Relativistic Harmonic Oscillator

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A relativistic formulation of the simple harmonic oscillator is discussed. It differs from the customary formulation in that it is derivable from a Lorentz-invariant variational principle. A consistent relativistic quantization procedure is thereby admitted.

I. INTRODUCTION

The special relativistic harmonic oscillator has a meager history; there has been only slight discussion[1-4] in recent years. This discussion does not include a quantum theory. The customary formulation is in terms of either the equation

$$\frac{d}{dt}\frac{m_0\dot{x}}{(1-v^2/c^2)^{1/2}}+kx=0 \tag{1}$$

or the energy integral which it admits. The particle rest mass is considered to be constant. Application of the identity $U^\alpha dU_\alpha/d\tau=0$ yields the customary result that F^4, the timelike component of the four-force, is just $i\gamma\vec{f}\cdot\vec{v}/c$ where $\vec{f}$ and $\vec{v}$ are the Newtonian three-force and velocity.

Equation (1) may be derived from a variational principle, but not in an unambiguously Lorentz-invariant fashion. This flaw is propagated to any quantized version. It is the purpose of this paper to present an alternative scheme which is possessed of a Lorentz-invariant variational principle and thereby admits quantization via the concomitant Hamilton-Jacobi and Klein-Gordon equations.

The procedure hinges essentially on relaxation of the requirement that the rest mass be constant. The theory thereby obtained is of the class, the prototype of which is the Lorentz-covariant theory of gravitation introduced by Nordström,[5] in which the rest mass is potential-dependent.

The following notation is used: m_0 is the rest mass of the oscillating particle when its spatial displacement from the attractive center is zero and k is the spring constant. Both are phenomenological scalars. Greek indices run from 1 to 4 and $x^4=ict$ so that there is no distinction between contravariant and covariant indices. The summation convention is used, and $U^\alpha=dx^\alpha/d\tau$, $d\tau=dt(1-v^2/c^2)^{1/2}$, so that $U^\alpha U_\alpha=-c^2$ and $\gamma=(1-v^2/c^2)^{-1/2}$.

II. EQUATIONS OF MOTION

The equations of motion take the form

$$\frac{d}{d\tau}mU^\alpha=-kr^\alpha$$
$$=-\phi_{,\alpha}, \tag{2}$$

where m is not necessarily constant, r^α are the components of the displacement of the oscillating particle, and $\phi=\frac{1}{2}kr^\alpha r_\alpha$. In its rest frame the attractive center has a world line given by $(0,0,0,ict)$. Hence, in this frame $r^4=0$ and r^i ($i=1,2,3$) is just x^i.

Inasmuch as m is not a constant, it follows that

$$m\frac{dU^\alpha}{d\tau}+U^\alpha\frac{dm}{d\tau}=-\phi_{,\alpha}. \tag{3}$$

The constraint $U_\alpha dU^\alpha/d\tau=0$ yields

$$-c^2\frac{dm}{d\tau}=-\frac{d\phi}{d\tau}, \tag{4}$$

which integrates immediately to

$$m=\frac{\phi}{c^2}+m_0. \tag{5}$$

Thus, m is potential-dependent.

By virtue of Eq. (4), Eq. (3) may be rewritten as

$$m\frac{dU^\alpha}{d\tau}=-\phi_{,\beta}\left(\delta^{\alpha\beta}+\frac{U^\alpha U^\beta}{c^2}\right). \tag{6}$$

The expression in parentheses is the operator which projects $\phi_{,\beta}$ parallel to $dU^\alpha/d\tau$.

It is readily verified that Eq. (6) may be obtained from the variational principle

$$\delta\int mc^2(-U^\alpha U_\alpha)^{1/2}d\tau=0, \tag{7}$$

where m is given by Eq. (5).

It may be observed[6] that ϕ is a solution of $\delta^{\alpha\beta}\phi_{,\alpha\beta}=k$.

III. ENERGY INTEGRAL

Solutions of equations of the type of Eq. (6) were first obtained by Behacker[7] in connection with Nordström's gravitation theory. The general techniques are applicable to any central potential. For the particular case at hand the results are as follows. For simplicity the calculation is made in the rest frame of the attractive center.

Because $\partial L/\partial x^4 = 0$ there is an energy integral which is independent of the dimensionality of the oscillator and is readily obtained. For $\alpha = 4$, Eq. (6) specializes to

$$\frac{dU^4}{d\tau} = -\frac{U^4 U^\beta}{mc^2}\frac{\partial\phi}{\partial x^\beta} = -\frac{U^4}{m}\frac{dm}{d\tau}. \tag{8}$$

This integrates immediately to

$$\ln U^4 = -\ln m + \ln\frac{iE}{c}, \tag{9}$$

where E is a constant and the factor ic is included for later convenience. Now, $U^4 = ic\gamma$ and Eq. (9) may be written compactly as

$$E = mc^2\gamma \tag{10a}$$

$$= \frac{\phi + m_0 c^2}{(1 - v^2/c^2)^{1/2}}. \tag{10b}$$

In anticipation of an oscillatory solution this may be written as

$$E = \tfrac{1}{2}ka^2 + m_0 c^2, \tag{11}$$

where a is the maximum excursion and corresponds to $v = 0$.

For the one-dimensional case it is particularly simple to integrate Eq. (10b). Solve first for

$$Ev/c = [E^2 - (\tfrac{1}{2}kx^2 + m_0 c^2)^2]^{1/2}. \tag{12}$$

This leads to an elliptic integral in the following way. With the use of Eq. (11) this may be written as

$$E\frac{dx}{cdt} = [(E + \tfrac{1}{2}kx^2 + m_0 c^2)(\tfrac{1}{2}ka^2 - \tfrac{1}{2}kx^2)]^{1/2}. \tag{13}$$

The exact solutions for the displacement and the period are readily obtainable[8] in terms of the Jacobian elliptic functions $\mathrm{sd}(u|m)$ and $K(m)$:

$$x = a\left(\frac{E + m_0 c^2}{2E}\right)^{1/2} \mathrm{sd}\left(\left(\frac{kc^2}{E}\right)^{1/2} t \,\middle|\, \frac{ka^2}{4E}\right) \tag{14}$$

and the period is

$$T = 4\left(\frac{E}{kc^2}\right)^{1/2} K\left(\frac{ka^2}{4E}\right). \tag{15}$$

For small displacements, i.e., $\frac{1}{2}ka^2 \ll m_0c^2$, Eq. (14) reduces to the classical limit

$$x = a\sin[(k/m_0)^{1/2}t].$$

IV. QUANTIZATION

With the Lagrangian function, $L = mc^2(-U^\alpha U_\alpha)^{1/2}$, the Hamilton-Jacobi equation is readily constructed as

$$p^\alpha p_\alpha + m^2c^4 = 0. \tag{16}$$

In one dimension this is

$$p_1^2 + p_4^2 + (\tfrac{1}{2}kx^2 + m_0c^2)^2 = 0. \tag{17}$$

The corresponding Klein-Gordon equation is

$$-\hbar^2c^2\frac{\partial^2\psi}{\partial x^2} + \hbar^2\frac{\partial^2\psi}{\partial t^2} + (\tfrac{1}{2}kx^2 + m_0c^2)^2\psi = 0. \tag{18}$$

The usual substitution $\psi(x,t) = \phi(x)\exp(-iEt/\hbar)$ reduces this to

$$-\hbar^2c^2\frac{d^2\phi}{dx^2} - E^2\phi + (\tfrac{1}{4}k^2x^4 + kx^2m_0c^2 + m_0{}^2c^4)\phi = 0. \tag{19}$$

Introduction of the dimensionless parameter $\rho = (m_0k/\hbar^2)^{1/4}x$ further reduces Eq. (18) to

$$\frac{d^2\phi}{d\rho^2} + \epsilon^0\phi - \rho^2\phi - \alpha\rho^4\phi = 0, \tag{20}$$

where

$$\epsilon^0 = (E^2 - m_0c^2)/\hbar\omega m_0c^2,$$

$$\omega = (k/m_0)^{1/2},$$

$$\alpha = \hbar\omega/4m_0c^2.$$

This is precisely the *form* of the nonrelativistic, biquadratic, anharmonic oscillator. The solutions are well known.

For energies which are sufficiently small, i.e., $\frac{1}{2}kx^2 \ll m_0c^2$, the ρ^4 term may be neglected and a lowest (relativistic) approximation for the eigenvalues is obtained:

$$\epsilon_n^0 = 2n + 1$$

and

$$E_n^0 = \pm m_0c^2\left[1 + 2(n + \tfrac{1}{2})\frac{\hbar\omega}{m_0c^2}\right]^{1/2} \approx \pm[m_0c^2 + (n + \tfrac{1}{2})\hbar\omega]. \tag{21}$$

In the regime where the binding energy is appreciable compared to the rest energy, the corrections to the energy levels are given by standard Rayleigh-Schrödinger perturbation methods. In the first order these are readily found to be

$$E_n = \pm\{m_0c^2 + \hbar\omega[n + \tfrac{1}{2} + \tfrac{3}{4}\alpha(n^2 + n + \tfrac{1}{2})]\}. \tag{22}$$

V. DISCUSSION

The formulation presented constitutes a consistent structure possessed of minimal assumptions. It is of special interest because it can be cast in the form of a variational principle. Being possessed of a Lagrangian function, the theory admits straightforward quantization. The theory is of the general class characterized by utilization of a scalar field, the gradient of which provides the four-force. In common with this class the mass is dependent upon the potential as well as the speed. The formulation may be of utility in the construction of hadron models.

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[6]This observation is due to P. G. Bergmann.
[7]M. Behacker, Z. Physik, 14, 989 (1913).
[8]See, e.g., L. M. Milne-Thomson, *Jacobian Elliptic Function Tables* (Dover, New York, 1950).

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Electromagnetic Fields and Massive Bodies

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We consider an example of the effects of massive bodies on static electromagnetic fields in general relativity which yields considerable insight into the fadeaway of multipole moments in nonspherical perturbations of gravitational collapse. We calculate the electromagnetic field of an electrostatic or magnetostatic multipole of fixed strength placed at the center of a massive, nonrotating, spherical shell. If we consider a sequence of static solutions in which the massive shell approaches its own Schwarzschild radius, we find that except in the monopole ($l=0$) case the value of the multipole moment measured by a distant observer goes to zero. Thus, for an arbitrary (but finite) stationary charge and current distribution inside the shell, in the limit as the shell approaches its Schwarzschild radius the only property of the distribution which can be measured by an external observer is the total electric charge.

I. INTRODUCTION

In the Newtonian theory of gravitation, a massive body can affect an electromagnetic field only by acting as a source, j^μ, of charge or current. Thus, a massive body with no electromagnetic sources has no influence upon an electromagnetic field. This statement is also true *locally* in general relativity. However, by curving the space-time geometry, a massive body in general relativity can have a significant effect upon an electromagnetic field. Probably the most familiar example of such an effect is the bending of light passing near a massive body. An extreme example of this sort of effect in a static problem occurs in the recently analyzed problem of a point charge placed near the horizon of a Schwarzschild black hole.[1] Although the charge is well "off center," it is found that as the charge approaches the horizon all the electrostatic multipole moments as measured at infinity go to zero. (Thus, lowering a charge toward a Schwarzschild black hole tends to produce a Reissner-Nordström black hole rather than a "naked singularity."[2]) The effects of rotating masses on electromagnetic fields have been treated by Cohen[3] and by Ehlers and Rindler.[4]

In this paper we investigate a further example of the effects of massive bodies on static electromagnetic fields by calculating the electromagnetic field of an electrostatic or magnetostatic multipole of fixed strength placed at the center of a massive spherical shell. We make no approximation concerning the strength of the gravitational field, but the electromagnetic field is assumed to be sufficiently weak that the effect of the electromagnetic stress-energy on the background space-time geometry is negligible (i.e., we consider test electromagnetic fields). An analogous dynamic problem