Solutions of a Nonlinear Integral Equation for High-Energy Scattering. I. An Existence Theorem*

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Ball and Zachariasen have proposed a model of high-energy diffraction scattering, which entails nonshrinking diffraction peaks. The model has some resemblance to the multiperipheral model, in that it emphasizes s-channel unitarity with a simple factored form of production amplitudes. It leads to an integral equation for the elastic amplitude which is relatively simple in appearance. The equation is analyzed here with the help of methods which may also be useful in the study of more realistic high-energy models. The Hankel transform of the original equation is studied as a nonlinear equation in a certain Banach space. Existence of an infinite class of solutions is proved by means of the contraction mapping principle. These solutions are constructed by iteration for small values of a parameter c, which measures the strength of particle production. The range of allowed values for the product of c and the elastic cross section does not include the physical value. One can try, however, to continue the solutions to the physical value, since they are analytic in c. In paper II, the solutions are calculated numerically, and the continuation to large c is attempted. The continuation stops short of its goal, because of a singularity of the Fréchet derivative of the nonlinear operator. This derivative becomes a linear integral operator of the "third kind," which has no inverse in a space of continuous functions. No physically acceptable solutions of the Ball-Zachariasen equation have been found. A proposed approximate solution appears to have some difficulties, as is explained in paper II.

I. INTRODUCTION

Models of strong-interaction processes almost always lead to nonlinear integral equations of one difficult type or another. Our understanding of the associated mathematical problems is at a very primitive level. On the one hand, theorems on existence and qualitative properties of solutions are usually lacking. On the other hand, attempts at numerical solution are often not convincing, since there is usually no theoretical basis for the numerical method used.

One can make a more systematic attack on nonlinear equations by combining the methods of nonlinear functional analysis with numerical studies. This is a very natural approach, which is under development in classical continuum mechanics and in other areas of applied mathematics.[1] In applying this method to problems of high-energy physics, we encounter some novel problems, since our equations are quite different in detail from those usually studied by applied mathematicians.

In this paper we study an integral equation proposed by Ball and Zachariasen,[2] in connection with a model of high-energy diffraction scattering. We shall not try to criticize or improve the physical basis of the model in this report, but rather regard it as an interesting special example from which we can learn some general lessons. For example, we learn how to use a fixed-point theorem in a Banach space of integral transforms. This technique should be useful in connection with various other models of high-energy processes.[3]

In order to explain our approach in general terms, we should first point out that the methods of functional analysis normally do not provide a basis for a complete qualitative analysis. That is, it is hard to get global existence theorems in which all solutions are cataloged. It is often possible, however, to prove existence of solutions for some restricted range of the physical parameters (for weak coupling, for instance), and to construct those solutions by a convergent iterative algorithm. This has been done for certain equations of S-matrix theory in the papers of Ref. 4; for a survey of such work, see Ref. 5. The restrictions on the physical parameters in the existence proofs are often due more to the limitations of technique than to some intrinsic properties of the equations. One is then free to attempt a continuation of the solutions with respect to the parameters, outside the original limited range. This continuation must be done numerically, generally speaking, but some guidance on what to expect in the calculation is available from functional analysis. For instance, one can expect that the continuation will proceed successfully at least to the first singularity of the Fréchet derivative of the operator studied.[1,6] The meaning of this remark is roughly as follows. Suppose that the equation to be solved is

$$F(\varphi, c) = 0\,, \tag{1.1}$$

where c represents the parameter or parameters to be varied in the continuation, and φ is the solution sought. The Fréchet derivative of F [evaluated at (φ, c)] is a linear operator $F_\varphi(\varphi, c)$ such that $F_\varphi(\varphi, c)\delta\varphi$ is the first-order change in $F(\varphi, c)$ when φ is changed to $\varphi + \delta\varphi$. Let φ_0 be a solution with parameter c_0, $F(\varphi_0, c_0) = 0$, and suppose that $F_\varphi(\varphi_0, c_0)$ is nonsingular (i.e., that it has an inverse). Then there will be a unique "solution curve" $\varphi(c)$ extending from $\varphi_0 = \varphi(c_0)$ to $\varphi_1 = \varphi(c_1)$, where c_1 is the first value of c at which $F_\varphi(\varphi(c), c)$ becomes singular. The curve may or may not go further, and if it does, the continuation is not necessarily unique. There might be "bifurcation" at c_1. In the calculation we must be on the lookout for singularities of F_φ, and if any are encountered we shall probably need a good *analytic* understanding of their character if we are to have a chance of passing beyond them. Since F_φ is a linear operator, an analysis of its singularities is not out of the question.

A less systematic approach, which is suitable when both the physical basis of the equation and one's physical intuition are sound, is simply to guess an approximate solution. Two serious questions then arise, however. First, how does one judge the goodness of the guess, and second, how may the guess be improved systematically? These questions are closely related. A possible answer to the second question is provided by the Newton-Kantorovich (NK) method.[6] Let φ_0 be the proposed approximate solution. We linearize about φ_0, with the hope of obtaining an improved approximation φ_1:

$$F(\varphi) \approx F(\varphi_0) + F_\varphi(\varphi_0)(\varphi - \varphi_0)\,, \tag{1.2}$$

$$F(\varphi_0) + F_\varphi(\varphi_0)(\varphi_1 - \varphi_0) = 0\,. \tag{1.3}$$

If $F_\varphi(\varphi_0)$ is nonsingular, there is a unique solution of (1.3):

$$\varphi_1 = \varphi_0 - F_\varphi(\varphi_0)^{-1} F(\varphi_0)\,. \tag{1.4}$$

The higher NK iterates,

$$\varphi_{n+1} = \varphi_n - F_\varphi(\varphi_n)^{-1} F(\varphi_n)\,, \tag{1.5}$$

will converge to a solution under conditions laid down by Kantorovich. According to Kantorovich, there will be convergence provided φ_1 is sufficiently close to φ_0, assuming that $F_\varphi{}^{-1}$ and the second derivative $F_{\varphi\varphi}$ are bounded in norm near φ_0. The closer F_φ is to being singular, however, the closer φ_1 must be to φ_0. At a singularity of the Fréchet derivative, the NK iteration fails completely.[7]

The matter of judging the goodness of an approximate solution is also strongly dependent on the "condition" of the Fréchet derivative (an operator which is close, in some sense, to singularity is called "ill-conditioned"). This is seen very clearly in the trivial case of a linear system of equations. Consider the ill-conditioned system

$$F(\varphi) = A\varphi - y = 0\,, \tag{1.6}$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1.01 \end{pmatrix}, \quad y = \begin{pmatrix} 2 \\ 2.01 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \tag{1.7}$$

The solution of (1.6) is $\varphi_1 = \varphi_2 = 1$. A small change in the matrix A will cause a big change in the solution, however. With A replaced by

$$\overline{A} = \begin{pmatrix} 1 & 1 \\ 1.001 & 1 \end{pmatrix}, \tag{1.8}$$

the solution $\overline{\varphi}_1 = 10$, $\overline{\varphi}_2 = -8$. Furthermore, the solution $\overline{\varphi}$ for $\overline{A}$ appears superficially to be a fairly good approximate solution of (1.6), in the sense that the "residuals," $A\overline{\varphi} - y$, are small compared to y. That is, $(A\overline{\varphi} - y)_1 = 0.09$, $(A\overline{\varphi} - y)_2 = 0$. In general, residuals are worthless as a test for the goodness of an approximate solution when the system is badly conditioned, simply because the approximate solution can have an incorrect component in the eigenspace of a small eigenvalue of the matrix, without having a big effect on the residual. The system is also sensitive to small inaccuracies in computation of the matrix. Essentially the same situation holds for a linear equation in an infinite-dimensional Banach space. Moreover, when a nonlinear equation may be regarded as locally linear, we can expect analogous phenomena in the case of nonlinear, infinite-dimensional equations. Representing the small nonlinear remainder by $R(\varphi)$, we may write $F(\varphi) = 0$ as

$$F_\varphi(\varphi_0)\varphi - y(\varphi) = 0\,,$$
$$y(\varphi) = -F(\varphi_0) + F_\varphi(\varphi_0)\varphi_0 - R(\varphi)\,. \quad (1.9)$$

If $F_\varphi(\varphi_0)$ is singular or nearly so, and φ_1 is a proposed approximate solution close to φ_0, then the residual $F_\varphi(\varphi_0)\varphi_1 - y(\varphi_1)$ may be a poor measure of the closeness of φ_1 to a solution.

We now give a brief recapitulation of the Ball-Zachariasen model.[2] The elastic scattering amplitude is assumed to factor as follows at high energy:

$$T(s,t) \sim is\,\mathcal{F}(t), \quad s \to \infty\,. \quad (1.10)$$

(The squared center-of-mass energy is s, and t is the invariant momentum transfer squared.) That is, the diffraction peak has constant shape asymptotically. The amplitude for two particles to produce n is also given a factorized form:

$$T_{2\to n} \sim iA_n(s)\mathcal{F}(t_1)\mathcal{F}(t_2)\cdots\mathcal{F}(t_{n-1}), \quad s\to\infty \quad (1.11)$$

where t_i is the invariant momentum transfer squared between an initial particle and a group of final particles. By some physical assumptions,[2] the form of $A_n(s)$ is determined as follows:

$$\left[\frac{A_n(s)}{s}\right]^2 = \frac{(\pi c)^{n-1}}{(\ln s)^{n-2}}\,, \quad (1.12)$$

where c is a constant. The s-channel unitarity condition now gives the Ball-Zachariasen equation[8]

$$\mathcal{F}(t) = \mathcal{G}(t)\, e^{c\mathcal{G}(t)}\,, \quad (1.13)$$

where

$$\mathcal{G}(t) = \frac{1}{16\pi^2}\iint \frac{dt_1\, dt_2\, \mathcal{F}(t_1)\,\mathcal{F}(t_2)}{[2(tt_1 + tt_2 + t_1t_2) - t^2 - t_1^{\,2} - t_2^{\,2}]^{1/2}}\,. \quad (1.14)$$

In (1.14), the integration is over the region where the argument of the square root is positive. The amplitude is normalized so that the total cross section is given by the optical theorem as

$$\sigma_{\text{tot}} = \mathcal{F}(0)\,. \quad (1.15)$$

Since (1.14) is the high-energy limit of the elastic unitarity integral, the asymptotic value of the elastic contribution to the total cross section is

$$\sigma_{\text{el}} = \mathcal{G}(0)\,. \quad (1.16)$$

By (1.13), we then obtain c as

$$c\sigma_{\text{el}} = \ln(\sigma_{\text{tot}}/\sigma_{\text{el}})\,. \quad (1.17)$$

If we use cross sections from p-p scattering at the highest available energies, we get approximately $c = 0.1\ \text{GeV}^2$.

In Sec. II, we discuss some general properties of Eq. (1.13), with the help of Hankel transforms. A formal Hankel transform of the equation is obtained, and in Appendix A we show that this is equivalent to the original equation. The transformed equation is studied in the remainder of the paper, since it is noticeably easier to handle. Our analysis follows the general pattern sketched above. First, in Sec. III, we give an existence proof for an infinite class of solutions at small values of c. We also find that these solutions are analytic in c, in a circle with center at $c=0$. The solutions may be constructed by iteration; in fact, the existence proof is based upon iteration. In Sec. IV, we discuss the continuation of the solutions to larger c. We examine the Fréchet derivative of the nonlinear operator, and find that it can develop a singularity when the solution becomes sufficiently large. The singularity arises because the Fréchet derivative becomes a so-called Fredholm operator of the third kind, i.e., an operator F_φ of the form

$$F_\varphi \delta\varphi = a(b)\delta\varphi(b) + \int_0^\infty K(b,b')\delta\varphi(b')db'\,, \quad (1.18)$$

where K is a Fredholm kernel, and $a(b)$ is a given function which has zeros. In general, such an operator does not have an inverse in a space of continuous functions. It may have an inverse in a space of generalized functions, however, as will be shown in another publication.[9]

In paper II,[10] we report the results of numerical calculations. We calculate solutions as obtained in the existence theorem, and attempt the continuation to larger c. One quickly encounters a singularity of the Fréchet derivative, however, which brings the continuation to a halt. We try to get around the singularity by an excursion into the complex c plane, but the attempt does not succeed. We also consider a proposed approximate solution of Ball and Zachariasen (second paper of Ref. 2), which, unlike the solutions obtained in the present paper, is continuous in the impact parameter. The Fréchet derivative evaluated at the Ball-Zachariasen function is again nearly singular, which makes it difficult to decide whether there is an actual solution in the neighborhood. Our conclusions are summarized at the end of paper II.

II. HANKEL TRANSFORMATION OF THE INTEGRAL EQUATION

It is convenient to change notation and write $\mathcal{F}((-t)^{1/2})$ and $\mathcal{G}((-t)^{1/2})$ for the functions that were called $\mathcal{F}(t)$ and $\mathcal{G}(t)$ previously. We define the variable $x=(-t)^{1/2}$, which is conjugate to the impact parameter b in the sense of Hankel transforms. The zeroth-order Hankel transform of $\mathcal{F}(x)$ is $H\mathcal{F}(b)$:

$$H\mathfrak{F}(b)=\int_0^\infty J_0(bx)\mathfrak{F}(x)x\,dx=\hat{\mathfrak{F}}(b). \tag{2.1}$$

It will be convenient to use the two notations $H\mathfrak{F}$ and $\hat{\mathfrak{F}}$ interchangeably. Here J_0 is the zeroth-order Bessel function. The integral appearing in Eq. (1.14) now reads

$$\mathcal{G}(x)=\frac{1}{(2\pi)^2}\int_0^\infty x_1\,dx_1\int_{|x-x_1|}^{x+x_1}x_2dx_2\,\frac{\mathfrak{F}(x_1)\mathfrak{F}(x_2)}{[2(x^2x_1^2+x^2x_2^2+x_1^2x_2^2)-x^4-x_1^4-x_2^4]^{1/2}}. \tag{2.2}$$

In order to compute the Hankel transform of (2.2), we apply a formula of Sonine[11]; namely, if $x_1, x_2, x>0$, then

$$\int_0^\infty J_0(x_1b)J_0(x_2b)J_0(xb)b\,db=\begin{cases}0, & x_2<|x-x_1|\\ \frac{2}{\pi}[2(x^2x_1^2+x^2x_2^2+x_1^2x_2^2)-x^4-x_1^4-x_2^4]^{-1/2}, & |x-x_1|<x_2<x+x_1\\ 0, & x_2>x+x_1\,.\end{cases} \tag{2.3}$$

By substitution of (2.3) in (2.2), we have

$$\mathfrak{F}(x)=\frac{1}{8\pi}\int_0^\infty x_1\mathfrak{F}(x_1)dx_1\int_0^\infty x_2\mathfrak{F}(x_2)dx_2\int_0^\infty J_0(x_1b)J_0(x_2b)J_0(xb)b\,db. \tag{2.4}$$

We shall now derive an equation for the Hankel transform $\hat{\mathfrak{F}}(b)$ by manipulations which are formal in the sense that integration orders are changed without justification, and Hankel's inversion formula[12] is assumed to hold in the form

$$H(Hu)=u, \tag{2.5}$$

whenever we need it. In Sec. III, we analyze the equation for $\hat{\mathfrak{F}}(b)$, showing that it has solutions which are such as to justify all the formal steps.

By evaluating the b integral last in (2.4), one finds

$$\mathcal{G}(x)=\frac{1}{8\pi}\int_0^\infty J_0(xb)\hat{\mathfrak{F}}(b)^2b\,db =\frac{1}{8\pi}H(\hat{\mathfrak{F}}(b)^2). \tag{2.6}$$

This result can be used to solve the Ball-Zachariasen equation (1.13) in the special case $c=0$, i.e., the equation

$$\mathfrak{F}(x)=\mathcal{G}(x). \tag{2.7}$$

By substituting (2.6) in (2.7), and taking the Hankel transform, we have

$$\hat{\mathfrak{F}}(b)=\frac{1}{8\pi}\hat{\mathfrak{F}}(b)^2. \tag{2.8}$$

According to this, $\hat{\mathfrak{F}}(b)$ can have only the values 8π and 0, so a solution of (2.8) is a step function

$$\hat{\mathfrak{F}}(b)=8\pi\sum_I\chi_I(b), \tag{2.9}$$

where $\chi_I(b)$ is the characteristic function of a closed interval I, i.e., $\chi_I(b)=1$ if $b\in I$, and $\chi_I(b)=0$, otherwise. The sum in (2.9) runs over any finite set of finite, disjoint intervals I. Equation (2.9) is not the general solution of (2.8), or even the general Hankel-transformable solution. Instead of (2.9), one could let $\hat{\mathfrak{F}}(b)$ be equal to 8π for all b greater than some b_0. Such solutions would not be transformable, except in the sense of generalized functions, and are probably not of physical interest. On the other hand, one can make transformable solutions by letting the sum in (2.9) run over an infinite sequence of intervals of decreasing length. If the intervals of the sequence converge to zero length with sufficient rapidity, then the resulting solution $\hat{\mathfrak{F}}(b)$ is Hankel-transformable. In this way, one gets an infinite set of transformable solutions, in addition to the infinite set that we already have in (2.9).

For reasons of simplicity alone, we shall deal only with a finite sum in (2.9). The function $\mathfrak{F}(x)$ for $c=0$, may be evaluated explicitly by means of the identity $xJ_0(x)=[xJ_1(x)]'$. Let the ith interval where (2.9) is nonzero be $s_i\leq b\leq r_i$. Then

$$\mathfrak{F}(x)=\frac{8\pi}{x}\sum_{i=1}^n[r_iJ_1(r_ix)-s_iJ_1(s_ix)] =O(x^{-3/2}),\quad x\to\infty. \tag{2.10}$$

A particularly simple choice of (2.9) is to put $\hat{\mathfrak{F}}(b)=8\pi$, $b\leq r$, and $\hat{\mathfrak{F}}(b)=0$, $b>r$. Then we have the simple result

$$\mathfrak{F}(x)=8\pi r\frac{J_1(rx)}{x}. \tag{2.11}$$

The scattering from a perfectly absorbing sphere, evaluated at high energy and small angle by the eikonal approximation, yields the amplitude[13]

$$T(s,t)=is\,\frac{4\pi aJ_1(ax)}{x}, \tag{2.12}$$

where a is the radius of the sphere. The formula (2.11) thus corresponds to the diffraction scattering from an absorbing sphere of radius r, except for the presence of an extra (and somewhat puzzling) factor of 2. The $c=0$ limit of the theory, although sensible mathematically, is rather curious physically. Putting $c=0$ turns off the particle production, which is the physical source of the absorption. Nevertheless, we have a picture of pure absorption at $c=0$.

To generate solutions to the full equation (1.13) for $c\neq 0$, we first take c to be very small, in the hope of finding solutions which are close to those for $c=0$. We continue to work in the space of Hankel transforms, in which the analysis is much easier, even when c is not zero. Before proceeding, it is convenient to renormalize f and c so as to eliminate factors of 8π. We put

$$f=\mathfrak{F}/8\pi, \quad c'=8\pi c, \tag{2.13}$$

and henceforth discard the prime. In view of (2.6), the Ball-Zachariasen equation now reads

$$f(x)=\int_0^\infty J_0(xb)\hat{f}(b)^2 b\,db \times\exp\left[c\int_0^\infty J_0(xb)\hat{f}(b)^2 b\,db\right], \tag{2.14}$$

or

$$f=H(\hat{f}^2)+H(\hat{f}^2)(e^{cH(\hat{f}^2)}-1), \tag{2.15}$$

where we have formally added and subtracted 1 in the exponential factor. Now take the Hankel transform of (2.15):

$$\hat{f}=\hat{f}^2+H(H(\hat{f}^2)(e^{cH(\hat{f}^2)}-1)). \tag{2.16}$$

There are some simple properties of solutions which are immediately evident from the equation. Suppose that $f(x)$, $g(x)$, $\hat{f}(b)$, $\hat{g}(b)$ are sufficiently well behaved so that their values at the origin may be obtained by putting $x=0$ or $b=0$ under the integral of their Hankel representations. Suppose also that $\hat{g}=\hat{f}^2$. In Appendix D we show that the following properties are then true:

(i) $|f(x)|\leq f(0), \quad |g(x)|\leq g(0);$ (2.17a)

(ii) if $f(x)\geq 0$ [hence $g(x)\geq 0$],

then $|\hat{f}(b)|\leq\hat{f}(0)\leq 1, \quad |\hat{g}(b)|\leq\hat{g}(0)\leq 1,$

$\hat{f}(0)\geq\sigma_{el}/\sigma_{tot}.$ (2.17b)

For the $c=0$ solution in b space [Eq. (2.9)], we use the notation

$$h(b)=\sum_I \chi_I(b). \tag{2.18}$$

It is convenient to define a function $\varphi(b)$, to take the place of $\hat{f}(b)$, as follows:

$$\hat{f}(b)=h(b)+\varphi(b)[1-2h(b)]. \tag{2.19}$$

Since $h^2=h$, $(1-2h)^2=1$, and $h(1-2h)=-h$, there is the identity

$$\hat{f}^2=(h-\varphi)^2,$$

and (2.16) becomes

$$\varphi=A(\varphi,c), \tag{2.20}$$

where

$$A(\varphi,c)=\varphi^2+H(H((h-\varphi)^2)(e^{cH((h-\varphi)^2)}-1)). \tag{2.21}$$

Equation (2.20) will be the object of our analysis, since the function $\varphi(b)$ turns out to have better continuity properties than $f(b)$ itself. In Sec. III, we shall find that (2.20) has solutions, and their behavior is such that one can take Hankel transforms, reverse integration orders, and thus get back to solutions of (1.13). The latter is demonstrated in Appendix A.

A naive approach to solving (2.20) is simple iteration. Beginning with $\varphi_0=0$, one forms the sequence $\{\varphi_n\}$, where

$$\varphi_n=A(\varphi_{n-1},c). \tag{2.22}$$

According to the work of Sec. III, this approach is successful for sufficiently small c, with any choice of the step function $h(b)$. The sequence $\{\varphi_n\}$ converges to a solution, which is the only solution in a certain subset of an appropriate function space.

The solutions for small c, obtained in Sec. III, are analytic in c inside a circle with center at $c=0$. This suggests analytic continuation to larger c. The circle mentioned is *not* the circle of convergence of the power series about $c=0$. It is merely a region to which we are confined by the technical limitations of the iterative method. In order to reach larger c we employ the numerical methods of paper II,[10] since we are not yet able to locate singularities in the c plane by analytic means.

It is possible to compute the initial terms in the power series for $\varphi(b)$. We put

$$\varphi(b;c)=\sum_{n=1}^\infty c^n\varphi^{(n)}(b), \tag{2.23}$$

and substitute in (2.20) to obtain

$$\varphi^{(1)}=H(\hat{h}^2),$$
$$\varphi^{(2)}=\varphi^{(1)}+H(\tfrac{1}{2}\hat{h}^3-4\hat{h}H(h\varphi^{(1)})). \tag{2.24}$$

In fact, any $\varphi^{(k)}$ may be expressed in terms of the lower terms $\varphi^{(l)}$, $l<k$. In the case of the simple step function $h(b)=\theta(r-b)$, we have $\hat{h}=rJ_1(rx)/x$, and, therefore,

$$\varphi^{(1)}(b) = r^2\int_0^\infty x^{-1}J_1{}^2(rx)J_0(bx)dx$$

$$= \begin{cases} -\frac{b}{4\pi}(4r^2-b^2)^{1/2}+r^2\left[\frac{1}{2}-\frac{1}{\pi}\sin^{-1}\left(\frac{b}{2r}\right)\right], & b<2r \\ 0, \quad b>2r \end{cases} \tag{2.25}$$

$$\frac{d\varphi^{(1)}(b)}{db} = \begin{cases} -\frac{1}{2\pi}(4r^2-b^2)^{1/2}, & b<2r \\ 0, & b>2r. \end{cases} \tag{2.26}$$

The first-order function $\varphi^{(1)}(b)$ decreases monotonically between $b=0$ and $b=2r$, with

$$\varphi^{(1)}(0)=r^2/2, \quad \varphi^{(1)}(2r)=0,$$

$$\varphi^{(1)\prime}(0)=-r/\pi, \quad \varphi^{(1)\prime}(2r)=0.$$

The second derivative of $\varphi^{(1)}$ has an inverse-square-root singularity at $b=2r$. This latter feature may occur also in the exact solutions obtained in Sec. III. We have no reason to expect that the power series (2.23) will provide a practical means of computing solutions for interesting values of c. Nevertheless, the formulas (2.25) and (2.26) will be useful later.

It is important to note the transformation properties of solutions of the integral equation under a change of scale.[14] The product bx is dimensionless, so a scale change $b\to\lambda b$ is accompanied by the change $x\to\lambda^{-1}x$. The constant c has the dimension of b^{-2}, while $f(x)$ has the dimension of b^2, and $\hat{f}(b)$ is dimensionless. Then it is easy to see that if $\hat{f}(b)$ is a solution of (2.17) with parameter $c\neq 0$, then $\hat{f}(\lambda b)$ is also a solution, but with parameter $\lambda^{-2}c$, where λ is any positive number. Similarly, if $f(x)$ is a solution of (1.13) with parameter $c\neq 0$, then $\lambda^2 f(\lambda^{-1}x)$ is also a solution with parameter $\lambda^{-2}c$.

The scaling property simplifies the problem of finding "all" solutions. Suppose, for example, that $h(b)=\theta(r-b)$. Then we have two parameters, r and c, on which solutions depend. We need not solve the equation for all r *and* all c to cover effectively the entire parameter space. Suppose that we can find a solution manifold of (1.13),

$$f(x;r_0,c), \quad 0<c<\infty \tag{2.27}$$

where $r_0>0$ has some fixed value. Then we immediately have a solution for any desired parameters (r_1, c_1), namely,

$$\tilde{f}(x;r_1,c_1)=\left(\frac{r_1}{r_0}\right)^2 f\left(\frac{r_0}{r_1}x;r_0,\left(\frac{r_1}{r_0}\right)^2 c_1\right). \tag{2.28}$$

Thus, when we calculate solutions by continuation in c, starting at $c=0$, which particular value of r we choose is irrelevant.

III. EXISTENCE THEOREM FOR ITERATIVE SOLUTIONS AT SMALL c

We intend to solve Eq. (2.20). Written out in full, it reads

$$\varphi(b)=\varphi^2(b)+\int_0^\infty x\,dx\,J_0(bx)E(x)\left[e^{cE(x)}-1\right], \tag{3.1}$$

$$E(x)=\int_0^\infty b\,db\,J_0(bx)[h(b)-\varphi(b)]^2, \tag{3.2}$$

where $h(b)$ is the step function (2.18). In order to simplify notation, we assume in the following that $h(b)$ has just a single step; i.e., that $h(b)=\theta(r-b)$. It will be clear that our proof applies as well to the general step function (2.18), with a finite number of finite closed intervals I. We employ some elementary methods of nonlinear analysis. A physicist's introduction to these methods may be found in the first paper of Ref. 5. In particular, we need the *contraction mapping principle*: Let A be a mapping of a complete metric space K into itself, and suppose that A is *contractive*; i.e.,

$$d(A\varphi, A\psi)\leq\beta d(\varphi,\psi), \qquad 0<\beta<1; \quad \text{all } \varphi,\psi\in K \tag{3.3}$$

where $d(\varphi,\psi)$ denotes the distance between φ and ψ. Then there is a unique fixed point of the mapping in K, which is to say a unique solution in K of the equation

$$\varphi=A\varphi. \tag{3.4}$$

The solution is the limit of the iterative sequence $\varphi_n=A\varphi_{n-1}$, which begins with any element φ_0 of K. The error at the nth iteration is bounded as follows:

$$d(\varphi,\varphi_n)\leq\frac{\beta^n}{1-\beta}d(\varphi_1,\varphi_0). \tag{3.5}$$

In the present problem, $A(\varphi)$ will be the right-hand side of Eq. (3.1), and the complete metric space will be a closed subset of a certain Banach space, the distance being provided by the norm in Banach space:

$$d(\varphi,\psi)=\|\varphi-\psi\|. \tag{3.6}$$

The Banach space B consists of all real functions $\varphi(b)$ on the line $0\leq b<\infty$ which are continuously differentiable, which have continuous second derivatives except at $b=0$ and $b=2r$, and for which the following quantity, the norm in B, exists:

$$\|\varphi\| = \sup_{0\le b<\infty} [\,|b_+^{5/2}\varphi(b)| + |b_+^{5/2}\varphi'(b)|\,] + \sup_{\substack{0<b<2r\\ 2r<b<\infty}} |b_+^{3/2} b^{1/2}(b-2r)^{1/2}\varphi''(b)|\,. \tag{3.7}$$

Here, $b_+ = b+1$, and "sup" means "least upper bound." All functions in B and their derivatives have bounds as follows:

$$|\varphi(b)| \le \frac{\|\varphi\|}{b_+^{5/2}},\quad |\varphi'(b)| \le \frac{\|\varphi\|}{b_+^{5/2}},\quad |\varphi''(b)| \le \frac{\|\varphi\|}{b_+^{3/2} b^{1/2}|b-2r|^{1/2}}\,. \tag{3.8}$$

The metric K will be merely a closed ball centered at the origin in B; i.e., it consists of all φ in B such that

$$\|\varphi\| \le \Phi\,, \tag{3.9}$$

where $\Phi>0$ is a fixed-ball radius. It will be shown that A maps K into itself and is contractive provided Φ and the parameter c of (3.1) are sufficiently small. Thus, there will be a unique solution of our integral equation satisfying (3.9). Of course, there may be other solutions which do not satisfy (3.9).

To show that A maps K into itself, we first analyze the integral (3.2), assuming that φ is any element of K. We must bound the asymptotic behavior of $E(x)$ and its first two derivatives, in order to show that the x integral in (3.1) belongs to K. This is done in Appendix B, where we obtain

$$E(x) = J(x) + K(x)\,, \tag{3.10}$$

$$J(x) = r(1-2\varphi(r))\frac{J_1(rx)}{x}\,, \tag{3.11}$$

$$K, K', K'' = O(x_+^{-5/2}\Phi)\,,\quad x_+ = x+1\,. \tag{3.12}$$

Also K'' is continuous (in fact, Hölder-continuous) at all x. Here and in the following, $F(x) = O(G(x))$ means $|F(x)| \le MG(x)$, for some fixed $M>0$ and *all* x.

To show that the x integral in (3.1) is in K, we must now treat the leading term of the integrand, which behaves at infinity as $[J_1(rx)/x]^2$, separately from the rest. We integrate that term explicitly, since upper bounds obtained by taking absolute values would not be sufficiently good. We isolate the leading term and bound the remainder as follows:

$$e^{cE} - 1 = cE + G\,, \tag{3.13}$$

$$|G| \le \tfrac{1}{2}(cE)^2 e^{c\,\sup|E|} = O(c^2 x_+^{-3})\,. \tag{3.14}$$

Similarly,

$$G' = cE'(cE+G) = O(c^2 x_+^{-3})\,,\quad G'' = cE''(cE+G) + cE'(cE'+G') = O(c^2 x_+^{-3})\,. \tag{3.15}$$

The integrand of (3.1) contains the factor

$$E(e^{cE}-1) = cJ^2 + cH\,, \tag{3.16}$$

where

$$cH = 2cJK + cK^2 + EG\,. \tag{3.17}$$

From (3.10) to (3.17), it follows that

$$H, H', H'' = O(x_+^{-4})\,. \tag{3.18}$$

When the first term of (3.16) is substituted in (3.1), we obtain the integral (2.25) that occurs in the first term of the power series for $\varphi(b)$. By (2.25) and (2.26), this integral is a member of the Banach space B.

To bound the contribution to (3.1) of the remainder cH, we can merely repeat our analysis of the integral E_2 (Appendix B), substituting x for b and H for φ^2; (the behavior of H is not quite as good as that of φ^2, but still good enough). The result is

$$I, I' = O(cb_+^{-5/2})\,,\quad I'' = O(cb^{-1/2}b_+^{-2})\,; \tag{3.19}$$

$$I(b) = c\int_0^\infty x\,dx\,J_0(bx)H(x,c)\,. \tag{3.20}$$

Also, $I''(b)$ is Hölder-continuous, except at $b=0$.

Now we may collect our results to see that the ball K of (3.9) is mapped into itself by the operator A, provided the constant c and the ball radius Φ are sufficiently small. The integral in (3.1) is in B, as we have shown above, and is sufficiently small by virtue of the factor of c in (3.16). The term φ^2 in (3.1) is in B, and is sufficiently small since its bound is proportional to the square of the ball radius.

It remains to show that the operator A is contractive. Let φ_1 and φ_2 be any two elements of K. To evaluate the first term in $\|A\varphi_1 - A\varphi_2\|$ we must treat

$$A\varphi_1 - A\varphi_2 = \varphi_1^2(b) - \varphi_2^2(b) + \int_0^\infty x\,dx\,J_0(bx)[F_1(x) - F_2(x)]\,, \tag{3.21}$$

where, according to (3.16),

$$F_1 - F_2 = cr^2[-4 + \varphi_1(r) + \varphi_2(r)][\varphi_1(r) - \varphi_2(r)] \times [J_1(rx)/x]^2 + c(H_1 - H_2)\,. \tag{3.22}$$

After some calculation (Appendix C), one finds that

$$H_1 - H_2, H_1' - H_2', H_1'' - H_2'' = O(x_+^{-4} \| \varphi_1 - \varphi_2 \|) . \tag{3.23}$$

From our previous analysis of integrals, it then follows that

$$\left\| \int_0^\infty x\,dx\,J_0(bx)[F_1(x) - F_2(x)] \right\| \le M_1 c \| \varphi_1 - \varphi_2 \| , \tag{3.24}$$

where M_1 is some constant. It is immediate to show that

$$\| \varphi_1^2 - \varphi_2^2 \| \le M_2 \Phi \| \varphi_1 - \varphi_2 \| , \tag{3.25}$$

so from (3.21) we have

$$\| A\varphi_1 - A\varphi_2 \| \le (M_1 c + M_2 \Phi) \| \varphi_1 - \varphi_2 \| . \tag{3.26}$$

The operator is contractive when c and Φ are sufficiently small. The proof is complete, so that we are assured of the existence of a unique solution in the ball K, when c and Φ are small.

The analysis above leads immediately to the result that $\varphi(b, c)$ is analytic in c for each b and $|c| < c_0$. Note that the proof can be done as well for complex c, if the space B is replaced by a similar, but complex, Banach space B_c. In the complex case, let c_0 be such that when $|c| < c_0$, the equation has a solution given by the limit of the iteration sequence $\{\varphi_n\}$. Since $\|\varphi_n\| < \Phi$ [Eq. (3.9)], the sequence of entire functions of c, $\{\varphi_n(b, c)\}$, is uniformly bounded and convergent for $|c| < c_0$, at each b. By Vitali's theorem of complex function theory, it follows that $\varphi(b, c)$ is analytic for $|c| < c_0$, at any b. An alternative proof of analyticity may be done by using the implicit function theorem[6] to show that the Cauchy-Riemann conditions hold.

IV. FRÉCHET DERIVATIVE OF THE INTEGRAL OPERATOR, AND CONTINUATION OF SOLUTIONS TO LARGER c

We rewrite our equation $\varphi = A(\varphi, c)$ as

$$F(\varphi, c) = \varphi - A(\varphi, c) = 0 . \tag{4.1}$$

For the question of dependence of solutions on c, we are interested in the Fréchet derivative (i.e., the "variational derivative") of F with respect to φ. This derivative, evaluated at (φ, c), $\varphi \in B$, is a linear operator in B, denoted by $F_\varphi(\varphi, c)$. Similarly, $F_c(\varphi, c)$ will be the derivative with respect to c. By formal variation of (3.1) and a change of integration order, we find

$$F_\varphi(\varphi, c)\delta\varphi = (1 - 2\varphi)\delta\varphi + \int_0^\infty K(b, b'; \varphi, c)\delta\varphi(b')db' , \tag{4.2}$$

where

$$K(b, b'; \varphi, c) = 2\int_0^\infty x\,dx\,J_0(bx)J_0(b'x) \times [e^{cE}(1 + cE) - 1][h(b') - \varphi(b')]b' . \tag{4.3}$$

It is not difficult to show that (4.2) is the actual (as well as formal) Fréchet derivative, and that $K(b, b'; \varphi, c)$ is the kernel of a completely continuous operator in B, when $\varphi \in B$. The Fréchet derivative with respect to c is an element of B having the form

$$F_c(\varphi, c) = \int_0^\infty x\,dx\,J_0(bx)E^2(x)e^{cE(x)} . \tag{4.4}$$

If $F_\varphi(\varphi_0, c_0)$ is nonsingular (i.e., it possesses an inverse in B), the implicit function theorem[6] guarantees that there is a solution $\varphi(c)$ which depends continuously on c, and which agrees with φ_0 at c_0; i.e.,

$$F(\varphi(c), c) = 0, \quad \varphi(c_0) = \varphi_0 . \tag{4.5}$$

This is true for c in a sufficiently small neighborhood of c_0. Furthermore, there are continuation theorems[6] to the effect that there exists a continuous "solution curve" $\varphi(c)$ which extends at least as far as the first singularity of F_φ. This curve does not cross itself: If $c_1 \ne c_2$, then $\varphi(c_1) \ne \varphi(c_2)$.

We may hope, then, to extend our solutions of Sec. III as far as the first singularity of the operator (4.2). As long as the factor $1 - 2\varphi(b)$ in (4.2) has no zero, the operator is effectively a standard Fredholm operator. We merely divide through by $1 - 2\varphi$ to put it in standard Fredholm form. Singularities will then occur only when the following kernel has a unit eigenvalue:

$$L(b, b') = \frac{-K(b, b', \varphi, c)}{1 - 2\varphi(b)} . \tag{4.6}$$

A second way to get a singularity arises when $1 - 2\varphi$ has a zero. Then we do not have a standard Fredholm problem, but rather an integral operator of the "third kind." Such an operator will not, in general, have an inverse in a space of continuous functions. By examining the finite dimensional analog of this case, one guesses that the operator does have an inverse in a space of generalized functions of the form

$$c\delta(b - b_0) + \psi(b) , \tag{4.7}$$

where b_0 is the point at which $1 - 2\varphi(b)$ vanishes, and $\psi(b)$ is continuous. Conditions for this guess to be true are given in Ref. 9. This space cannot be used in analysis of our nonlinear equation, since we would encounter squares of the delta function.

In paper II we shall find by numerical calculation that a singularity of the derivative does arise. We have not been able to pass through the singularity, which appears long before the physical value of $c\sigma_{el}$ is reached. Thus, we are not able to attach any physical significance to the class of solutions found in this paper, at least in the case where $h(b)$ is the simple step $\theta(r-b)$. The numerical calculation was done with the simple step; it is conceivable that some more complicated choice of $h(b)$ would be more successful.

In the second paper of Ref. 2, the existence of a different sort of solution of (1.13) is conjectured. This solution is continuous in b, in contrast to our $\hat{f}(b)$ which contains step functions. Unfortunately, we are not able to say much about this conjecture by analytic means. The only solution continuous in b that we know of is $\hat{f}(b)\equiv 0$; (the contraction mapping argument carried out in a space of *small* continuous functions would lead to this trivial solution). We do try to study the conjectured continuous solution numerically in paper II.

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APPENDIX A

In Sec. III, we have obtained solutions of Eq. (3.1) which have continuous second derivatives (except at $b=0$ and $b=2r$) and which obey the bounds (3.8). We now prove that any such solution yields a solution of the original Ball-Zachariasen Eq. (1.13). First recall that the following equation is the same as (3.1):

$$\hat{f}(b)=\hat{f}^2(b)+H(H(\hat{f}^2)(e^{cH(\hat{f}^2)}-1)), \tag{A1}$$

$$\hat{f}(b)=h(b)+[1-2h(b)]\varphi(b). \tag{A2}$$

To go back to x space, we may use Hankel's inversion theorem,[12] namely, if the integral

$$\int_0^\infty \psi(x)x^{1/2}\,dx \tag{A3}$$

exists and converges absolutely, then

$$\int_0^\infty bdbJ_0(by)\int_0^\infty xdx\,J_0(bx)\psi(x) =\tfrac{1}{2}[\psi(y+0)+\psi(y-0)]. \tag{A4}$$

Apply (A4) to the continuous function

$$\psi(x)=H(\hat{f}^2)(e^{cH(\hat{f}^2)}-1), \tag{A5}$$

which is $O(x_+^{-3})$, according to (3.39) and (3.40). By (A4), $H(H\psi)=\psi$, so the Hankel transform of (A1) reads as follows:

$$H\hat{f}=H(\hat{f}^2)e^{cH(\hat{f}^2)}. \tag{A6}$$

Thus, $H\hat{f}$ solves Eq. (1.13) provided that $H(\hat{f}^2)$ is equal to the function g of (2.2) evaluated with $f=H\hat{f}$. In view of (2.4), this means that we must prove the identity

$$\int_0^\infty bdbJ_0(bx)\left[\int_0^\infty xdx\,J_0(bx)H\hat{f}(x)\right]^2 =\int_0^\infty x_1dx_1H\hat{f}(x_1)\int_0^\infty x_2dx_2H\hat{f}(x_2) \times\int_0^\infty bdb\,J_0(bx_1)J_0(bx_2)J_0(bx). \tag{A7}$$

We write $f(x)$ for $H\hat{f}(x)$, and use the following properties of f and $\hat{f}$ obtained from Sec. III:

$$f(x)=a\frac{J_1(rx)}{x}+O(x_+^{-5/2}),$$

$$f'(x)=a\left(\frac{J_1(rx)}{x}\right)'+O(x_+^{-5/2}),\quad a=\text{const} \tag{A8}$$

$$\hat{f}(b)=O(b_+^{-5/2}),\quad \hat{f}'(b)=O(b_+^{-5/2});$$

$$f'\text{ continuous},\quad \hat{f}'\text{ piecewise continuous}. \tag{A9}$$

These conditions are a good deal stronger than necessary for the proof of (A7).

Since the b integral on the right-hand side of (A7) is only marginally convergent, the problem is somewhat delicate. Our technique will be to use partial integrations to improve the convergence, so that the integration order may be reversed on grounds of absolute convergence.

We first justify a reversal in order of the x_2 and b integrations in (A7); i.e., we prove that the following integrals are equal:

$$I_1=\int_0^\infty x_2dx_2f(x_2)\int_0^\infty bdbJ_0(bx_2)J_0(bx_1)J_0(bx), \tag{A10}$$

$$I_2=\int_0^\infty bdbJ_0(bx_1)J_0(bx)\int_0^\infty x_2dx_2f(x_2)J_0(bx_2). \tag{A11}$$

After a partial integration using (3.12), the b integral of (A10) takes the form

$$-\frac{1}{x_2}\int_0^\infty bdbJ_1(bx_2)\frac{d}{db}[J_0(bx_1)J_0(bx)]. \tag{A12}$$

After substituting (A12), we *formally* reorder the integrals in (A10) to obtain

$$I_3=-\int_0^\infty bdb\,\frac{d}{db}[J_0(bx_1)J_0(bx)]\int_0^\infty dx_2 f(x_2)J_1(bx_2)\,. \tag{A13}$$

We now show that $I_3=I_2$, and later that $I_3=I_1$. By partial integration I_3 becomes

$$I_3=\int_0^\infty db\,J_0(bx_1)J_0(bx)\frac{d}{db}\left[b\int_0^\infty dx_2 f(x_2)J_1(bx_2)\right]. \tag{A14}$$

We can prove that the derivative indicated in (A14) is piecewise continuous, and may be evaluated by differentiating under the integral sign. In that case,

$$\frac{d}{db}\left[b\int_0^\infty dx_2 f(x_2)J_1(bx_2)\right]=b\int_0^\infty x_2\,dx_2 f(x_2)J_0(bx_2)\,, \tag{A15}$$

because of the identity $J_1(z)+zJ_1{}'(z)=zJ_0(z)$. The desired result $I_3=I_2$ follows. To justify the differentiation under the integral, we first remark that there is no difficulty for the part of $f(x)$ which is $O(x_+{}^{-5/2})$; the justification is made by standard means. The remainder of $f(x)$, proportional to $J_1(rx)/x$, is handled by explicit evaluation of the integral. The relevant integrals are in standard books, and the result is that

$$\frac{d}{db}\int_0^\infty \frac{dx}{x}J_1(rx)J_1(bx)=\int_0^\infty dx\,J_1(rx)J_1{}'(bx) \tag{A16}$$

$$=\begin{cases}\dfrac{1}{2r}, & 0<b\le r\\[4pt] -\dfrac{r}{2b^2}, & r\le b<\infty\,.\end{cases} \tag{A17}$$

For the proof that $I_3=I_1$, first integrate by parts on x_2:

$$\int_0^\infty dx_2 f(x_2)J_1(bx_2)=\frac{f(0)}{b}+\frac{1}{b}\int_0^\infty dx_2 f'(x_2)J_0(bx_2)\,. \tag{A18}$$

Since $J_0(0)=1$, we obtain

$$I_3=f(0)-\int_0^\infty db\,\frac{d}{db}[J_0(bx_1)J_0(bx)]\times\int_0^\infty dx_2 f'(x_2)J_0(bx_2)\,. \tag{A19}$$

Reordering of the integrals in (A19) is now permitted, since the repeated integral converges absolutely [cf. (A8)]. We reorder, then, and integrate by parts on b to obtain

$$I_3=-\int_0^\infty x_2\,dx_2 f'(x_2)\int_0^\infty J_1(bx_2)J_0(bx_1)J_0(bx)db\,. \tag{A20}$$

By introducing the known value of the b integral in (A20) (Ref. 11, p. 411), and by a final partial integration on x_2, we find

$$I_3=\frac{2}{\pi}\int_{|x-x_1|}^{x+x_1}\frac{x_2dx_2f(x_2)}{K^{1/2}}\,, \tag{A21}$$

where the denominator $K^{1/2}$ is the same that occurs in Sonine's formula (2.3). By (2.3), it follows that $I_3=I_1$.

To complete the proof of (A7), we are concerned with showing that the following integrals are equal:

$$I_4=\int_0^\infty x_1dx_1 f(x_1)\int_0^\infty bdb\,J_0(bx_1)J_0(bx)\hat{f}(b)\,, \tag{A22}$$

$$I_5=\int_0^\infty bdb\,J_0(bx)\hat{f}(b)^2\,. \tag{A23}$$

This is done by essentially the same means used to prove that $I_1=I_2$. That is, we integrate partially on b in (A22), and reverse the order formally to obtain

$$I_6=-\int_0^\infty bdb\,\frac{d}{db}[J_0(bx)\hat{f}(b)]\times\int_0^\infty dx_1 f(x_1)J_1(bx_1)\,. \tag{A24}$$

We then prove that $I_6=I_4$ and $I_6=I_5$, following the above pattern. The details are simpler than before.

APPENDIX B

We must majorize the integral of (3.2) which consists of three terms:

$$E(x)=E_0(x)-2E_1(x)+E_2(x)=\frac{rJ_1(rx)}{x}-2\int_0^r bdb\,J_0(bx)\varphi(b)+\int_0^\infty bdb\,J_0(bx)\varphi^2(b)\,. \tag{B1}$$

We suppose that φ belongs to the ball K, so that (3.8) and (3.9) hold. For all b and x we have the bound

$$x^{-n}J_n(bx)=O(b_+{}^{-1/2}x_+{}^{-1/2-n}),\quad n\ge 1$$
$$b_+=b+1,\quad x_+=x+1\,. \tag{B2}$$

The following identities will also be needed:

$$s^nJ_{n-1}(s)=[s^nJ_n(s)]'\,, \tag{B3}$$

$$\frac{d}{dx}[x^{-2}J_2(bx)] = -bx^{-2}J_3(bx)\,, \tag{B4}$$

$$\frac{d}{dx}[x^{-2}J_3(bx)] = \tfrac{1}{6}bx^{-2}[5J_4(bx) - J_2(bx)]\,. \tag{B5}$$

We do two partial integrations with the help of (B3) to obtain

$$\begin{aligned} E_1(x) &= \frac{rJ_1(rx)}{x}\varphi(r) - \frac{rJ_2(rx)}{x^2}\varphi'(r) \\ &\quad - \frac{1}{x^2}\int_0^r db\,J_2(bx)\varphi'(b) \\ &\quad + \frac{1}{x^2}\int_0^r b\,db\,J_2(bx)\varphi''(b)\,. \end{aligned} \tag{B6}$$

From (B2) and (3.8), it then follows that

$$E_1(x) = \frac{rJ_1(rx)}{x}\varphi(r) + O(x_+^{-5/2}\Phi)\,. \tag{B7}$$

By appealing to (B4) and (B5), we have similar bounds for derivatives:

$$\begin{aligned} E_1'(x) &= \left[\frac{rJ_1(rx)}{x}\right]'\varphi(r) + O(x_+^{-5/2}\Phi)\,, \\ E_1''(x) &= \left[\frac{rJ_1(rx)}{x}\right]''\varphi(r) + O(x_+^{-5/2}\Phi)\,. \end{aligned} \tag{B8}$$

For E_2 the partial integrations give

$$\begin{aligned} E_2(x) &= -\frac{1}{x^2}\int_0^\infty db\,J_2(bx)\varphi^{2\prime}(b) \\ &\quad + \frac{1}{x^2}\int_0^\infty b\,db\,J_2(bx)\varphi^{2\prime\prime}(b)\,. \end{aligned} \tag{B9}$$

From (3.8) and (3.9), we have

$$\begin{aligned} \varphi^2(b) &= O(b_+^{-5}\Phi^2), \\ \varphi^{2\prime}(b) &= O(b_+^{-5}\Phi^2)\,, \\ \varphi^{2\prime\prime}(b) &= O(b_+^{-4}b^{-1/2}|b-2r|^{-1/2}\Phi^2)\,. \end{aligned} \tag{B10}$$

When (B10) is used with (B9), (B4), and (B5), we obtain

$$E_2, E_2', E_2'' = O(x_+^{-5/2}\Phi^2)\,. \tag{B11}$$

Differentiation under the integrals in (B9) was justified by the uniform convergence of the differentiated integrals.

When we apply the above analysis to the integral (3.17), we have to use the weaker bounds (3.15) in place of (B10). The bounds are still good enough to evaluate $I''(b)$ by differentiation under the integral, except near $b=0$. Since $J_1'(s) = O(s^{-1/2})$, the behavior near $b=0$ is bounded as follows:

$$|I''(b)| \leq \left| c\int_0^\infty x^3 dx\,J_1'(bx)H(x,c)\right| = O(cb^{-1/2})\,. \tag{B12}$$

Equation (3.19) is then established.

Finally, we note that $E''(x)$ is continuous at all x, and that $I''(b)$ is continuous except at $b=0$. For E_0'' and E_1'' this is clear, and for E_2'' we use the formula

$$E_2''(x) = -\int_0^\infty b^3\,db\,J_1'(bx)\varphi^2(b)\,. \tag{B13}$$

By the mean-value theorem,

$$\begin{aligned} |E_2''(x) - E_2''(y)| &\leq \int_0^\infty b^3\,db\,\varphi^2(b)|J_1'(bx) - J_1'(by)|^{1-\delta} \\ &\quad \times |J_1''(b\xi)|^\delta b^\delta |x-y|^\delta\,. \end{aligned} \tag{B14}$$

Since J_1' and J_1'' are uniformly bounded, (B10) gives

$$|E_2''(x) - E_2''(y)| \leq A|x-y|^\delta, \quad 0<\delta<1\,, \tag{B15}$$

for all x, y. For $I''(b)$ a similar argument is used, but convergence from $J_1'(bx)$, $J_1''(bx) = O((bx)^{-1/2})$ must be invoked to give

$$|I''(b) - I''(b')| \leq A|b-b'|^\delta, \quad 0<\delta<\tfrac{1}{2}\,, \tag{B16}$$

for all $b, b' > \epsilon > 0$.

APPENDIX C

To establish the bounds (3.20), we use the definition of H, (3.17), to calculate the following difference:

$$\begin{aligned} c(H_1 - H_2) &= 2c(J_1 - J_2)K_2 + 2c(K_1 - K_2)J_1 \\ &\quad + c(K_1 - K_2)(K_1 + K_2) \\ &\quad + (E_1 - E_2)G_2 + (G_1 - G_2)E_1\,. \end{aligned} \tag{C1}$$

From Sec. III, we know that

$$\begin{aligned} J_1 - J_2 &= O(x_+^{-3/2}\|\varphi_1 - \varphi_2\|)\,, \\ K_1 - K_2 &= O(x_+^{-5/2}\|\varphi_1 - \varphi_2\|)\,, \\ E_i &= O(x_+^{-3/2}), \quad G_i = O(x_+^{-3}c^2), \\ K_i &= O(x_+^{-5/2})\,, \end{aligned} \tag{C2}$$

and that similar bounds hold for the derivatives of these quantities. Hence the first four terms of (C1), and the derivatives thereof, are $O(cx_+^{-4}\|\varphi_1 - \varphi_2\|)$. The last term is handled by noting the identity

$$G_1 - G_2 = c(E_1 - E_2)\int_0^1 [e^{c[uE_1 + (1-u)E_2]} - 1]\,du\,. \tag{C3}$$

The integral in (C3) and its first two derivatives are $O(cx_+^{-3/2})$; Eq. (3.20) follows.

APPENDIX D

To prove the statements of Eqs. (2.17a) and (2.17b), we first note that $|J_0(bx)| \leq 1$, so that

$$g(0) - |g(x)| \geq \int_0^\infty bdb[1 - |J_0(bx)|][Hf]^2 \geq 0. \tag{D1}$$

From this assertion (2.17a) follows. If $f(x) \geq 0$ [and, hence, $g(x) \geq 0$], then the same argument gives

$$|\hat{f}(b)| \leq \hat{f}(0), \quad |\hat{g}(b)| \leq \hat{g}(0). \tag{D2}$$

Also,

$$\hat{f}(0) = \int_0^\infty x dx\, g(x) e^{cg(x)} \geq \int_0^\infty x dx\, g(x) = \hat{f}(0)^2. \tag{D3}$$

Therefore,

$$\hat{f}(0) \leq 1, \quad \hat{g}(0) = \hat{f}(0)^2 \leq 1. \tag{D4}$$

Finally, note that

$$\hat{f}(0) \leq e^{cg(0)} \int_0^\infty x dx\, g(x) = \hat{f}(0)^2 e^{cg(0)} \tag{D5}$$

or

$$\hat{f}(0) \geq e^{-cg(0)} = \sigma_{el}/\sigma_{tot}. \tag{D6}$$

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[1]See, for instance, P. M. Anselone and R. H. Moore, J. Math. Anal. Appl. 13, 476 (1966).

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[14]We thank Dr. G. R. Bart for pointing this out.