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Description for the Reactions $a+b \to c+d+e$ *

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A formalism for describing the reactions $a+b \to c+d+e$ is presented in detail and the experiments necessary for the reconstruction of the transition amplitude described. Experiments involving polarized targets are discussed carefully and a partial-wave analysis is made which is particularly suitable for application at low c.m. energies. It is shown that in order to resolve parity ambiguities it is necessary to observe the final particle polarizations.

I. INTRODUCTION

In this paper we exhaustively develop a formalism[1] to describe reactions of the type

$$a+b \to c+d+e \tag{1.1}$$

which is particularly suitable for partial-wave analyses at low energies where there is appreciable overlap of resonances in the Dalitz plot.

The reaction is described in terms of the Dalitz-plot variables, the spin components of the individual particles normal to the three-particle plane and the Euler angles describing the orientation of this plane with respect to a fixed c.m. coordinate system. In the partial-wave analysis these Euler angles are replaced by an equivalent set of quantum numbers – J the total angular momentum, M its Z component in the fixed coordinate system, and Λ the component parallel to the normal of the three-particle plane. This formalism then leads easily to a method of recording the angular correlations as a function of Dalitz-plot variables in a manner analogous to the description of elastic scattering in terms of Legendre coefficients.

We give explicit formulas for the restrictions imposed on these states by parity (correcting earlier versions[1-3]) and by the presence of two identical particles together with the resulting properties of the transition amplitudes. Armed with this formalism we then discuss all possible types of polarization experiments and point out the existence of a parity ambiguity in the analysis of the unpolarized cross section which can only be resolved by measurements of final particle polarizations.

We have not considered the imposition of three-particle unitarity constraints nor have we identified all the kinematic singularities[4] within the formalism. It is also clear that the type of three-body analysis presented here is particularly useful in the low-energy resonance region. A natural question which then arises is what formalism is best suited to what energy region. All these questions are very important phenomenologically and they are at present under investigation.

The plan of the paper is as follows. In Sec. II we describe the method of construction of the two- and three-particle states, the projection of angular momentum states, and the consequences of parity and the presence of two identical particles for these states. We then calculate the transition amplitudes and cross-section formulas for the processes (1.1) together with their partial-wave decompositions.

In Sec. III we specialize our formulas to the case

$$MB \to MMB \quad (\text{e.g., } \pi N \to \pi\pi N,\ KN \to \pi\pi\Lambda, \text{ etc.}), \tag{1.2}$$

where M is a pseudoscalar meson and B is a $\frac{1}{2}^+$ baryon, and discuss all the experiments which can be performed when B is both a stable particle (e.g., proton or neutron) or an unstable particle (e.g., Λ, Σ) whose decay distribution gives information on the parent polarization. We give specific formulas in the case of scattering from unpolarized targets where the final baryon polarization is unobserved and comment on the application of our method to all possible polarization experiments.

In Sec. IV we discuss the properties of this formulation, its advantages and disadvantages, and compare it with present methods used in analyzing reactions of the type (1.1) and (1.2).

The Appendix describes the interpretation of the density matrix we use for describing the polarization properties of $2 \to 3$ reactions.

II. THEORETICAL FORMALISM

A. Definitions of Coordinate Systems and Kinematical Quantities

We consider the production process 2 → 3 shown in Fig. 1. Our metric is such that $p_i^2 = M_i^2$. We define

$$s = (k_1 + k_2)^2 = (p_1 + p_2 + p_3)^2 , \tag{2.1}$$

$$s_i = (p_j + p_k)^2 , \tag{2.2}$$

and we have the relation

$$s = s_1 + s_2 + s_3 - (M_1^2 + M_2^2 + M_3^2) .$$

We consider the process in the c.m. system

$$\sum_i \vec{k}_i = \sum_i \vec{p}_i = 0 \tag{2.3}$$

and

$$|\vec{k}_i| = k ,$$

$$W = \epsilon_1 + \epsilon_2 = \omega_1 + \omega_2 + \omega_3 = \sqrt{s} , \tag{2.4}$$

where ϵ_i and ω_i are the energies of the particles in the c.m. system.

The three outgoing momenta define a plane, and the final configuration may be specified by the momenta $\vec{p}_i$ [nine variables and four constraints (Eqs. (2.3) and (2.4)], or by the following procedure. We define a system of axes $Oxyz$ fixed in space, and define a "standard orientation" of the three-particle final state to be when all momenta are in the xy plane. This set of standard momenta we denote by $\vec{\pi}_i$, where we choose (Fig. 2) $\vec{\pi}_1 + \vec{\pi}_2 = -\vec{\pi}_3$ to be along the x axis and $\vec{\pi}_1 \times \vec{\pi}_2$ to be along the z axis. (Such a choice simplifies our discussion of systems containing two identical particles.) The relative orientation of the three particles is now specified by two variables which may be chosen to be two energies ω_1, ω_2 or, equivalently, s_1, s_2. As for a rigid body, a general orientation of the three-particle final state is specified by the rotation[5] $R(\alpha, \beta, \gamma)$ from the standard orientation. The momenta $\vec{p}_i$ are obtained from $\vec{\pi}_i$ by this rotation $R(\alpha, \beta, \gamma)$. If we define a set of moving axes $OXYZ$ fixed with respect to the particle momenta, and initially parallel to the set $Oxyz$, it is clear that OZ defines the normal to the production plane of the three particles.[6]

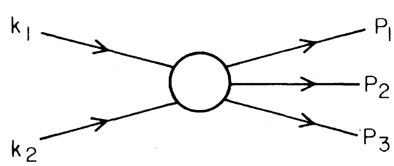

FIG. 1. Notation for the reaction $a + b \rightarrow c + d + e$.

B. The Incident Two-Particle States

We specify the initial two-particle states using the helicities of each particle.

A two-particle helicity state in the c.m. system is defined in the following manner:

$$|\vec{P} = \vec{0}, W; \theta, \phi, \mu_i\rangle = R(\phi, \theta, 0) |\vec{P} = \vec{0}, W; 0, 0, \mu_i\rangle \tag{2.5}$$

and

$$|\vec{P} = \vec{0}, W; 0, 0, \mu_i\rangle = |k, \mu_1\rangle |-k, \mu_2\rangle , \tag{2.6}$$

where $\vec{P}$ is the total momentum of the two-particle system, k is the c.m. momentum of each particle, W the total energy of the two-particle system, θ, ϕ the polar coordinates of particle 1, μ_1, μ_2 the helicities of particles 1 and 2, and we have used the phase conventions of Werle.[7] The normalization is

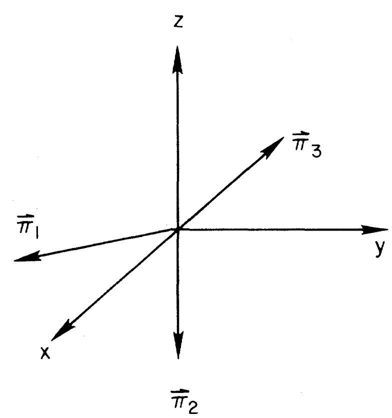

FIG. 2. Orientation of momentum vectors in the standard state.

$$\langle \vec{P}', P_0'; \theta', \phi', \mu_i' | \vec{P}=\vec{0}, W; \theta, \phi, \mu_i\rangle = (4W/P)\delta(W-P_0')\delta^3(\vec{P}')\delta(\cos\theta'-\cos\theta)\delta(\phi'-\phi)\prod_{i=1}^{2}\delta_{\mu_i'\mu_i}. \quad (2.7)$$

Angular momentum states $|\vec{P}=\vec{0}, W; \mu_i; JM\rangle$ are constructed in standard fashion.

$$|\vec{P}=\vec{0}, W; \mu_i; JM\rangle = \frac{N_J}{2\pi}\int d^3R\, D^{J*}_{M\mu}(R) R|\vec{P}=\vec{0}, W; 0, 0, \mu_i\rangle, \quad (2.8)$$

where

$$N_J = \left(\frac{2J+1}{4\pi}\right)^{1/2},$$

$$\mu = \mu_1 - \mu_2.$$

We obtain the usual parity condition

$$\mathcal{P}|\vec{P}=\vec{0}, W; \mu_i; JM\rangle = \eta(-1)^{J-\nu_1-\nu_2}|\vec{P}=\vec{0}, W; -\mu_i; JM\rangle, \quad (2.9)$$

with $\eta=\eta_1\eta_2$, the product of the intrinsic parities of particles 1 and 2, and ν_1, ν_2 the intrinsic spins of particles 1 and 2.

C. The Final Two-Particle States

Our standard single-particle states are constructed as follows:

$$|\vec{\pi}_i, \tau_i\rangle = L(\vec{\pi}_i)|\tau_i\rangle, \quad (2.10)$$

where $L(\vec{\pi}_i)$ is the relevant boost operator in the xy plane. For the general orientation

$$|\vec{p}_i, \tau_i\rangle = R(\alpha, \beta, \gamma)|\vec{\pi}_i, \tau_i\rangle, \quad (2.11)$$

τ_i specifies the spin degree of freedom for each particle: They are not helicities. Here, the τ_i specify the Z component of spin in the rest frame of the particle. Now, Eqs. (2.10) and (2.11) are usually regarded in the active sense; i.e., as a prescription for generating a complete set of states in the frame $Oxyz$. However, they may also be regarded in the passive sense; as describing the same state in two different reference systems: one in which the particle has momentum $\vec{p}_i$ and the other a rest system. Thus Eqs. (2.10) and (2.11) amount to a definition of the orientation of a set of "rest-frame axes" for each particle, relative to the frame $Oxyz$. With the above conventions, the rest-frame axes for the final-state particles are just the moving axes $OXYZ$. The τ_i may be called "transversities": spin in the rest frame quantized along the normal to the production plane.

We define three-particle states:

$$|\vec{P}=\vec{0}, W; \alpha, \beta, \gamma; s_i, \tau_i\rangle = R(\alpha, \beta, \gamma)\prod_i |\vec{\pi}_i, \tau_i\rangle. \quad (2.12)$$

The single-particle states are normalized as follows:

$$\langle \vec{p}_i', \tau_i' | \vec{p}_i, \tau_i\rangle = 2\omega_i\delta^3(\vec{p}_i'-\vec{p}_i)\delta_{\tau_i'\tau_i}. \quad (2.13)$$

By a standard change of variables,

$$\prod_i \frac{d^3p_i}{2\omega_i} = \frac{W}{8P^0}d^3P\,dW\,d^3R\,d\omega_1 d\omega_2 = \frac{1}{32s}\frac{W}{P^0}d^3P\,dW\,d^3R\,ds_1 ds_2, \quad (2.14)$$

where $P^\mu = \sum_i p_i^\mu$ and in the c.m. system $P^0 = W$, and $d^3R = d\alpha\, d\cos\beta\, d\gamma$. We find the three-particle states are normalized as follows:

$$\langle \vec{P}', P^{0\prime}; R'; s_i', \tau_i' | \vec{P}=\vec{0}, W; R; s_i, \tau_i\rangle = 32s\delta^3(\vec{P}')\delta(W-P^{0\prime})\delta(\alpha'-\alpha)\delta(\cos\beta'-\cos\beta)\delta(\gamma'-\gamma) \times \prod_{i=1}^{2}\delta(s_i'-s_i)\prod_{i=i}^{3}\delta_{\tau_i'\tau_i}. \quad (2.15)$$

Angular momentum states may be defined in the usual way:

$$|\vec{P}=\vec{0}, W; s_i, \tau_i; J, \Lambda, M\rangle = \frac{2J+1}{8\pi^2}\int d^3R\; D^{J*}_{M\Lambda}(\alpha, \beta, \gamma)|\vec{P}=\vec{0}, W; \alpha, \beta, \gamma; s_i, \tau_i\rangle \quad (2.16)$$

and this may be inverted to give

$$|\vec{P}=\vec{0}, W; \alpha, \beta, \gamma; s_i, \tau_i\rangle = \sum_{JM\Lambda} D^J_{M\Lambda}(\alpha, \beta, \gamma)|\vec{P}=\vec{0}, W; s_i, \tau_i; J, \Lambda, M\rangle . \tag{2.17}$$

We now consider the effect of the parity operation on these three-particle states and the modifications necessary when there are two identical particles.

Parity. For our standard single-particle states, it is useful to define the operator for reflection in the xy plane:

$$Z = e^{-i\pi J_z}\mathcal{P} . \tag{2.18}$$

Clearly, since $L(\vec{\pi}_i)$ generates boosts in this plane,

$$ZL(\vec{\pi}_i) = L(\vec{\pi}_i)Z,$$

and consequently

$$Z|\vec{\pi}_i, \tau_i\rangle = \eta_i e^{-i\pi\tau_i}|\vec{\pi}_i, \tau_i\rangle , \tag{2.19}$$

where η_i is the appropriate intrinsic parity. Thus we obtain for the three-particle states

$$\mathcal{P}|\vec{P}=\vec{0}, W; \alpha, \beta, \gamma; s_i, \tau_i\rangle = \eta' e^{-i\pi\tau}|\vec{P}=\vec{0}, W; \alpha, \beta, \gamma-\pi; s_i, \tau_i\rangle , \tag{2.20}$$

where $\tau = \sum_i \tau_i$, and we have written η' for $\prod_i \eta_i$.

As is physically obvious, the parity operation relates one three-particle configuration to another, without changing the "transversities." For our definition of the angular momentum states (2.16) we obtain

$$\mathcal{P}|\vec{P}=\vec{0}, W; s_i, \tau_i; J, \Lambda, M\rangle = (-1)^{\Lambda-\tau}\eta'|\vec{P}=\vec{0}, W; s_i, \tau_i; J, \Lambda, M\rangle \tag{2.21}$$

and we see that these states are eigenstates of the parity operator.

Identical Particles. In this section particles 1 and 2 are assumed to be identical. If P_{12} is the operator which exchanges particles 1 and 2, we have

$$P_{12}|\vec{0}; s_i, \tau_i\rangle = P_{12}|\vec{\pi}_1, \tau_1\rangle_{a1}|\vec{\pi}_2, \tau_2\rangle_{a2}|\vec{\pi}_3, \tau_3\rangle_{a3} = |\vec{\pi}_1, \tau_1\rangle_{a2}|\vec{\pi}_2, \tau_2\rangle_{a1}|\vec{\pi}_3, \tau_3\rangle_{a3} , \tag{2.22}$$

where

$$|\vec{\pi}, \tau\rangle_a = L(\vec{\pi})|\tau\rangle , \tag{2.23}$$

i.e., our standard single-particle basis states. Now the state (2.22) is no longer a standard state of the type $|\vec{0}; s_i', \tau_i'\rangle$ since the body-fixed axes Z and Y are opposite to the z and y axes of the frame $Oxyz$. A rotation of π about the x axis of a standard state does not, however, lead to (2.23) since the rest-frame axes of each particle are also rotated. Thus we need to consider states $|\vec{\pi}, \tau'\rangle_b$ with the rest-frame axes rotated by π from those of states $|\vec{\pi}, \tau\rangle_a$:

$$|\vec{\pi}, \tau\rangle_b = L(\vec{\pi})R_x(\pi)|\tau\rangle . \tag{2.24}$$

These states are related to $|\vec{\pi}, \tau\rangle_a$ by

$$|\vec{\pi}, \tau\rangle_a = e^{i\pi\tau}(-1)^{\sigma+\tau}|\vec{\pi}, -\tau\rangle_b , \tag{2.25}$$

where σ is the intrinsic spin of the particle. Then

$$\begin{aligned} R_x(\pi)|\vec{0}; s_2, -\tau_2, s_1, -\tau_1, s_3, -\tau_3\rangle &\equiv R_x(\pi)|\vec{0}; \bar{s}_i, -\bar{\tau}_i\rangle \\ &= |-\tfrac{1}{2}\pi, \pi, \tfrac{1}{2}\pi; \bar{s}_i, -\bar{\tau}_i\rangle \\ &= |\vec{\pi}_2, -\tau_2\rangle_{b1}|\vec{\pi}_1, -\tau_1\rangle_{b2}|\vec{\pi}_3, -\tau_3\rangle_{b3} . \end{aligned} \tag{2.26}$$

Then

$$\begin{aligned} P_{12}|\vec{0}, s_i, \tau_i\rangle &= e^{i\pi(\Sigma_j\tau_j)}(-1)^{\Sigma_j(\sigma_j+\tau_j)} \times |\vec{\pi}_2, -\tau_2\rangle_{b1}|\vec{\pi}_1, -\tau_1\rangle_{b2}|\vec{\pi}_3, -\tau_3\rangle_{b3} \\ &= \rho R_x(\pi)|\vec{0}, \bar{s}_i, -\bar{\tau}_i\rangle , \end{aligned} \tag{2.27}$$

where

$$\rho = e^{i\pi(\Sigma_j\tau_j)}(-1)^{\Sigma_j(\sigma_j+\tau_j)} . \tag{2.28}$$

This result may then be used in (2.16) to give

$$P_{12}|s_i, \tau_i; J, \Lambda, M\rangle = \xi|\bar{s}_i, -\bar{\tau}_i; J, -\Lambda, M\rangle , \tag{2.29}$$

where

$$\xi = (-1)^{J+\sigma_1+\sigma_2+\sigma_3} \tag{2.30}$$

and

$$\sigma_1 = \sigma_2 .$$

Note that all angular momentum quantum numbers referred to the normal to the reaction plane have changed sign as expected.

D. The Transition Amplitude and Its Partial-Wave Decomposition

The transition amplitude is

$$\langle \vec{P}=\vec{0}, W; \alpha, \beta, \gamma; s_i, \tau_i | T | \vec{P}=\vec{0}, W; \phi, \theta, \mu_j\rangle . \tag{2.31}$$

For simplicity we take $\theta=\phi=0$, and then the partial-wave decomposition of the T-matrix element can be written

$$\langle \alpha, \beta, \gamma; s_i, \tau_i | T(s) | 0, 0, \mu_j\rangle = \sum_{J\Lambda} N_J D^{J*}_{\mu\Lambda}(\alpha, \beta, \gamma)\langle s_i, \Lambda, \tau_i | T_J(s) | \mu_j\rangle , \tag{2.32}$$

where $\mu=\mu_1-\mu_2$. For convenience we define

$$B^{\mu_j\tau_i}_{J\Lambda}(s, s_i) = \langle s_i, \Lambda, \tau_i | T_J(s) | \mu_j\rangle \tag{2.33}$$

and then we have

$$F_{\tau_i\mu_j}(s, s_i; R) = \langle \alpha, \beta, \gamma; s_i, \tau_i | T(s) | 0, 0, \mu_j\rangle = \sum_{J\Lambda} N_J D^{J*}_{\mu\Lambda} B^{\mu_j\tau_i}_{J\Lambda}(s, s_i) . \tag{2.34}$$

Using our results for the parity operation gives the condition

$$B^{\mu_j\tau_i}_{J\Lambda}(s, s_i) = \eta\eta'(-1)^{\Lambda-\tau}(-1)^{J-\nu_1-\nu_2} B^{-\mu_j\tau_i}_{J\Lambda}(s, s_i) . \tag{2.35}$$

This parity constraint was given incorrectly by Branson, Landshoff, and Taylor[4] (now corrected in a recent erratum[2]) and this incorrect version was used by Arnold and Uretsky[3] in an analysis of the reaction

$$\pi N \to \pi\pi N .$$

They associated the partial-wave amplitudes $B^{\mu\tau}_{J\Lambda}$ and $B^{\mu-\tau}_{J\Lambda}$ with the same parity initial state. Since the parity of the final angular momentum state is $\eta'(-1)^{\Lambda-\tau}$, this is clearly incorrect. Moreover, this error completely invalidates their conclusions concerning constraints on the absorption parameters η^{JP} of elastic phase-shift analyses from the presently-available unpolarized production angular distributions. In this paper, we show that unless final-state polarization experiments are performed, a complete parity ambiguity must exist in the analysis. Thus bounds on the inelasticities for each J^P *cannot* be obtained from unpolarized angular distributions and the analysis of Ref. 3 should be disregarded.

In order to make a partial-wave decomposition of the transition amplitude (2.34), we note that $B^{\mu_j\tau_i}_{J\Lambda}(s, s_i)$ represents a transition to the final partial wave $|s_i, \tau_i; J\Lambda M\rangle$ which is an eigenstate of parity with eigenvalue $\eta'(-1)^{\Lambda-\tau}$. This means we can project out states of opposite parity by writing (2.34) as

$$F_{\tau_i\mu_j}(s, s_i; R) = \sum_{J\Lambda} N_J D^{J*}_{\mu\Lambda}(\alpha, \beta, \gamma) B^{\mu_j\tau_i}_{J\Lambda}(s, s_i)\tfrac{1}{2}[1+\eta'(-1)^{\Lambda-\tau}+1-\eta'(-1)^{\Lambda-\tau}]$$

$$= F^{+}_{\tau_i\mu_j}(s, s_i; R) + F^{-}_{\tau_i\mu_j}(s, s_i; R) , \tag{2.36}$$

where

$$F^{\pm}_{\tau_i\mu_j}(s, s_i; R) = \tfrac{1}{2}\sum_{J\Lambda} N_J D^{J*}_{\mu\Lambda}(\alpha, \beta, \gamma) B^{\mu_j\tau_i}_{J\Lambda}(s, s_i)[1\pm\eta'(-1)^{\Lambda-\tau}] \tag{2.37}$$

and the $\pm$ refers to parity of the transition. Thus (2.36) and (2.37) are our partial-wave decomposition of the transition amplitude.

Finally if the final state contains two identical particles then the transition to the correctly symmetrized final state must be considered. This then imposes the restriction that

$$B^{\mu_j-\bar{\tau}_i}_{J\Lambda}(s, \bar{s}_i) = (-1)^{2\sigma}\xi B^{\mu_j\tau_i}_{J-\Lambda}(s, s_i) , \tag{2.38}$$

where σ is the intrinsic spin of one of the identical particles and ξ is defined in (2.30).

E. Cross Section and the Contributions of States of Different Parity

The differential cross section can now easily be calculated and we obtain in the c.m. system,

$$\frac{d^5\sigma(\tau_i\mu_j)}{ds_1 ds_2 d^3R} = \frac{\pi^2}{32 s^{3/2} k} |\langle R; s_i, \tau_i | T(s) | 0, \mu_j\rangle|^2 \tag{2.39}$$

$$= \beta^2 |F_{\tau_i\mu_j}(s, s_i; R)|^2$$
$$= \beta^2 |F^{+}_{\tau_i\mu_j}(s, s_i; R) + F^{-}_{\tau_i\mu_j}(s, s_i; R)|^2 , \tag{2.40}$$

where the equations define β.

If we now substitute our partial-wave expansion for $F_{\tau_i\mu_j}(s, s_i; R)$, we have

$$\frac{d^5\sigma(\tau_i\mu_j)}{ds_1 ds_2 d^3R} = \beta^2 \sum_{\substack{J\Lambda \\ J'\Lambda'}} N_J N_{J'} D^{J}_{\mu\Lambda}(\alpha, \beta, \gamma) D^{J'*}_{\mu\Lambda'}(\alpha, \beta, \gamma) \times B^{\mu_j\tau_i\,*}_{J\Lambda}(s, s_i) B^{\mu_j\tau_i}_{J'\Lambda'}(s, s_i) , \tag{2.41}$$

and we note that transitions from definite helicity states give no dependence on α. This is not true if the initial polarization vectors have nonzero transverse components. We also note that the α, β, and γ distributions do not depend on τ.

Finally we may integrate (2.41) over α, β, and γ to give

$$\frac{d^2\sigma(\tau_i\mu_j)}{ds_1ds_2} = 2\pi\beta^2 \sum_{J\Lambda} |B^{\mu_j\tau_i}_{J\Lambda}(s, s_i)|^2 \tag{2.42}$$

$$= \pi\beta^2 \sum_{J\Lambda} |B^{\mu_j\tau_i}_{J\Lambda}|^2 [1 + \eta'(-1)^{\Lambda-\tau} + 1 - \eta'(-1)^{\Lambda-\tau}]$$

$$= \frac{d^2\sigma^+(\tau_i\mu_j)}{ds_1ds_2} + \frac{d^2\sigma^-(\tau_i\mu_j)}{ds_1ds_2}, \tag{2.43}$$

where

$$\frac{d^2\sigma^\pm(\tau_i\mu_j)}{ds_1ds_2} = \pi\beta^2 \sum_{J\Lambda} |B^{\mu_j\tau_i}_{J\Lambda}|^2 [1 \pm \eta'(-1)^{\Lambda-\lambda}].$$

We thus obtain the well-known result that waves of different angular momentum and parity do not interfere in the Dalitz plot.

If the situation obtains in which the three-particle final state is the only inelastic channel (as approximately realized in low-energy πN scattering) the measurement of these partial-wave cross sections can provide a valuable constraint on the inelasticity parameters of elastic phase-shift analyses.

The expressions for polarizations and polarization tensors may now be written down. However, it is more instructive and useful if we specialize to the case

$$MB \to MMB, \tag{2.44}$$

where M is a pseudoscalar meson and B a spin-$\frac{1}{2}^+$ baryon. This is dealt with in the next section.

III. THE REACTION $MB \to MMB$

In this section we specialize all of our previous formulas to the process

$$MB \to MMB,$$

where M is a pseudoscalar (0^-) meson and B a spin-$\frac{1}{2}$ baryon with positive parity ($\frac{1}{2}^+$). We take the baryon to be particle 1 in the initial two-particle state, with helicity μ. The final baryon transversity is labeled τ, and it may be taken as any one of the three final-state particles.

A. Transition Amplitude and Differential Cross Section

The partial-wave decomposition of the transition amplitude is now written as

$$F_{\tau\mu}(s, s_i; R) = \langle \alpha, \beta, \gamma, s_i, \tau | T(s) | 0, 0, \mu\rangle = \sum_{J\Lambda} N_J D^{J*}_{\mu\Lambda} B^{\mu\tau}_{J\Lambda}(s, s_i) \tag{3.1}$$

and the differential cross section from a nucleon of helicity μ to a state with final baryon transversity τ is

$$\frac{d^5\sigma(\tau\mu)}{ds_1ds_2d^3R} = \beta^2 |F_{\tau\mu}(s, s_i; R)|^2 \equiv |f_{\tau\mu}(s, s_i; R)|^2, \tag{3.2}$$

where we define

$$f_{\tau\mu}(s, s_i; R) = \beta F_{\tau\mu}(s, s_i; R). \tag{3.3}$$

The unpolarized cross section is immediately obtained by averaging over initial helicities and summing over final transversities,

$$\frac{d^5\bar{\sigma}}{ds_1ds_2d^3R} = \frac{1}{2} \sum_{\mu\tau} |f_{\tau\mu}(s, s_i; R)|^2. \tag{3.4}$$

This may then be expressed in terms of the partial-wave amplitudes $B^{\mu\tau}_{J\Lambda}(s, s_i)$, and we discuss this in a later section.

B. Experiments with Polarized Particles

In this section we obtain expressions for all measurable quantities; the formulas look most transparent in terms of the transition amplitudes $f_{\tau\mu}(s, s_i; R)$, and we omit the arguments $(s, s_i; R)$.

For $MB \to MMB$, there are four possible types of experiments:

(a) Unpolarized differential cross section.

(b) Polarization "asymmetry" – i.e., cross section from polarized target.

(c) Measurement of final polarization.

(d) "Depolarization tensor" – measurement of final baryon polarization from polarized target.

It is also convenient in this section to discuss the equivalent experiments:

(e) In which the final baryon can undergo weak decay to an MB system whose angular distribution can define the density matrix of the decaying baryon.

These four experiments, (a), (b), (c), and (d), constitute a "complete set" of experiments for reconstruction of the scattering amplitude $f_{\tau\mu}(s, s_i; R)$ at a given s, and final-state configuration (R, s_i). The problem of direct reconstruction of the elastic T matrix for the nucleon-nucleon system has been extensively discussed.[8] Here we outline the extension of these ideas to reconstruction of the inelastic T-matrix elements for the reaction $MB \to MMB$. We make extensive use of the density matrix and our interpretation of this is described in the Appendix. The polarizations of the initial baryon (taken as particle 1 in construction of the initial two-body state) are referred to the fixed axes $Oxyz$, but polarizations of final particles are referred to the moving axes $OXYZ$.[9] With this convention, the formalism becomes as simple as for the nonrelativistic case. We briefly consider each type of experiment.

(a) Unpolarized cross section. The initial density matrix is

$$\rho^{(i)}=\tfrac{1}{2}1\,. \tag{3.5}$$

The differential cross section may be written

$$\frac{d^5\bar{\sigma}}{d^3R\,ds_1ds_2}\equiv I_0=\mathrm{Tr}(f\rho^{(i)}f^\dagger)=\tfrac{1}{2}\sum_{\tau\mu}|f_{\tau\mu}|^2\,. \tag{3.6}$$

(b) Polarized target. The initial density matrix is now

$$\rho^{(i)}=\tfrac{1}{2}(1+\vec{\mathrm{P}}_a\cdot\vec{\sigma}_a) \tag{3.7}$$

(labeling the initial baryon as particle a). The cross section now includes an "asymmetry" term $\vec{\mathrm{A}}$:

$$\left(\frac{d^5\sigma}{d^3R\,ds_1ds_2}\right)_{\rho^{(i)}}\equiv I_P=\mathrm{Tr}(f\rho^{(i)}f^\dagger)=I_0(1+\vec{\mathrm{P}}_a\cdot\vec{\mathrm{A}})\,, \tag{3.8}$$

where

$$I_0\vec{\mathrm{A}}=\tfrac{1}{2}\,\mathrm{Tr}(f\vec{\sigma}_af^\dagger)\,. \tag{3.9}$$

(c) Final polarization. The initial density matrix is Eq. (3.5), and a final density matrix may be defined:

$$I_0\rho^{(f)}=\tfrac{1}{2}ff^\dagger\,, \tag{3.10}$$

where

$$\mathrm{Tr}(\rho^{(f)})=1\,.$$

The final baryon polarization is given by

$$I_0\vec{\mathrm{P}}_c^{(0)}=\tfrac{1}{2}\,\mathrm{Tr}(ff^\dagger\vec{\sigma}_c)\,, \tag{3.11}$$

where the final baryon is particle c and the superscript (0) for the polarization signifies that the initial state was unpolarized.

(d) Depolarization tensor. The initial density matrix is given by Eq. (3.7), and the final-state density matrix is

$$I_P\rho^{(f)}=f\rho^{(i)}f^\dagger\,. \tag{3.12}$$

The final polarization is obtained from

$$I_PP_{k_c}=I_0(P_{k_c}^{(0)}+P_{i_a}D_{i_ak_c})\,, \tag{3.13}$$

where

$$I_0D_{i_ak_c}=\tfrac{1}{2}\,\mathrm{Tr}(f\sigma_{i_a}f^\dagger\sigma_{k_c})\,. \tag{3.14}$$

In these formulas, directions of polarizations for particle a are referred to $Oxyz$; for particle c to $OXYZ$. With this understanding, the 2×2 σ matrices have the standard representation of the Pauli spin matrices.

For completeness the results of all these experiments in terms of the amplitudes $f_{\tau\mu}$ are listed in Table I. In principle, the four complex amplitudes may be reconstructed up to an arbitrary over-all phase.

(e) Reactions in which the final baryon undergoes a weak decay to an MB system.

In this case the decay angular distribution of the final decay products of the baryon (e.g., a Λ) can serve to analyze the polarization of the baryon. If we consider the reaction occurring from an incident target proton described by the density matrix $\rho^{(i)}$, then the density matrix of the baryon in the three-particle state is given by

$$\rho^{(f)}=\frac{f\rho^{(i)}f^\dagger}{\mathrm{Tr}(f\rho^{(i)}f^\dagger)} \tag{3.15}$$

or $I\rho^{(f)}=f\rho^{(i)}f^\dagger$ and $I=\mathrm{Tr}(f\rho^{(i)}f^\dagger)$.

This $\rho^{(f)}$ then describes the baryon (Λ) in its rest system with the axes parallel to $OXYZ$. If C is the matrix which describes the decay of this baryon leading to another spin-$\frac{1}{2}$ particle (e.g., $\Lambda\to p\pi^-$), then the density matrix of this particle can be written as

$$\rho^d=\frac{C\rho^{(f)}C^\dagger}{\mathrm{Tr}(C\rho^{(f)}C^\dagger)}\,. \tag{3.16}$$

If the spin states of the final baryon are defined by

$$|\vec{\mathrm{P}}Ms\rangle=L(\vec{\mathrm{P}})|0Ms\rangle,$$

TABLE I. Expressions for all observable quantities in the reaction $MB\to MMB$. Amplitudes $f_{\tau\mu}(s,s_i;R)$ with $\tau,\mu=\pm\frac{1}{2}$ are written as $\tau,\mu=+,-$.

I_0	$=\frac{1}{2}(\lvert f_{++}\rvert^2+\lvert f_{+-}\rvert^2+\lvert f_{-+}\rvert^2+\lvert f_{--}\rvert^2)$	$=\rho_{11}^0+\rho_{22}^0$
I_0A_x	$=\mathrm{Re}[f_{++}f_{+-}^*]+\mathrm{Re}[f_{-+}f_{--}^*]$	$=\rho_{11}^x+\rho_{22}^x$
I_0A_y	$=\mathrm{Im}[f_{++}f_{+-}^*]+\mathrm{Im}[f_{-+}f_{--}^*]$	$=\rho_{11}^y+\rho_{22}^y$
I_0A_z	$=\frac{1}{2}(\lvert f_{++}\rvert^2+\lvert f_{-+}\rvert^2-\lvert f_{+-}\rvert^2-\lvert f_{--}\rvert^2)$	$=\rho_{11}^z+\rho_{22}^z$
$I_0P_X^{(0)}$	$=\mathrm{Re}[f_{++}f_{-+}^*]+\mathrm{Re}[f_{+-}f_{--}^*]$	$=2\,\mathrm{Re}\rho_{12}^0$
$I_0P_Y^{(0)}$	$=-\mathrm{Im}[f_{++}f_{-+}^*]-\mathrm{Im}[f_{+-}f_{--}^*]$	$=-2\,\mathrm{Im}\rho_{12}^0$
$I_0P_Z^{(0)}$	$=\frac{1}{2}(\lvert f_{++}\rvert^2+\lvert f_{+-}\rvert^2-\lvert f_{-+}\rvert^2-\lvert f_{--}\rvert^2)$	$=\rho_{11}^0-\rho_{22}^0$
I_0D_{xX}	$=\mathrm{Re}[f_{+-}f_{-+}^*]+\mathrm{Re}[f_{++}f_{--}^*]$	$=2\,\mathrm{Re}\rho_{12}^x$
I_0D_{xY}	$=-\mathrm{Im}[f_{+-}f_{-+}^*]-\mathrm{Im}[f_{++}f_{--}^*]$	$=-2\,\mathrm{Im}\rho_{12}^x$
I_0D_{xZ}	$=\mathrm{Re}[f_{++}f_{+-}^*]-\mathrm{Re}[f_{-+}f_{--}^*]$	$=\rho_{11}^x-\rho_{22}^x$
I_0D_{yX}	$=-\mathrm{Im}[f_{+-}f_{-+}^*]+\mathrm{Im}[f_{++}f_{--}^*]$	$=2\,\mathrm{Re}\rho_{12}^y$
I_0D_{yY}	$=\mathrm{Re}[f_{++}f_{--}^*]-\mathrm{Re}[f_{+-}f_{-+}^*]$	$=-2\,\mathrm{Im}\rho_{12}^y$
I_0D_{yZ}	$=\mathrm{Im}[f_{++}f_{+-}^*]-\mathrm{Im}[f_{-+}f_{--}^*]$	$=\rho_{11}^y-\rho_{22}^y$
I_0D_{zX}	$=\mathrm{Re}[f_{++}f_{-+}^*]-\mathrm{Re}[f_{+-}f_{--}^*]$	$=2\,\mathrm{Re}\rho_{12}^z$
I_0D_{zY}	$=-\mathrm{Im}[f_{++}f_{-+}^*]+\mathrm{Im}[f_{+-}f_{--}^*]$	$=-2\,\mathrm{Im}\rho_{12}^z$
I_0D_{zZ}	$=\frac{1}{2}(\lvert f_{++}\rvert^2+\lvert f_{--}\rvert^2-\lvert f_{-+}\rvert^2-\lvert f_{+-}\rvert^2)$	$=\rho_{11}^z-\rho_{22}^z$

where $\vec{P}$ is the momentum vector of the final baryon in the frame $OXYZ$ and $L(\vec{P})$ is the corresponding boost operator, then the rest-frame axes of the final baryon coincide with the axes $OXYZ$, and all observables are referred to these axes.

The angular distribution for the decay baryon is

$$I_d = \mathrm{Tr}(C\rho^{(f)}C^\dagger) . \qquad (3.17)$$

If the decay matrix C is expressed in terms of the usual parameters s and p, the amplitudes for the s- and p-wave decays, the angular distribution then becomes

$$I_d = \frac{1}{4\pi}[(\rho_{11}^{(f)} + \rho_{22}^{(f)}) + (\rho_{11}^{(f)} - \rho_{22}^{(f)})\alpha \cos\theta + 2\alpha \,\mathrm{Re}\rho_{12}^{(f)} \sin\theta \cos\phi - 2\alpha \,\mathrm{Im}\rho_{12}^{(f)} \sin\theta \sin\phi] , \qquad (3.18)$$

where θ and ϕ are the polar angles of the decay nucleon with respect to $OXYZ$ and

$$\alpha = \frac{\mathrm{Re}(s^*p)}{|s|^2 + |p|^2} .$$

For comparison with the formulas derived for experiments (a)–(d) it is convenient to introduce an unnormalized final density matrix for the remainder of this section:

$$\rho = f\rho^{(i)}f^\dagger .$$

The trace of ρ is now the appropriate differential cross section

$$\mathrm{Tr}\rho = I .$$

Thus

$$II_d = \frac{1}{4\pi}[(\rho_{11} + \rho_{22}) + (\rho_{11} - \rho_{22})\alpha \cos\theta + 2\alpha \,\mathrm{Re}\rho_{12} \sin\theta \cos\phi - 2\alpha \,\mathrm{Im}\rho_{12} \sin\theta \sin\phi] . \qquad (3.19)$$

Two types of experiments can be performed in this case: (i) from an unpolarized target and (ii) from a polarized target. We will see that these are equivalent (as they must be) to (a), (b), (c), and (d) discussed earlier.

(i) Unpolarized target.

$$\rho^{(i)} = \tfrac{1}{2}1$$

and we have that

$$\rho^0 = \tfrac{1}{2}ff^\dagger . \qquad (3.20)$$

(ii) Polarized target.

$$\rho^{(i)} = \tfrac{1}{2}(1 + \vec{P}_a \cdot \vec{\sigma}_a)$$

and

$$\rho = \tfrac{1}{2}f(1 + \vec{P}_a \cdot \vec{\sigma}_a)f^\dagger = \rho^0 + \vec{P}_a \cdot \vec{\rho}^a , \qquad (3.21)$$

where

$$\vec{\rho}^a = \tfrac{1}{2}f\vec{\sigma}_a f^\dagger . \qquad (3.22)$$

The values of ρ^0 and $\vec{\rho}^a$ in terms of the f's are also included in Table I and we see that their measurement is indeed equivalent to (a), (b), (c), and (d).

Before proceeding to the derivation of the explicit formulas in the case of the unpolarized cross sections, we would like to make some comments concerning the formulas we have derived.

(i) The partial-wave expansion of the $f_{\tau\mu}$ can be inserted into these formulas to give them in terms of the partial-wave amplitudes.

(ii) The parity of any given partial-wave amplitude $B_{J\Lambda}^{\mu\tau}(s, s_i)$ is $\eta'(-1)^{\Lambda-\tau}$. It is important to note in (a) and (b) that all observables are sums over τ, i.e.,

$$I_0 = \tfrac{1}{2}\sum_\tau (|f_{\tau+}|^2 + |f_{\tau-}|^2) ,$$

$$I_0 A_x = \sum_\tau \mathrm{Re}(f_{\tau+}f_{\tau-}^*) ,$$

$$I_0 A_y = \sum_\tau \mathrm{Im}(f_{\tau+}f_{\tau-}^*) ,$$

$$I_0 A_z = \sum_\tau (|f_{\tau+}|^2 - |f_{\tau-}|^2) .$$

This means that a complete parity ambiguity must exist in solutions derived only from experiments (a) and (b).

(iii) In these experiments the measurement of the final baryon polarization is not equivalent to the measurement of the asymmetry from a polarized target, as it is in elastic scattering. This is due to the fact that a simple relation does not exist between $f_{\tau\mu}(R)$ and $f_{-\tau\,-\mu}(R)$. Clearly measurements of the final polarization allow the resolution of the parity ambiguity noted above.

(iv) The final baryon polarization is defined with respect to axes in its rest frame which are defined with respect to the three-particle production plane. This is not directly measured in rescattering experiments. However, in the case in which the final baryon undergoes decay, the decay distribution measures this polarization. In fact, the measurement of reaction (1.2) from polarized targets offers the easiest way of determining a depolarization tensor.

C. Explicit Formulas for the Unpolarized Cross Section

From Table I we have that

$$\frac{d^5\sigma}{ds_1 ds_2 d^3R} = \tfrac{1}{2}\sum_\tau (|f_{\tau+}|^2 + |f_{\tau-}|^2) \qquad (3.23)$$

and we can substitute the partial-wave expansion (2.32) to give

$$\frac{d^5\sigma}{ds_1 ds_2 d^3R}=\tfrac{1}{2}\sum_\tau\sum_L\sum_{\substack{JJ'\\ \Lambda\Lambda'}}F(J,\Lambda,J',\Lambda',L)B^{\mu\tau *}_{J\Lambda}B^{\mu\tau}_{J'\Lambda'}Y^{\Lambda-\Lambda' *}_L(\beta,\gamma),\qquad \mu=\tfrac{1}{2} \tag{3.24}$$

where

$$F(J,\Lambda,J',\Lambda',L)=(-1)^{1/2-\Lambda}\frac{N_J N_{J'}}{N_L}\,C(J,\tfrac{1}{2},J',-\tfrac{1}{2}|L,0)C(J,\Lambda,J',-\Lambda\,|\,L,\Lambda-\Lambda')[1+(-1)^{\Lambda-\Lambda'-L}]. \tag{3.25}$$

Finally we can write

$$\frac{d^5\sigma}{ds_1 ds_2 d^3R}=\sum_{Lm}W^m_L Y^{m*}_L(\beta,\gamma), \tag{3.26}$$

where

$$W^m_L=\sum_\tau\sum_{\substack{JJ'\\ \Lambda\Lambda'}}F(J,\Lambda,J',\Lambda',L)B^{\mu\tau *}_{J\Lambda}B^{\mu\tau}_{J'\Lambda'}\delta_{m,\Lambda-\Lambda'}. \tag{3.27}$$

Several features of this differential cross section deserve comment.

(1) Formula (3.26) is analogous to the expansion of the elastic scattering differential cross section in terms of Legendre polynomials.

(2) If L is odd (even) then $\Lambda-\Lambda'$ must be odd (even).

(3) The expression for W^m_L contains a sum over τ and thus, as we stressed earlier, a parity ambiguity necessarily exists. However, $B^{\mu\tau}_{J\Lambda}$ represents a transition to a state of parity $\eta'(-1)^{\Lambda-\tau}$ and hence if terms with $\Lambda-\Lambda'$ odd (even) appear we have waves of opposite (same) parity. Coupled with remark (2) this means that waves of opposite (same) parity lead to terms with L odd (even).

(4) If $J_{\max}$ is the maximum angular momentum contributing in the reaction, and only one parity is present corresponding to this value, then $L_{\max}=2J_{\max}-1$ (for J half-integral).

(5) Ignoring the sum over τ there are still insufficient measurable quantities to allow determination of the $B^{\mu\tau}_{J\Lambda}$. Thus some form of polarization data is necessary to determine the amplitudes and as we remarked earlier, measurements from polarized targets will still not resolve the parity ambiguity.

D. Qualitative Deductions from Present Data

At this point we can summarize the qualitative deductions made from present data[10-13] using this formalism. As we have stressed, in the absence of polarization measurements, a quantitative analysis is not feasible and hence a close scrutiny of the predictions of elastic phase-shift analyses for inelastic cross sections is impossible. However, one can make valuable qualitative statements using points (3) and (4) of Sec. III C. We consider separately π^+p and π^-p collisions.

π^+p. (a) The absence of moments W^m_L with $L>6$ (Refs. 10, 13) for $E_{\text{c.m.}}<1.70$ GeV indicates that waves with $j>\frac{5}{2}$ are not important. Furthermore the small size of the $L=5$ moments demonstrate that large contributions from $j=\frac{5}{2}$ waves of opposite parities are unlikely.

(b) The large sizes of W^1_1 moments[10] in the region $E_{\text{c.m.}}\sim 1.45$ GeV are due to the presence of waves of opposite parity. At present no elastic phase-shift analysis predicts large p-wave inelasticities at these energies.

π^-p. (a) For $E_{\text{c.m.}}<1.535$ GeV, moments W^m_L with $L\geq 4$ are consistent with zero.[11-13] Thus inelasticity in waves with $j\geq\frac{5}{2}$ is not required but the presence of $j=\frac{3}{2}$ waves of both parities is necessary.

(b) In the region of $E_{\text{c.m.}}\sim 1.7$ GeV the presence of the inelastic decays of the F_{15} and D_{15} resonances is signalled by the $L=4$ and 5 moments.[12] However, the values, although significant, are not large, and we must conclude that strong cancellations are occurring between the two waves in this case.

(c) For $E_{\text{c.m.}}<2.0$ GeV the moments W^m_L are consistent with zero for $L\geq 6$. We conclude that there is little evidence for strong contributions of $j=\frac{7}{2}$ states to the final states $(\pi\pi N)^0$. [The $F_{37}(1920)$ coupling to the π^-p channel is suppressed by isospin Clebsch-Gordan coefficients.]

IV. SUMMARY AND DISCUSSION

In this section we summarize the main features of this analysis and discuss its advantages and limitations.

Methods of analyzing inelastic processes (other than two-body reactions) are clearly needed to supplement the understanding of elastic scattering. The method described in detail in this paper specifies a three-body final state by means of two Dalitz-plot variables, and three Euler angles which describe the orientation of the final-state c.m. momentum triangle with respect to a fixed set of axes. The s-channel partial-wave decomposition is then given in terms of three discrete angular momentum variables, J, M, and Λ, which replace the three continuous angular variables. The quantum numbers J, M, and Λ are easily interpreted as being the total angular momentum and its projections onto the space-fixed z axis and body-fixed Z axis, respectively, in the same manner as for a symmetric top. Our partial-wave amplitudes $B_{J\Lambda}^{\tau\mu}(s, s_i)$ then contain all possible information about the reaction and are model-independent parameters. However, the number of amplitudes $B_{J\Lambda}^{\mu\tau}(s, s_i)$ necessary to describe a $2 \to 3$ reaction is large and growing rapidly with energy: For $MB \to MMB$, if J_M is the maximum angular momentum present, then there are $\frac{1}{2}(2J_M+1)(2J_M+3)$ amplitudes compared to $(2J_M+1)$ for elastic scattering. Furthermore these partial-wave amplitudes are functions of the invariant masses s_i and thus should be determined at every point in the Dalitz plot. Clearly the most one can hope for is a measurement of these quantities over small regions of the Dalitz plot, which would require a large amount of data in each region. Optimistically one would hope to see variations of the partial-wave amplitudes as a function of the Dalitz-plot variables indicating the association of J^P state with a particular decay channel. (However, to extract couplings to these decay channels requires a detailed model of the variation of the $B_{J\Lambda}^{\mu\tau}$ with the Dalitz-plot variables.) A unique determination of the $B_{J\Lambda}^{\mu\tau}$ (up to an over-all phase) requires, as we have seen explicitly for the parity ambiguity, data on polarization experiments as well as the unpolarized differential cross section.

There are, however, useful features. The introduction of the partial-wave expansion into the formulas of Table I (a lengthy and tedious task) means that the moments of the functions $D^J(R)$ can serve as a permanent model-independent record of the data, containing all the correlations between production angles and position in the Dalitz plot. These moments therefore contain much more information than, for example, a Dalitz-plot distribution averaged over all production angles. Moreover, these experimental parameters are free from the approximations and assumptions of parameters derived in the usual isobar-model analysis.[14]

Qualitatively, the values of the moments integrated over the Dalitz plot can be useful guides as to the partial waves present,[10-13] e.g., the observation of nonzero moments with odd L in unpolarized cross sections is a clear indication of the presence of waves of opposite parity, and the maximum value of L can limit the value of the total angular momentum considered in any analysis.

In the reaction $\bar{K}p \to \pi\pi\Lambda$, for example, the measurement of the Λ decay (together with the use of polarized targets) allows the complete set of experiments to be performed with relative ease, and allows the partial-wave amplitudes and transition amplitudes to be reconstructed.

It is also interesting to note that the formalism is applicable to zero-mass particles and the versatility of photon beams should prove very useful in the analysis of photoproduction of two mesons.

Conventionally reactions of the type (1.2) have been analyzed using the isobar model or its modifications. The reason is clear: In general the number of amplitudes necessary to describe the process is very large and the isobar model reduces this number dramatically by two main features:

(1) It limits the partial-wave amplitudes allowed.

(2) The variation of the partial-wave amplitudes as a function of Dalitz-plot variables is specified within a very definite model.

However, it must be stressed that the isobar model contains several approximations and *ad hoc* assumptions[14] and these are weakest in regions of resonance overlap. The parameters derived using the BLT formalism are free from such arbitrariness and may be critically compared with the predictions of the various models.

It is worth pointing out that the connection of the BLT three-particle states to the three-particle states constructed by Wick[15] is of course explicit, but unfortunately a little involved. This latter method constructs the three-particle states by first coupling particles 1 and 2 in their c.m. system and then the (12) subsystem to particle 3 in the over-all c.m. system. Clearly the restriction to a quasi-two-body final state is very simple in this formalism. These states are connected to our states by Wigner rotations arising from transforming the states from the (12) c.m. system to the over-all c.m. system (apart from a simple rotation relating helicity states to our transversity states: Appendix A of Morgan[14] contains further details).

In conclusion we have developed in detail a formalism which may be used to analyze reactions (1.1) and have given explicitly the restrictions implied by parity conservation and the presence of two identical particles in the final state. A careful discussion of polarization experiments is given and the question of a "complete set" of experiments considered. We have shown that a parity ambiguity exists and indicated its resolution in terms of measurement of final particle polarizations.

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APPENDIX: INTERPRETATION OF THE DENSITY MATRIX

In order to describe polarization properties of the particles we use the density matrix. For a particle at rest this may be written as

$$\rho(0)=\sum_{MM'} |M'\rangle \rho_{M'M}\langle M|, \quad (A1)$$

where M' and M are the z components of spin in the rest frame of the particle. If nonzero momentum states are constructed according to some definite prescription,

$$|\vec{P}M\rangle = H(\vec{P})|M\rangle, \quad (A2)$$

then the density matrix in this basis is

$$\rho(\vec{P})=H(\vec{P})\rho(0)H^{-1}(\vec{P}) = \sum_{MM'} |\vec{P}M'\rangle \rho_{M'M}\langle \vec{P}M|, \quad (A3)$$

where the elements $\rho_{MM'}$ are clearly unchanged. However, as we have stressed, Eq. (A2) may be regarded as defining the relative orientation of the axes in the rest frame with respect to the system in which the particle has momentum $\vec{P}$. The elements $\rho_{MM'}$ refer to quantization along these rest-frame axes. Therefore, the density matrix for a moving particle described by the states (A2) may be written exactly as for a particle at rest, providing the spin operators and polarizations are interpreted as referring to the rest-frame axes.[16] Thus polarization formulas for relativistic particles have exactly the nonrelativistic form except that the directions must be referred to rest-frame directions for each particle. The use of these rest-frame axes is useful in discussions of polarization experiments, where they provide an easy way to avoid the technical problems associated with the polarization of relativistic particles. This formalism is immediately generalized to multiparticle states and we use it extensively in discussing reactions of type (1.1) and (1.2).

For a spin-$\frac{1}{2}$ particle at rest the general spin state is described by a density matrix of the form

$$\rho(0)=\tfrac{1}{2}(1+\vec{P}\cdot\vec{\sigma}), \quad (A4)$$

where $\vec{P}$ is the polarization and $\mathrm{Tr}(\rho)=1$. As discussed above, a similar form may be used for the particle in motion providing we interpret the spin operators $\vec{\sigma}$ and polarization $\vec{P}$ as referring to the appropriate rest-frame axes.

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