason for the small quantitative difference is that a resonance in the elastic region largely determines the form factor in that region and a small inelasticity has very little influence on it, in this region.

It should be emphasized that the diagonalization of $\underline{F}$ does not in general lead to a decoupling of G_i's $(i = 1, 2)$. The cases where the diagonalization would indeed decouple the G_i's are where $R(s)$ is energy-independent or either M_{11} or M_{22} is zero.

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APPENDIX

We want to solve the inhomogeneous integral equation for $J_1(s)$, Eq. (2.16),

$$J_1(s) = J_1(0) + \frac{s}{\pi} \int_{4m^2}^{\infty} ds' \frac{H(s') \operatorname{Re} J_1(s')}{s'(s' - s - i\epsilon)} + f(s), \tag{A1}$$

with

$$\tan\phi_1(s) = H(s). \tag{A2}$$

We define

$$F(s_\pm) = \frac{s}{2\pi i} \int_{4m^2}^{\infty} ds' \frac{\tan\phi_1(s') \operatorname{Re} J_1(s')}{s'(s' - s \mp i\epsilon)} + \frac{1}{2i} J_1(0) \tag{A3}$$

$$= \frac{1}{2i} J_1(0) \pm \tfrac{1}{2} \tan\phi_1(s) \operatorname{Re} J_1(s) + \frac{s}{2\pi i} P \int_{4m^2}^{\infty} ds' \frac{\tan\phi_1(s') \operatorname{Re} J_1(s')}{s'(s' - s)} \tag{A4}$$

so that

$$F(s_+) - F(s_-) = \tan\phi_1(s) \operatorname{Re} J_1(s). \tag{A5}$$

We know from (A1) that

$$\operatorname{Re} J_1(s) = J_1(0) + \frac{s}{\pi} P \int_{4m^2}^{\infty} ds' \frac{\tan\phi_1(s') \operatorname{Re} J_1(s')}{s'(s - s)} + \operatorname{Re} f(s). \tag{A6}$$

Combining Eqs. (A3) to (A6) we get

$$\operatorname{Re} J_1(s) = i[F(s_+) + F(s_-)] + \operatorname{Re} f(s). \tag{A7}$$

Therefore from Eq. (A5)

$$F(s_+) - F(s_-) = i \tan\phi_1(s)[F(s_+) + F(s_-)] + \tan\phi_1(s) \operatorname{Re} f(s) \tag{A8}$$

or

$$F(s_+) e^{-2i\phi_1(s)} - F(s_-) = \frac{\tan\phi_1(s)}{1 + i\tan\phi_1(s)} \operatorname{Re} f(s). \tag{A9}$$

Let us look for a solution of the kind

$$F(s) = \Omega(s) \psi(s), \tag{A10}$$

where $\Omega(s)$ satisfies the homogeneous equation

$$\Omega(s_+)e^{-2i\phi_1(s)} - \Omega(s_-) = 0$$

or

$$\ln\Omega(s_+) - \ln\Omega(s_-) = 2i\phi_1(s). \tag{A11}$$

This gives a once-subtracted solution

$$\Omega(s_\pm) = \Omega(0)\exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{\phi_1(s')ds'}{s'(s'-s\mp i\epsilon)}\right) \equiv \Omega(0)\exp[\rho(s)+i\phi_1(s)], \tag{A12}$$

where

$$\rho(s) = \frac{s}{\pi}P\int_{4m^2}^{\infty}\frac{\phi_1(s')ds'}{s'(s'-s)} . \tag{A13}$$

We choose

$$\Omega(0) = J_1(0). \tag{A14}$$

Putting (A12) in (A9) we get

$$\Omega(0)[\psi(s_+)e^{\rho(s)-i\phi_1(s)} - \psi(s_-)e^{\rho(s)-i\phi_1(s)}] = \frac{\tan\phi_1(s)}{1+i\tan\phi_1(s)}\ \mathrm{Re}f(s) \tag{A15}$$

or

$$\psi(s_+) - \psi(s_-) = e^{-\rho(s)}\sin\phi_1(s)\,\mathrm{Re}f(s)/\Omega(0) \tag{A16}$$

with a solution $[\psi(0) = 1/2i]$

$$\psi(s_\pm) = \frac{1}{2i}\left(1+\frac{s}{\pi}\int_{4m^2}^{\infty}ds'\,\frac{e^{-\rho(s')}\sin\phi_1(s')\,\mathrm{Re}f(s')}{\Omega(0)s'(s'-s-i\epsilon)}\right) . \tag{A17}$$

Thus

$$F(s_+) = \Omega(s_+)\psi(s_+)$$

$$= \frac{J_1(0)}{2i}\left(1+\frac{s}{J_1(0)\pi}\int_{4m^2}^{\infty}ds'\,\frac{e^{-\rho(s')}\sin\phi_1(s')\,\mathrm{Re}f(s')}{s'(s'-s-i\epsilon)}\right)\exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}ds'\,\frac{\phi_1(s')}{s'(s'-s-i\epsilon)}\right) \tag{A18}$$

and

$$J_1(s_+) = 2i\Omega(s_+)\psi(s_+) + f(s)$$

$$= f(s) + \left(J_1(0)+\frac{s}{\pi}\int_{4m^2}^{\infty}ds'\,\frac{e^{-\rho(s')}\sin\phi_1(s')\,\mathrm{Re}f(s')}{s'(s'-s-i\epsilon)}\right)\exp\left(\frac{s}{\pi}\int_{4m^2}^{\infty}ds'\,\frac{\phi_1(s')}{s'(s'-s-i\epsilon)}\right) , \tag{A19}$$

which is the solution to Eq. (A1).

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