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Fourth-Order Contribution to Z_3^{-1} in Scalar Electrodynamics

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Two previous calculations of Z_3^{-1} to fourth order in scalar electrodynamics were in disagreement. We show that, with slight modifications to one of these calculations, they are brought into agreement. A simpler calculation, in which the Ward identities are explicit, is presented.

I. INTRODUCTION

The work of Johnson, Baker, and Willey[1] has revived interest in the calculation of the divergent part of Z_3^{-1} in both spinor and scalar electrodynamics to determine whether it might vanish for particular value(s) of the bare charge e_0.

The fourth-order contribution to Z_3^{-1} in scalar electrodynamics has been calculated independently by Kim and Hagen[2] and by Białynicka-Birula.[3] Their results, however, do not agree. We indicate that the calculation of Kim and Hagen performed using a different choice of integration variables yields the Białynicka-Birula result. We also indicate how the calculation can be simplified using the method of Rosner,[4] and the ambiguities in the definition of Z_1 and Z_2 can be completely avoided.

In Sec. II we indicate a choice of variables for the Kim-Hagen calculation which avoids the Kim-Hagen ambiguities[5] and yields the Białynicka-Birula result. Section III is an exposé of our simpler calculation which makes maximum use of the Ward identities.

II. THE KIM-HAGEN CALCULATION

Following Kim and Hagen[2] and Fry[6] we work in the Yennie gauge[7] in which the scalar propagator Δ_F, 3-vertex Γ_μ, and 4-vertex are finite, after mass renormalization. To zeroth order in this gauge, the photon propagator is

$$D^{(0)}_{F\mu\nu} = -\frac{i(g_{\mu\nu} + 2q_\mu q_\nu/q^2)}{q^2} . \tag{2.1}$$

The integrals defining Δ_F and Γ_μ to second order are intrinsically linearly divergent, being rendered finite only by symmetric integration. Thus, since their value depends on the choice of origin, they depend on the choice of integration variables. Only gauge-invariant quantities such as Z_3^{-1} are unambiguously defined. However, once we choose a definition of either Γ or Δ_F, the other is uniquely defined by requiring the Ward identity

$$\Gamma_\mu(p,p) = \frac{\partial}{\partial p_\mu} \Delta_F^{-1}(p) \tag{2.2}$$

or its generalization

$$(p-p')\cdot\Gamma(p,p') = \Delta_F^{-1}(p) - \Delta_F^{-1}(p') \tag{2.3}$$

to hold.

The second-order calculation of Δ_F requires evaluation of the graphs of Fig. 1. The Kim-Hagen calculation of Δ_F with integration variable k, the photon momentum, gives

$$\Delta_F^{-1}(p) = \left[1 - \frac{9}{4}\left(\frac{\alpha_0}{2\pi}\right)\right](p^2 - \mu^2) . \tag{2.4}$$

In calculating the graphs for Z_3^{-1} which involve the second-order part of Γ_μ we change variables such that the photon momentum is no longer k but rather $p-k$. This circumvents the Kim-Hagen error.[5] Since the contributions from each of these sets of graphs is the same as the original Kim-Hagen result we will not quote this result here. Such a change, since it redefines Γ_μ, is clearly inadmissible unless one makes the corresponding change of photon momentum in the definition of

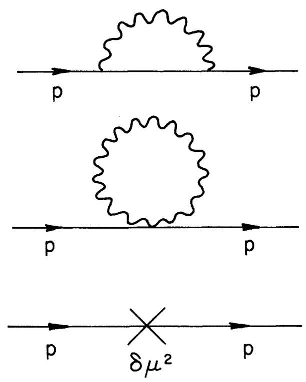

FIG. 1. Second-order contribution to Δ_F.

Δ_F, as required by the Ward identity.

Evaluating Δ_F with photon momentum $p-k$ gives

$$\Delta_F^{-1}(p)=\left[1-\frac{3}{2}\left(\frac{\alpha_0}{2\pi}\right)\right](p^2-\mu^2) . \tag{2.5}$$

This gives for the fourth-order contribution to the divergent part of Z_3^{-1}

$$(Z_3^{-1})^{(4)}_{\text{divergent}}=\left(\frac{\alpha_0}{2\pi}\right)^2\ln\left(\frac{\Lambda^2}{\mu^2}\right) , \tag{2.6}$$

where Λ is the momentum cutoff, which agrees with the result of Białynicka-Birula.

III. THE ROSNER METHOD

In calculating $(Z_3^{-1})^{(4)}$ we use the method used by Rosner[4] for quantum electrodynamics. As in quantum electrodynamics, subtractions to ensure manifest gauge invariance do not contribute to the divergent part of Z_3^{-1}. We note also that those graphs where the emitted photon emerges from the same vertex as the incident photon do not contribute and are hence neglected.[2]

Following Rosner, since the vacuum polarization tensor can be written

$$\Pi_{\mu\nu}(q)=(q^2g_{\mu\nu}-q_\mu q_\nu)\rho(q^2) \tag{3.1}$$

with $\rho(0)$ defined, then

$$Z_3^{-1}=1+\rho(0) , \tag{3.2}$$

and thus

$$Z_3^{-1}-1=\frac{1}{24}\frac{\partial^2}{\partial q_\alpha\partial q^\alpha}\Pi^\mu{}_\mu(q^2)\bigg|_{q=0} . \tag{3.3}$$

We have seen that to second order

$$\Delta_F^{-1}(p)=(p^2-\mu^2)(1+B^{(2)}) , \tag{3.4}$$

where $B^{(2)}$, which is second order in e_0, is a constant [Eqs. (2.4) and (2.5)]. Hence, by the Ward identity [Eq. (2.2)],

$$\Gamma_\mu(p,p)=2p_\mu(1+B^{(2)}) . \tag{3.5}$$

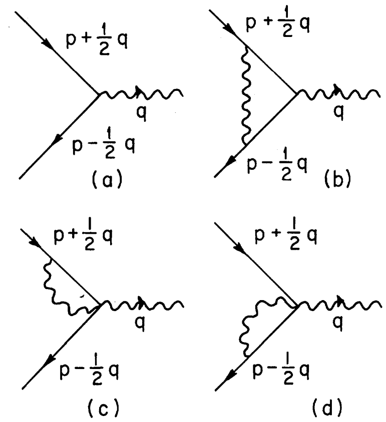

FIG. 2. Graphs contributing to Γ_μ up to second order. (a) $2p$. (b) $K^{(2)}G^{(0)}2p$. (c) and (d) $V^{(2)}$.

Symbolically we write to second order

$$\Gamma=2p+K^{(2)}G^{(0)}2p+V^{(2)} , \tag{3.6}$$

where $2p$ is the contribution of Fig. 2(a), $K^{(2)}G^{(0)}2p$ that of Fig. 2(b), and $V^{(2)}$ that of Figs. 2(c) and 2(d). K is the Bethe-Salpeter kernel, which to second order is simply $-e_0^2$ times the zeroth-order photon propagator multiplied by the two zeroth-order vertices at which it interacts. G is the product of the two scalar propagators Δ_F, while the superscripts indicate the order in e_0 of the term considered. Thus, using this symbolic notation,

$$\Pi^{(4)}=ie_0^2[(2pG\Gamma)^{(2)}+(VG2p)^{(2)}+ie_0^2(2gG2gD)^{(0)}] . \tag{3.7}$$

The three terms are shown in Figs. 3(a)–3(c), respectively. Hence

$$\Pi^{(4)}=ie_0^2[2pG^{(2)}2p+2pG^{(0)}V^{(2)}+V^{(2)}G^{(0)}2p+2pG^{(0)}K^{(2)}G^{(0)}2p+ie_0^2 2gG^{(0)}2gD^{(0)}] . \tag{3.8}$$

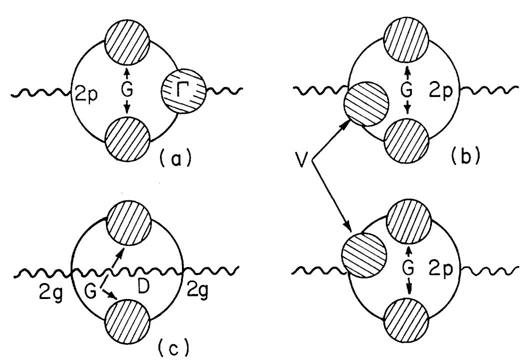

FIG. 3. Graphs contributing to Π in fourth order. (a) $ie_0^2(2pG\Gamma)^{(2)}$. (b) $ie_0^2(VG2p)^{(2)}$. (c) $-e_0^4(2gG2gD)^{(0)}$.

We define

$$\frac{\partial A}{\partial q_\alpha}\bigg|_{q=0} \equiv A' , \quad \frac{\partial^2 A}{\partial q_\alpha \partial q^\alpha}\bigg|_{q=0} \equiv A'' \tag{3.9}$$

and note that

$$G = G^{(0)}(1-2B^{(2)}) , \tag{3.10a}$$

$$\Gamma = 2p(1+B^{(2)}) , \tag{3.10b}$$

$$G^{(0)\prime} = G^{(2)\prime} = 0 , \tag{3.10c}$$

$$\Gamma^{(2)} = K^{(2)}G^{(0)}2p + V^{(2)}$$
$$= 2pG^{(0)}K^{(2)} + V^{(2)}$$
$$= 2pB^{(2)} , \tag{3.10d}$$

$$G^{(2)} = -2G^{(0)}B^{(2)} . \tag{3.10e}$$

Equation (3.10a) is a consequence of (3.4); (3.10b) is just (3.5), since after differentiation we put $q=0$; (3.10c) is a result of choosing momenta such that G is an even function of q; (3.10d) is a consequence of (3.6) and (3.10b); and (3.10e) comes from (3.10a). Hence Π'' reduces to

$$\Pi'' = ie_0^2(2pG^{(0)}V^{(2)\prime\prime} + V^{(2)\prime\prime}G^{(0)}2p + 2pG^{(0)}K^{(2)\prime\prime}G^{(0)}2p + ie_0^2 2gG^{(0)\prime\prime}2gD^{(0)}) , \tag{3.11}$$

which is independent of $B^{(2)}$ (and hence of $Z_1 = Z_2$) to this order, as in the case of quantum electrodynamics.

First we ignore all terms containing $\Delta^{(0)\prime\prime}$. Then explicit evaluation yields

$$V_\mu^{(2)\prime\prime} = -e_0^2 \int \frac{d^4k}{(2\pi)^4} 2g_{\alpha\sigma} 2(p-k)^\alpha [i\Delta_F^{(0)}(p-k)]^2 2g_{\lambda\mu} iD_F^{(0)\sigma\lambda}(k), \tag{3.12}$$

with $V^{(2)}$ given in Figs. 2(c) and 2(d). Therefore

$$ie_0^2(2pG^{(0)}V^{(2)\prime\prime} + V^{(2)\prime\prime}G^{(0)}2p) = -32e_0^4 \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} [\Delta_F^{(0)}(p-k)]^2 [\Delta_F^{(0)}(p)]^2 (p-k)_\sigma D_F^{(0)\sigma\lambda}(k) p_\lambda , \tag{3.13}$$

which we note is just, up to a factor, the graph of Fig. 4(a).

Now

$$K^{(2)} = -e_0^2 (2p+q+k)^\sigma i D_{\sigma\lambda}^{(0)}(k)(2p-q+k)^\lambda . \tag{3.14}$$

Thus

$$K^{(2)\prime\prime} = 2ie_0^2 g_{\alpha\sigma} D_F^{(0)\sigma}{}_\lambda(k) g^{\lambda\alpha} . \tag{3.15}$$

Since

$$G^{(0)} = \Delta_F^{(0)}(p+\tfrac{1}{2}q-k)\Delta_F^{(0)}(p-\tfrac{1}{2}q) , \tag{3.16}$$

then

$$G^{(0)\prime\prime} = -\tfrac{1}{2}[2p_\alpha 2(p-k)^\alpha][\Delta_F^{(0)}(p)]^2[\Delta_F^{(0)}(p-k)]^2 . \tag{3.17}$$

Hence by explicit substitution

$$ie_0^2(2pG^{(0)}K^{(2)\prime\prime}G^{(0)}2p + ie_0^2 2gG^{(0)\prime\prime}2gD^{(0)}) = 16e_0^4 \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} p\cdot(p-k)[\Delta_F^{(0)}(p)]^2[\Delta_F^{(0)}(p-k)]^2 D_F^{(0)\mu}{}_\mu(k) , \tag{3.18}$$

which is seen to correspond to Fig. 4(b).

The divergent parts of these integrals can be calculated, either by using a change of variables from p and k to p and $q=p-k$, valid since they are no longer intrinsically linearly divergent, or alternatively by taking the scalar mass to be zero, and using an infrared cutoff. This latter method is valid, since their infrared divergence is only logarithmic. We see that the ambiguities associated with the evaluation of Δ_F and Γ have disappeared because the Ward identities are now explicit. The remaining integrals are evaluated by Wick rotating the integration contour, writing the Euclidean integral in 4-dimensional spherical polar coordinates, and using the expansion of $(p\cdot q)^m/[(p-q)^2]^n$ in terms of Tchebycheff polynomials of the cosine of the angle between p and q.[2,4,8]

The contribution to Π'' is

$$24\left(\frac{\alpha_0}{2\pi}\right)^2 \ln\left(\frac{\Lambda^2}{\mu^2}\right) . \tag{3.19}$$

It remains to consider those terms containing Δ''. We make use of the zero-mass limit in which

$$\frac{\partial^2}{\partial p_\alpha \partial p^\alpha}\Delta_F^{(0)}(p) = \frac{\partial^2}{\partial p_\alpha \partial p^\alpha}\frac{1}{p^2+i\epsilon}$$
$$= -(2\pi)^2 i\delta^4(p). \tag{3.20}$$

These terms are easily evaluated, and are seen to cancel.

Hence

$$\Pi_{\mu}^{(4)\mu\nu} = 24\left(\frac{\alpha_0}{2\pi}\right)^2 \ln\left(\frac{\Lambda^2}{\mu^2}\right) \qquad (3.21)$$

and

$$(Z_3^{-1})^{(4)}_{\text{divergent}} = \left(\frac{\alpha_0}{2\pi}\right)^2 \ln\left(\frac{\Lambda^2}{\mu^2}\right), \qquad (3.22)$$

in agreement with Sec. II, Eq. (2.6).

This approach, as well as being simpler, completely avoids the ambiguities involved in defining Δ_F and Γ_μ by enforcing the Ward identities from the beginning.

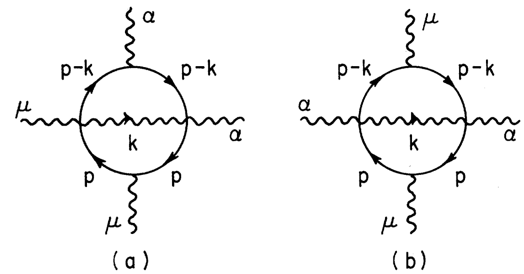

FIG. 4. (a) Graph corresponding to $ie_0^2(2pG^{(0)}V^{(2)\prime\prime}+V^{(2)\prime\prime}G^{(0)}2p)$.
(b) Graph corresponding to $ie_0^2(2pG^{(0)}K^{(2)\prime\prime}G^{(0)}2p+ie_0^2 2gG^{(0)\prime\prime}2gD^{(0)})$.

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[5]Since submitting this article we have received a preprint by Hagen and Kim [C. R. Hagen and I.-J. Kim, following paper, Phys. Rev. D 6, 1185 (1972)], indicating the precise nature of their initial error to be a result of an illegal translation of the two electron momenta p and $p-k$ to $p-k$ and p in evaluating certain graphs. We note that, by choosing the photon momentum to be $p-k$ (in Sec. II) where electron momenta are p and k, the problem is avoided, since the interchange of p and k is a legal operation. This choice of momenta also justified the formal manipulations of Sec. III.

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