Weak κ-Dominance Model and Bounds on K_{l3} Decay Parameters*

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The previously found bounds on the K_{l3} form factors have been improved without assuming the validity of unsubtracted dispersion relations for two-point functions and without assuming specific SW(3)-symmetry-breaking Hamiltonians. As a price which we pay for this generality, it is necessary to assume a weak form of κ dominance for the two-point function (but not for the three-point functions). Under this assumption the two-point function may still have a nonzero and sizable contribution from the high-energy region. The derived bounds still give results inconsistent with the present experiments if we utilize the soft-pion theorem as input information.

I. INTRODUCTION AND SUMMARY OF RESULTS

Li and Pagels have derived[1] a rigorous bound for a derivative of the scalar K_{l3} form factor $D(t)$, defined by

$$D(t)=(m_K{}^2-m_\pi{}^2)f_+(t)+tf_-(t)\,. \tag{1.1}$$

Since then, their method has been considerably refined[2,3] to obtain better bounds for $D(t)$ and $D'(t)$; these techniques have been applied to the K_{l3} problem[4,5] as well as to some other problems.[6,7] In particular, it now appears[3,5] that the theoretical bounds for $D(t)$ and $D'(t)$ disagree with the present experimental data. Therefore, it is interesting to investigate the validity of the theoretical assumptions used to derive these bounds. Apart from standard *Ansätze* such as analyticity, unitarity, and crossing symmetry, the most crucial assumptions are the following three:

(i) exact validity of the K_{l3} soft-pion theorem[8];

(ii) validity of the unsubtracted spectral representation for $\Delta(t)$ (see below);

(iii) an estimate of $\Delta(0)$,[9]

$$[\Delta(0)]^{1/2}\le\frac{1}{\sqrt{2}}(m_Kf_K-m_\pi f_\pi)\,. \tag{1.2}$$

Let $V_\mu^\alpha(x)$ $(\alpha=1,2,\ldots,8)$ be the octet of weak vector currents. Then $\Delta(t)$ is defined by

$$\Delta(t)=\tfrac{1}{2}i\int d^4x\,e^{iq(x-y)}\times\langle 0|(\partial_\mu V_\mu^{(4-i5)}(x),\partial_\nu V_\nu^{(4+i5)}(y))_+|0\rangle\,,$$
$$t=-q^2\,. \tag{1.3}$$

Now, condition (ii) is equivalent to the assumption that we can write

$$\Delta(t)=\int_{t_0}^\infty dt'\,\frac{\rho(t')}{t'-t}\,,\quad t_0=(m_K+m_\pi)^2\,, \tag{1.4}$$

where the spectral weight $\rho(t)$ is given by

$$\rho(t)=\tfrac{1}{2}(2\pi)^3\sum_n|\langle 0|\partial_\mu V_\mu^{(4-i5)}(0)|n\rangle|^2\delta^{(4)}(p_n-q)\,,$$
$$t=-q^2\,. \tag{1.5}$$

In particular, the validity of Eq. (1.4) implies

$$\Delta(0)=\int_{t_0}^\infty dt\,\frac{\rho(t)}{t}<\infty\,. \tag{1.6}$$

On the basis of Eq. (1.5), Li and Pagels[1] derived the inequality

$$|D(t)|^2\le\tfrac{64}{3}\pi^2\frac{t}{(t-t_0)^{1/2}(t-t_1)^{1/2}}\rho(t) \tag{1.7}$$

on the cut $t\ge t_0$, where t_0 and t_1 are given by

$$t_0=(m_K+m_\pi)^2\,,\quad t_1=(m_K-m_\pi)^2\,. \tag{1.8}$$

Therefore, we find

$$\frac{1}{\pi}\int_{t_0}^\infty dt\,k(t)|D(t)|^2\le\Delta(0)\,,$$
$$k(t)=\frac{3}{64\pi}\,\frac{(t-t_0)^{1/2}(t-t_1)^{1/2}}{t^2}\,,$$

which was the starting point of the previous analyses.

It should be remarked[1] that Eqs. (1.6) and (1.7) do necessarily imply[10] that $D(t)$ satisfies an unsubtracted dispersion relation.

Finally, the estimate (1.2) is obtained on the basis of the chiral model[11] of Gell-Mann, Oakes, and Renner and of Glashow and Weinberg (GMORGW), together with some technical assumptions[9] which will not be specified here. However, there is good reason to believe[9] that assumption (iii) is reasonable within the framework of the GMORGW model.

Now, as we remarked already, the present experimental situation appears to be in conflict with the theoretical prediction; therefore, we have to give up some of the assumptions made above. First, if we give up the soft-pion theorem, we can still derive an exact bound for $D'(0)$, which is barely compatible[2,4] with the present experimental value. However, the far weaker assumptions are (ii) and (iii), and it is the purpose of this note to investigate the problem without making these last two *Ansätze*.

To achieve that, we first observe the following point. Suppose that $\rho(t)$ or its upper bound is somehow known on the cut. Then we can rewrite Eq. (1.7) as

$$|D(t)| \leq w(t) \tag{1.9}$$

on the cut $t \geq t_0$. Then it is known that we must have rigorous inequalities[7,12] such as

$$\ln|D(0)| \leq \epsilon\,, \tag{1.10}$$

$$\left| D'(0) - \left(\eta - \frac{\epsilon}{2t_0}\right) D(0) \right| \leq \frac{1}{4t_0}\left[e^{\epsilon} - e^{-\epsilon}|D(0)|^2\right], \tag{1.11}$$

where ϵ and η are given by

$$\epsilon = \frac{t_0^{1/2}}{\pi}\int_{t_0}^{\infty} dt\, \frac{1}{t(t-t_0)^{1/2}}\ln w(t)\,,$$
$$\eta = \frac{t_0^{1/2}}{\pi}\int_{t_0}^{\infty} dt\, \frac{1}{t^2(t-t_0)^{1/2}}\ln w(t)\,. \tag{1.12}$$

Therefore, we can obtain bounds for $D'(0)$ if $w(t)$ is given. As we shall show in Sec. II, we can also derive a better bound for $D'(0)$, if we use the soft-pion theorem[8]

$$\frac{D(\delta)}{D(0)} = \frac{f_K}{f_+(0) f_\pi} \approx 1.28\,, \tag{1.13}$$

where δ is given by

$$\delta = m_K^2 - m_\pi^2\,.$$

Thus, the problem is reduced to estimating $\rho(t)$ or its upper bound on the cut without using *Ansätze* (ii) and (iii). Unfortunately this is not so easy; we must assume here a weak form of κ dominance for the spectral weight $\rho(t)$. To explain the precise meaning of this assumption, we note that $\rho(t)$ will have a kind of Breit-Wigner shape around $t \approx m_\kappa^2$ where m_κ is the mass of the κ meson. First we tentatively assume that $\rho(t)$ satisfies

$$\rho(t) \leq \bar\rho(t) \equiv N\frac{t^n}{(t-m_\kappa^2)^2 + \frac{1}{4}m_\kappa^2\Gamma^2} \tag{1.14}$$

for $t \geq t_0$, where N and n are some positive numbers. If we choose $n=0$, then Eq. (1.14) reproduces the usual κ dominance result. However, for a large positive value of n (say $n=2$), $\bar\rho(t)$ will become constant instead of zero as $t \to \infty$. In that case, Eq. (1.14) describes a possible violation of *Ansatz* (ii); i.e., we can have, for $\rho(t) = \bar\rho(t)$,

$$\int_{t_0}^{\infty} dt\,\frac{\rho(t)}{t} = \infty\,,$$

when $n \geq 2$. Roughly speaking we may say that the spectral representation for $\Delta(t)$ will require one or two subtractions for $n=2$ and $n=3$, respectively. In Eq. (1.14), the constant N can be computed as follows. It is related to the coupling parameter f_κ which is the analog of f_π and f_K. For a reasonable range of f_κ with $|f_\kappa/f_\pi| \leq 2.0$, we will show in Sec. II that we still have results inconsistent with experiments provided that we use Eq. (1.14) and the soft-pion theorem. In Sec. III, we relax our requirement Eq. (1.14); instead, we assume the existence of a broad Breit-Wigner form for $\rho(t)$ only around $t \approx m_\kappa^2$, without specifying its explicit behavior at infinity. Nevertheless we find that the inconsistency still persists, even though it becomes somewhat less blatant. At any rate, we do not believe that the source of the trouble is easily removed if we relax our assumptions (ii) and (iii). Unless f_κ is unreasonably large or the soft-pion theorem is incorrect, the present conflict appears to be avoidable only if there are many low-lying 0^+ resonance states (other than the single κ meson we are discussing) in the K-π scattering channel.

It should be emphasized that our weak form of κ dominance does not necessarily imply a similar pole dominance of $D(t)$. Notice that $\Delta(t)$ is a two-point function while $D(t)$ is a three-point function. It is possible that some kind of pole dominance approximation is better for two-point functions than for three-point functions.

Finally, a similar method has been used to derive a bound for λ_+, and we find

$$\lambda_+ \lesssim 0.059$$

by means of weak forms of K^* dominance.

II. WEAK κ DOMINANCE

We assume here the inequality (1.14). Then, $w(t)$ given in Eq. (1.9) is chosen to be

$$[w(t)]^2 = \tfrac{64}{3}\pi^2\frac{t}{(t-t_0)^{1/2}(t-t_1)^{1/2}}\bar\rho(t)\,, \tag{2.1}$$

$$\bar\rho(t) = N\frac{t^n}{(t-m_\kappa^2)^2 + \frac{1}{4}m_\kappa^2\Gamma^2} \tag{2.2}$$

in view of Eqs. (1.7) and (1.14). To evaluate the unknown multiplicative constant N, we first note that in the zero-width limit $\Gamma \to 0$ the spectral weight $\rho(t)$ has the form

$$\rho(t) = \tfrac{1}{2}f_\kappa^2 m_\kappa^4\delta(t - m_\kappa^2)\,,$$

where f_κ is defined by

$$\langle 0|V_\mu^{(4-i5)}(0)|\kappa^+(p)\rangle = \frac{i}{(2p_0V)^{1/2}}\, p_\mu f_\kappa\,.$$

Therefore, it is natural to define the effective $\bar{f}_\kappa$ for the case of nonzero width Γ by

$$\int_{t_0}^{\infty} dt\,\frac{\bar{\rho}(t)}{t^m} = \tfrac{1}{2}(m_\kappa^{\;2})^{2-m}\bar{f}_\kappa^{\;2} \tag{2.3}$$

for some non-negative integer m (provided that the integral exists). Of course, this definition of $\bar{f}_\kappa$ is not unambiguous and may not correspond exactly to the standard f_κ since it depends upon the value of the integer m and also upon the asymptotic form of $\bar{\rho}(t)$ as $t\to\infty$. However, once we fix the value of $\bar{f}_\kappa$, then N is calculable by means of Eqs. (2.2) and (2.3). Hereafter, we assume that the value of $\bar{f}_\kappa$ is limited to the range

$$|\bar{f}_\kappa| \le 2.0 f_\pi\,, \tag{2.4}$$

in view of the standard analysis which gives $|f_\kappa| \le 0.60 f_\pi$ for most models.[13] Notice that in the exact SU(3) limit we expect to have $\bar{f}_\kappa = 0$; therefore, we feel that our estimate Eq. (2.4) is reasonable.

Also, throughout this paper, we assume

$$0.9 \lesssim f_+(0) \lesssim 1.0 \tag{2.5}$$

in conformity to the Ademollo-Gatto theorem.[14] Then

$$D(0) = (m_K^{\;2} - m_\pi^{\;2}) f_+(0) \tag{2.6}$$

is a known quantity. Now, the value of the integer m in the integral of Eq. (2.3) must be taken as $n = m$ by the following reasoning. First, we notice that the integral Eq. (2.3) is convergent only for $m \ge n$. Then the inequality (1.10), i.e.,

$$\ln|D(0)| \le \epsilon$$

gives a constraint. We can easily check that, for values of $\bar{f}_\kappa$ and $f_+(0)$ given by Eqs. (2.4) and (2.5), this constraint is satisfied only for $m=n$, for integer values of m and n. Therefore, hereafter we set $m=n$. Then, Eq. (1.11) leads to a bound for $D'(0)$, if we specify the values of n, m_κ, $\bar{f}_\kappa$, and Γ. We have calculated these bounds for the cases $\bar{f}_\kappa/f_\pi = 0.3$, 0.5, 1.0, and 2.0; $m_\kappa = 1000$, 1100, 1200 MeV and $\Gamma = 300$, 400, 500 MeV with $n = 0, 1, 2, 3$. The results are tabulated in Table I, where Λ_0 is defined by

$$\Lambda_0 = m_\pi^{\;2}\frac{D'(0)}{D(0)} = \lambda_+ + \frac{m_\pi^{\;2}}{m_K^{\;2} - m_\pi^{\;2}}\,\xi\,. \tag{2.7}$$

First, we remark that the interesting cases are $n = 2$ and 3, since then the integral

$$\int_{t_0}^{\infty} dt\,\frac{\bar{\rho}(t)}{t}$$

is divergent [thus violating the assumption (ii) of the Introduction]. However, for the sake of completeness, we have also computed bounds for $n=0$ and 1. Even in these cases we see from Table I that the small value $\bar{f}_\kappa = 0.3 f_\pi$ leads to a contradiction with inequality (1.10). Also, the values of Λ_0 are mostly positive even for $\bar{f}_\kappa = 0.5 f_\pi$, contradicting the present experimental value[15] of

$$\Lambda_0 = -0.11 \pm 0.03\,. \tag{2.8}$$

We see from Table I that only the cases $n=2$ or 3 with large $\bar{f}_\kappa = 2.0 f_\pi$ can give negative values of Λ_0 which is consistent with Eq. (2.8).

In the above discussion, we did not exploit the validity of the soft-pion theorem Eq. (1.13). As is expected, the use of Eq. (1.13) makes the situation even worse. To derive the improved bound, we follow the method of Ref. 12 and set

$$G(t) = \exp\left[\frac{(t_0 - t)^{1/2}}{\pi} \times \int_{t_0}^{\infty} dt'\,\frac{1}{(t'-t)(t'-t_0)^{1/2}}\ln w(t')\right]. \tag{2.9}$$

Then $G(t)$ is a real analytic function of t with a cut at $t_0 \le t < \infty$; on the boundary it satisfies

$$|G(t)| = w(t) \quad (t \ge t_0)\,. \tag{2.10}$$

Since $G(t)$ has no zero point in the cut plane, the function given by

$$R(t) = \frac{D(t)}{G(t)} \tag{2.11}$$

is also a real analytic function of t with a cut at $t_0 \le t < \infty$. Moreover, because of Eqs. (1.9) and (2.10), it satisfies

$$|R(t)| \le 1 \tag{2.12}$$

on the cut $t \ge t_0$. Now $w(t)$ behaves as some power of t as $t\to\infty$. Therefore, $G(t)$ behaves as a power of t as t approaches the infinite point in the complex t plane. Hence, if we assume polynomial boundedness of $|D(t)|$ at infinity, then $|R(t)|$ is also polynomially bounded[16] at infinity. Then, the Phragmén-Lindelöf theorem[17] demands the validity of Eq. (2.12) also at the infinite point in the complex t plane. Thus by the maximum modulus theorem, we conclude that $R(t)$ must satisfy Eq. (2.12) in the entire complex t plane. In particular, setting $t=0$, this gives Eq. (1.10).

To derive a bound for $D'(0)$, we map the cut t plane into the interior of the unit circle $|z|=1$ by the conformal mapping

$$(t_0 - t)^{1/2} = t_0^{1/2}\,\frac{1+z}{1-z}\,. \tag{2.13}$$

Now, if we set

$$R(t) \equiv B(z)\,, \tag{2.14}$$

$B(z)$ is a real analytic function of z for $|z|<1$ and satisfies

$$|B(z)| \le 1 \tag{2.15}$$

inside the unit circle. Define another function $A(z)$ by[18]

$$A(z) = \frac{1-\lambda^* z}{z-\lambda}\,\frac{B(z)-B(\lambda)}{1-B^*(\lambda)B(z)} \tag{2.16}$$

for λ satisfying $|\lambda|<1$. From Eqs. (2.15) and (2.16)

TABLE I. Bounds for Λ_0 without the soft-pion theorem. The four pairs of numbers for each m_K and Γ correspond to the four choices $\bar{f}_K/f_\pi = 0.3$, 0.5, 1.0, and 2.0, respectively. The blank items indicated by three dots imply that there is no solution at all consistent with our inequalities.

$\Gamma \backslash m_K$	1000 MeV		1100 MeV		1200 MeV	
(a) $n=0$						
300 MeV	$\cdots$ $\le\Lambda_0\le$	$\cdots$	$\cdots$ $\le\Lambda_0\le$	$\cdots$	$\cdots$ $\le\Lambda_0\le$	$\cdots$
	0.0232	0.0236	0.0186	0.0224	0.0149	0.0216
	0.0058	0.0410	0.0007	0.0403	−0.0035	0.0400
	−0.0203	0.0672	−0.0271	0.0681	−0.0328	0.0692
400 MeV	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	0.0209	0.0258	0.0162	0.0250	0.0123	0.0244
	0.0028	0.0439	−0.0026	0.0438	−0.0071	0.0439
	−0.0255	0.0722	−0.0330	0.0741	−0.0393	0.0761
500 MeV	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	0.0193	0.0272	0.0144	0.0268	0.0105	0.0264
	0.0006	0.0459	−0.0051	0.0463	−0.0099	0.0468
	−0.0293	0.0758	−0.0374	0.0786	−0.0444	0.0813
(b) $n=1$						
300 MeV	$\cdots$ $\le\Lambda_0\le$	$\cdots$	$\cdots$ $\le\Lambda_0\le$	$\cdots$	$\cdots$ $\le\Lambda_0\le$	$\cdots$
	0.0083	0.0173	0.0059	0.0139	0.0041	0.0111
	−0.0106	0.0361	−0.0128	0.0326	−0.0143	0.0295
	−0.0410	0.0666	−0.0427	0.0625	−0.0436	0.0588
400 MeV	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	0.0059	0.0196	0.0034	0.0165	0.0016	0.0139
	−0.0140	0.0394	−0.0163	0.0362	−0.0179	0.0334
	−0.0471	0.0726	−0.0491	0.0691	−0.0502	0.0657
500 MeV	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	0.0041	0.0211	0.0016	0.0184	−0.0003	0.0159
	−0.0165	0.0417	−0.0190	0.0389	−0.0207	0.0363
	−0.0516	0.0768	−0.0540	0.0740	−0.0554	0.0710
(c) $n=2$						
300 MeV	$\cdots$ $\le\Lambda_0\le$	$\cdots$	$\cdots$ $\le\Lambda_0\le$	$\cdots$	$\cdots$ $\le\Lambda_0\le$	$\cdots$
	−0.0069	0.0112	−0.0069	0.0055	−0.0066	0.0005
	−0.0278	0.0321	−0.0265	0.0250	−0.0251	0.0190
	−0.0636	0.0679	−0.0588	0.0574	−0.0545	0.0484
400 MeV	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	−0.0094	0.0136	−0.0095	−0.0081	−0.0092	0.0033
	−0.0317	0.0359	−0.0303	0.0289	−0.0287	0.0229
	−0.0709	0.0750	−0.0658	0.0645	−0.0611	0.0553
500 MeV	0.0012	0.0027	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	−0.0113	0.0152	−0.0113	0.0100	−0.0110	0.0054
	−0.0346	0.0385	−0.0331	0.0318	−0.0315	0.0258
	−0.0762	0.0801	−0.0711	0.0698	−0.0663	0.0606

TABLE I (Continued)

(d) $n=3$

$\Gamma \backslash m_K$	1000 MeV			1100 MeV			1200 MeV		
300 MeV	−0.0098	$\leq\Lambda_0\leq$	−0.0072	···	$\leq\Lambda_0\leq$	···	···	$\leq\Lambda_0\leq$	···
	−0.0223		0.0053	−0.0198		−0.0030	−0.0173		−0.0100
	−0.0460		0.0290	−0.0404		0.0176	−0.0358		0.0085
	−0.0885		0.0715	−0.0754		0.0527	−0.0654		0.0380
400 MeV	−0.0122		−0.0050	···		···	···		···
	−0.0251		0.0080	−0.0224		−0.0002	−0.0200		−0.0072
	−0.0506		0.0335	−0.0445		0.0218	−0.0395		0.0124
	−0.0971		0.0800	−0.0831		0.0604	−0.0721		0.0449
500 MeV	−0.0138		−0.0035	−0.0119		−0.0107	···		···
	−0.0271		0.0098	−0.0243		0.0017	−0.0218		−0.0052
	−0.0540		0.0366	−0.0475		0.0249	−0.0423		0.0154
	−0.1035		0.0861	−0.0888		0.0662	−0.0772		0.0503

it is easy to check that on the boundary, $A(z)$ satisfies

$$|A(e^{i\theta})|\leq 1\,.$$

Since $A(z)$ is constructed to be analytic for $|z|<1$, the maximum modulus theorem demands that

$$|A(z)|\leq 1 \quad \text{for} \quad |z|\leq 1\,. \tag{2.17}$$

In particular, if we set $z=0=\lambda$, then Eq. (2.17) leads to

$$|B'(0)|\leq 1-|B(0)|^2\,. \tag{2.18}$$

Noting that

$$B(z)\equiv R(t)=\frac{D(t)}{G(t)}\,, \tag{2.19}$$

Eq. (2.18) reproduces the inequality (1.11).

Now repeating the same argument for $A(z)$ instead of $B(z)$, we must also have

$$|A'(0)|^2\leq 1-|A(0)|^2\,, \tag{2.20}$$

which can be rewritten as

$$\left|\frac{B'(0)}{B(0)-B(\lambda)}+\frac{B'(0)B^*(\lambda)}{1-B^*(\lambda)B(0)}+\frac{1-|\lambda|^2}{\lambda}\right| \leq \left|\frac{\lambda[1-B^*(\lambda)B(0)]}{B(0)-B(\lambda)}\right|-\left|\frac{B(0)-B(\lambda)}{\lambda[1-B^*(\lambda)B(0)]}\right|\,. \tag{2.21}$$

We choose λ to be

$$\frac{1+\lambda}{1-\lambda}=\left(\frac{t_0-\delta}{t_0}\right)^{1/2}, \tag{2.22}$$

so that the corresponding value of t is exactly the soft-pion point $\delta=m_K^2-m_\pi^2$. In this case,

$$B(\lambda)\equiv\frac{D(\delta)}{G(\delta)} \tag{2.23}$$

is calculable with the aid of the soft-pion theorem Eq. (1.13); then, by means of Eqs. (2.19) and (2.21), we can find a stronger bound on $D'(0)$, which now takes into account the soft-pion theorem. Before going into details, we remark that for $z=0$, Eq. (2.17) gives the constraint

$$\frac{B(0)-|\lambda|}{1-|\lambda|B(0)}\leq B(\lambda)\leq\frac{B(0)+|\lambda|}{1+|\lambda|B(0)}\,, \tag{2.24}$$

assuming that λ, $B(0)$, and $B(\lambda)$ are real. Equation (2.24) is an improvement over the bound $|B(\lambda)|\leq 1$.

Now, we again compute the bounds for Λ_0 under the same assumptions as before, and the results are tabulated in Table II for various parameter choices. We notice from the table that the constraint Eq. (2.24) together with Eq. (1.10) is very strong and that it is not possible to find any solution at all unless the soft-pion theorem is badly violated or unless $\bar{f}_K$ is very large. Even assuming a 10% error for the soft-pion theorem with $D(\delta)/D(0)=1.15$ with $\bar{f}_K=1.0f_\pi$, the value of Λ_0 is still positive in contradiction to the experimental value Eq. (2.8). Only for $\bar{f}_K=2.0f_\pi$, $D(\delta)/D(0)=1.15$, Λ_0 can be negative, but still too small as we see, for example, $\Lambda_0\geq -0.0096$ even for $n=3$.

In conclusion, we find that our bounds for Λ_0 with $\bar{f}_K\leq 2.0f_\pi$ are incompatible with the present experimental data, unless we give up the soft-pion theorem.

Ending this section, we briefly remark that a similar technique is applicable to the evaluation of λ_+. In this case, we work with $f_+(t)$ instead of $D(t)$. Then, the vector spectral weight factor $\rho^{(1)}(t)$ is defined by

TABLE II. The same calculation with the soft-pion theorem. Also, the values for $D(\delta)/D(0)$ are a test of the soft-pion theorem on the basis of Eq. (2.24). ··· and xxx indicate violation of the constraints Eq. (1.10) and Eq. (2.24), respectively.

(a) $n=0$

Γ \ m_K	1000 MeV			1100 MeV			1200 MeV		
300 MeV	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	1.45		1.46	1.33		1.43	1.24		1.41
	0.94		1.87	0.84		1.84	0.76		1.82
	0.25		2.54	0.14		2.52	0.05		2.51
	···	$\le\Lambda_0$ [a] $\le$	···	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···
	xxx		xxx	xxx		xxx	0.0157		0.0166
	0.0141		0.0205	0.0126		0.0205	0.0115		0.0204
	0.0082		0.0260	0.0066		0.0263	0.0054		0.0265
400 MeV	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	1.38		1.50	1.26		1.48	1.17		1.47
	0.86		1.94	0.76		1.92	0.67		1.91
	0.11		2.67	−0.01		2.67	−0.11		2.68
	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···
	xxx		xxx	0.0166		0.0171	0.0150		0.0171
	0.0133		0.0210	0.0118		0.0211	0.0107		0.0212
	0.0070		0.0270	0.0053		0.0275	0.0039		0.0279
500 MeV	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	1.33		1.54	1.21		1.53	1.12		1.51
	0.81		2.00	0.69		1.99	0.61		1.98
	0.02		2.77	−0.12		2.79	−0.23		2.81
	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···
	xxx		xxx	0.0159		0.0174	0.0145		0.0175
	0.0127		0.0213	0.0111		0.0216	0.0100		0.0217
	0.0061		0.0277	0.0043		0.0283	0.0028		0.0281

(b) $n=1$

Γ \ m_K	1000 MeV			1100 MeV			1200 MeV		
300 MeV	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	1.10		1.30	1.06		1.24	1.04		1.18
	0.66		1.69	0.64		1.60	0.64		1.53
	−0.01		2.35	0.01		2.22	0.04		2.11
	···	$\le\Lambda_0$ [b] $\le$	···	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···
	0.0095		0.0106	0.0085		0.0101	0.0084		0.0096
	0.0050		0.0146	0.0046		0.0140	0.0044		0.0134
	−0.0014		0.0209	−0.0017		0.0202	−0.0017		0.0195
400 MeV	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	1.04		1.35	1.00		1.29	0.98		1.23
	0.59		1.76	0.57		1.68	0.56		1.61
	−0.15		2.48	−0.12		2.36	−0.10		2.25
	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···
	0.0087		0.0110	0.0079		0.0106	0.0076		0.0101
	0.0042		0.0151	0.0037		0.0147	0.0036		0.0142
	−0.0028		0.0221	−0.0031		0.0215	−0.0032		0.0209
500 MeV	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	1.00		1.38	0.96		1.32	0.94		1.27
	0.53		1.82	0.51		1.74	0.50		1.67
	−0.24		2.58	−0.23		2.47	−0.20		2.37
	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···	···	$\le\Lambda_0\le$	···
	0.0081		0.0112	0.0074		0.0109	0.0071		0.0105
	0.0036		0.0155	0.0031		0.0151	0.0029		0.0147
	−0.0038		0.0229	−0.0042		0.0224	−0.0044		0.0219

TABLE II *(Continued)*

(c) $n=2$

Γ \ m_K	1000 MeV			1100 MeV			1200 MeV		
300 MeV	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	0.83		1.17	0.85		1.07	0.87		0.99
	0.43		1.54	0.49		1.39	0.53		1.28
	−0.23		2.19	−0.09		1.96	0.03		1.78
	···	$\le \Lambda_0$ [b] $\le$	···	···	$\le \Lambda_0 \le$	···	···	$\le \Lambda_0 \le$	···
	0.0093		0.0103	xxx		xxx	xxx		xxx
	0.0030		0.0148	0.0046		0.0136	0.0065		0.0126
	−0.0048		0.0223	−0.0029		0.0205	−0.0013		0.0190
400 MeV	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	0.78		1.21	0.80		1.11	0.82		1.04
	0.36		1.61	0.42		1.46	0.47		1.35
	−0.36		2.33	−0.21		2.09	−0.09		1.90
	···	$\le \Lambda_0 \le$	···	···	$\le \Lambda_0 \le$	···	···	$\le \Lambda_0 \le$	···
	0.0078		0.0107	xxx		xxx	xxx		xxx
	0.0020		0.0155	0.0034		0.0144	0.0050		0.0133
	−0.0064		0.0237	−0.0046		0.0219	−0.0029		0.0203
500 MeV	1.01	$\le D(\delta)/D(0)\le$	1.04	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	0.75		1.24	0.76		1.15	0.78		1.07
	0.31		1.66	0.37		1.52	0.42		1.40
	−0.46		2.42	−0.30		2.18	−0.17		1.99
	xxx	$\le \Lambda_0 \le$	xxx	···	$\le \Lambda_0 \le$	···	···	$\le \Lambda_0 \le$	···
	0.0071		0.0110	xxx		xxx	xxx		xxx
	0.0012		0.0159	0.0026		0.0149	0.0040		0.0139
	−0.0076		0.0247	−0.0058		0.0230	−0.0042		0.0214

(d) $n=3$

Γ \ m_K	1000 MeV			1100 MeV			1200 MeV		
300 MeV	0.83	$\le D(\delta)/D(0)\le$	0.88	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	0.62		1.05	0.67		0.93	0.73		0.83
	0.24		1.41	0.36		1.22	0.45		1.08
	−0.41		2.06	−0.16		1.73	0.02		1.49
	xxx	$\le \Lambda_0$ [b] $\le$	xxx	···	$\le \Lambda_0 \le$	···	···	$\le \Lambda_0 \le$	···
	xxx		xxx	xxx		xxx	xxx		xxx
	0.0046		0.0161	0.0100		0.0141	xxx		xxx
	−0.0060		0.0255	−0.0012		0.0222	0.0032		0.0195
400 MeV	0.79	$\le D(\delta)/D(0)\le$	0.91	···	$\le D(\delta)/D(0)\le$	···	···	$\le D(\delta)/D(0)\le$	···
	0.57		1.09	0.63		0.97	0.69		0.87
	0.17		1.48	0.30		1.28	0.39		1.14
	−0.55		2.19	−0.27		1.85	−0.08		1.60
	xxx	$\le \Lambda_0 \le$	xxx	···	$\le \Lambda_0 \le$	···	···	$\le \Lambda_0 \le$	···
	xxx		xxx	xxx		xxx	xxx		xxx
	0.0030		0.0170	0.0078		0.0150	xxx		xxx
	−0.0081		0.0272	−0.0033		0.0238	0.0010		0.0209
500 MeV	0.76	$\le D(\delta)/D(0)\le$	0.93	0.81	$\le D(\delta)/D(0)\le$	0.83	···	$\le D(\delta)/D(0)\le$	···
	0.54		1.12	0.60		1.00	0.66		0.90
	0.12		1.53	0.25		1.33	0.35		1.20
	−0.64		2.29	−0.36		1.94	−0.15		1.67
	xxx	$\le \Lambda_0 \le$	xxx	xxx	$\le \Lambda_0 \le$	xxx	···	$\le \Lambda_0 \le$	···
	xxx		xxx	xxx		xxx	xxx		xxx
	0.0020		0.0176	0.0064		0.0156	0.0118		0.0138
	−0.0096		0.0284	−0.0048		0.0249	−0.0005		0.0220

[a] With $D(\delta)/D(0)=1.28$.

[b] With $D(\delta)/D(0)=1.15$.

$$\tfrac{1}{2}(2\pi)^3\sum_n \langle 0|V_\mu^{(4-i5)}(0)|n\rangle \times\langle n|V_\nu^{(4+i5)}(0)|0\rangle\,\delta^{(4)}(p_n-q) = \left(\delta_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\rho^{(1)}(t)-\frac{q_\mu q_\nu}{(q^2)^2}\rho(t),\quad t=-q^2. \tag{2.25}$$

The positivity of the Hilbert space demands that

$$|f_+(t)|^2 \leq 128\pi^2\frac{t^2}{(t-t_0)^{3/2}(t-t_1)^{3/2}}\rho^{(1)}(t)\quad (t\geq t_0). \tag{2.26}$$

Now we assume a weak form of K^* dominance, i.e.,

$$\rho^{(1)}(t)\leq\bar\rho^{(1)}(t)=N\frac{t^n}{(t-m_{K^*}{}^2)^2+\frac14 m_{K^*}{}^2\Gamma^2}. \tag{2.27}$$

Again the constant N is calculable by

$$\int_{t_0}^\infty dt\,\frac{\bar\rho^{(1)}(t)}{t^m}=\frac{1}{(m_{K^*}{}^2)^m}\bar G_{K^*}{}^2, \tag{2.28}$$

where

$$\langle 0|V_\mu^{(4-i5)}(0)|K^{*+}(p)\rangle=\sqrt2\,G_{K^*}(2p_0V)^{-1/2}\epsilon_\mu(p) \tag{2.29}$$

in the narrow-width limit. As before, only the assignment $m=n$ for the integer m in the integrand of Eq. (2.28) leads to a consistent result, if we evaluate G_{K^*} by Weinberg's first sum rule[19]:

$$\frac{G_{K^*}{}^2}{m_{K^*}{}^2}=\frac{G_\rho{}^2}{m_\rho{}^2}=f_\pi{}^2. \tag{2.30}$$

Then λ_+ can be found from an equation analogous to Eq. (1.11). The results are

$n=0$, $0.044\leq\lambda_+\leq 0.059$,

$n=1$, $0.026\leq\lambda_+\leq 0.056$,

$n=2$, $0.006\leq\lambda_+\leq 0.055$,

$n=3$, $-0.016\leq\lambda_+\leq 0.055$.

It is interesting to note that the upper bounds for λ_+ are relatively insensitive to the specific behavior of $\rho^{(1)}(t)$ at high energy. A similar phenomenon has also been observed in the evaluation of the electromagnetic radius of the pion.[20] Perhaps, the upper bound $\lambda_+\leq 0.059$ is trustworthy in view of this fact. We remark that for the case $n=2, 3$ the integral

$$\int_{t_0}^\infty dt\,\frac{\bar\rho^{(1)}(t)}{t}=\infty,$$

so that we need subtractions in the Kamefuchi-Umezawa-Lehmann-Källén representations of two-point functions.

III. WEAKER κ DOMINANCE MODEL

In Sec. II we encountered difficulty when we assumed the specific bound Eq. (1.14) for $\rho(t)$ $(t\geq t_0)$. So it is desirable to relax this condition a little more. Here we will not assume Eq. (1.14). The only thing we will need is the assumption that $\rho(t)$ may have a large Breit-Wigner form around $t\simeq m_\kappa{}^2$ due to κ dominance. The asymptotic behavior of $\rho(t)$ as $t\to\infty$ need not be specified. To achieve this end, we follow the method utilized by Mathur[21] for the evaluation of λ_+. Let $k(t)$ be an arbitrary non-negative function defined on the cut $t\geq t_0$. Then, multiplying $k(t)$ by both sides of Eq. (1.7) and integrating with respect to t, we find

$$\frac1\pi\int_{t_0}^\infty dt\,k(t)|D(t)|^2\leq I^2, \tag{3.1}$$

$$I^2\equiv\tfrac13 64\pi\int_{t_0}^\infty dt\,\frac{t}{(t-t_0)^{1/2}(t-t_1)^{1/2}}\rho(t)k(t). \tag{3.2}$$

Suppose for a moment that I^2 is known. Then we know[2,3] that Eq. (3.1) leads to the inequalities

$$|D(0)|^2\leq\frac{1}{4t_0}I^2\exp(-\bar\epsilon), \tag{3.3}$$

$$|D(0)|^2+|(2+\bar\epsilon-2t_0\bar\eta)D(0)-4t_0D'(0)|^2 \leq\frac{1}{4t_0}I^2\exp(-\bar\epsilon), \tag{3.4}$$

where $\bar\epsilon$ and $\bar\eta$ are defined by Eq. (1.12) [replacing $w(t)$ there by $k(t)$].

Since $\bar\epsilon$, $\bar\eta$, and I^2 are functions of the $k(t)$, the best bound for $D(0)$ and $D'(0)$, given $\rho(t)$, can be achieved by suitably choosing an explicit form of $k(t)$. As we shall prove in the Appendix, this procedure indeed reproduces Eqs. (1.10) and (1.11), with (1.12). However, the spirit of this section is different from that of the above remarks. We do not choose $k(t)$ to be the optimal function, since we don't demand exact knowledge of $\rho(t)$. Now, if the κ meson exists, then $\rho(t)$ will be peaked around $t\approx m_\kappa{}^2$. Hence, if we choose $k(t)$ to be a similarly peaked function of t around $t\approx m_\kappa{}^2$, then we may replace $\rho(t)$ by its resonant part without introducing much error. Thus we can set

$$\rho(t)=\frac{(\Gamma/4\pi)f_\kappa{}^2m_\kappa{}^5}{(t-m_\kappa{}^2)^2+\frac14 m_\kappa{}^2\Gamma^2}, \tag{3.5}$$

if we choose, for example,

$$k(t)=\frac{3}{64\pi}\,\frac{(t-t_0)^{1/2}(t-t_1)^{1/2}}{t}\times[(t-m_\kappa^2)^2+\tfrac{1}{4}m_\kappa^2\Gamma^2]^{-p}t^{-q}. \qquad (3.6)$$

Since $k(t)$ is peaked around $t\approx m_\kappa^2$ for positive integer values of p, we expect that using Eq. (3.5) introduces only a small error in I^2. As we noted already, this choice will give bounds worse than those given in the previous sections. However, an advantage of this approach is that we need not know the precise form of $\rho(t)$ at infinity. At any rate, I^2 is now evaluated as

$$I^2\approx\frac{\Gamma}{4\pi}f_\kappa^{\,2}m_\kappa^{\,5}\int_{t_0}^{\infty}dt\,\frac{1}{t^q[(t-m_\kappa^2)^2+\frac{1}{4}m_\kappa^2\Gamma^2]^{p+1}} \qquad (3.7)$$

and we can find bounds for $D'(0)$ from Eq. (3.4). The results are tabulated in Table III for various choices of p and q as well as of f_κ. So far, we have not utilized the soft-pion theorem. However, it is possible to exploit the information of the soft-pion theorem.

The explicit analytical results have been given elsewhere,[3] and we will not reproduce them here. The corresponding numerical results are tabulated in Table III. As is expected, the value of Λ_0 can be negative if we don't take into account the soft-pion theorem. However, if we utilize the latter information, then Λ_0 is in general positive. Even in those cases in which Λ_0 becomes negative, its magnitude is not large enough to be consistent with the experiment. We remark that perhaps the best choices for p and q are $p=1$ and $q=0$ or 1. The reason is that the value $p=1$ produces the same resonant behavior in $k(t)$ and $\rho(t)$ around $t\approx m_\kappa^2$; therefore, the replacement of $\rho(t)$ by the form in Eq. (3.5) is reasonable. The large value of q is perhaps unphysical since it suppresses the contribution from high t values and emphasizes the contribution from the low-energy t region. However, for the sake of comparison, we computed the bounds for various choices of p and q.

This method has been used by Mathur[21] for the evaluation of λ_+; he obtains

$$\lambda_+\leq 0.084$$

by using K^*-dominated form factors for $k(t)$.

In conclusion, it appears that our results do not agree with the present experimental value of $\Lambda_0=-0.11\pm0.03$. However, it should be kept in mind that various experimental values of Λ_0 are mutually conflicting. For example, a recent Rochester experiment[22] gives

$$\Lambda_0=-0.004\pm0.035,$$

which can be consistent even with a positive Λ_0.

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APPENDIX

Here we shall prove that Eqs. (1.10) and (1.11) can be obtained from Eq. (3.1) if we choose a suitable form for $k(t)$. More generally, let us suppose that a real analytic function $D(t)$ with a cut at $t_0\leq t<\infty$ satisfies

$$|D(t)|\leq w(t)\quad(t\geq t_0) \qquad (A1)$$

on the cut, where $w(t)$ is a given non-negative function. Let $k(t)$ be an arbitrary non-negative function defined on the cut. Then from Eq. (A1) we obtain

TABLE III. Bounds for Λ_0 with and without the soft-pion theorem when we assume the weaker κ-dominance model described in Sec. III, with $m_\kappa=1100$ MeV and $\Gamma=400$ MeV.

$\bar{f}_\kappa/f_\pi$	(p,q)	Without soft-pion theorem			With soft-pion theorem		
0.3	(1, 0)	−0.0603	$\leq\Lambda_0\leq$	0.0900	−0.0045	$\leq\Lambda_0\leq$	0.0267
	(1, 1)	−0.0822		0.0889	−0.0079		0.0276
	(1, 2)	−0.1063		0.0898	−0.0092		0.0308
	(1, 3)	−0.1329		0.0933	−0.0081		0.0367
0.4	(1, 0)	−0.0858		0.1156	−0.0098		0.0320
	(1, 1)	−0.1112		0.1179	−0.0139		0.0336
	(1, 2)	−0.1394		0.1229	−0.0162		0.0377
	(1, 3)	−0.1710		0.1314	−0.0163		0.0449
0.5	(1, 0)	−0.1113		0.1411	−0.0151		0.0373
	(1, 1)	−0.1401		0.1468	−0.0199		0.0396
	(1, 2)	−0.1724		0.1560	−0.0231		0.0447
	(1, 3)	−0.2089		0.1693	−0.0244		0.0530

$$\frac{1}{\pi}\int_{t_0}^{\infty} dt\, k(t)\,|D(t)|^2 \le I^2\,, \tag{A2}$$

$$I^2 = \frac{1}{\pi}\int_{t_0}^{\infty} dt\, k(t)[w(t)]^2\,. \tag{A3}$$

The validity of Eq. (A2) implies that we have[2,7]

$$|D(0)|^2 \le \frac{1}{4t_0}\, I^2 e^{-\bar{\epsilon}}\,, \tag{A4}$$

$$|D(0)|^2 + |(2+\bar{\epsilon}-2t_0\bar{\eta})D(0) - 4t_0 D'(0)|^2 \le \frac{1}{4t_0}\, I^2 e^{-\bar{\epsilon}}\,, \tag{A5}$$

where $\bar{\epsilon}$ and $\bar{\eta}$ are given by

$$\bar{\epsilon} = \frac{t_0^{1/2}}{\pi}\int_{t_0}^{\infty} dt\, \frac{1}{t(t-t_0)^{1/2}}\, \ln k(t)\,,$$
$$\bar{\eta} = \frac{t_0^{1/2}}{\pi}\int_{t_0}^{\infty} dt\, \frac{1}{t^2(t-t_0)^{1/2}}\, \ln k(t)\,. \tag{A6}$$

Now we shall first prove that Eq. (A4) leads to Eq. (1.10) if we choose a suitable $k(t)$.

In fact the correct choice is

$$k(t) = I^2 t_0^{1/2}\, \frac{1}{t(t-t_0)^{1/2}}\, \frac{1}{[w(t)]^2}\,. \tag{A7}$$

We notice that Eq. (A3) is identically satisfied by this choice. Also, Eq. (A4) leads to the result

$$\ln|D(0)| \le \epsilon\,, \tag{A8}$$

where ϵ is given by Eq. (1.12). This can be seen if we use the integral formula

$$\frac{(t_0-t)^{1/2}}{\pi}\int_{t_0}^{\infty} dt'\, \frac{\ln|t'-\alpha|}{(t'-t)(t'-t_0)^{1/2}} = 2\ln[(t_0-t)^{1/2} + (t_0-\alpha)^{1/2}] \tag{A9}$$

for real values of α satisfying $\alpha \le t_0$. Actually the method of Lagrange multipliers can be used to prove that the choice equation (A7) gives the optimal value of the upper bound on $|D(0)|$.

Next we shall prove that Eq. (A5) leads to Eq. (1.11) if we choose a suitable $k(t)$. This case is a bit more complicated. First, let us define $\xi(t)$ by

$$k(t)[w(t)]^2 = I^2 t_0^{1/2}\, \frac{1}{t(t-t_0)^{1/2}}\, \xi(t)\,. \tag{A10}$$

Then the constraint equation (A3) is written as

$$\frac{t_0^{1/2}}{\pi}\int_{t_0}^{\infty} dt\, \frac{\xi(t)}{t(t-t_0)^{1/2}} = 1\,. \tag{A11}$$

Now, after some calculation, we can rewrite Eq. (A5) as

$$|D(0)|^2 + \left|\left\{4t_0\left(\eta - \frac{\epsilon}{2t_0}\right) - K\right\}D(0) - 4t_0 D'(0)\right|^2 \le e^{2\epsilon - A}\,, \tag{A12}$$

where A and K are given by

$$K = \frac{t_0^{1/2}}{\pi}\int_{t_0}^{\infty} dt\, \frac{(2t_0-t)}{t^2(t-t_0)^{1/2}}\, \ln\xi(t)\,,$$
$$A = \frac{t_0^{1/2}}{\pi}\int_{t_0}^{\infty} dt\, \frac{1}{t(t-t_0)^{1/2}}\, \ln\xi(t)\,, \tag{A13}$$

and ϵ and η are the same quantities as in the text. Equation (A12) is now rewritten as

$$\alpha \le 4t_0\left(\eta - \frac{\epsilon}{2t_0}\right) D(0) - 4t_0 D'(0) \le \beta\,, \tag{A14}$$

$$\alpha = KD(0) - [e^{2\epsilon - A} - |D(0)|^2]^{1/2}\,,$$
$$\beta = KD(0) + [e^{2\epsilon - A} - |D(0)|^2]^{1/2}\,. \tag{A15}$$

Now suppose that $D(0)$ is known. Then we want to find the best bounds on $D'(0)$ by varying the arbitrary function $\xi(t)$ subject to the subsidiary constraints Eqs. (A11) and (A13). By using the Lagrange multiplier technique, the solution is easily found to be

$$t\,\xi(t) = \frac{1}{1-(1/2t_0)\gamma}\,(t-\gamma) \tag{A16}$$

for some constant γ. Then A and K are given by

$$K = -\frac{t_0^{1/2} - (t_0-\gamma)^{1/2}}{t_0^{1/2} + (t_0-\gamma)^{1/2}}\,,$$
$$e^{-A} = \left(1 - \frac{\gamma}{2t_0}\right)\frac{4t_0}{[t_0^{1/2} + (t_0-\gamma)^{1/2}]^2}\,. \tag{A17}$$

Inserting these expressions in Eq. (A15) and optimizing α and β by a suitable choice of γ, we find that Eq. (A14) reduces to

$$\left|4t_0\left(\eta - \frac{\epsilon}{2t_0}\right)D(0) - 4t_0 D'(0)\right|^2 \le e^{\epsilon} - e^{-\epsilon}|D(0)|^2\,.$$

This last result is the same as Eq. (1.11).

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[10]Consider the case in which $\rho(t)$ behaves as $C_1(\ln t)^{-2}$as $t\to\infty$. In that case, $D(t)$ may behave as $C_2(\ln t)^{-1}$ at infinity without any conflict with our inequality. Here C_1 and C_2 are some constants. Notice that $\int_{t_0}^\infty dt\ \rho(t)/t$ is finite in that case but $\int_{t_0}^\infty dt\ D(t)/t$ diverges. However, $\int_{t_0}^\infty dt\ \mathrm{Im}D(t)/t$ is always convergent.

[11]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968); S. L. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968).

[12]I. Raszillier, Lett. Nuovo Cimento 2, 349 (1971) and Bucharest Institute of Physics report, 1971 (unpublished). See also, B. V. Geshkenbein, Yad. Fiz. 9, 1232 (1969) [Sov. J. Nucl. Phys. 9, 720 (1969)]; E. E. Radescu, Phys. Rev. D 5, 135 (1972).

[13]See Ref. 9 for some details. For example, Glashow and Weinberg (see Ref. 11) give $f_K\approx 0.58f_\pi$, while asymptotic SW(3) or $SU_W(6)$ symmetry suggests $f_K\approx 0.30f_\pi$.

[14]M. Ademollo and R. Gatto, Phys. Rev. Letters 13, 264 (1965); H. R. Quinn and J. D. Bjorken, Phys. Rev. 171, 1660 (1968).

[15]M. K. Gaillard and L. M. Chounet, CERN Report No. 70-14, 1970 (unpublished); L. M. Chounet, J. M. Gaillard, and M. K. Gaillard, CERN Report, 1971 (unpublished). Even if we use the older small value $\Lambda_0=-0.024\pm0.02$, it is still in conflict with our bounds.

[16]This condition is necessary although it is not explicitly stated in the literatures. For example, consider the function $R(t)=\exp[\alpha(t_0-t)^{1/2}]$ $(\alpha>0)$, which satisfies $|R(t)|=1$ on the cut. This function, however, diverges as $t\to-\infty$ and obeys the inequality $|R(t)|>1$ instead of $|R(t)|\le 1$ in the entire cut plane.

[17]Here we use a version of the Phragmên-Lindelöf theorem due to Nevanlinna. R. Nevanlinna, *Eindeutige Analytische Funktionen* (Springer, Berlin, 1953), p. 44; E. Hill, *Analytic Function Theory* (Ginn, Boston, 1962), Vol. II, p. 412.

[18]This method is known as the Pick-Nevanlinna interpolation procedure; see C. Carathéodory, *Funktionentheorie* (Birkhäuser, Basel, 1950), Vol. 2, p. 14. It has been applied by Raszillier and Radescu (see Ref. 12) to the problem we are considering now.

[19]See, e.g., T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters 19, 470 (1967).

[20]D. N. Levin, V. S. Mathur, and S. Okubo, Phys. Rev. D 5, 912 (1972).

[21]V. S. Mathur, 1971 (unpublished).

[22]I-Hung Chiang, University of Rochester Report No. C00-3065-2, 1972 (unpublished).