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Radiation from a Free Electron Interacting with a Circularly Polarized Laser Pulse*

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The radiation from an electron driven by an intense circularly polarized electromagnetic wave of finite duration and smoothly varying intensity is analyzed. The effect of radiative reaction on the electron orbit is accounted for self-consistently. Expressions are derived for the total scattering cross section, the angular and spectral distributions, and the polarization tensor of the scattered radiation. The results are relativistically correct, and exhibit the dependence of the characteristics of the scattered radiation on the intensity profile of the incident radiation.

In the present communication we analyze the properties of the radiation emitted by a free electron interacting with an intense circularly polarized electromagnetic wave of finite duration. First we derive relativistically correct expressions for the total cross section associated with the scattering of the incident wave by the moving electron. We then calculate the angular and spectral distributions of the scattered radiation and discuss the polarization of its spectral components.

Sarachik and Schappert[1] have dealt with essentially all of these characteristics of the scattered radiation for the case of an elliptically polarized incident wave. Although radiation damping is neglected in their calculations, they point out that for a sufficiently long pulse it may induce a significant modification of the electron motion and consequently of the emitted radiation. Several authors[2-5] have discussed such a modification of the particle motion. For a linearly polarized incident wave, Sen Gupta[5] has employed an expansion technique valid for pulses of short duration to calculate the modified electron orbit and the associated change of the spectrum of the scattered radiation.

The analysis we are presenting is applicable to pulses of arbitrary duration. We assume the pulse of the incident electromagnetic radiation to be characterized by the vector potential

$$\vec{A}(\eta) = \mathrm{Re}[A_0(\eta)e^{-i\eta}(\hat{e}_1 + i\lambda\hat{e}_2)], \tag{1}$$

where λ denotes the helicity, the (real) amplitude $A_0(\eta)$ is a smooth function of the phase

$$\eta = \omega_0 t - k_0 z, \tag{2}$$

and where the propagation vector $\vec{k}_0 = k_0\hat{e}_3$ lies along the z axis of the Cartesian coordinate system (x, y, z). The amplitude $A_0(\eta)$ is different from zero only in the given interval $0 \le \eta \le \eta_f$. Since the electric field associated with the vector potential (1) is

$$\vec{E}(\eta) = -\frac{1}{c}\frac{\partial\vec{A}(\eta)}{\partial t} = -\frac{\omega}{c}\frac{d\vec{A}(\eta)}{d\eta},$$

it follows that

$$\vec{\mathrm{E}}(\eta) = \frac{\omega}{c}\,\mathrm{Re}\left[\left(iA_0(\eta) - \frac{dA_0(\eta)}{d\eta}\right)e^{-i\eta}(\hat{e}_1 + i\lambda\hat{e}_2)\right]. \tag{3}$$

The amplitude of the electric field is therefore

$$E_0(\eta) = \frac{\omega}{c}\left[A_0^2(\eta) + \left(\frac{dA_0(\eta)}{d\eta}\right)^2\right]^{1/2}, \tag{4}$$

and Eq. (3) can be replaced by

$$\vec{\mathrm{E}}(\eta) = \mathrm{Re}[E_0(\eta)e^{-i(\eta-\pi/2-\alpha)}(\hat{e}_1 + i\lambda\hat{e}_2)], \tag{5}$$

where

$$\alpha = \tan^{-1}\left[\left(\frac{1}{A_0(\eta)}\right)\frac{dA_0(\eta)}{d\eta}\right]$$

is the phase by which the electric vector $\vec{\mathrm{E}}$ lags behind the vector potential $\vec{\mathrm{A}}$.

In a previous paper[2] we show that, for smoothly varying pulses for which

$$\left(\frac{1}{A_0(\eta)}\right)\frac{dA_0(\eta)}{d\eta} \ll 1,$$

an electron initially at rest acquires the longitudinal velocity $c\beta_z$ determined by the relations

$$\beta_z(\eta_e) = \frac{[1+\epsilon(\eta_e)]^2[1+\mu^2(\eta_e)]-1}{[1+\epsilon(\eta_e)]^2[1+\mu^2(\eta_e)]+1}, \tag{6}$$

$$\begin{aligned}\epsilon(\eta) &= \sigma_0\left(\frac{1}{mc^2}\right)\left(\frac{1}{\omega_0}\right)\int_0^{\eta} I_0(\eta')d\eta' \\ &= \epsilon_0\int_0^{\eta}\mu^2(\eta')d\eta',\end{aligned} \tag{7}$$

and

$$\mu(\eta) = \left(\frac{4\pi e^2 I_0(\eta)}{m^2c^3\omega_0^{\,2}}\right)^{1/2}, \tag{8}$$

where

$$\eta_e = \omega_0 t - k_0 z_e \tag{9}$$

is the value of the phase at the coordinate z_e of the electron,

$$\epsilon_0 = \left(\frac{2e^2}{3c^3}\right)\left(\frac{\omega_0}{m}\right) \tag{10}$$

is a dimensionless parameter relating to the radiative reaction,

$$\sigma_0 = \left(\frac{8\pi}{3}\right)\left(\frac{e^2}{mc^2}\right)^2 \tag{11}$$

is the Thomson cross section, and

$$I_0(\eta) = \frac{c}{4\pi}E_0^{\,2} \tag{12}$$

is the intensity of the incident radiation. The orbit of the electron is a helical curve of slowly varying slope and slowly varying radius,

$$\begin{aligned}\rho(\eta_e) &= \left(\frac{c}{\omega_0}\right)\left(\frac{1+\beta_z(\eta_e)}{1-\beta_z(\eta_e)}\right)^{1/2}\frac{\mu(\eta_e)}{[1+\mu^2(\eta_e)]^{1/2}} \\ &= \frac{c}{\omega_0}\,\mu(\eta_e)[1+\epsilon(\eta_e)].\end{aligned} \tag{13}$$

In a system of cylindrical coordinates (θ, ϕ, z) the relation between the azimuthal coordinate ϕ_e of the electron and the phase η_e is[6]

$$\phi(\eta_e) = \lambda(\eta_e - \tfrac{1}{2}\pi), \tag{14}$$

where $\frac{1}{2}\pi$ is the angle between the vector potential $\vec{\mathrm{A}}$ and the radial coordinate vector $\rho_e\hat{e}_\rho$; it is also the angle between the azimuthal component of the particle velocity and the electric vector $\vec{\mathrm{E}}$. Thus the orbit of the electron is described by the position vector

$$\vec{\mathrm{r}}_e(\eta_e) = \rho(\eta_e)(\hat{e}_1\sin\eta_e - \lambda\,\hat{e}_2\cos\eta_e) + z(\eta_e)\hat{e}_3, \tag{15}$$

where

$$z(\eta_e) = \int_0^t c\beta_z\,dt'. \tag{16}$$

The radial and azimuthal velocity components $c\beta_\rho$ and $c\beta_\phi$ may be derived from the relations

$$c\beta_\rho = \frac{d\rho_e}{dt} = \omega_0(1-\beta_z)\frac{d\rho_e}{d\eta_e}$$

and

$$c\beta_\phi = \rho_e\frac{d\phi_e}{dt} = \omega_0(1-\beta_z)\frac{d\phi_e}{d\eta_e}.$$

By virtue of the relations (6), (13), and (14) we obtain

$$\beta_\rho(\eta_e) = \frac{2\{[1+\epsilon(\eta_e)]d\mu(\eta_e)/d\eta_e + \epsilon_0\mu^3(\eta_e)\}}{[1+\epsilon(\eta_e)]^2[1+\mu^2(\eta_e)]+1} \tag{17}$$

and

$$\beta_\phi(\eta_e) = \frac{2\lambda\mu(\eta_e)[1+\epsilon(\eta_e)]}{[1+\epsilon(\eta_e)]^2[1+\mu^2(\eta_e)]+1}. \tag{18}$$

The instantaneous rate at which the electron scatters radiation from the incident wave and the spectral characteristics of the emitted radiation depend on the phase η_e, which is a measure of how far the electron lags behind the leading edge of the pulse. To proceed with the calculations we must choose a particular phase η_{es} for the spectral analysis, and express the electron orbit in a somewhat simpler form, which is valid for a complete azimuthal cycle $\eta_e - \eta_{es} = 2\pi$, and which retains all of the essential features of the motion. We first note that for optical frequencies

$$\epsilon_0 = \left(\frac{2e^2}{3c^3}\right)\frac{\omega_0}{m}$$

is of the order 10^{-8}, while μ^3 is of order 10^2 for a radiation intensity of 10^{20} W/cm^2. Thus the term $2\epsilon_0\mu^3$ in the numerator of expression (17) is of order 10^{-6}. Furthermore, for the radiation intensity 10^{20} W/cm^2 and for a pulse duration of 1 nsec $\epsilon(\eta)$ does not exceed unity. Since our analysis of the electron orbit pertains to smoothly varying pulses for which $d\mu(\eta)/d\eta \ll 1$, the radial component $c\beta_\rho$ of the velocity is much less than the components $c\beta_\phi$ and $c\beta_z$, and may be neglected in calculating the radiation emitted by the electron. Next, we express the longitudinal position of the electron as the linear approximation

$$z_e(\eta_e) = z_e(\eta_{es}) + c\beta_z(\eta_{es})(t-t_s) = z_{es} + \frac{c\beta_{zs}}{\omega_0(1-\beta_{zs})}(\eta_e - \eta_{es}), \tag{19}$$

where we have used the relation $\eta_{es} = \omega_0 t_s - k_0 z_{es}$ and the definitions $\beta_{zs} = \beta_z(\eta_{es})$, $z_{es} = z_e(\eta_{es})$. With these substitutions we may now express the position vector as

$$\vec{r}_e(\eta_e) = \rho_{es}(\hat{e}_1 \sin\eta_e - \lambda\hat{e}_2\cos\eta_e) + \hat{e}_3\left(z_{es} + \frac{c\beta_{zs}}{\omega_0(1-\beta_{zs})}(\eta_e - \eta_{es})\right). \tag{20}$$

The corresponding velocity is $c\vec{\beta}_s$ with

$$\vec{\beta}(\eta_e) = \beta_{\phi s}(\lambda\hat{e}_1\cos\eta_e + \hat{e}_2\sin\eta_e) + \beta_{zs}\hat{e}_3\,. \tag{21}$$

In these expressions ρ_{es}, z_{es}, $\beta_{\phi s}$, and β_{zs} are to be regarded as constants, whose values are all affected by the radiative reaction.

We now determine the rate of radiation of energy $d\mathcal{E}^{\text{rad}}/dt$ and momentum $d\vec{P}^{\text{rad}}/dt$ by the electron. Since $(1/c)d\mathcal{E}^{\text{rad}}$ is the time component of the differential four-momentum we utilize the well-known expressions [Ref. 7, Eq. (73.3)]

$$\frac{1}{c}\,d\mathcal{E}^{\text{rad}} = \left(\frac{2e^2}{3c}\right)\frac{w^2}{c^4}\,(c\,dt) \tag{22}$$

and

$$d\vec{P}^{\text{rad}} = \left(\frac{2e^2}{3c}\right)\frac{w^2}{c^4}\,c\vec{\beta}\,dt \tag{23}$$

in which

$$\frac{w^2}{c^4} = \frac{\gamma^4}{c^2}\left[\left(\frac{d\vec{\beta}}{dt}\right)^2 + \gamma^2\left(\vec{\beta}\cdot\frac{d\vec{\beta}}{dt}\right)^2\right], \tag{24}$$

with $\gamma = 1/(1-\vec{\beta}^2)^{1/2}$, and $\vec{\beta}$ given by Eq. (21). By virtue of the connection

$$\frac{d}{dt} = \omega_0[1-\beta_z(\eta_e)]\frac{d}{d\eta_e}, \tag{25}$$

Eq. (24) may be written as

$$\frac{w^2(\eta_e)}{c^4} = \omega_0^2\left(\frac{\gamma^4(\eta_e)}{c^2}\right)[1-\beta_z(\eta_e)]^2 \times\left[\left(\frac{d\vec{\beta}(\eta_e)}{d\eta_e}\right)^2 + \gamma^2(\eta_e)\left(\vec{\beta}(\eta_e)\cdot\frac{d\vec{\beta}(\eta_e)}{d\eta_e}\right)^2\right]. \tag{26}$$

By substituting in Eq. (26) the expression (21) for $\vec{\beta}$, we obtain

$$\frac{w^2(\eta_{es})}{c^4} = \frac{\omega_0^2}{c^2}\left(\frac{\beta_\phi(\eta_{es})[1-\beta_z(\eta_{es})]}{1-[\beta_\phi^2(\eta_{es})+\beta_z^2(\eta_{es})]}\right)^2 = \frac{\omega_0^2}{c^2}\left(\frac{\mu(\eta_{es})}{1+\epsilon(\eta_{es})}\right)^2, \tag{27}$$

and the radiated power becomes

$$\frac{d\mathcal{E}^{\text{rad}}(\eta_{es})}{dt} = \left(\frac{2e^2}{3c}\right)\omega_0^2\left(\frac{\mu(\eta_{es})}{1+\epsilon(\eta_{es})}\right)^2. \tag{28}$$

By decomposing the vector equation (23) into a perpendicular part

$$dP_\perp^{\text{rad}}(\eta_{es}) = [(dP_x^{\text{rad}}(\eta_{es}))^2 + (dP_y^{\text{rad}}(\eta_{es}))^2]^{1/2}$$

and a longitudinal part $dP_z^{\text{rad}}(\eta_{es})$, we arrive at the relations for the momentum radiated per unit time,

$$\frac{dP_\perp^{\text{rad}}(\eta_{es})}{dt} = \frac{4}{3}e^2\left(\frac{\omega_0}{c}\right)^2\frac{[\mu(\eta_{es})]^3}{\{[1+\epsilon(\eta_{es})]^2[1+\mu^2(\eta_{es})]+1\}[1+\epsilon(\eta_{es})]}, \tag{29}$$

$$\frac{dP_z^{\text{rad}}(\eta_{es})}{dt} = \frac{2}{3}e^2\left(\frac{\omega_0}{c}\right)^2\left(\frac{\mu(\eta_{es})}{1+\epsilon(\eta_{es})}\right)^2\frac{[1+\epsilon(\eta_{es})]^2[1+\mu^2(\eta_{es})]-1}{[1+\epsilon(\eta_{es})]^2[1+\mu^2(\eta_{es})]+1}. \tag{30}$$

The total cross section for the radiation scattered by the moving electron,

$$\sigma^{\text{rad}}(\eta_{es}) = \frac{1}{I_0}\frac{d\mathcal{E}^{\text{rad}}(\eta_{es})}{dt},$$

where I_0 denotes the incident flux defined by Eq. (12), is then

$$\sigma^{\text{rad}}(\eta_{es}) = \sigma_0[1+\epsilon(\eta_{es})]^{-2}, \tag{31}$$

where σ_0 is the Thomson cross section, given in Eq. (11). This equation shows that the effect of radiative reaction is to reduce the cross section for the scattering of the incident radiation by the factor $[1+\epsilon(\eta_{es})]^{-2}$. If radiative reaction is neglected, the longitudinal motion of the electron has no effect on the total cross section. Similarly, the ratio of the momentum radiated per unit time to the momentum per unit area per unit time of the incident wave is given by

$$\frac{1}{I_0/c}\,\frac{dP_\perp^{\rm rad}(\eta_{es})}{dt}=2\sigma_0\mu(\eta_{es})[1+\epsilon(\eta_{es})]^{-1}\{[1+\epsilon(\eta_{es})]^2[1+\mu^2(\eta_{es})]+1\}^{-1} \tag{32}$$

and

$$\frac{1}{I_0/c}\,\frac{dP_z^{\rm rad}(\eta_{es})}{dt}=\sigma_0[1+\epsilon(\eta_{es})]^{-2}\left(\frac{[1+\epsilon(\eta_{es})]^2[1+\mu^2(\eta_{es})]-1}{[1+\epsilon(\eta_{es})]^2[1+\mu^2(\eta_{es})]+1}\right) \tag{33}$$

for the perpendicular and longitudinal components, respectively.

The angular distribution of the scattered radiation intensity may be calculated from the equation [Ref. 7, Eq. (73.8)]

$$\vec{\mathrm{E}}^{\rm rad}(r,\theta,\phi,t+R/c)=\frac{e}{c}\,\frac{\hat{k}\times\{[\hat{k}-\vec{\beta}(t)]\times[d\vec{\beta}(t)/dt]\}}{R(t)[1-\hat{k}\cdot\vec{\beta}(t)]^3} \tag{34}$$

for the electric vector of the wave field generated by the moving electron. Here $R(t)$ is the distance between the particle and the observer, $\hat{k}$ denotes the unit vector in the direction of propagation of the radiated wave, and $c\vec{\beta}(t)$ is the velocity of the electron at the time t which is retarded with respect to the observer time $t+R/c$. As standard approximations for radiation in the wave zone we replace R by r, and $\hat{k}$ by $\hat{e}_r$, where

$$\hat{e}_r=\hat{e}_1\sin\theta\cos\phi+\hat{e}_2\sin\theta\sin\phi+\hat{e}_3\cos\theta\,. \tag{35}$$

If the electron is at the phase $\eta_e=\eta_{es}$, we then obtain for the differential power radiated into the solid angle $d\Omega=\sin\theta\,d\theta d\phi$

$$\begin{aligned}dW(\theta,\phi,t+R/c,\eta_{es})&=\frac{c}{4\pi}[\vec{\mathrm{E}}^{\rm rad}(r,\theta,\phi)]^2r^2\,d\Omega\\&=\frac{c}{4\pi}\left(\frac{e\omega_0}{c}\right)^2[\beta_\phi(\eta_{es})]^2[1-\beta_z(\eta_{es})]^2[1-\cos\theta\,\beta_z(\eta_{es})-\lambda\sin\theta\cos(\lambda\eta_{es}-\phi)]^{-6}\\&\quad\times(\sin^2(\lambda\eta_{es}-\phi)[\cos\theta-\beta_z(\eta_{es})]^2+\{\lambda\cos(\lambda\eta_{es}-\phi)[1-\cos\theta\,\beta_z(\eta_{es})]-\sin\theta\,\beta_\phi(\eta_{es})\}^2)\,d\Omega\,.\end{aligned} \tag{36}$$

We now proceed to the spectral resolution of the radiation emitted by the electron. The Fourier components of the vector potential in the wave zone are given by[8]

$$\vec{\mathrm{A}}_n^{\rm rad}=e^{i(\omega_n/c)r}\,\frac{e}{2\pi r}\oint\vec{\beta}\,e^{i[\omega_n t-(\omega_n/c)\hat{k}\cdot\vec{\mathrm{r}}_e(\eta_e)]}\,d\eta_e\,, \tag{37}$$

where the integration is over one azimuthal period of the electron motion. If we substitute for $\vec{\mathrm{r}}_e$ and $\vec{\beta}$ the expressions (20) and (21), and if ω_n assumes certain discrete values to be determined, the integrand in Eq. (37) is periodic. The corresponding Fourier components of the electric field are

$$\begin{aligned}\vec{\mathrm{E}}_n^{\rm rad}&=i\,\frac{\omega_n}{c}\,\hat{k}\times(\vec{\mathrm{A}}_n\times\hat{k})\\&=i\,\frac{\omega_n}{c}[\vec{\mathrm{A}}_n-\hat{k}(\vec{\mathrm{A}}_n\cdot\hat{k})]\,,\end{aligned}$$

and therefore

$$\vec{\mathrm{E}}_n^{\rm rad}=e^{i(\omega_n/c)r}\,\frac{ie\omega_n}{2\pi cr}\int_{\eta_{es}}^{\eta_{es}+2\pi}\vec{\beta}_t(\eta_e)\,e^{i[\omega_n t-(\omega_n/c)\hat{k}\cdot\vec{\mathrm{r}}_e(\eta_e)]}\,d\eta_e\,, \tag{38}$$

where $c\vec{\beta}_t(\eta_e)$ is the component of the electron velocity perpendicular to the direction of propagation $\hat{k}$. Since $\hat{k}=\hat{e}_r$ [see Eq. (35)], we may write

$$\vec{\beta}_t=\vec{\beta}-\hat{e}_r(\vec{\beta}\cdot\hat{e}_r)\,. \tag{39}$$

From Eq. (21) and the transformation[9] between the Cartesian unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ and the unit vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$ for the spherical coordinate system (r, θ, ϕ), we find

$$\vec{\beta}_t(\eta_e) = \hat{e}_\theta[\lambda\beta_{\phi s}\cos\theta\cos(\lambda\eta_e - \phi) - \beta_{zs}\sin\theta] + \hat{e}_\phi\lambda\beta_{\phi s}\sin(\lambda\eta_e - \phi) . \tag{40}$$

By virtue of Eqs. (9), (19), and (20) the argument of the exponential in the integrand of Eq. (38) may be written as

$$\omega_n t - \frac{\omega_n}{c}\hat{k}\cdot\vec{r}_e = \frac{\omega_n\lambda(1-\beta_{zs}\cos\theta)}{\omega_0(1-\beta_{zs})}(\lambda\eta_e - \phi) - \frac{\omega_n}{c}\lambda\rho_{es}\sin\theta\sin(\lambda\eta_e - \phi) + \delta_n , \tag{41}$$

where

$$\delta_n = \frac{\omega_n(1-\cos\theta)}{c} z_{es} - \frac{\omega_n\beta_{zs}(1-\cos\theta)}{\omega_0(1-\beta_{zs})}\eta_{es} + \frac{\omega_n\lambda(1-\beta_{zs}\cos\theta)}{\omega_0(1-\beta_{zs})}\phi . \tag{42}$$

By utilizing the expansions

$$e^{-iQ\sin\eta} = \sum_{m=-\infty}^{+\infty} J_m(Q)\, e^{-im\eta} , \tag{43}$$

$$-i\sin\eta\, e^{-iQ\sin\eta} = \sum_{m=-\infty}^{+\infty} J_m'(Q) e^{-im\eta} , \tag{44}$$

and

$$iQ\cos\eta\, e^{-iQ\sin\eta} = \sum_{m=-\infty}^{+\infty} im\, J_m(Q)\, e^{im\eta} , \tag{45}$$

Eq. (38) may be brought to the form

$$\vec{E}_n^{\,\mathrm{rad}} = e^{i[(\omega_n/c)r+\delta_n]}\frac{ie\omega_n}{2\pi c r}\sum_{m=-\infty}^{+\infty}\left(\hat{e}_\theta\frac{\lambda m\beta_{\phi s}\cos\theta}{Q_n}J_m(Q_n) - \hat{e}_\theta\beta_{zs}\sin\theta\, J_m(Q_n) + \hat{e}_\phi i\beta_{\phi s}J_m'(Q_n)\right)\int_0^{2\pi} e^{i(n-m)\eta}\, d\eta \tag{46}$$

in which

$$Q_n = \frac{\omega_n}{c}\rho_{es}\sin\theta , \tag{47}$$

$$n = \frac{\omega_n}{\omega_0}\,\frac{1-\beta_{zs}\cos\theta}{1-\beta_{zs}} , \tag{48}$$

and where $J'(Q) = dJ(Q)/dQ$. The period of the integrand in Eq. (46) equals 2π (and integral multiples of 2π) only if n, as given by Eq. (48), is an integer, and in this case the integral equals $2\pi\delta_{mn}$. Thus the frequency of the radiation from the electron assumes the discrete set of values

$$\omega_n = \omega_n(\eta_{es}) = \frac{n\omega_0[1-\beta_z(\eta_{es})]}{1-\beta_z(\eta_{es})\cos\theta} , \tag{49}$$

and the Fourier components of the electric field

$$\vec{E}^{\,\mathrm{rad}}(r, \theta, \phi, t, \eta_{es}) = \sum_{n=-\infty}^{\infty} \vec{E}_n^{\,\mathrm{rad}}(\eta_{es}) e^{-i\omega_n t}$$

become

$$\vec{E}_n^{\,\mathrm{rad}}(\eta_{es}) = e^{i[(\omega_n/c)r+\delta_n]}\frac{e\omega_n}{cr}\left\{\left[i\hat{e}_\theta\left(\frac{c\lambda n\beta_{\phi s}\cos\theta}{\omega_n\rho_{es}\sin\theta} - \beta_{zs}\sin\theta\right)\right]J_n(Q_n) - \hat{e}_\phi\beta_{\phi s}J_n'(Q_n)\right\} . \tag{50}$$

Equations (49) and (50) show that in the direction of propagation of the incident wave ($\cos\theta = 1$) the spectrum of the emitted radiation contains only one component, whose frequency is $\omega_1 = \omega_0$. In the opposite direction ($\cos\theta = -1$) the single component which is present has the frequency $\omega_1 = \omega_0(1-\beta_{zs})/(1+\beta_{zs})$. For all other directions the frequency ω_1 lies between the two values given above, and in general all the higher harmonics are present.

We now utilize Eq. (50) to discuss the characteristics of the spectral components of the emitted radiation.

The differential power radiated into the element of solid angle $d\Omega$ is given for the nth spectral component by

$$dW_n(\theta, \phi, \eta_{es}) = \frac{cr^2}{2\pi} \vec{E}_n^{\text{rad}}(\eta_{es}) \cdot \vec{E}_n^{\text{rad}*}(\eta_{es}) d\Omega$$

$$= \frac{e^2\omega_n^2}{2\pi c} \left[[J_n(Q_n)]^2 \left(\frac{c\lambda n \beta_{\phi s} \cos\theta}{\omega_n \rho_{es} \sin\theta} - \beta_{zs} \sin\theta \right)^2 [J_n'(Q_n)]^2 \beta_{\phi s}{}^2 \right] d\Omega . \quad (51)$$

The polarization properties of the scattered radiation are represented by the tensor[10]

$$\tau(n, \eta_{es}) = \frac{c}{2\pi} \vec{E}_n^{\text{rad}}(\eta_{es}) \otimes \vec{E}_n^{\text{rad}*}(\eta_{es}) . \quad (52)$$

In accordance with Eq. (50) the components of this tensor are

$$\tau_{\theta\theta}(n, \eta_{es}) = \frac{e^2\omega_n{}^2}{2\pi c r^2} \left(\frac{c\lambda n \beta_{\phi s} \cos\theta}{\omega_n \rho_{es} \sin\theta} - \beta_{zs} \sin\theta \right)^2 [J_n(Q_n)]^2 , \quad (53)$$

$$\tau_{\theta\phi}(n, \eta_{es}) = -i \frac{e^2\omega_n{}^2}{2\pi c r^2} \left(\frac{c\lambda n \beta_{\phi s} \cos\theta}{\omega_n \rho_{es} \sin\theta} - \beta_{zs} \sin\theta \right) \beta_{\phi s} J_n(Q_n) J_n'(Q_n) , \quad (54)$$

$$\tau_{\phi\theta} = \tau^*_{\theta\phi} = -\tau_{\theta\phi} , \quad (55)$$

and

$$\tau_{\phi\phi}(n, \eta_{es}) = \frac{e^2\omega_n{}^2}{2\pi c r^2} \beta_\phi{}^2 [J_n'(Q_n)]^2 . \quad (56)$$

The normalization of $\tau(n, \eta_{es})$ has been so chosen that the intensity of the nth spectral component radiated in the direction θ, ϕ is given by

$$I_n^{\text{rad}}(\theta, \phi, \eta_{es}) = \text{tr}\,\tau(n, \eta_{es})$$

$$= \frac{e^2\omega_n{}^2}{2\pi c r^2} \left[\left(\frac{c\lambda n \beta_{\phi s} \cos\theta}{\omega_n \rho_{es} \sin\theta} - \beta_{zs} \sin\theta \right)^2 [J_n(Q_n)]^2 + \beta_{\phi s}{}^2 [J_n'(Q_n)]^2 \right] . \quad (57)$$

In order to determine the degree of circular polarization of the nth harmonic we introduce the scalar

$$C(n, \eta_{es}) = i[\text{tr}\,\tau(n, \eta_{es})]^{-1}[\tau_{\theta\phi}(n, \eta_{es}) - \tau_{\phi\theta}(n, \eta_{es})]$$

$$= \frac{2\omega_n \rho_{es} \sin\theta (c\lambda n \beta_{\phi s} \cos\theta - \omega_n \rho_{es} \beta_{zs} \sin^2\theta) \beta_{\phi s} J_n(Q_n) J_n'(Q_n)}{(c\lambda n \beta_{\phi s} \cos\theta - \omega_n \rho_{es} \beta_{zs} \sin^2\theta)^2 [J_n(Q_n)]^2 + [\omega_n \rho_{es} \sin\theta \beta_{\phi s} J_n'(Q_n)]^2} . \quad (58)$$

The range of $C(n, \eta_{es})$ is $-1 \leq C(n, \eta_{es}) \leq +1$, with $+1$, -1, and 0 corresponding to right circular, left circular, and linear polarization, respectively. It is straightforward to show that $C(n, \eta_{es}) = +\lambda$ if $\theta = 0$ and $C(n, \eta_{es}) = -\lambda$ if $\theta = \pi$.

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[2]A. D. Steiger and C. H. Woods, Phys. Rev. D 5, 2912 (1972).

[3]J. J. Sanderson, Phys. Letters 18, 114 (1965).

[4]T. W. B. Kibble, Phys. Letters 20, 627 (1966).

[5]N. D. Sen Gupta, Z. Physik 196, 385 (1966).

[6]In the presence of intensity gradients and radiation damping Eq. (14) can be generalized to

$$\phi = \lambda \left(\eta - \tfrac{1}{2}\pi + \frac{1}{\mu} \frac{d\mu}{d\eta} + \frac{\epsilon_0[1 + 2\mu^2(\eta)]}{1 + \epsilon(\eta)} \right).$$

For reasons stated in the text after Eq. (18), the last two terms are negligible.

[7]L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, Mass., 1971), 3rd ed.

[8]The conventional way of writing Eq. (37) is

$$\vec{A}_n = e^{i(\omega_n/c)r} \frac{e}{r T_{es}} \oint \vec{\beta}\, e^{i[\omega_n t - (\omega_n/c)\hat{k}\cdot\vec{r}_e(t)]} dt ,$$

where the azimuthal period for our present problem is $T_{es} = 2\pi[\omega_0(1 - \beta_{zs})]^{-1}$.

[9]The transformation from the spherical unit vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$ to the Cartesian unit vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ is given by the expressions

$$\hat{e}_1 = \hat{e}_r \sin\theta \cos\phi + \hat{e}_\theta \cos\theta \cos\phi - \hat{e}_\phi \sin\phi ,$$

$$\hat{e}_2 = \hat{e}_r \sin\theta \sin\phi + \hat{e}_\theta \cos\theta \sin\phi + \hat{e}_\phi \cos\phi ,$$

$$\hat{e}_3 = \hat{e}_r \cos\theta - \hat{e}_\theta \sin\theta .$$

[10]In Eq. (52) the symbol $\otimes$ denotes the tensor product of the vector $\vec{E}_n^{\text{rad}}$ and its adjoint $\vec{E}_n^{\text{rad}}{}^*$.