theless, the good agreement with experimental results that we have obtained suggests that our ideas concerning the structure of a proton have some validity.

ACKNOWLEDGMENT

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Bose Distribution and Scaling Law*

T. F. Hoang

Argonne National Laboratory, Argonne, Illinois 60439

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Properties of a two-parameter Bose-type distribution have been investigated using data on π and K mesons of inclusive p-p collisions from 12 to 30 GeV/c. It is found that the temperature T is characteristic of the particle emission: $T_\pi > T_K$, and T for π^- is about 5 MeV higher than that of π^+. The scaling behavior is found to hold for both π and K for incident lab momentum above 20 GeV/c.

In a recent work on the inclusive $\bar{p}$-p reactions,[1] it has been found that the salient features of the double-differential cross sections in terms of the c.m. longitudinal, P_L, and the transverse, P_T, momentum, can be accounted for by the following Bose-type distribution:

$$\frac{d\sigma}{dP_T{}^2 dP_L} \sim \frac{1}{e^{\epsilon(\lambda)/T} - 1} , \tag{1}$$

where T is the temperature, the Boltzmann constant being set $k = 1$, and

$$\epsilon(\lambda) = (P_T{}^2 + \lambda^2 P_L{}^2 + m^2)^{1/2} , \tag{2}$$

m being the mass of the secondary meson. The dimensionless parameter λ is related to the scaling law,[2,3] according to which we should expect

$$\lambda \sim 1/P_{\max} , \tag{3}$$

$P_{\max}$ being the maximum of the c.m. momentum of the secondary meson.[4] It should be mentioned that another interpretation for λ is also possible.[5]

The purpose of this paper is to investigate the intrinsic properties of the two parameters T and λ. In particular, we propose to investigate the scaling behavior according to (3).

We recall that the temperature T, as in the Bose distribution, i.e., $\lambda = 1$, is determined entirely by the average of P_T, and that the parameter λ depends on the ratio of averages of P_T and P_L. Referring to formula (33) of Ref. 4, we write

$$\langle P_T \rangle / \langle P_L \rangle = \tfrac{1}{2}\pi\lambda . \tag{4}$$

In the present work, we shall apply the distribution to the currently available data on π and K production by p-p collisions. We begin with the inclusive reactions

$$p + p \to \pi^{\pm} + \text{anything}$$

from the following experiments[6]: Berkeley,[7] Brookhaven,[8] Scandinavian Collaboration,[9] and Day *et al.*[10] The parameters T and λ are estimated by fitting the experimental data with the distribution (1), except the Berkeley data. In this case, the

TABLE I. Estimates of parameters T and λ.

Experiment	P_{lab} (GeV/c)	Secondary	T (GeV)	λ
Day *et al.*	12.5	π^-	0.119± 0.001	0.644± 0.002
		K^-	0.105± 0.002	0.680± 0.004
Berkeley	12.88	π^-	0.121± 0.003	0.650± 0.016
		π^+	0.114± 0.003	0.655± 0.017
	18.0	π^-	0.129± 0.003	0.618± 0.015
		π^+	0.123± 0.003	0.592± 0.013
	21.08	π^-	0.130± 0.003	0.600± 0.015
		π^+	0.124± 0.003	0.550± 0.012
	24.12	π^-	0.132± 0.005	0.588± 0.015
		π^+	0.127± 0.005	0.527± 0.011
	28.44	π^-	0.134± 0.005	0.565± 0.010
		π^+	0.126± 0.005	0.503± 0.011
Scandinavia	19.2	π^-	0.128± 0.003	0.681± 0.004
Brookhaven	28.0	π^-	0.136± 0.005	0.540± 0.005
CERN	24.5	$\pi^\pm$	0.142± 0.015	0.60 ± 0.10
		K	0.134± 0.015	0.69 ± 0.10

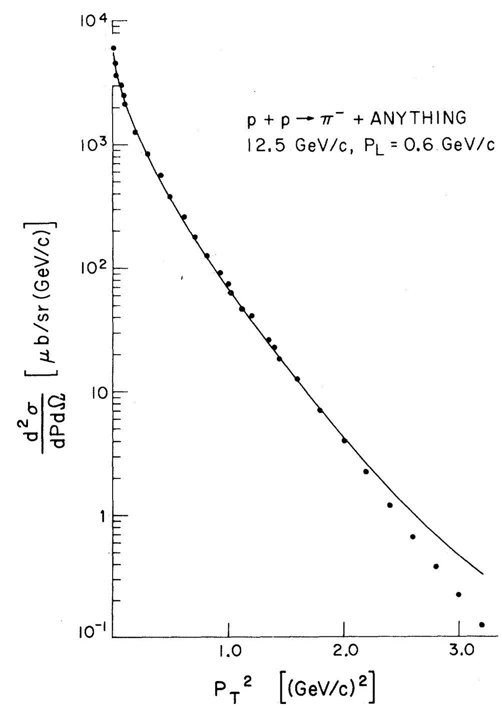

FIG. 1. Comparison of two-parameter Bose-type distribution fit with experimental data. The parameters T and λ, estimated by the least-squares fit, are listed in Table I.

parameters are estimated by means of $\langle P_T \rangle$ and $\langle P_L \rangle$. The results are summarized in Table I.

We note that λ is rather critical for the fitting. As an illustration, we present in Fig. 1 our fit to the P_T^2 distribution at fixed $P_L = 0.6$ of the counter experiment at 12.5 GeV/c by Day *et al.*[10] It is readily seen that our fit is excellent in a range of $P_T^2 \leq 2$ (GeV/c)^2 covering almost 4 decades of measured cross sections. In particular, we note that the drastic fall near the origin has been well reproduced and that this can only be achieved through the parameter λ. Indeed, if we set $\lambda = 1$ and try the similar fit, we find the χ^2 increased by about 40.

As regards the temperature dependence, Fig. 2 shows the log plot of T, for π^+ and π^- vs the available energy W in the p-p c.m. system, $W = \sqrt{s} - 2M$, M being the proton mass. We note that at a given W, there is a systematic difference between the two temperatures for π^+ and π^-. This is due to the effect of the "leading particle." Experimentally, we know that in p-p collisions, $\langle P_L \rangle$ for π^+ is larger than that for π^-, whereas the reverse is true for $\langle P_T \rangle$.[11] Therefore we have $T_+ < T_-$.

If we assume a power-law dependence

$$T \sim W^{\alpha} \tag{5}$$

we find for π^- and π^+

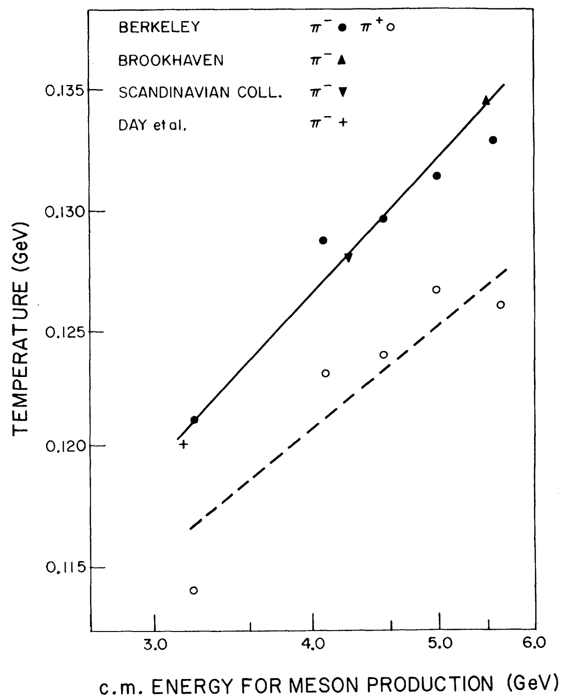

FIG. 2. Dependence of temperature T (GeV) on available energy W (GeV) in p-p c.m. system.

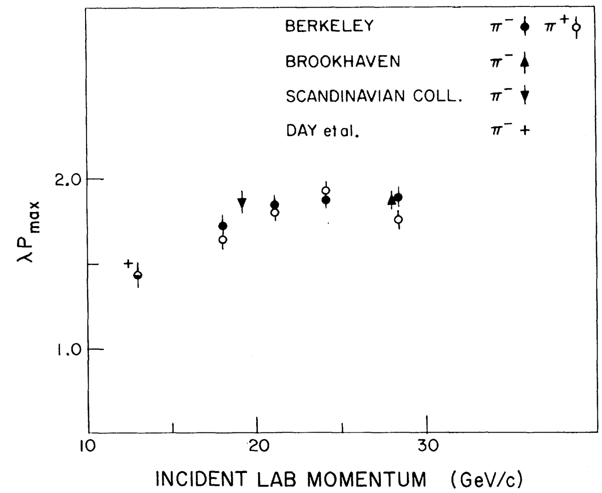

FIG. 3. Plot of λP_{max} vs incident momentum in P_{lab} system. P_{max} is the maximum c.m. momentum of π. The scaling law requires λP_{max} = const.

$$\alpha_- = 0.203 \pm 0.014, \quad \alpha_+ = 0.187 \pm 0.056 .$$

The fits are shown by the straight lines in Fig. 2. Although these values of α_+ and α_- are comparable to the power $\alpha = \frac{1}{4}$ of the statistical model,[12] nonetheless we bear in mind that such a dependence as (5) will not hold at higher energies. Because the temperature T has an upper bound of 0.160 GeV,[13] this characteristic feature cannot be accounted for by (5).

We now turn to the parameter λ, which describes the scaling behavior according to (3). We investigate this property with the data listed in Table I. For this purpose, we have plotted in Fig. 3 the products λP_{max} against the incident lab momentum P_{inc}. First, we notice that, in contrast to the plot in Fig. 2, i.e., T versus W, here we find no systematic difference between π^+ and π^-; therefore no distinction will be made of the positive and the negative π. Next, we note that the points show a definite rise in the region of $P_{inc} = 12$ to 20 GeV/c, beyond which there is a plateau. This suggests that the scaling law holds for $P_{inc} > 20$ GeV/c and that we may write

$$\lambda P_{max} \to \text{const.} \tag{6}$$

The constant is estimated to be 1.86 ± 0.06 GeV/c by taking the average of points with $P_{inc} > 20$ GeV/c.

It is interesting to investigate this important property of scaling behavior with the K meson. In spite of the scarcity of data, we are tempted to make a preliminary study with the following two experiments: the counter experiment at 12.5 GeV/c by Day *et al.*[10] and the CERN bubble-chamber experiment at 24.5 GeV/c.[14] The values of T and λ are listed in Table I. If we compute P_{max} assuming the $K\bar{K}$ production by p-p collision, we find λP_{max} equal to

$$1.405 \pm 0.008 \text{ and } 2.10 \pm 0.31 ,$$

for the 12.5- and 24.5-GeV/c experiments, respectively. Here also it seems that we observe a trend of rise as in the previous case of π production. If we compare these values with those corresponding to π, namely

$$1.492 \pm 0.002 \text{ and } 1.92 \pm 0.32 ,$$

we note that, at a given incident momentum, the products λP_{max} are equal within the statistical errors for both π and K. We tentatively write

$$(\lambda P_{max})_\pi = (\lambda P_{max})_K . \tag{7}$$

This indicates that λ_K satisfies also the scaling behavior (6) as in the case of π.

It is worth noting that if the constant of the scaling law turns out to be the same for π and K, say c, then the expression (1) of our two-parameter Bose distribution can be written as a universal function for both π and K in terms of the following variables:

$$m_T = (P_T{}^2 + m^2)^{1/2} , \quad x = P_L / P_{max} , \tag{8}$$

m_T being the "transverse mass" and x the Feynman variable of the reduced longitudinal momen-

TABLE II. Three-parameter fits to 12.5-GeV/c experiment by Day *et al.*

Second	T (GeV)	λ	μ (GeV)
π^-	0.121 ± 0.001	0.644 ± 0.001	0.102 ± 0.001
K^-	0.105 ± 0.001	0.679 ± 0.001	0.110 ± 0.004

tum.[2] Indeed, the expression $\epsilon(\lambda)$ becomes

$$\epsilon=(m_T^2+c^2x^2)^{1/2}\ , \qquad (9)$$

the same for π and K. It would be interesting to test if these important properties conjectured thus far turn out to be true, and in particular, if c be also the same for other hadron collisions.[15]

Finally, let us compare the temperatures for π and K listed in Table I. We observe a notable difference of ~10 MeV between the temperature for K production and that for π. We have investigated whether it is possible by another parametrization to obtain the same temperature for both particles, K and π. For instance, with a new parameter μ, positive and less than m, we may write our distribution (1) in a more general form as follows[16]:

$$\frac{d\sigma}{dP_T^2dP_L}\sim\frac{1}{e^{[\epsilon(\lambda)-\mu]/T}-1}\ . \qquad (10)$$

We have refitted the data of Day *et al.*[10] The results are summarized in Table II. A comparison with the previous two-parameter fits listed in Table I indicates that μ has almost no effect on the estimates of T and λ; both values remain the same as in the case of two-parameter fits.

We are unable to remove the temperature difference by this artifice. Thus we propose to regard the parameter T of the distribution (1) as a characteristic of the mechanism of particle emission. This may be a shortcoming from the point of view of the thermodynamical model. Nevertheless, this does not affect the intrinsic properties of the other parameter λ, as is shown by the plot of λP_{max} in Fig. 3. Consequently, the modified Bose distribution (1) here considered remains to be useful especially for the investigation of the scaling behavior, which is one of the most important properties of inclusive reactions.

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[15]For the inclusive reaction $\bar{p}+p\rightarrow\pi^++$anything at 2.32

GeV/c, Ref. 1, we find $\lambda P_{max} = 1.198 \pm 0.001 \pm 0.003$ GeV/c for the forward π^+ and 1.111 ± 0.003 GeV/c for the backward π^+.

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Low-Energy Theorem for $\gamma\gamma \to 3\pi$ and the Reaction $\gamma + Z \to 3\pi + Z^*$

J. Smith and N. Stanko

The Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, New York 11790

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Recently a low-energy theorem was proposed by Adler, Lee, Treiman, and Zee and by Terent'ev for the reaction $2\gamma \to 3\pi$. This theorem is supposed to be valid at an unphysical point in the production amplitude. One possible way to test its reliability is to measure the reaction $e^+e^- \to e^+e^- + 3\pi$ which unfortunately has a very small cross section. In this paper we investigate another reaction which may allow a more satisfactory check of the low-energy theorem, namely, the photoproduction of three pions off a target nucleus. Cross sections, angular and energy distributions, and invariant-mass plots are given for several beam energies assuming a lead target. The η-meson background is also examined.

I. INTRODUCTION

Since the work of Adler[1] and Bell and Jackiw[2] the subject of perturbation-theory anomalies has aroused a great deal of interest. The anomalies have experimental consequences for the decay $\pi^0 \to 2\gamma$, which was originally thought to be suppressed through first order in the pion mass. In a more careful investigation, it was discovered that the axial-vector Ward identity has an anomalous term, which leads to the conclusion that the $\pi^0 \to 2\gamma$ decay is not suppressed. Any low-energy theorem connecting the $\pi^0 \to 2\gamma$ amplitude to another amplitude is therefore important because it gives one an opportunity of testing these arguments in another system. The recent low-energy theorem of Adler, Lee, Treiman, and Zee[3] (which was also discovered by Terent'ev[4]) relates the $\gamma \to 3\pi$ amplitude to the $\pi^0 \to 2\gamma$ decay amplitude in the following way:

$$eF^{3\pi} = F^{\pi}/f^2. \tag{1.1}$$

Here $F^{3\pi}$ and F^{π} are coupling constants for the $\gamma \to 3\pi$ and $\pi^0 \to 2\gamma$ amplitudes, while f is related to the charged pion decay constant. As has been stressed by Aviv and Zee,[5] this theorem is based upon the supposedly sacred principles of gauge invariance, current algebra, partial conservation of axial-vector current (PCAC), and the fact that the electromagnetic current commutes with the neutral axial-vector charge at equal times. It is independent, however, of the exact nature of chiral-symmetry breaking. The experimental verification of relation (1.1) is important because it directly checks that the terms omitted in deriving the theorem are in fact negligible compared to the terms retained. Possible reactions which can be used to check (1.1) are $e^+e^- \to 3\pi$ and $\pi + Z \to 2\pi + Z$. The first reaction is not very good because the photon propagator must be far enough off its mass shell to be above the three-pion production threshold. This means that the amplitude is being measured in a region where the theorem has no reason to be valid. The second reaction is more promising so long as the invariant mass of the two-pion state is kept low and we stay below the region of the ρ-meson resonance. We refer to the recent paper of Zee[6] for details.

Relation (1.1) can be derived by considering the amplitude for the reaction $2\gamma \to 3\pi$. This amplitude is completely determined up to terms of second order in the particle momenta by the requirement that it be gauge-invariant in the two photons and vanish in the limit that the π^0 momentum vanishes. The unique chiral and gauge completions which perform this task unfortunately (or fortunately for the chiral-symmetry-breaking enthusiasts) introduces one new parameter. Specifically the theorem contains an unknown constant which controls the size of the chiral-symmetry breaking, i.e., the isotensor component of the σ term. One can therefore investigate reactions which will check the low-energy theorem and perhaps shed some light on the magnitude of this constant. A possible reaction is $e^+e^- \to e^+e^-3\pi$. However, we[7] have already cal-