Broken Scale Invariance, Current Algebra, and Massive "Gravitation." I. General Formulation*

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A general analysis is given of the interaction of mesons of $J^P=0^\pm$, $1^\pm$, and 2^+ obeying the principles of broken scale invariance in the tree and seagull approximations. In analogy with current algebra, where one assumes that the vector and axial-vector currents are dominated by $J^P=1^\pm$ and $0^\pm$ mesons in a field-current identity, we assume that the stress tensor $\Theta^{\mu\nu}$ is dominated mainly by the 2^+ and 0^+ f, f', σ, and σ' mesons in a field–stress-tensor identity. A consistent formalism is seen to require also certain nonpole f-meson mass terms in $\Theta^{\mu\nu}$. With the usual smoothness conditions, the dynamics can be conveniently characterized by introducing an effective Lagrangian. The conservation law $\partial_\nu\Theta^{\mu\nu}=0$ and the Poincaré-group conditions then imply that (i) the $f^i_{\mu\nu}(x)$ fields ($i=1, 2, \ldots$) of the f, f', etc., mesons couple with all other "matter fields" (e.g., $J^P=0^\pm$, $1^\pm$ mesons) by making the usual matter Lagrangian of current algebra "generally covariant" by replacing the Lorentz metric $\eta_{\mu\nu}$ by a "metric" formed from $g_{\mu\nu a}\equiv\eta_{\mu\nu}+\sum_i\lambda_{ai}f^i_{\mu\nu}$ ($a=1, 2, \ldots$). (ii) The kinetic-energy part of the f-meson self-couplings must have the form of Einstein Lagrangians formed using the $g_{\mu\nu a}$, e.g., $\sqrt{-g_a}\,g^{\mu\nu}_a R_{\mu\nu a}$ where $R_{\mu\nu}$ is the contracted curvature tensor. (iii) Improvement for the spin-zero parts of the stress tensor is obtained by including "curvature" couplings, e.g., $\pi^2\sqrt{-g_a}\,R_a$, where R_a is the curvature scalar formed from $g_{\mu\nu a}$. In general, then, the f-meson couplings are analogous to very strong gravitational couplings, with the f-meson mass terms breaking the gravitational gauge invariance. For the situation where one has only one f meson present our "metric space" is analogous to that of Zumino. However, the "metric space" considered here is considerably more complicated than such a Riemannian space as more than one metric, $g_{\mu\nu a}$, $a=1, 2, \ldots$, is defined on it, and hence by algebraic combinations an infinite number of "metrics" exist. (We note that in general these metrics will depend nonlinearly on the f-meson fields.)

Broken scale invariance is introduced through a new postulate which requires that the improved Belinfante stress tensor and its trace play a fundamental role as sources of the $J^P=2^+$, 0^+ mesons with a universal coupling strength. The universality also leads to new relations of the type $g_f=F_\sigma m_f^2$, etc., between the f- and σ-meson interpolating constants which resemble the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin-type relations in current algebra. The form of the vector current in the presence of broken scale invariance is derived. The condition of scale breaking implies that the vector current has canonical scale dimension 3, and the apparent conflict of conservation of vector current with the f couplings is resolved. Experimental tests of the present formalism are indicated here and will be examined in detail in a subsequent paper.

I. INTRODUCTION

During the past two years, there has been a great deal of interest generated in the possibility that strong interactions possess a broken scale invariance.[1] This interest has in part arisen as a consequence of the observed scaling in the electroproduction data, for the hypothesis that physical laws are scale-invariant at high energies gives a natural explanation of the scaling obeyed by the electroproduction form factors in the deep-inelastic region. While this result suggests the attractive possibility that scale invariance holds rigorously at asymptotic energies, hadron interactions are certainly not scale-invariant at intermediate and low energies. Here dimensioned constants (masses and coupling constants) enter in an important way. Further, should it turn out that asymptotic scale invariance actually is of fundamental significance, the manner of its breakdown as one proceeds to lower energies is also of importance. In order to discuss this latter problem, it is thus necessary to examine hadron interactions in the intermediate- and low-energy regions.

The purpose of this paper is to construct a general formalism which attempts to describe meson interactions at intermediate and lower energies obeying the conditions of broken scale invariance and chiral current algebra. In previous analyses in this energy domain, much information about the symmetries and partial symmetries of current algebra could be obtained by making the single-meson-dominance approximation.[2] Thus in this procedure one dominates the $I=1$ vector current by the ρ meson, the $I=1$ axial-vector current by the A_1 and π mesons, the divergence of the axial-vector current by the π meson, etc. S-matrix elements are then reduced to calculating a specific set of tree and seagull diagrams (appropriately unitarized when necessary). The technique has been applied to a wide variety of phenomena with considerable success.

A similar program for broken-scale-invariant interactions would deal analogously with the stress tensor $\Theta^{\mu\nu}$. Thus one would dominate $\Theta^{\mu\nu}$ by the $J^P=2^+$ mesons [$f(1260)$ and $f'(1514)$], as well as the $J^P=0^+$ σ and σ' mesons [which we take tentatively as the $\eta_{0^+}(700)$ and $\eta_{0^+}(1060)$]. Scale breaking might be described by dominating the trace of the stress tensor[3] $\Theta\equiv\eta_{\mu\nu}\Theta^{\mu\nu}$ by the σ and σ' mesons. Much work along these lines has already been done.[4] However, analyses up to now have not been sufficiently complete to bring out the full nature of the couplings, implied by broken scale invariance, of the $J^P=2^+, 0^+$ mesons to themselves and to the other hadrons. Thus in previous work only a single f meson is assumed to exist, or if both f mesons are included they are treated only in the first (i.e., linearized) approximation. Also, the spin-0 parts of $\Theta^{\mu\nu}$ are neglected (or inaccurately treated). As will be seen below, the presence of more than one f meson leads to a much more complex formalism than the single f-meson case for the nonlinear f-meson interactions. A correct inclusion of the spin-0 pole parts of $\Theta^{\mu\nu}$ leads in a natural fashion to dilatonlike scale transformation properties for the σ and σ'. Most importantly, it is seen that when the nonlinear f interactions are included, the simple assumption above that the trace of the stress tensor is pole-dominated by the σ and σ' mesons becomes inconsistent. *A new principle of scale breaking is proposed based on a universal coupling of the $J^P=2^+,0^+$ mesons to the (improved) Belinfante stress tensor and its trace.* Thus while scale invariance is lost at lower energies, the breaking occurs in a universal fashion. The universality requirement leads to new "Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin- (KSRF)-type" relations between the spin-2 and spin-0 interpolating constants

$$g_f=F_\sigma m_f^{\,2},\quad g_{f'}=F_\sigma m_{f'}^{\,2},\quad F_\sigma=F_{\sigma'},\qquad(1.1)$$

where g_f is the coupling strength of $\Theta^{\mu\nu}$ to the f meson, F_σ is the coupling strength of $\Theta^{\mu\nu}$ to the σ meson, etc. These are in quite good agreement with the f-meson data (which also imply $F_\sigma\cong F_\pi$).

To carry out the above program, we make use of the technique of introducing an effective Lagrangian. As is well known, with the usual smoothness assumptions, this method is equivalent to the Ward-identity technique.[2] For our considerations here it is more convenient than the latter, however, in that the construction of the single nonlinear Lagrangian corresponds in effect to the solution of an infinite number of Ward identities. Further, a Lagrangian approach allows more easily physical insights into questions involving symmetries and symmetry breakdowns. In Sec. II the effective Lagrangian is set up for an arbitrary number of $J^P=2^+$ mesons interacting with other hadrons. The full conditions of the conservation of $\Theta^{\mu\nu}$ and the Poincaré-group commutators are then imposed on this Lagrangian. As mentioned above, the allowed nonlinear interactions so obtained are quite complicated, and correspond to a Riemannian space with an *infinite number* of "metrics" defined upon it (formed from the f-meson fields). Section III illustrates these results for a simple example involving f, π, and σ couplings. The improvement of the usual Belinfante couplings arises by coupling the pion to the curvature scalar formed from the f-meson "metric". The dilaton nature of the σ meson and noncanonical dimensions of the interacting pion field are seen to arise naturally from the formalism. Section IV introduces the scale-breaking condition for the general situation of more than one f meson and more than one σ meson being present. The scale breaking is also characterized there in terms of a breakdown of Weyl gauge invariance. When f-meson interactions were neglected, it was a valid approximation to apply pole dominance to the vector and axial-vector currents of current algebra.[2] However, just as the nonlinear f couplings prevent the pole dominance of Θ, the nonlinear f and σ couplings (the latter arising from the scale-breaking conditions) force specific nonpole terms involving f and σ mesons in the currents. These terms are briefly discussed in Sec. V, where it is seen that they can account for the sustaining of the e^+-e^- annihilation cross section in the 2–3-GeV region.

II. FIELD-CURRENT IDENTITY FOR $\Theta^{\mu\nu}$, CONSERVATION, AND POINCARÉ-GROUP CONSTRAINTS

A. Field-Current Identity

Any appropriately constructed symmetric stress tensor must obey the local conservation law

$$\partial_\mu\Theta^{\mu\nu}=0\qquad(2.1)$$

as well as leading to $P^\mu\equiv\int d^3x\,\Theta^{0\mu}$ and $M^{\mu\nu}\equiv\int d^3x[x^\mu\Theta^{0\nu}-x^\nu\Theta^{0\mu}]$ obeying the Poincaré-group relations. We refer to Eq. (2.1) as the CTC condition (conservation of the "tensor current") and it plays a role analogous to the CVC condition for vector currents. For the latter case, the conservation condition can be achieved by first choosing the vector current $V_a^\mu(x)$ as the interpolating field for the ρ meson:

$$V_a^\mu(x) = g_\rho \rho_a^\mu(x), \quad g_\rho = \text{const.} \tag{2.2}$$

The effective Lagrangian for ρ-meson interactions (in first-order formalism) is

$$\mathcal{L} = -\tfrac{1}{2}\rho_a^{\mu\nu}(\partial_\mu \rho_{\nu a} - \partial_\nu \rho_{\mu a}) - \tfrac{1}{2}m_\rho^2 \rho_a^\mu \rho_{\mu a} + \tfrac{1}{4}\rho_{\mu\nu a}\rho_a^{\mu\nu} + \mathcal{L}_I, \tag{2.3}$$

where $\mathcal{L}_I$ is the interaction Lagrangian. The field equations are

$$\partial_\nu \rho_a^{\mu\nu} + m_\rho^2 \rho_a^\mu = J_a^\mu; \quad J_a^\mu \equiv \delta\mathcal{L}_I/\delta\rho_a^\mu. \tag{2.4}$$

The field-current identity (2.2) then leads to a conserved vector current provided $\mathcal{L}_I$ is constructed so that J_a^μ is conserved. Thus CVC constrains the form of $\mathcal{L}_I$.

One may proceed similarly for the stress tensor. Consider first the part of $\Theta^{\mu\nu}$ that can be used as an interpolating field for the f meson. If $f_{\mu\nu}(x)$ is the phenomenological f-meson field, $\Theta^{\mu\nu}$ could in general be a linear combination of $f^{\mu\nu}$ and $\eta^{\mu\nu}f$, where $f \equiv \eta^{\alpha\beta}f_{\alpha\beta}$. To find the proper linear combination we note that in the vector case, Eq. (2.2) reads $V^\mu = -(g_\rho/m_\rho^2)(\partial\mathcal{L}_m/\partial\rho_\mu)$, where $\mathcal{L}_m$ is the ρ-meson mass term in Eq. (2.3). For the free f-meson field, the mass term leading to an irreducible spin-2 representation of the Lorentz group may be taken to be[5]

$$\mathcal{L}_m^{(2)} = -\tfrac{1}{2}m^2(f^{\mu\nu}f_{\mu\nu} - f^2), \tag{2.5}$$

where m is the f-meson mass. The f-meson contribution to $\Theta^{\mu\nu}$ may be chosen then to be $-(g_f/m^2) \times(\partial\mathcal{L}_m^{(2)}/\partial f_{\mu\nu})$, and leads to a field-current f-meson piece of

$$g_f(f^{\mu\nu} - \eta^{\mu\nu}f), \quad g_f = \text{const.} \tag{2.6}$$

Here g_f plays the same role as g_ρ did for the vector current. A term similar to Eq. (2.6) will appear in $\Theta^{\mu\nu}$ for each f meson.

While the conserved vector current of Eq. (2.2) can have only a spin-1 ρ field in its field-current identity, the existence of two tensor indices in $\Theta^{\mu\nu}$ allows for the presence of spin-zero σ fields as well as the spin-2 terms of Eq. (2.6). The CTC condition (2.1) requires that these fields enter $\Theta^{\mu\nu}$ with the structure

$$\tfrac{1}{3}F_\sigma(\eta^{\mu\nu}\Box^2 - \partial^\mu\partial^\nu)\sigma, \quad F_\sigma = \text{const.} \tag{2.7}$$

Equations (2.6) and (2.7) represent the f- and σ-meson pole parts of $\Theta^{\mu\nu}$. We now choose as our field-current identity for $\Theta^{\mu\nu}$ the general form

$$\Theta^{\mu\nu} = \sum_i g_i(f^{\mu\nu i} - \eta^{\mu\nu}f^i) + \tfrac{1}{3}\sum_a F_a(\eta^{\mu\nu}\Box^2 - \partial^\mu\partial^\nu)\sigma_a + M^{\mu\nu}, \tag{2.8}$$

where g_i is the interpolating constant for the ith $J^P = 2^+$ f meson, and F_a the interpolating constant for the ath $J^P = 0^+$ σ meson. Thus

$$\langle 0|\Theta^{\mu\nu}(0)|f^i, p, \lambda\rangle = g_i \epsilon_i^{\mu\nu}(p, \lambda)N_i, \tag{2.9}$$

where[6] $\epsilon_i^{\mu\nu}$ is the polarization tensor of the ith f meson of helicity λ, and

$$\langle 0|\Theta^{\mu\nu}(0)|\sigma_a, k\rangle = \tfrac{1}{3}F_a(k^\mu k^\nu + \eta^{\mu\nu}m_a^2)N_a. \tag{2.10}$$

The function $M^{\mu\nu}$ of Eq. (2.8) represents any terms nonlinear in the f-meson fields that may be needed for consistency. For the vector and axial-vector currents such nonpole terms may be set to zero. We will see below, however, that the CTC condition forces the presence of a nonzero $M^{\mu\nu}$ to correctly account for the f-meson mass contribution to $\Theta^{\mu\nu}$. Thus $M^{\mu\nu}$ is proportional to the f-meson masses squared m_i^2. The appearance of $M^{\mu\nu}$ is due to the basic nonlinearity of f-meson self-interactions required by CTC, and would not show up in any linearized treatment of the f-meson system.

B. CTC and Poincaré-Group Constraints

We examine first the case of a single f meson being present, and will generalize to the case of many f mesons in Sec. II C below. The total effective Lagrangian describing both the self-interaction of the f meson and its interaction with other fields has the general form

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_m + \mathcal{L}_M(f_{\mu\nu}, \chi_A). \tag{2.11}$$

The quantities $\mathcal{L}_f$ and $\mathcal{L}_m$ involve only the f-meson fields, with $\mathcal{L}_f$ representing the "kinetic energy" part of the f-meson Lagrangian and $\mathcal{L}_m$ the mass terms. One may write in general

$$\mathcal{L}_m = \mathcal{L}_m^{(2)} + \mathcal{L}_m' \tag{2.12}$$

where $\mathcal{L}_m^{(2)}$ is the free-field mass term of Eq. (2.5) and $\mathcal{L}_m'$ involves any cubic or higher mass interaction terms which may be present. Similarly one may write $\mathcal{L}_f = \mathcal{L}_f^{(2)} + \mathcal{L}_f'$, where $\mathcal{L}_f^{(2)}$ is the usual spin-2 free-field term.[5] In first-order formalism (with $f_{\mu\nu}$ and $\Lambda^\alpha{}_{\mu\nu}$ to be varied independently) one has

$$\mathcal{L}_f^{(2)} = -2(f^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu}f)[\partial_\alpha \Lambda^\alpha{}_{\mu\nu} - \partial_\nu \Lambda^\alpha{}_{\mu\alpha}] + 2\eta^{\mu\nu}[\Lambda^\alpha{}_{\mu\nu}\Lambda^\beta{}_{\alpha\beta} - \Lambda^\alpha{}_{\mu\beta}\Lambda^\beta{}_{\nu\alpha}]. \tag{2.13}$$

The term $\mathcal{L}_M$ contains the free Lagrangians of all fields $\chi_a(x)$ other than the f-meson, the interaction between the χ_a fields, and the interaction between the f and χ_a systems. We will collectively refer to the χ_a fields as the "matter fields".

Varying Eq. (2.11) yields the f-meson equations

$$-\frac{\delta \mathcal{L}_f}{\delta f_{\mu\nu}} + m^2(f^{\mu\nu} - \eta^{\mu\nu} f) = \frac{\delta \mathcal{L}_M}{\delta f_{\mu\nu}} + \frac{\delta \mathcal{L}'_m}{\delta f_{\mu\nu}} . \quad (2.14)$$

Using the field-current identity Eq. (2.8) (for the single f-meson case), one obtains an expression for $\Theta^{\mu\nu}$:

$$\Theta^{\mu\nu} = g_f m^{-2} \frac{\delta \mathcal{L}_M}{\delta f_{\mu\nu}} + g_f m^{-2} \frac{\delta \mathcal{L}_f}{\delta f_{\mu\nu}} + \left[M^{\mu\nu} + (g_f m^{-2}) \frac{\delta \mathcal{L}'_m}{\delta f_{\mu\nu}} \right] + \sum F_a H_a^{\mu\nu} , \quad (2.15)$$

where we have introduced the abbreviation

$$H_a^{\mu\nu} \equiv \tfrac{1}{3}[\eta^{\mu\nu}\Box^2 - \partial^\mu \partial^\nu]\sigma_a . \quad (2.16)$$

Now if $\Theta^{\mu\nu}$ is to be correctly conserved and yield a P^μ and $M^{\mu\nu}$ which generate the Poincaré group, it clearly can differ from the usual symmetric Belinfante stress tensor $\Theta_B^{\mu\nu}$ by at most a "superpotential" term.[7] As is well known, the Belinfante stress tensor for any Lagrangian $\mathcal{L}$ can be conveniently constructed by the following device: Replace the Lorentz metric $\eta_{\mu\nu}$ in $\mathcal{L}$ by a *fictitious external gravitational metric* $\bar{g}_{\mu\nu}(x)$, multiplying where necessary by factors of $(-\bar{g})^{1/2}$ (where $\bar{g} = \det \bar{g}_{\mu\nu}$) to form a new Lagrangian $\bar{\mathcal{L}}$ which is a scalar density under general coordinate transformations. ($\bar{\mathcal{L}}$ just represents the correct Lagrangian of the original system in the presence of the gravitational field.) Then the $\Theta_B^{\mu\nu}$ associated with Lagrangian $\mathcal{L}$ is

$$\Theta_B^{\mu\nu} = 2\left[\frac{\delta \bar{\mathcal{L}}}{\delta \bar{g}_{\mu\nu}(x)}\right]_{\bar{g}_{\mu\nu} = \eta_{\mu\nu}} . \quad (2.17)$$

In the following we will introduce the convenient abbreviation $2(\delta\mathcal{L}/\delta\eta_{\mu\nu})$ for the right-hand side of Eq. (2.17).

Returning now to $\Theta^{\mu\nu}$, the first three terms of Eq. (2.15) can be associated with the contributions to the Belinfante stress tensor arising from $\mathcal{L}_M$, $\mathcal{L}_f$, and $\mathcal{L}_m$, respectively. Thus if we write

$$g_f m^{-2} \frac{\delta \mathcal{L}_M}{\delta f_{\mu\nu}} = 2 \frac{\delta \mathcal{L}_M}{\delta \eta_{\mu\nu}} , \quad (2.18a)$$

$$g_f m^{-2} \frac{\delta \mathcal{L}_f}{\delta f_{\mu\nu}} = 2 \frac{\delta \mathcal{L}_f}{\delta \eta_{\mu\nu}} , \quad (2.18b)$$

$$M^{\mu\nu} + g_f m^{-2} \frac{\partial \mathcal{L}'_m}{\partial f_{\mu\nu}} = 2 \frac{\delta \mathcal{L}_m}{\delta \eta_{\mu\nu}} , \quad (2.18c)$$

the total $\Theta^{\mu\nu}$ will coincide, to within a superpotential, with the total Belinfante stress tensor, and hence automatically obey the conservation and Lorentz group conditions. Equations (2.18a) and (2.18b) may be directly integrated. To do this define an "f-meson metric" $g_{\mu\nu}(x)$ according to

$$g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + \lambda f_{\mu\nu} , \quad \lambda \equiv 2m^2/g_f . \quad (2.19)$$

Equations (2.18a) and (2.18b) then imply that $\eta_{\mu\nu}$ and $f_{\mu\nu}(x)$ enter into $\mathcal{L}_M$ and $\mathcal{L}_f$ *only in the combination* $g_{\mu\nu}(x)$. More precisely, the f-meson interactions in $\mathcal{L}_M$ are to be constructed by starting with the matter Lagrangian in the absence of f mesons (i.e., with $f_{\mu\nu}$ set to zero) *and introducing f mesons by formally forming a generally covariant scalar density using the f-meson metric* $g_{\mu\nu}(x)$ *of Eq. (2.19)*. Thus one replaces $\eta_{\mu\nu}$ by $g_{\mu\nu}(x)$ everywhere and multiplies by the appropriate factor of $(-g)^{1/2}$ to form a scalar density. For example, to introduce f-meson couplings into the pion mass and kinetic energy terms one modifies them in the following manner:

$$\begin{aligned} -\tfrac{1}{2} m_\pi{}^2 \pi_a \pi_a &\to -\tfrac{1}{2} m_\pi{}^2 \sqrt{-g}\, \pi_a \pi_a , \\ -\tfrac{1}{2} \eta^{\mu\nu} \partial_\mu \pi_a \partial_\nu \pi_a &\to -\tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu} \partial_\mu \pi_a \partial_\nu \pi_a , \end{aligned} \quad (2.20)$$

where $g^{\mu\nu}$ is the contravariant f-meson metric defined by $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$. The above result thus determines all the f-meson couplings to the matter variables to arbitrary order in terms of one coupling parameter λ.

In a similar fashion, Eq. (2.18b) implies that $\mathcal{L}_f$ is a generally covariant scalar density constructed purely from the f-meson metric $g_{\mu\nu}(x)$. Furthermore, to correctly represent the free spin-2 f meson, it must have a quadratic piece equal to $\mathcal{L}_f^{(2)}$ of Eq. (2.13). As is well known from Riemannian geometry, only the curvature scalar density satisfies these conditions.[8] Thus, define the "contracted curvature tensor" $R_{\mu\nu}$ by

$$R_{\mu\nu}(\Gamma^\alpha{}_{\beta\gamma}) = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} + \Gamma^\alpha{}_{\mu\nu}\Gamma^\beta{}_{\alpha\beta} - \Gamma^\alpha{}_{\mu\beta}\Gamma^\beta{}_{\nu\alpha} , \quad (2.21)$$

where $\Gamma^\alpha{}_{\beta\gamma}$ is an "affinity." Then in first-order formalism $\mathcal{L}_f$ must have the Palatini form

$$\mathcal{L}_f = (2/\lambda^2)\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma^\alpha{}_{\beta\gamma}) , \quad (2.22)$$

where $g_{\mu\nu}$ and $\Gamma^\alpha{}_{\beta\gamma}$ are to be varied independently. If one expands $\sqrt{-g}\, g^{\mu\nu}$ to terms linear in $f_{\mu\nu}$, one easily sees that the quadratic parts of $\mathcal{L}_f$ have precisely the form of Eq. (2.13) with the association $\Gamma^\alpha{}_{\beta\gamma} = \lambda\Lambda^\alpha{}_{\beta\gamma}$. The quantity $\mathcal{L}_f$ of course is highly nonlinear, the cubic and higher terms representing f-meson self-interactions.

The above discussion shows that the interaction of the matter fields χ_a with the f meson and the self-interactions of the f meson contained in $\mathcal{L}_f$ are formally identical to the corresponding gravitational interactions. Both $\mathcal{L}_M$ and $\mathcal{L}_f$ are "generally covariant" structures if one assumes that the f-meson metric $g_{\mu\nu}(x)$ formally transforms under general coordinate transformations as the gravitational metric $\bar{g}_{\mu\nu}(x)$ does. However, the mass terms breaks this covariance, as there is no scalar density formed from $g_{\mu\nu}(x)$ with quadratic part equal to the $\mathcal{L}_m^{(2)}$ of Eq. (2.5). Thus the f meson acts as a massive strong gravitation. In general, the mass terms in the Lagrangian are restricted by Eq. (2.18c). Since $\mathcal{L}_m$ is a Lorentz scalar formed from the Lorentz tensors $f_{\mu\nu}$, $\eta_{\mu\nu}$, $\eta^{\mu\nu}$, Eq. (2.18c) can be written as

$$M^{\mu\nu} + g_f m^{-2} \frac{\partial \mathcal{L}_m'}{\partial f_{\mu\nu}} = \eta^{\mu\nu}\mathcal{L}_m - 2f^{\mu}{}_{\alpha}\frac{\partial \mathcal{L}_m}{\partial f_{\alpha\nu}}. \qquad (2.23)$$

We first note that Eq. (2.23) implies that $M^{\mu\nu}$ cannot vanish, i.e., *the field-current identity Eq. (2.8) must contain nonpole terms to correctly account for the f-meson mass contributions to* $\Theta^{\mu\nu}$. This may be seen most easily by attempting to solve Eq. (2.23) with $M^{\mu\nu}=0$ by an iteration procedure. Thus, inserting the $\mathcal{L}_m^{(2)}$ of Eq. (2.5) on the right-hand side of Eq. (2.23) gives a set of partial differential equations to determine the cubic terms in $\mathcal{L}_m'$. However, one easily verifies that these equations are not integrable, and so $M^{\mu\nu}$ cannot be zero. Rather Eq. (2.23) can be viewed as an equation that determines $M^{\mu\nu}$ for arbitrary $\mathcal{L}_m'$. Thus the formalism leaves $\mathcal{L}_m$ *completely arbitrary except for its quadratic part* which is given by Eq. (2.5), i.e., the nonderivative f-meson self-couplings are undetermined. The simplest possibility would be to set $\mathcal{L}_m'=0$, i.e., choose $\mathcal{L}_m=\mathcal{L}_m^{(2)}$. Equation (2.23) then yields

$$M^{\mu\nu} = -\tfrac{1}{2}m^2\eta^{\mu\nu}(f^{\alpha\beta}f_{\alpha\beta} - f^2) + 2m^2(f^{\mu}{}_{\alpha}f^{\alpha\nu} - f^{\mu\nu}f), \qquad (2.24)$$

which is, of course, just the mass terms of the stress tensor arising from the Lagrangian of Eq. (2.5).

As we have now determined the form of $\mathcal{L}_f$ and $\mathcal{L}_M$, it is interesting to see how the field-current identity combined with the f-meson field equations yield the correct stress tensor. Thus varying Eq. (2.11) with respect to $f_{\mu\nu}$ and using Eqs. (2.5), (2.19), and (2.22) one obtains the "massive" Einstein equations[9]

$$\frac{2}{\lambda}(-g)^{1/2}G^{\mu\nu} + m^2(f^{\mu\nu} - \eta^{\mu\nu}f) = \lambda\frac{\delta\mathcal{L}_M}{\delta g_{\mu\nu}} + \frac{\delta\mathcal{L}_m'}{\delta f_{\mu\nu}}, \qquad (2.25)$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Ricci tensor ($R \equiv g^{\mu\nu}R_{\mu\nu}$) formed from the f-meson metric Eq. (2.19). We note that the first term on the right-hand side in Eq. (2.25) is just $\frac{1}{2}\lambda\Theta_M^{\mu\nu}$, where $\Theta_M^{\mu\nu}$ is just the Belinfante stress tensor formed from $\mathcal{L}_M$. Varying $\mathcal{L}$ with respect to[10] $\Gamma^{\alpha}{}_{\mu\nu}$ and following the usual analysis of general relativity yields[11]

$$\Gamma^{\alpha}{}_{\mu\nu} = \tfrac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}), \qquad (2.26)$$

i.e., the "affinity" is the usual Christoffel symbol constructed from the f metric. Equation (2.8) now gives

$$\Theta^{\mu\nu} = -4\lambda^{-2}\sqrt{-g}\,G^{\mu\nu} + \Theta_M^{\mu\nu} + \Theta_m^{\mu\nu} + \sum F_a H_a^{\mu\nu}, \qquad (2.27)$$

where

$$\Theta_m^{\mu\nu} \equiv M^{\mu\nu} + g_f m^{-2}\frac{\delta\mathcal{L}_m'}{\delta f_{\mu\nu}}$$

is just the Belinfante stress tensor contribution due to the f-meson mass terms $\mathcal{L}_m$ [by Eq. (2.18c)]. The first term is, of course, the Belinfante stress tensor arising from $\mathcal{L}_f$. Its physical content becomes clearer, however, by noting from Eqs. (2.21) and (2.26) and the definition of $G^{\mu\nu}$ that $\sqrt{-g}\,G^{\mu\nu}$ begins with a structure *linear* in the f-meson fields. Thus one may write

$$\sqrt{-g}\,G^{\mu\nu} = L^{\mu\nu} + Q^{\mu\nu}, \qquad (2.28)$$

where $L^{\mu\nu}$ is the linear piece and $Q^{\mu\nu}$ the quadratic and higher nonlinear parts. One finds directly

$$\lambda^{-1}L^{\mu\nu} = \tfrac{1}{2}(\partial_\alpha\partial^\nu f^{\alpha\mu} + \partial_\alpha\partial^\mu f^{\alpha\nu} - \Box^2 f^{\mu\nu} - \partial^\mu\partial^\nu f) - \tfrac{1}{2}\eta^{\mu\nu}(\partial_\alpha\partial_\beta f^{\alpha\beta} - \Box^2 f). \qquad (2.29)$$

It is easy to verify that $L^{\mu\nu}$ is just a superpotential term, i.e., $\partial_\nu L^{\mu\nu} \equiv 0$ and $L^{\mu\nu}$ gives zero contribution to P^μ and $M^{\mu\nu}$. In fact $L^{\mu\nu}$ plays a role in $\Theta^{\mu\nu}$ analogous to the linear superpotential term $\partial_\nu \rho_a^{\mu\nu}$ of the vector current obtained from Eqs. (2.2) and (2.4). Thus the real f-meson contributions to the stress tensor come from the nonlinear parts of $Q^{\mu\nu}$ which represent the f-meson kinetic-energy contributions to $\Theta^{\mu\nu}$. Indeed, the quadratic parts of $-4\lambda^{-2}Q^{\mu\nu} + \Theta_m^{\mu\nu}$ are just the usual free-field massive spin-2 stress tensor, while the additional nonlinear pieces represent the self-interaction stress tensor.[12]

C. Generalization for Many f Mesons—"Multimetric" Spaces

We now generalize the results of the previous section to the case where there is more than one f meson present. Each part of the Lagrangian of Eq. (2.11) now depends upon all the f-meson fields $f_{\mu\nu}^i$, $i=1,2,\ldots,N$. The quadratic part of the f-mass Lagrangian is $\mathcal{L}_m^{(2)} = \sum m_i^2[f^{\mu\nu i}f_{\mu\nu}^i - (f^i)^2]$, and so the field equations read

$$\frac{\delta \mathcal{L}_f}{\delta f^i_{\mu\nu}} + m_i^2(f^{\mu\nu i} - \eta^{\mu\nu} f^i) = \frac{\delta \mathcal{L}_M}{\delta f^i_{\mu\nu}} + \frac{\delta \mathcal{L}'_m}{\delta f^i_{\mu\nu}} . \tag{2.30}$$

Using the field current identity Eq. (2.8), the conditions that $\Theta^{\mu\nu}$ obey the conservation and Poincaré-group constraints [the generalizations of Eqs. (2.18)] are

$$\sum g_i m_i^{-2} \frac{\delta \mathcal{L}_M}{\delta f^i_{\mu\nu}} = 2 \frac{\delta \mathcal{L}_M}{\delta \eta_{\mu\nu}} , \tag{2.31a}$$

$$\sum g_i m_i^{-2} \frac{\delta \mathcal{L}_f}{\delta f^i_{\mu\nu}} = 2 \frac{\delta \mathcal{L}_f}{\delta \eta_{\mu\nu}} , \tag{2.31b}$$

$$M^{\mu\nu} + \sum g_i m_i^{-2} \frac{\partial \mathcal{L}'_m}{\partial f^i_{\mu\nu}} = 2 \frac{\delta \mathcal{L}_m}{\delta \eta_{\mu\nu}} . \tag{2.31c}$$

In order to integrate Eqs. (2.31a) and (2.31b) we define an f-meson metric $g_{\mu\nu i}$ for each f meson:

$$g_{\mu\nu i}(x) \equiv \eta_{\mu\nu} + \lambda_i f^i_{\mu\nu} , \quad \lambda_i \equiv 2 m_i^2 / g_i . \tag{2.32}$$

[The index i is *not* summed in Eq. (2.32).] The full content of Eq. (2.31a) is that *the f mesons enter $\mathcal{L}_M$ only in the combinations $g_{\mu\nu i}$ and that $\mathcal{L}_M$ is a generally covariant scalar density constructed by using these metrics*. Similarly, Eq. (2.31b) implies that $\mathcal{L}_f$ is a generally covariant scalar density constructed purely from the $g_{\mu\nu i}(x)$. Finally, Eq. (2.31c) gives no constraints on $\mathcal{L}'_m$ (which is arbitrary other than the requirement that it has the correct quadratic $\mathcal{L}_m^{(2)}$ piece).

When more than one f meson is present there is more than one metric which may be used to construct $\mathcal{L}_M$ and $\mathcal{L}_f$. The mathematical substructure needed to describe the many f-meson interactions is much more complicated than for the single f meson and corresponds to a Riemannian space with N independent metrics defined upon it. Spaces of this type do not appear to have been studied in any detail by relativists or mathematicians. We give here a brief indication of the complexity of such "multimetric" spaces. Once more than one metric exists in a Riemannian space, an *infinite number of metrics* can be formed from algebraic functions of the fundamental metrics. The new metrics $g_{\mu\nu A}$ are restricted only in that they be second-order symmetric tensors constructed from the fundamental $g_{\mu\nu i}$ and that they be normalized to the "flat-space" limit

$$g_{\mu\nu A} \xrightarrow[f^i_{\mu\nu} \to 0]{} \eta_{\mu\nu} . \tag{2.33}$$

Essentially two characteristic types of algebraic functions can be formed. First, one may construct new metrics out of polynomials of the type

$$g_{\mu\nu A} = \sum_i c_{Ai} g_{\mu\nu i} + \sum c_{Aijk} g_{\mu\alpha i} g_j^{\alpha\beta} g_{\beta\nu k} + \cdots . \tag{2.34}$$

Second, one may multiply any metric by ratios of $\sqrt{-g_i}$, since each of these transforms as a scalar density, e.g.,

$$g_{\mu\nu A} = \left(\frac{-g_i}{-g_j}\right)^{\alpha_1/2} \left(\frac{-g_k}{-g_l}\right)^{\alpha_2/2} \cdots g_{\mu\nu m} . \tag{2.35}$$

Of course one can also combine the two types of operations. Any of the new metrics can be used in forming the covariant parts of the Lagrangian, $\mathcal{L}_M$ and $\mathcal{L}_f$. Thus in the example of Eq. (2.20) the metric in the pion mass in general may be a different *nonlinear* function of the $g_{\mu\nu i}$ than the one used in the kinetic part. Similarly, the metrics in the pion Lagrangian need not be the same as the one in the kaon part of the Lagrangian or in the interaction part of the Lagrangian, etc. Thus when more than one f meson is present, the nonlinear f couplings become very complicated and are not uniquely determined (as in the single-f case, where we saw that all the f couplings depended only on one parameter λ).

If one treats the f meson to the linearized approximation, matters again simplify considerably. Thus keeping only terms linear in $f^i_{\mu\nu}$, Eq. (2.34) reduces to

$$g_{\mu\nu A} \cong \sum_i \beta_{Ai} g_{\mu\nu i} , \quad \sum_i \beta_{Ai} = 1 , \tag{2.36a}$$

or alternately, using Eq. (2.32), one can write

$$\begin{aligned} &g_{\mu\nu A} \cong \eta_{\mu\nu} + \sum_i \lambda_{Ai} f^i_{\mu\nu} , \\ &\lambda_{Ai} \equiv \beta_{Ai} \lambda_i , \\ &\sum_i (g_i / 2 m_i^2) \lambda_{Ai} = 1 . \end{aligned} \tag{2.36b}$$

Structures of the type Eq. (2.35) linearize to

$$\begin{aligned} &g_{\mu\nu A} \cong g_{\mu\nu m} + \eta_{\mu\nu} \sum_i \gamma_{Ai} f^i , \\ &\sum_i (g_i / 2 m_i^2) \gamma_{Ai} = 0 . \end{aligned} \tag{2.37}$$

As can be seen, the general metrics even in the linearized approximation are not diagonal in the f-meson fields, and mixing between the different f-meson fields can occur.

The arbitrary metrics of course also enter into $\mathcal{L}_f$. If $g_{\mu\nu A}$, $A = 1 \ldots N$, are N algebraically independent metrics, then the N-meson generalization of Eq. (2.22) is

$$\mathcal{L}_f = \sum_A (2/\lambda_A^2) \sqrt{-g_A}\, g_A^{\mu\nu} R_{\mu\nu}(\Gamma^\alpha{}_{\beta\gamma A}) , \tag{2.38}$$

where λ_A is the analog of λ in Eq. (2.22) (and is determined below). If the $f^i_{\mu\nu}$ are to correctly annihilate and create f mesons in the "in" and "out" states, the quadratic part of Eq. (2.38) must diagonalize to be a sum of structures of type Eq. (2.13), one for each f meson. One can verify that this condition (that the theory linearize correctly) excludes algebraic functions of type (2.35), and thus to the linear approximation $g_{\mu\nu A}$ has the form of Eq. (2.36). Writing

$$\Gamma^{\alpha}{}_{\beta\gamma A}=\sum_j \lambda_{Aj}\,\Lambda^{\alpha}{}_{\beta\gamma j}\,,$$

the requirement that the quadratic parts of $\mathcal{L}_f$ diagonalize (i.e., that $\mathcal{L}_f^{(2)}=\sum_i \mathcal{L}_{fi}^{(2)}$) becomes

$$\sum_A (\lambda_{Ai}\lambda_{Aj})/\lambda_A{}^2=\delta_{ij}\,. \tag{2.39}$$

Alternately, Eq. (2.39) implies that the matrix $\mu_{Ai}\equiv\lambda_{Ai}/\lambda_A$ is orthogonal. In addition to Eq. (2.39) the λ_{Ai} obey the constraints given in Eq. (2.36b), which allows one to relate λ_A back to $\lambda_i\equiv 2m_i{}^2/g_i$. For the physically interesting case of two f mesons (the f and f'), one has that μ_{Ai} is given in terms of one "mixing angle." The fact that the quadratic parts of $\mathcal{L}_f$ have been diagonalized does not imply that the cubic and higher parts will be. Thus f-f-f', etc. vertices can still exist. Such interactions would be forbidden in the quark model, which would correspond here to the choice $\mu_{Ai}=\delta_{Ai}$ and $g_{\mu\nu A}=g_{\mu\nu i}$.

D. Summary

In this section we have obtained the full conditions imposed upon the effective Lagrangian by requiring that the field-current identity stress tensor obey the conservation and Poincaré-group constraints. The analysis has been carried out to arbitrary order in f couplings, with an arbitrary number of f and σ mesons interacting with an arbitrary hadron system. It is interesting that the assumption that the stress tensor is a smooth interpolating field for the $J^P=2^+, 0^+$ mesons (which is what the field-current identity implies) automatically leads in the single-f-meson case to dynamical interactions between the f meson and other particles and f self-couplings which are identical to graviton couplings. In the many-f-meson case (which is the physically interesting situation) the coupling structure is more complicated than in the Einstein theory; it corresponds to a Riemannian space with more than one metric (and hence an infinite number of metrics) defined on it.

We should like to emphasize that the assumptions that have been made in this section are very few. We have started by choosing the stress tensor as a smooth interpolating field for the $J^P=2^+$ and 0^+ mesons. The stress tensor must of course satisfy the conservation of energy and momentum as well as obey the constraints imposed by the algebra of the Poincaré group. These results then automatically imply that the hadrons interact with the f mesons by inserting f-meson metrics into the hadron Lagrangian to form "covariant" structures. Furthermore, the presence of more than one f meson implies that the choice of metric is not unique, and may in general be different in each term in the hadron Lagrangian. This in turn leads to the result that the f-f' mixing will *a priori* be different at each vertex. One can of course force this mixing to be the same by an appropriate choice of the coupling constants. However, this would be an additional *ad hoc* assumption for which there is no justification, and in fact is in contradiction with present experiment. One may also argue from a purely theoretical point of view that mixing should be different at different vertices. We must keep in mind that the development presented here is a phenomenological one, with the field operators creating and annihilating physical particles, and the Lagrangian an effective one to be treated in the tree-seagull approximation. If, in fact, there did exist an underlying fundamental theory in which the f and f' couplings were in a fixed ratio at all vertices (e.g., "ideal" mixing) then renormalization effects at the vertices would in general result in the ratio of the *renormalized* coupling constants (with which we work here) being different. (One could obtain the same renormalized mixing at all vertices only if there existed some deeper principle such as a gauge invariance to guarantee it.)

The nonlinear self-interactions of the single f meson have been treated by Wess and Zumino and by Isham *et al.*[4] in a different manner. Wess and Zumino do not examine f couplings to other hadrons and choose a particular class of f-meson mass terms.[13] Raman and Renner[4] have considered the many-f-meson case interacting with hadrons in the linearized approximation. In addition we have also included the spin-zero components of the stress tensor, which will be seen to play an important role in the next sections.

III. IMPROVEMENT, ANOMALOUS DIMENSIONS, AND DILATONS

In this section we illustrate some of the properties implied by the previous results for the model of a π meson interacting with a σ and an f meson. For simplicity we consider only the case of a single f meson and a single σ meson. The matter Lagrangian is chosen to be (in second-order formalism)

$$\mathcal{L}_M=\mathcal{L}_{0\pi}+\mathcal{L}_{0\sigma}+\mathcal{L}_I+\mathcal{L}_R\,, \tag{3.1a}$$

where

$$\mathcal{L}_{0\pi}=-\tfrac{1}{2}\sqrt{-g}\;g^{\mu\nu}\partial_\mu\pi_a\partial_\nu\pi_a-\tfrac{1}{2}\sqrt{-g}\;m_\pi{}^2\pi_a{}^2\,, \tag{3.1b}$$

$$\mathcal{L}_{0\sigma}=-\tfrac{1}{2}\sqrt{-g}\;g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma-\tfrac{1}{2}\sqrt{-g}\;m_\sigma{}^2\sigma^2\,, \tag{3.1c}$$

$$\mathcal{L}_I = \tfrac{1}{2} g_{\sigma\pi\pi}\sqrt{-g}\;\pi_a\pi_a\sigma + \tfrac{1}{2}\lambda_{\sigma\pi\pi}\sqrt{-g}\;g^{\mu\nu}\partial_\mu\pi_a\partial_\nu\pi_a\sigma + \mu_{\sigma\pi\pi}\sqrt{-g}\;g^{\mu\nu}\pi_a\partial_\mu\pi_a\partial_\nu\sigma\,. \quad (3.1\text{d})$$

As discussed in Sec. II the use of the metric $g_{\mu\nu}(x)$ of Eq. (2.19) introduces the f couplings in a fashion guaranteeing the Poincaré-group constraints, as $\mathcal{L}_M$ is a generally covariant scalar density. The three σ-π-π couplings are precisely the ones considered in previous current-algebra analyses.[2] $\mathcal{L}_R$ is the "improvement" term defined below in Eq. (3.5).

The dilatation current is given by $D^\mu = -x_\nu\Theta^{\mu\nu}$. The dilatation charge,

$$D(t) = -\int d^3x\, x_\nu\Theta^{0\nu}\,, \quad (3.2)$$

generates changes in fields $\phi(x)$ of fixed scale dimension d_ϕ according to

$$i[D(t), \phi(x)] = [d_\phi + x^\mu\partial_\mu]\phi(x)\,. \quad (3.3)$$

On the other hand we shall call a "dilaton" any field $\tau(x)$ which transforms under scale changes according to

$$i[D(t), \tau(x)] = 1/b + x^\mu\partial_\mu\tau(x)\,; \quad (3.4)$$

the stress-tensor components $\Theta^{0\nu}$ are obtained from Eq. (2.27).

The "canonical" dimension of the pion field (as obtained from the Noether's construction of the dilatation charge), is $d_\pi = 1$. As is well known,[14] this dimension is not obtained from Eq. (3.3) unless the stress tensor $\Theta^{\mu\nu}$ is "improved." An improved stress tensor can automatically be obtained by adding a "curvature" coupling $\mathcal{L}_R$ between the π and f mesons:

$$\mathcal{L}_R = -\tfrac{1}{6}\pi_a\pi_a\sqrt{-g}\;g^{\mu\nu}R_{\mu\nu}(\Gamma^\alpha{}_{\beta\gamma})\,, \quad (3.5)$$

where $R_{\mu\nu}$ is given in Eq. (2.21) and $\Gamma^\alpha{}_{\beta\gamma}$ is the Christoffel symbol [Eq. (2.26)] constructed from the f metric. (We are using second-order formalism.) The total matter Belinfante stress-tensor term $\Theta_M^{\mu\nu}$ of Eq. (2.27) thus contains the usual terms from $\mathcal{L}_{0\pi} + \mathcal{L}_{0\sigma} + \mathcal{L}_I$ plus an additional term from $\mathcal{L}_R$. One finds that $\Theta_R^{\mu\nu} \equiv 2(\partial\mathcal{L}_R/\partial g_{\mu\nu})$ is given by

$$\Theta_R^{\mu\nu} = \tfrac{1}{6}\sqrt{-g}\,[-(\pi^2)^{;\mu\nu} + g^{\mu\nu}(\pi^2)^{;\alpha}{}_\alpha] + \tfrac{1}{6}\,\pi^2\sqrt{-g}\;G^{\mu\nu}\,, \quad (3.6)$$

where the covariant derivatives[9] in Eq. (3.6) are formed in the usual fashion from the f-meson metric. In the limit that one neglects the f meson, the first term is just the usual Huggins term.[15] The second term represents an additional interaction energy between pions and f mesons.

In the following analyses, we will neglect the f-meson couplings by taking the "flat-space" limit $g_{\mu\nu} \to \eta_{\mu\nu}$ (i.e., $\lambda \to 0$). A more complete discussion of the points considered below will be given in the second paper of this series.[16] The dimension of the pion field is most conveniently obtained by evaluating Eq. (3.3) at $x^\mu = 0$. Then $D(0) = -\int x_i\Theta^{0i}$. Using Eqs. (2.27), (3.1), and (3.6) one has (in the flat-space approximation[17])

$$\Theta^{0i} = \Theta_M^{0i} - \tfrac{1}{3}F_\sigma\partial^i\partial^0\sigma\,, \quad (3.7)$$

where

$$\Theta_M^{0i} = [\partial^0\pi_a\partial^i\pi_a + \partial^0\sigma\partial^i\sigma - \lambda_{\sigma\pi\pi}\partial^0\pi_a\partial^i\pi_a\sigma - \mu_{\sigma\pi\pi}\pi_a(\partial^0\pi_a\partial^i\sigma + \partial^i\pi_a\partial^0\sigma)] - \tfrac{1}{6}\partial^i\partial^0(\pi_a\pi_a)\,. \quad (3.8)$$

The last term in Eq. (3.8) comes from the curvature coupling (3.5). To calculate the commutator of Eq. (3.3), one eliminates the time derivatives $\partial_0\pi_a$ and $\partial_0\sigma$ in terms of the canonical momenta $p_\pi(x)_a = \partial\mathcal{L}_M/\partial(\partial_0\pi_a)$ and $p_\sigma(x) = \partial\mathcal{L}_M/\partial(\partial_0\sigma)$. One finds

$$\partial_0\pi_a = [p_{\pi a} + \mu_{\sigma\pi\pi}\pi_a p_\sigma]/\Delta\,, \quad (3.9\text{a})$$

$$\partial_0\sigma = [(1 - \lambda_{\sigma\pi\pi}\sigma)p_\sigma + \mu_{\sigma\pi\pi}\pi_a p_{\pi a}]/\Delta\,, \quad (3.9\text{b})$$

where

$$\Delta \equiv 1 - \lambda_{\sigma\pi\pi}\sigma - (\mu_{\sigma\pi\pi})^2\pi_a\pi_a\,. \quad (3.9\text{c})$$

The only terms in Θ^{0i} which contribute to the pion dimension are the σ-superpotential term and the pion Huggins term. One finds

$$d_\pi = 1 + [F_\sigma\mu_{\sigma\pi\pi} + \lambda_{\sigma\pi\pi}\sigma + (\mu_{\sigma\pi\pi})^2\pi_a\pi_a]/\Delta\,. \quad (3.10)$$

It is convenient to distinguish between the "canonical" dimension of a field and its "anomalous" dimension. The canonical scale dimension is the dimension one would obtain from a Lagrangian possessing only nonderivative couplings, and coincides with the usual dimension of the field (i.e., unity for the pion field). In our phenomenological approach, we see that the pion has anomalous scale dimension as well, arising from the derivative couplings. In a nonderivative coupling theory, anomalous dimensions arise only due to the singular nature of closed-loop diagrams.[18] We see that the *derivative couplings produce anomalous dimensions even at the tree and seagull approximations.* The anomalous dimensions of Eq. (3.10) are of two types. First there is a c-number piece $F_\sigma\mu_{\sigma\pi\pi}$. In addition, however, Eq. (3.10) exhibits more complicated *q-number anomalous dimension.* This is precisely the type of structures found by Coleman and Jackiw[18] in their study of the closed-loop diagrams of simple models. There the anomalous dimension could be represented by a c-number shift of d only to lowest order. In the phenomenological analysis used here, the derivative couplings arise naturally in the current-algebra interactions, and thus the anomalous dimensions arise naturally. We see therefore that the *effective-Lagrangian approach in the tree seagull approximation is sufficiently powerful to simulate in the low- and intermediate-energy ranges the anomalous effects characteristic of closed-loop diagrams.* Most significant is the fact that since the effective Lagrangian gives an approximately realistic description of the low-energy phenomena, the parameters appearing in the anomalies can be related to real experimental phenomena, i.e., one is not merely dealing with a model.[19]

We next turn to the scale-transformation properties of the σ meson, which again can be calculated using the Θ^{0i} of Eqs. (3.7) and (3.8). The significance of the σ-meson superpotential term becomes clearer in that it implies that the σ meson has dilatonlike transformation properties. Thus evaluating the commutator of Eq. (3.4) for $\sigma(x)$ explicitly at $x^\mu=0$ gives

$$1/b_\sigma = F_\sigma + \mu_{\sigma\pi\pi}(1+F_\sigma\mu_{\sigma\pi\pi})\pi_a\pi_a/\Delta\,. \tag{3.11}$$

Again we see that there is a normal or canonical piece to $1/b_\sigma$ and an additional anomalous q-number part due to the derivative couplings. One may thus call the σ an *anomalous dilaton.* Note that the *canonical part of* b_σ *is not arbitrary, but in fact it is deduced to be* $1/F_\sigma$. It is important to realize that the dilaton nature of the σ field does not arise in attempting to realize some scale-invariance condition, but rather is a direct consequence of the presence of a spin-zero part of the stress-tensor field current identity[20] (2.8).

IV. SCALE-BREAKING CONDITIONS

In this section we shall investigate the dynamics of scale breaking. As pointed out in the Introduction it appears possible that the physical laws are scale-invariant at high energies, as for example demonstrated by the scaling of the structure functions of the electroproduction in the deep-inelastic region. At the low- and the intermediate-energy regions significant deviations from the scale-invariance limit would naturally arise. Thus if the study of scale invariance is to be a useful idea one must discover the manner in which scale invariance breaks.

We propose in this work a new principle of scale breaking which occurs as a dynamical equation of motion. For the first part of this discussion we assume the existence of only one σ and one f meson. The generalization to more than one meson is given below. That scale breaking is intimately connected with the existence of the σ meson and its interactions with other particles has already been noted. Thus it has been suggested[4] that scale breaking can be characterized by dominating the trace of the total stress tensor by the σ meson itself, i.e.,

$$\eta_{\mu\nu}\Theta^{\mu\nu} = F_\sigma m_\sigma^2\sigma\,. \tag{4.1}$$

This condition, unfortunately, turns out to be in conflict with the CTC and the Poincaré-group conditions when f-meson couplings are included. Thus it was seen in Sec. II that the f meson couples to all "matter" fields in a "generally covariant" way, while Eq. (4.1) is a noncovariant "flat-space" trace. Specific problems arise in the f-σ couplings. Using the stress tensor of Eq. (2.8), the covariant f-σ couplings of Eq. (3.1c) can be seen to be inconsistent with Eq. (4.1). Thus while Eq. (4.1) represents a valid possible scale-breaking condition in the absence of f mesons, it must be modified to take f couplings into account. These modifications will have to produce a "covariant" condition to eliminate the inconsistencies with CTC, which implies covariant f-meson couplings.

At this point we should like to remark on a difficulty usually associated with Eq. (4.1). In the event that we deal with a single σ meson, one finds that it leads to the well-known problem that[1] $m_\pi^2 = \frac{1}{2}F_\sigma g_{\sigma\pi\pi} = O(m_\sigma^2)$. However, if there exist both a σ and a σ' meson, then one finds instead $m_\pi^2 = \frac{1}{2}(F_\sigma g_{\sigma\pi\pi} + F_{\sigma'}g_{\sigma'\pi\pi})$. This last result is consistent with experiment since it only requires that $g_{\sigma\pi\pi}$ and $g_{\sigma'\pi\pi}$ be of opposite sign and cancel to $O(m_\pi^2)$. The scale-breaking conditions with two mesons present will be given below, and the detailed verification that the resultant equations are consistent with the experimental σ and σ' widths will be discussed in paper II.

It is instructive to examine the question of scale breaking in terms of fields and their sources. The sources of the gauge fields are the currents. For instance, the source of the photon equation is the electromagnetic current

$$\partial_\nu F^{\mu\nu} = e j^\mu_{\text{em}}\,, \tag{4.2}$$

and the source of the ρ-meson equation is the vector current

$$\partial_\nu \rho_a^{\mu\nu} + m_\rho^2\rho_a^\mu = J_a^\mu\,. \tag{4.3}$$

For the graviton and the $J^P=2^+$ fields, the Belinfante stress tensor of matter $\Theta_M^{\mu\nu}$ $(=2\delta\mathcal{L}_M/\delta\eta_{\mu\nu})$ plays a role analogous to that which the electromagnetic and the vector currents play for the photon and the $J^P=1^-$ meson. Thus $\Theta_M^{\mu\nu}$ acts as a source of both the graviton and the f meson. For the f meson the field equations read

$$\frac{g_f}{m^2}(-g)^{1/2}G^{\mu\nu}+m^2(f^{\mu\nu}-\eta^{\mu\nu}f)=\frac{m^2}{g_f}\Theta_M^{\mu\nu}\,. \quad (4.4)$$

The above discussion exhibits the fundamental role that $\Theta_M^{\mu\nu}$ plays as a source and a "current." However, unlike the vector case where CVC forbids the vector J_a^μ having a spin-zero part, CTC does not prevent $\Theta_M^{\mu\nu}$ from having a spin-zero piece, i.e., its trace, $\Theta_M \equiv g_{\mu\nu}\Theta_M^{\mu\nu}$. If Θ_M is also to be a source, it could only be a source of the spin-zero fields. We therefore postulate that *the "curved space" trace of the Belinfante matter stress tensor* Θ_M *is the source of the* σ *meson.* We will see that this assumption represents a scale-breaking condition which is a natural generalization of Eq. (4.1).

We now have

$$\sqrt{-g}\,(-\sigma^{;\alpha}{}_{;\alpha}+m_\sigma{}^2\sigma)=\gamma\Theta_M \quad (4.5)$$

where $\sigma^{;\alpha}{}_{;\alpha}$ is the generally covariant d'Alembertian

$$\sigma^{;\alpha}{}_{;\alpha}=(1/\sqrt{-g}\,)\partial_\mu(\sqrt{-g}\;g^{\mu\nu}\partial_\nu\sigma) \quad (4.6)$$

formed from the f-meson metric and arising from varying $\mathfrak{L}_{0\sigma}$ of Eq. (3.1c). Our postulate implies that the scale breaking is governed by the *dynamical* equation (4.5) rather than Eq. (4.1). The matter Lagrangian may be written as $\mathfrak{L}_M=\mathfrak{L}_{0\sigma}+\mathfrak{L}_{I\sigma}$, where $\mathfrak{L}_{I\sigma}$ is the σ interaction Lagrangian. Equation (4.5) thus implies that $\delta\mathfrak{L}_{I\sigma}/\delta\sigma=\gamma\Theta_M$. The constant γ is thus far undetermined. It represents the effective charge or strength with which the σ meson couples to matter via the trace of the Belinfante stress tensor. However, in the approximation of neglecting the f meson, the total stress tensor of Eq. (2.27) has a trace of $\Theta=\Theta_M+F_\sigma\Box^2\sigma$. Requiring Eq. (4.5) to reduce to Eq. (4.1) in this approximation evaluates the constant γ to be $\gamma=1/F_\sigma$. Hence the scale-breaking condition is

$$F_\sigma\frac{\delta\mathfrak{L}_{I\sigma}}{\delta\sigma}=\Theta_M\equiv 2g_{\mu\nu}\frac{\delta\mathfrak{L}_M}{\delta g_{\mu\nu}}\,. \quad (4.7)$$

Equation (4.7) may be viewed as a functional differential equation to determine the σ-coupling structure in $\mathfrak{L}_M$, and an example of this will be given in Sec. V and also paper II. It represents a generalization of Eq. (4.1) to include the nonlinear f- and σ-meson couplings in a consistent fashion.

Equations (4.4) and (4.5) are two fundamental equations of the formalism. The two constants m^2/g and $\gamma=1/F_\sigma$ represent the coupling constants of the matter stress tensor to the spin-2 and spin-0 mesons, respectively. It is possible now to carry the discussion one step further. Just as the nonlinearity of the current algebra normalizes the amplitudes of the currents so that it is meaningful to talk about a universal Fermi interaction, the Poincaré group normalizes the amplitude of the stress tensor, and so it is possible to talk about a universal coupling of $\Theta_M^{\mu\nu}$ to the f and σ mesons. We now propose a *universal coupling of the* $J^P=2^+$, 0^+ *mesons to the Belinfante stress tensor and its trace.* This universality demands that $\gamma=m^2/g_f$, and hence one obtains a "KSRF-type" condition relating the spin-2 and spin-0 interpolating constants:

$$g_f=F_\sigma m^2\,. \quad (4.8)$$

It is instructive to derive the scale-breaking condition (4.5) from another viewpoint which is mathematically equivalent to our first postulate. This approach characterizes the scale breaking as a broken-gauge invariance of the *second kind.* To motivate the argument let us consider first an analogous derivation of the PCAC (partial conservation of axial-vector current) condition of current algebra. Consider the A_1 field $a_a^\mu(x)$ interacting with a set of other fields $\chi_A=\pi_a(x),\ \rho_a(x),\ldots$. We may write the total Lagrangian as $\mathfrak{L}=\mathfrak{L}_{0A}+\mathfrak{L}_M(\chi_A, a_a^\mu)$, where $\mathfrak{L}_{0A}$ is the free A_1 Lagrangian and $\mathfrak{L}_M$ is the remainder. We define a (chiral) gauge transformation of the second kind by

$$\delta a_a^\mu(x)=(g_A m_A{}^{-2})\partial_\mu\delta\lambda_a(x)\,,$$

$$\delta\pi_a(x)=F_\pi\delta\lambda_a(x)\,,$$

etc., where $\delta\lambda_a(x)$ is the infinitesimal gauge function. One has then

$$\delta A_M=\int d^4x\left[\frac{\delta\mathfrak{L}_M}{\delta a_{\mu a}}\,\delta a_{\mu a}+\sum\frac{\delta\mathfrak{L}_M}{\delta\chi_a}\,\delta\chi_A\right]\,, \quad (4.9)$$

where

$$\frac{\delta\mathfrak{L}_M}{\delta\chi_A}\equiv\frac{\delta\mathfrak{L}_M}{\partial\chi_A}-\partial_\mu\frac{\partial\mathfrak{L}_M}{\partial(\partial_\mu\chi_A)}\,.$$

We now postulate that chiral breakdown arises due to the lack of invariance of the free pion Lagrangian, i.e.,

$$\delta A_M=\int\frac{\delta\mathfrak{L}_{0\pi}}{\delta\pi_a}\,\delta\pi_a\,.$$

(That it is the total $\mathfrak{L}_{0\pi}$ rather than only the mass part of $\mathfrak{L}_{0\pi}$ that enters into this characterization of chiral breakdown is due to the fact we are dealing with gauge transformations of the second kind.) One has then that

$$\frac{\delta\mathfrak{L}_M}{\delta a_{\mu a}}\,\delta a_{\mu a}+\sum\frac{\delta\mathfrak{L}_M}{\delta\chi_A}\,\delta\chi_A=\frac{\delta\mathfrak{L}_{0\pi}}{\delta\pi_a\delta\pi_a}\,.$$

Using the field equations this implies

$$(g_A m_A{}^{-2})\partial_\mu\frac{\delta\mathfrak{L}_M}{\delta a_{\mu a}}=-F_\pi\frac{\delta\mathfrak{L}_{0\pi}}{\delta\pi_a}=F_\pi J_{\pi a}\,, \quad (4.10)$$

where $J_{\pi a}(x)$ is the source of the pion field. Equation (4.10) is just the PCAC relation $\partial_\mu A^\mu_a = F_\pi m_\pi{}^2 \pi_a$ for the axial-vector current $A^\mu_a = g_A a^\mu_a + F_\pi \partial^\mu \pi_a$.

Returning now to the scale-breaking condition, the analog of the chiral transformations are the Weyl transformations, which are also gauge transformations of the second kind. Under the Weyl transformations the fields transform as

$$\begin{aligned} \delta g_{\mu\nu} &= -2 g_{\mu\nu}(x)\delta\lambda(x)\,, \\ \delta\chi_A &= d_A \chi_A(x)\delta\lambda(x)\,, \quad \chi_A \neq \sigma \\ \delta\sigma &= F_\sigma \delta\lambda(x)\,, \end{aligned} \tag{4.11}$$

where d_A is the Weyl dimension of the field χ_A. The σ meson has a dilatonlike transformation. The Weyl dimension of $g_{\mu\nu}$ is -2. We write

$$\mathcal{L} = \mathcal{L}_{0f} + \mathcal{L}_M(g_{\mu\nu}, \chi_A)\,,$$

where $\mathcal{L}_{0f} = \mathcal{L}_f + \mathcal{L}_m$ is the f-meson Lagrangian (which now contains f-meson self-interactions due to CTC), and χ_A are the matter fields, including the σ meson. As has been shown by Wess and Zumino,[4] invariance of $\mathcal{L}_M$ under the Weyl and the Einstein transformations guarantees invariance under the scale transformations. In Wess and Zumino's analysis, the metric was a fictitious external gravitational field. However, the argument works equally well with the f-meson metric. The change in the action under the Weyl transformation is

$$\delta A_M = \int \left(\frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}} \delta g_{\mu\nu} + \sum_A \frac{\delta \mathcal{L}_M}{\delta \chi_A} \delta\chi_A \right). \tag{4.12}$$

We now postulate that it is the free-σ Lagrangian $\mathcal{L}_{0\sigma}$ that prevents A_M from being Weyl-invariant (and hence causes the breakdown of scale invariance); i.e., we assume as our basic scale-breaking condition

$$\delta A_M = \int \frac{\delta \mathcal{L}_{0\sigma}}{\delta\sigma} \delta\sigma\,. \tag{4.13}$$

Thus it is the dilaton field that causes scale breaking in A_M. Using Eqs. (4.11), (4.12), and (4.13) we have [since $\delta\lambda(x)$ is arbitrary]

$$-2 \frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu}} g_{\mu\nu} \delta\lambda + \sum_A \frac{\delta \mathcal{L}_M}{\delta \chi_A} \delta\chi_A = -\sqrt{-g}\,(-\sigma^{;\alpha}{}_{;\alpha} + m_\sigma{}^2 \sigma) F_\sigma \delta\lambda\,. \tag{4.14}$$

The matter field equations read $\delta\mathcal{L}_M/\delta\chi_A = 0$ and

$$\sqrt{-g}\,(-\sigma^{;\alpha}{}_{;\alpha} + m_\sigma{}^2\sigma) = J_\sigma \equiv \frac{\delta \mathcal{L}_{I\sigma}}{\delta\sigma}\,. \tag{4.15}$$

From Eqs. (4.14) and (4.15) we obtain

$$F_\sigma J_\sigma = g_{\mu\nu} \Theta^{\mu\nu}_M\,. \tag{4.16}$$

Eq. (4.16) is identical to Eq. (4.7).

Let us now consider the extension of our result to the multi-f and multi-σ cases. We consider again the action A_M, where

$$A_M = \int d^4x\, \mathcal{L}_M(g_{\mu\nu i}, \chi_A)\,, \tag{4.17}$$

and $g_{\mu\nu i}$ is the metric of the ith f meson of Eq. (2.32). For the case of more than one metric the Weyl transformation must be generalized. We define

$$\delta\sigma_a = F_a \delta\lambda_a(x)\,, \tag{4.18}$$

$$\delta g_{\mu\nu i} = -2 \sum_{j,c} \lambda_{ijc} g_{\mu\nu j}\, \delta\lambda_c(x)\,, \tag{4.19}$$

since it is possible that the Weyl transformation mixes the $g_{\mu\nu i}$. If one is now to extend the theorem of Wess and Zumino[4] and achieve scale invariance, as a consequence of Einstein and Weyl invariance it is necessary that $\delta g_{\mu\nu i} = -2 g_{\mu\nu i}\epsilon$ for the special case $\delta\lambda_c = \epsilon$, where $x'^\mu = (1+\epsilon)x^\mu$ is the infinitesimal scale transformation. This leads to the condition

$$\sum_c \lambda_{ijc} = \delta_{ij}\,. \tag{4.20}$$

The change in A_M under the general Weyl transformation is

$$\delta A_M = \int d^4x \left(\sum_i \frac{\delta \mathcal{L}_M}{\delta g_{\mu\nu i}} g_{\mu\nu i} + \sum_A \frac{\delta \mathcal{L}_M}{\delta_A} \delta\chi_A \right). \tag{4.21}$$

The extension of the postulate Eq. (4.13) implies that for the multi-σ-meson case the change δA_M under the Weyl transformation be given by

$$\delta A_M = \int d^4x \frac{\delta \mathcal{L}_{0\sigma}}{\delta\sigma_a} \delta\sigma_a\,, \tag{4.22}$$

where $\mathcal{L}_{0\sigma} = \sum_a \mathcal{L}_{0\sigma a}$. From Eqs. (4.21) and (4.22) and the equations of motion we get

$$F_a J_a + \sum_{i,j} \lambda_{ija} \Theta_{Mji} = 0\,, \tag{4.23}$$

where J_a is the source of the ath σ meson;

$$J_a = \sqrt{-g_\sigma}\,(-\sigma_a{}^{;\alpha}{}_{;\alpha} + m_{\sigma_a}{}^2 \sigma_a) \tag{4.24}$$

and

$$\Theta_{Mji} \equiv 2 g_{\mu\nu j} \frac{\partial \mathcal{L}_M}{\partial g_{\mu\nu i}}\,. \tag{4.25}$$

Summing over the index a in Eq. (4.23) and using Eq. (4.20) we obtain the result

$$\sum_a F_a J_a = \sum_i \Theta_{Mii}\,. \tag{4.26}$$

Equation (4.26) is the generalization of Eq. (4.1) to include f-meson couplings, when more than one f and σ meson is present.

The above formalism allows for a particularly simple possibility if the numbers of f and σ mesons are equal (as at present appears to be the case experimentally). Then the indices a, c, i, and j of Eqs. (4.18) and (4.19) run over the same range, $1, 2, \ldots N$. As discussed in Sec. II, one is free to introduce new metrics which are linear combinations of the fundamental $g_{\mu\nu i}$. Thus a new set of N algebraically independent metrics is

$$g_{\mu\nu a}=\sum_i c_{ai}g_{\mu\nu i}\,, \quad \sum_i c_{ai}=1\,. \tag{4.27}$$

From Eqs. (4.19) and (4.27) we obtain

$$\delta g_{\mu\nu a}=-2\sum \lambda_{abc}g_{\mu\nu b}\delta\lambda_c\,, \tag{4.28}$$

where

$$\lambda_{abc}=\sum_{i,j}c_{ai}\lambda_{ijc}(c^{-1})_{jb}\,. \tag{4.29}$$

Let us now assume that there exists a particular set of $g_{\mu\nu a}$ such that the λ_{abc} are completely diagonal. Then

$$\lambda_{abc}=\delta_{ab}\delta_{ac}\,, \tag{4.30}$$

since Eq. (4.20) requires in any basis $\sum_c\lambda_{abc}=\delta_{ab}$. This implies that the multidimensional Weyl group of Eqs. (4.18) and (4.19) is a direct product of one-dimensional groups. In the special frame of Eq. (4.30), $\delta g_{\mu\nu a}=-2g_{\mu\nu a}\delta\lambda_a$, i.e., the $g_{\mu\nu a}$ metrics have Weyl dimension -2 with respect to the transformations generated by $\delta\lambda_a$. From Eqs. (4.23) and (4.30) we find

$$\sqrt{-g_a}\,(-\sigma_a{}^{;\alpha}{}_{;\alpha}+m_{\sigma_a}{}^2\sigma_a)=J_a=F_a{}^{-1}\Theta_{Ma}\,, \tag{4.31}$$

where

$$\Theta_{Ma}=2\frac{\partial \mathfrak{L}_M}{\partial g_{\mu\nu a}}g_{\mu\nu a}\,. \tag{4.32}$$

We note that Eq. (4.31) is the direct analog of Eq. (4.7) for the multi-f multi-σ case. Equation (4.31) tells us that the source of any given σ meson is the trace of the subpiece of the Belinfante tensor, formed from the $g_{\mu\nu a}$ metric. Analogous to Eq. (4.31) we also have the f-meson equations in the diagonal form

$$\frac{g_i}{m_i^2}(-g)^{1/2}G_i^{\mu\nu}+m_i^2(f_i^{\mu\nu}-\eta^{\mu\nu}f_i)=\frac{m_i^2}{g_i}\Theta_{Mi}^{\mu\nu}\,, \tag{4.33}$$

where $\Theta_{Mi}^{\mu\nu}=2\partial\mathfrak{L}_M/\partial g_{\mu\nu i}$. Thus the $g_{\mu\nu i}$ of Eq. (2.32) are the fundamental metrics for the f-meson couplings, while the $g_{\mu\nu a}$ are the fundamental metrics of the σ-meson couplings. The σ-meson metrics are "rotated" relative to the f-meson metrics.

The existence of the special basis of Eq. (4.30) leads to Eq. (4.31). This allows one to generalize Eq. (4.8) to the multimeson case with the assumption that all f and σ couplings to the Belinfante stress-tensor sources are universal. From Eq. (4.31) and (4.33), this implies that all the constants m_i^2/g_i and F_a are equal. For the case of two f mesons and two σ mesons the universality condition reads

$$F_\sigma=F_{\sigma'}=g_f/m_f{}^2=g_{f'}/m_{f'}{}^2\,. \tag{4.34}$$

As will be discussed in paper II, Eq. (4.34) is in agreement with the present data.

In summary, we have emphasized the role that currents play as sources and have postulated that the scale-breaking condition is given by using the trace of the Belinfante stress tensor as the source of the σ meson. A universality of the strengths with which the Belinfante stress tensor and its trace couple to the source of the f meson and the σ meson is assumed and new KSRF-type relations are derived. Equation (4.5) for the single-f and single-σ cases, and Eqs. (4.26) and (4.31) for the multi-f, multi-σ cases, are powerful conditions which represent dynamical equations for scale breaking. Some consequences of these equations are examined in the next section and others will be examined in paper II.

V. EFFECTS OF SCALE BREAKING ON THE CHIRAL CURRENTS

In this section we focus our attention on the development of the chiral algebra in a manner consistent with the constraints imposed by the field-current identity for $\Theta^{\mu\nu}$, the CTC condition, and the scale-breaking conditions. The general procedure for invoking the current-algebra conditions in the absence of f couplings was as follows.[2] The vector and axial-vector currents were introduced via the field-current identity, and the constraint variables eliminated in favor of the canonical variables. One was then in a position to impose the Gell-Mann equal-time commutation relations on these currents, thus obtaining algebraic equations determining the coupling constants. However, some interesting new features surface when f and σ couplings are introduced consistent with CTC and scale breaking. We consider here the $I=1$ vector current, which serves as a nice example for illustrating some of these points.

In the *absence* of f-meson couplings, the vector current is proportional to the ρ field, i.e., $V_a^\mu(x)=g_\rho\eta^{\mu\nu}\rho_{\nu a}(x)$. Such a form cannot be correct, however, when f-meson couplings are included, for as was seen in Sec. II the CTC conditions require that all interactions be formed using the f-meson metric (rather than the Lorentz metric) in a generally covariant way. This suggests that the vector current be modified so that $V^\mu_a \sim \sqrt{-g}\, g^{\mu\nu} \times \rho_{\nu a}(x)$, where $g_{\mu\nu}$ is one of the metrics of Eqs. (2.34) and (2.35). (The factor $\sqrt{-g}$ implies that V^μ_a is a vector *density*. Thus the CVC condition $\partial_\mu V^\mu_a = 0$ is then a covariant condition since the ordinary and covariant divergences coincide for vector densities.) As was seen in Sec. IV, the scale-breaking condition plays a role analogous to CTC in determining the σ-meson couplings. Thus just as there is f-meson "clothing" of the ρ field in V^μ_a from CTC, one may expect σ-meson "clothing" also arising from scale breaking. We will therefore assume that the vector current has the general form

$$V^\mu_a(x) = g_\rho \lambda(\sigma_a)\sqrt{-g}\; g^{\mu\nu}\rho_{\nu a}(x)\,, \tag{5.1}$$

where $\lambda(\sigma_a)$ is a function of the σ fields and obeys $\lambda(0) = 1$.

To illustrate the above points more explicitly, we limit our consideration to the case of a single-f and single-σ meson coupling to the ρ meson. The effective Lagrangian of the ρ-f-σ system (in first-order formalism for the ρ field) reads

$$\begin{aligned}\mathcal{L}_M = &-\tfrac{1}{2}\lambda_1(\sigma)\,\rho^{\mu\nu}_a \partial_{[\mu}\rho_{\nu]a} + \tfrac{1}{4}\bar\lambda_1(\sigma)\rho^{\mu\nu}_a\rho^{\alpha\beta}_a(-g)^{-1/2}g_{\mu\alpha}g_{\nu\beta} - \tfrac{1}{2}m_\rho^{\,2}\lambda_2(\sigma)\sqrt{-g}\; g^{\mu\nu}\rho_{\mu a}\rho_{\nu a}\\ &+ \tfrac{1}{2}\lambda_3(\sigma)\rho^{\mu\nu}_a(\rho_{\mu a}\partial_\nu\sigma_b - \rho_{\nu a}\partial_\mu\sigma) + \tfrac{1}{2}\lambda_4(\sigma)\epsilon_{abc}\rho_{\mu a}\rho_{\nu b}\rho^{\nu\mu}_c - \tfrac{1}{2}\lambda_5(\sigma)\sqrt{-g}\, g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \tfrac{1}{2}m_\sigma^{\,2}\lambda_6(\sigma)\sqrt{-g}\;.\end{aligned} \tag{5.2}$$

Here $\rho_{\mu a}$ and the tensor density $\rho^{\mu\nu}_a$ are the fundamental ρ-field variables, and $\bar\lambda_1$ and $\lambda_{1,2}$ obey the boundary conditions $\bar\lambda_1(0) = \lambda_1(0) = 1 = \lambda_2(0)$ so that $\mathcal{L}_M$ correctly contains the free ρ-field Lagrangian.

We first note that by a point transformation $\rho_{\mu a} = \Lambda(\sigma)\bar\rho_{\mu a}$, with $\partial(\ln\Lambda)/\partial\sigma = -\lambda_3/\lambda_1$, one can always set λ_3 to zero. With this convention, the field equations read

$$\partial_\nu(\lambda_1\rho^{\mu\nu}_a) + m_\rho^{\,2}\lambda_2\sqrt{-g}\;\rho^\mu_a = \lambda_4\epsilon_{abc}\rho_{\nu b}\rho^{\nu\mu}_c\,, \tag{5.3}$$

$$\bar\lambda_1\rho_{\mu\nu a} = \lambda_1\sqrt{-g}\;\partial_{[\mu}\rho_{\nu]a} + \lambda_4\sqrt{-g}\;\epsilon_{abc}\rho_{\mu b}\rho_{\nu c}\,, \tag{5.4}$$

$$\partial_\nu(\lambda_5\sqrt{-g}\;\partial^\mu\sigma) - m_\sigma^{\,2}\sqrt{-g}\;\frac{\partial\lambda_6}{\partial\sigma} = -\frac{\delta\mathcal{L}_\rho}{\delta\sigma}\,, \tag{5.5}$$

where $\mathcal{L}_\rho$ is the first four terms on the right-hand side of Eq. (5.2). From Eqs. (5.1), (5.3), and (5.4), the CVC condition $\partial_\mu V^\mu_a = 0$ implies

$$\lambda_2 = \lambda\,, \quad \frac{\partial(\lambda_4/\lambda_1)}{\partial\sigma} = 0\,. \tag{5.6}$$

We next impose the scale-breaking condition (4.7). This, along with Eq. (5.5) gives

$$\frac{\partial\lambda_1}{\partial\sigma} = \frac{\partial\bar\lambda_1}{\partial\sigma} = 0 = \frac{\partial\lambda_4}{\partial\sigma}\,, \tag{5.7}$$

and

$$\frac{\partial\lambda_2}{\partial\sigma} = \frac{2}{F_\sigma}\lambda_2\lambda_5\,, \tag{5.8}$$

$$\frac{1}{2\lambda_5}\frac{\partial\lambda_5}{\partial\sigma} = -\frac{\lambda_5}{F_\sigma}\,, \tag{5.9}$$

$$\sigma - \frac{1}{2\lambda_5}\frac{\partial\lambda_6}{\partial\sigma} = -\frac{2}{F_\sigma}\lambda_6\,. \tag{5.10}$$

Using the boundary conditions $\lambda_1(0) = \bar\lambda_1(0) = 1$, we then have $\lambda_1 = 1 = \bar\lambda_1$. Equation (5.9) gives $\lambda_5 = 1/(1 + 2\sigma/F_\sigma)$ since $\lambda_5(0) = 1$. Equation (5.8) now yields $\lambda_2 = 1 + 2\sigma/F_\sigma$ upon using the boundary condition $\lambda_2(0) = 1$. Finally, Eq. (5.10) gives $\lambda_6 = \sigma^2$. From Eq. (5.7) one has also that λ_4 is a constant and hence it can be evaluated by going to the limit where the f couplings are neglected. The current commutation relations then determines[2] $\lambda_4 = m_\rho^{\,2}/g_\rho$.

We see from the above that the combination of CVC, CTC, and scale breaking has determined the f-ρ-ρ and σ-ρ-ρ couplings explicitly. The vector current has also been determined to be

$$V^\mu_a(x) = g_\rho(1 + 2\sigma/F_\sigma)\sqrt{-g}\,g^{\mu\nu}\rho_{\nu a}\,. \tag{5.11}$$

Note that if we define the f and σ "clothed" mass term by

$$\mathcal{L}_{m_\rho} \equiv -\tfrac{1}{2}m_\rho^{\,2}\lambda_2(\sigma)\sqrt{-g}\,g^{\mu\nu}\rho_{\mu a}\rho_{\nu a}\,,$$

then

$$V^\mu_a = -(g_\rho/m_\rho^{\,2})(\partial\mathcal{L}_{m_\rho}/\partial\rho_{\mu a})\,,$$

and so Eq. (5.11) is the natural generalization of the usual ρ-dominance current in the presence of f and σ couplings obeying broken scale invariance. The remarkable feature about Eq. (5.11), however, is that *CTC and scale breaking require that there be non-ρ-pole f-ρ and σ-ρ terms in the vector current.* This leads to several immediate consequences which we now discuss.

(i) As was seen in Sec. II, the CTC condition requires that the f mesons have generally covariant couplings so that there must exist a three-point f-ρ-ρ vertex in Eq. (5.2). If one had chosen the usual ρ-dominance form $g_\rho\eta^{\mu\nu}\rho_\nu$ for the vector current, then CVC (and the current algebra) would require that the ρ field couple only to the isospin current.[2] Thus CTC and CVC would be inconsistent. Similarly, scale breaking and CVC would be inconsistent (since the former requires a σ-ρ-ρ vertex, i.e., $\lambda_2 \neq 1$). It is precisely the nonpole form of Eq. (5.11) that removes this inconsistency, and so the more complicated form of Eq. (5.11) is actually required for a consistent formalism to exist.

(ii) The usual ρ-dominance current, $g_\rho \eta^{\mu\nu}\rho_\nu$, has canonical scale dimension 1, which is to be contrasted with the quark currents which possess canonical dimension 3. This may be viewed as a serious difficulty in the ρ-dominance current, for if asymptotically any anomalous-dimension contributions due to derivative couplings (as discussed in Sec. III) disappear and the interactions become scale-invariant, then a scale dimension 3 is precisely the value needed by Bjorken[21] to deduce the experimentally observed scaling of the electroproduction form factors. (The scale dimension 3 also implies a $1/q^2$ falloff for the total e^+-e^- annihilation cross section.) However, the σ factor in Eq. (5.11) precisely modifies V_a^μ so that it has canonical scale dimension 3, i.e., *the current of Eq. (5.11) implies asymptotic scaling of the electroproduction form factors* (provided, of course, the anomalous-dimension terms vanish asymptotically).

To see this we note that the canonical coordinates of the ρ field are ρ_{ia} ($i=1, 2, 3$), while the conjugate momenta are given by

$$P_a^i = \frac{\delta \mathcal{L}}{\delta \partial_0 \rho_{ia}} = -\rho_a^{0i}. \tag{5.12}$$

Equations (5.3) and (5.11) then yield

$$V_a^0 = \frac{g_\rho}{m_\rho^2}\partial_i P_a^i + \epsilon_{abc}\rho_{ib}P_c^i. \tag{5.13}$$

(Note that V_a^0 correctly generates isotopic rotations.) Now the canonical scale dimension of P_a^i is 2, and hence V_a^0 has scale dimension 3. For the spatial components of $V_{\mu a}$ we have

$$V_{ia} = g_\rho(1+2\sigma/F_\sigma)\sqrt{-g}\,\rho_{ia}. \tag{5.14}$$

Since ρ_{ia} has canonical scale dimension 1 and $e^{2\sigma/F_\sigma}$ has canonical scale dimension 2, V_{ia} has scale dimension 3. Making use of the fact that $g_{\mu\nu}$ has canonical scale dimension zero, and using Eqs. (5.13) and (5.14), we may determine V_{0a} and V_a^i to both have scale dimension 3 as well. In summary, CVC and the scale-breaking condition lead to the conclusion that the scale dimensions of all components of the vector current of Eq. (5.11) are 3.

(iii) If one expands Eq. (5.11) one obtains

$$V_a^\mu = g_\rho \eta^{\mu\nu}\rho_{\nu a} - 2(m_f^2/g_f)f^{\mu\nu}\rho_{\nu a} + (2/F_\sigma)\sigma\eta^{\mu\nu}\rho_{\nu a} + \cdots . \tag{5.15}$$

The first term is the usual ρ-pole contribution. The additional terms show that there exists a direct coupling between the photon ρ and f mesons and between the photon ρ and σ mesons. These couplings will have one less ρ pole than the ρ-dominance term in the electromagnetic current. Thus the e^+-e^- annihilation cross section into pions will contain additional structures that will sustain in the 1.5–2.5-GeV region. These additional effects turn out to be in agreement with present colliding-beam experiments and are discussed in paper II.

VI. CONCLUSIONS

In this paper we have examined some of the consequences of setting up a formalism for a stress tensor obeying broken-scale-invariance conditions in a fashion analogous to the one that has been used successfully in current-algebra considerations.[2] Thus just as one assumed in the latter case that the currents are smooth interpolating fields for $J^P = 1^\pm, 0^\pm$ mesons, we have assumed that the stress tensor $\Theta^{\mu\nu}$ is a smooth interpolating field for the $J^P = 2^+, 0^+$, $I=0$ mesons ($f, f', \ldots, \sigma, \sigma', \ldots$). When one imposes the usual physical constraints on $\Theta^{\mu\nu}$ a number of remarkable features emerge. Thus the conservation condition $\partial_\mu \Theta^{\mu\nu} = 0$ and the Poincaré-group constraints strongly restrict the form of the f-meson couplings to itself and to other hadrons. When all the nonlinear effects are correctly included one finds that the f mesons couple in a fashion identical to the graviton except that the former possess mass. Indeed, one may define "f-meson metrics" $g_{\mu\nu i}$, $i=1\ldots N$ (one for each f meson) which play a role analogous to the gravitational metric of Einstein's theory. The existence of many metrics due to the physical presence of more than one f meson, however, makes the geometry of these "multimetric Riemannian spaces" much more complicated than the conventional Riemannian geometry. The parameter governing the strength of the deviation of the f-meson metrics from "flat space" is $\lambda \equiv 2m_f^2/g_f$. This quantity is about 10^{20} times larger than the corresponding gravitational parameter $\sqrt{2\kappa}$, and so the f meson does indeed represent a "strong" gravitation. As is well known in Einstein theory, the nonlinearity of the equations eventually produces very large gravitational fields having dramatic effects, but only at very small distances (e.g., at $\approx 10^{-53}$ cm for elementary particles). For f-meson couplings these would occur at distances of the order $\lambda^2/\kappa \approx 10^{40}$ larger, i.e., at 10^{-13} cm. However, the f-meson potentials are not long-range due to the f-meson mass, and so the potentials are effectively reduced (i.e., they act over a limited region of space-time). Whether this latter effect is sufficient to prevent the nonlinearity from being significant is not *a priori* obvious, and it might be interesting to take the nonlinear f-meson Lagrangian sufficiently seriously to see if any of the striking effects of general relativity remain.

At asymptotic energies, one might assume that all interactions are scale-invariant. The form of the f-meson couplings complicates the nature of the breakdown of scale invariance that must occur at intermediate and low energies. Thus the usual assumption made in the absence of f mesons, that $\Theta \equiv \eta_{\mu\nu}\Theta^{\mu\nu}$ is pole-dominated by the σ mesons, is seen to be inconsistent with the above f-meson couplings. For the case in which the number of f mesons equals the number of σ mesons (which appears to be experimentally reasonable) a particularly simple and elegant statement of scale-invariance breaking can be given: The f mesons have as their source the Belinfante stress tensor of the matter fields $\Theta_M^{\mu\nu}$ (i.e., of all fields other than the f-meson itself) and the σ mesons the "curved-space" trace, $g_{\mu\nu}\Theta_M^{\mu\nu}$, of the matter stress tensor. The coupling constants are all equal and given by a single universal constant F_σ. The possibility of a universal coupling of the f and σ mesons to the Belinfante stress tensor is a meaningful idea, for just as the nonlinear current algebra normalizes the Cabibbo currents and allows for the concept of a universal Fermi coupling constant, the Poincaré group normalizes the amplitude of $\Theta_M^{\mu\nu}$. The above scale-breaking hypothesis reduces in the zeroth approximation to the usual pole dominance of Θ and allows one to treat higher-order processes. The universality condition leads to a number of "KSRF-like" relations on the interpolating constants of the field-current identity (2.8) [i.e., Eq. (1.1)]. In paper II these latter are tested in the f-meson decay sum rules and found to check quite well. The discussion there shows that the data imply that the universal scale-breaking constant F_σ is equal to the PCAC constant F_π within experimental accuracy. This last result, combined with the fact that the *breakdown* of a symmetry (scale invariance) can be characterized in terms of a universal coupling, suggests the possibility of a deeper origin of some of the results of this paper. In particular, it should be noted that the f-meson couplings arise from the very fundamental requirements of Poincaré-group invariance, while the σ-meson couplings arise from the *a priori* unrelated scale-breaking condition. (Some not-so-deep possibilities involving hypotheses on the breaking of Weyl invariance are considered in the text.)

The scale-breaking condition combined with the "strong gravitation" form of the f couplings forces modifications on the structure of the axial-vector and vector currents. In particular, specific nonpole f-ρ and σ-ρ terms must appear in the vector currents. This gives rise to a number of consequences. On the theoretical side, the inconsistency between CVC and scale breaking (one forbidding and one requiring a σ-ρ-ρ coupling) is removed. Further, all components of the vector current are forced to have canonical scale dimension 3. Thus the field-current-identity current has the same (canonical) scale dimension as is usually associated with the quark currents. If one is willing to extend the analysis to higher energies and if the anomalous dimensions are absent asymptotically, the f-meson and scale-breaking couplings considered here *deduce* the scaling of the electroproduction data and the $1/q^2$ falloff of the e^+-e^- annihilation cross section.[21] At lower energies the f-ρ and σ-ρ nonpole terms make direct contributions to the $e^+ + e^- \to 4\pi$ cross section over and above the usual vector-dominance terms. These are calculated in paper II and are seen to give contributions of the right size in the high-energy region [i.e., $\sigma_{e^+e^-} \simeq (1-2)\times 10^{-32}$ cm^2 for energies of 1.5–2.5 GeV]. A number of additional successes in comparison with experiment (f-nucleon couplings, σ-nucleon couplings, etc.) will be discussed in paper II. Paper II also discusses a number of formal applications of the theory, i.e., anomalous dimensions, the resolution of the problem that the σ couplings $g_{\sigma aa}$ are proportional to $m_a{}^2$, and a more complicated discussion of the current-algebra conditions in the presence of f and σ mesons.

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[1]G. Mack, Nucl. Phys. B5, 499 (1968); H. A. Kastrup, Phys. Rev. 150, 1183 (1966). Applications of scale invariance to the study of short-distance behavior of field theories has been emphasized by K. Wilson [Phys. Rev. 179, 1499 (1969); Phys. Rev. D 2, 1473 (1970)]. A survey of some of the work on broken scale invariance can be found in P. Carruthers, Phys. Reports 1C, 1 (1970), and in B. Zumino, rapporteur's talk, in *Proceedings of the Fifteenth International Conference on High Energy Physics, Kiev, U.S.S.R., 1970* (Atomizdat, Moscow, 1971).

[2]See, e.g., R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Rev. Letters 19, 1085 (1967); R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. D 3, 594 (1971); H. Schnitzer and S. Weinberg, Phys. Rev. 164, 1828 (1967); I. S. Gerstein, H. J. Schnitzer, and S. Weinberg, *ibid.* 175, 1873 (1968); S. G. Brown and G. B. West, Phys. Rev. Letters 19, 812 (1967).

[3]We use the Lorentz metric $\eta_{\mu\nu}$ with signature +2.

[4]See, e.g., B. Zumino, in *Brandeis Summer Institute Lectures, 1970* (MIT Press, Cambridge, Mass., 1970), Vol. 2; S. P. de Alwis and P. J. O'Donnell, Phys. Rev. D2, 1023 (1970); C. J. Isham, A. Salam, and J. Strathdee, *ibid.* 3, 867 (1971); K. Raman, *ibid.* 2, 1577 (1970); Phys. Rev. Letters 26, 1069 (1971); Phys. Rev. D 3, 2900 (1971); B. Renner, Phys. Letters 33B, 599 (1970);

P. Carruthers, *ibid.* 2, 2265 (1970); *ibid.* 3, 959 (1971); J. Ellis, Nucl. Phys. B26, 536 (1971); CERN Report No. CERN-TH-1289 (unpublished); H. Kleinert, L. P. Staunton, and P. H. Weisz, Nucl. Phys. B38, 87 (1972); and other references found in these papers.

[5]See, e.g., S. Deser, J. Trubatch, and S. Trubatch, Can. J. Phys. 44, 1715 (1966); S. J. Chang, Phys. Rev. 148, 1259 (1966). Since the f meson is described by a symmetric second-order Lorentz tensor, it is possible to set up many formalisms differing by point transformations. We use here the second of the Chang formulations (given in Sec. III of that paper), since it is closest to the (massless) gravitational field formalism.

[6]We normalize states so that $N_i = [2\omega_i(2\pi)^3]^{-1/2}$, where $\omega_i \equiv (p^2 + m_i^2)^{1/2}$. The polarization tensors obey $\epsilon_i^{\mu\nu}(\lambda)^* \times \epsilon_{\mu\nu i}(\lambda') = \delta_{\lambda\lambda'}$ and $p_\mu \epsilon_i^{\mu\nu} = 0 = \eta_{\mu\nu}\epsilon_i^{\mu\nu}$.

[7]By a superpotential we mean a term which is *identically* conserved and gives zero contribution to P^μ and $M^{\mu\nu}$. The Huggins structures of Eq. (2.16) are of course examples of such quantities.

[8]We, of course, impose the usual smoothness requirements here as well.

[9]In Eq. (2.25) and all the following equations, indices of all generally covariant quantities will be raised and lowered by the full f-meson metric $g_{\mu\nu}$, while the indices of the f fields of noncovariant mass terms will be raised and lowered by the usual Lorentz metric. Thus in Eq. (2.25) $G^{\mu\nu} \equiv g^{\mu\alpha} g^{\nu\beta} G_{\alpha\beta}$, while $f^{\mu\nu} \equiv \eta^{\mu\alpha}\eta^{\nu\beta} f_{\alpha\beta}$. The usual nomenclature of Riemannian geometry will be used for the generally covariant structures.

[10]We assume here for simplicity that $\mathcal{L}_M$ depends only on $g_{\mu\nu}$ and not on $\Gamma^\alpha{}_{\mu\nu}$. When improvement terms are included, this is no longer true. The additional contributions to $\mathcal{L}$ needed to generate improved stress tensors are discussed in Sec. III.

[11]See, e.g., E. Schrödinger, *Space Time Structure* (Cambridge Univ. Press, Cambridge, England, 1960), p. 107.

[12]For the gravitational case there are no mass terms, of course. That the nonlinear parts of $\sqrt{-g}\, G^{\mu\nu}$ should represent the stress tensor of the gravitational field was first proposed by A. Papapetrou [Proc. Roy. Irish Acad. 52A, 11 (1948)].

[13]As was discussed above, the nonlinear parts of the f-meson mass terms are completely undetermined by the CTC and Poincaré-group constraints.

[14]C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970).

[15]E. R. Huggins, Ph.D. thesis, Caltech, 1962 (unpublished).

[16]M. H. Friedman, P. Nath, and R. Arnowitt (unpublished), hereafter referred to as paper II.

[17]It will be shown in paper II that the linear superpotential term $\mathcal{L}^{\mu\nu}$ of Eq. (2.29) does not contribute in the limit $\lambda \to 0$.

[18]K. G. Wilson, Phys. Rev. D 2, 1473 (1970); S. Coleman and R. Jackiw, Ann. Phys. (N.Y.) 67, 552 (1971).

[19]The situation is analogous to the well-known PCAC anomalies. Thus these can be thought of as arising from singular closed-loop diagrams [S. L. Adler, Phys. Rev. 177, 2426 (1969)], or can be characterized phenomenologically in the tree-seagull approximation [R. Arnowitt, M. H. Friedman, and P. Nath, Phys. Letters 27B, 657 (1968)]. The latter procedure has the advantage of correctly *predicting* the $\eta \to 2\gamma$ rate.

[20]In this way the treatment of the σ field here is different from, e.g., the analyses of Zumino and Ellis (Ref. 4), who introduce a dilation field as a mechanism of maintaining scale invariance. No conditions as to whether the Lagrangian is scale-invariant or not have yet been assumed here. (The scale-breaking postulate is introduced in Sec. IV of the present paper.)

[21]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).