distance singularity for the vacuum matrix element of $\bar{\psi}\psi$, where ψ is the proton field of an "equivalent" elementary-proton theory. This need not contradict canonical singularities for certain other combinations, particularly if no reference is made to an "equivalent" elementary-particle theory.

[13]S. D. Drell and T. D. Lee, Phys. Rev. D 5, 1738 (1972).

PHYSICAL REVIEW D VOLUME 6, NUMBER 4 15 AUGUST 1972

Gribov's Triple-Pomeranchukon Vertex and Inclusive Reactions

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(Received 22 February 1972)

We show that in the nonplanar model of Gordon and of Mueller and Trueman the triple-Pomeranchukon vertices of Gribov's Reggeon calculus and inclusive reactions are identical in the forward direction. We therefore connect the zero of the forward vertex with the zero of Gribov's vertex, and the collision of the Pomeranchukon poles and cuts. We use a modification of the Reggeon unitary amplitudes of Gribov, Pomeranchuk, and Ter-Martirosyan to discuss inclusive reactions for $t \neq 0$. Like Goddard and White, we find that the inclusive vertex vanishes linearly in t. However, we disagree with the argument of Goddard and White.

I. INTRODUCTION

Some time ago it was shown very generally that the triple-Pomeranchukon vertex involved in inclusive reactions must vanish in the forward direction.[1] To see this, consider the inclusive reaction $A+B \to C+X$ in the part of phase space defined by the relations

$$(p_A+p_B)^2 = s \gg m^2 ,$$
$$(p_A+p_B-p_C)^2 = M^2 \gg m^2 , \qquad (1.1)$$
$$s \gg M^2 .$$

m^2 is a typical hadron mass. In this region, the leading contribution to the cross section is provided by the triple-Pomeranchukon coupling shown in Fig. 1, after use of the optical theorem.[2,3] (We assume that $\bar{A}C$ has vacuum quantum numbers.) It follows that just this contribution, integrated over its corner of phase space, already grows more rapidly with s than the total cross section unless either $\alpha(0)<1$ or the triple-Pomeranchukon coupling vanishes at $t=0$.

Several suggestions have been put forward to deal with this constraint. One of them is $\alpha(0)<1$, so that the problem never arises. If one rejects this possibility as being inconsistent with experimental data, the next simplest suggestion is that the amplitude for Fig. 1 has the form

$$F(s,M^2,t) \sim s^{2\alpha(t)}(-M^2-i\epsilon)^{\alpha(0)-2\alpha(t)}\beta(t) \qquad (1.2)$$

in the region (1.1). The cross section is proportional to the discontinuity of F in M^2,[2,3] so a factor $\sin\pi[2\alpha(t)-\alpha(0)]$ produces the required zero at $t=0$. That is, no special dynamical mechanism is required because of the presence of a kinematic zero introduced by taking the absorptive part. Chang, Gordon, Low, and Treiman showed that the form (1.2) is realized when the Pomeranchukons are joined by the planar connection of Fig. 2.[4]

Recently Gordon[5] and Mueller and Trueman[6] have shown that the crucial phase given in Eq. (1.2) is changed when the coupling has the nonplanar topology of Fig. 3. The modulus of F is unchanged, but there are both left and right cuts in M^2:

$$F(s,M^2,t) \sim s^{2\alpha(t)}(M^2)^{\alpha(0)/2-\alpha(t)}(-M^2-i\epsilon)^{\alpha(0)/2-\alpha(t)}\beta(t) . \qquad (1.3)$$

Upon taking the discontinuity, the sine factor is $\sin\frac{1}{2}\pi[2\alpha(t)-\alpha(0)]$, which does not vanish at $t=0$. In terms of the partial-wave amplitude in the $B\bar{B}$ channel, this finite result is due to a wrong-signature nonsense pole canceling the zero in the sine. More specifically, each produced Pomeranchukon in the $B\bar{B}$ channel has complex helicity $\alpha(t)$, and for nonplanar couplings there are poles at the nonsense wrong-signature points $\alpha(0)=2\alpha(t)-n$, $n=1,3,5,\ldots$. In Sec. III we will employ the analysis of Gribov, Pomeranchuk, and Ter-Martirosyan to explore this view of the inclusive reaction problem.[7]

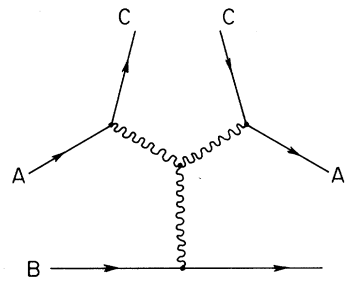

FIG. 1. The triple-Pomeranchukon coupling for inclusive reactions.

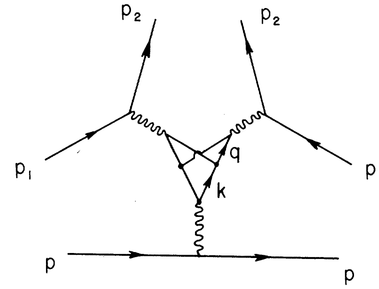

FIG. 3. The nonplanar coupling of Gordon and Mueller and Trueman.

Since the vanishing of the triple-Pomeranchukon coupling cannot be understood "kinematically," we must turn to dynamics. One possibility has been mentioned by Chang *et al.*,[4] namely, that the zero in β is related to the zero in the triple-Pomeranchukon vertex that appears in Gribov's Reggeon diagrams.[8] In Gribov's theory, Reggeons appear as elements in perturbation graphs, much as ordinary particles do in Feynman diagrams. Gribov and his collaborators have argued in a series of papers that there is no consistent solution of the Schwinger-Dyson equations unless the triple-Pomeranchukon vertex vanishes in the forward direction.[9] In this paper we will call the triple-Pomeranchukon coupling in Gribov's graphs "Gribov's vertex." It is discussed in Ref. 8, and includes the Pomeranchukon rescattering graphs depicted in Eq. (49b) of that paper. The rescattering graphs are alleged to renormalize Gribov's vertex to zero when all the Reggeons are on the spin shell and have momentum zero.[9]

Linking the two triple-Pomeranchukon vertices provides a dynamical mechanism for the vanishing of the triple-Pomeranchukon coupling in inclusive reactions at $t=0$, namely, pole-cut collisions. However, this linkage is only possible at $t=0$ because planar couplings like that of Fig. 2 do not contribute to Gribov's vertex. In the terminology of Refs. 7 and 10, the planar coupling does not contribute to the residues of nonsense poles. Therefore, the Gribov and inclusive vertices can be identified only when one is on the fixed pole, as at $t=0$.

In this paper we shall study the identity of the two vertices at $t=0$, and the behavior of the inclusive vertex near $t=0$. In Sec. II we use the model of Gordon[5] and Mueller and Trueman[6] to calculate Gribov's vertex at $t=0$. Comparing with the results of these authors, we find that the Gribov and inclusive vertices are equal, provided $\alpha(0)=1$. The condition $\alpha(0)=1$ puts the nonsense pole at $t=0$. This result is not surprising, and can be obtained in many ways including the formalism we use in Sec. III. However, it does complete the line of investigation begun with the perturbative calculation of the inclusive vertex in Ref. 4. There it was shown that the inclusive vertex vanishes at $t=0$ when one uses the coupling of Fig. 2. Then in Refs. 5 and 6 it was pointed out that a finite coupling ensues when one uses the nonplanar coupling of Fig. 3 instead. Our further result is that this finite coupling is equal to Gribov's triple-Pomeranchukon vertex. One is therefore entitled to apply the arguments of Refs. 9 and 10 to the inclusive vertex at $t=0$, and to conclude that it vanishes there.

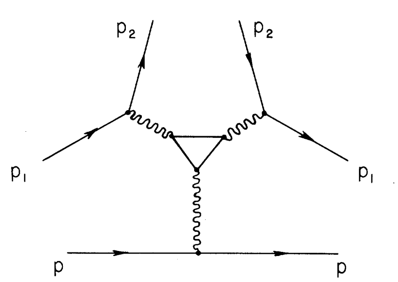

FIG. 2. A planar model of the triple-Pomeranchukon coupling.

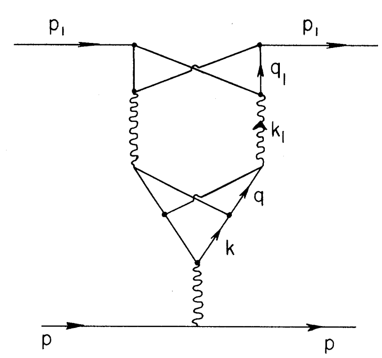

FIG. 4. A Pomeranchukon pole plus cut contribution to forward elastic scattering. Gribov's triple-Pomeranchukon vertex is in the center.

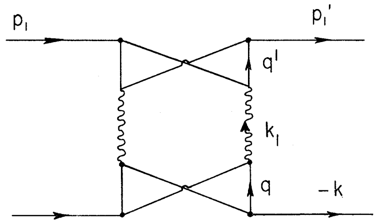

FIG. 5. The upper (cut) portion of Fig. 4.

In Sec. III we examine the behavior of the inclusive vertex near $t=0$ by using a formalism that includes the effect of pole-cut collisions. We modify the Reggeon unitary amplitudes of Refs. 7 and 10 in a manner that permits us to get off the nonsense pole and examine the inclusive vertex near $t=0$. Our procedure is directly comparable to that of Goddard and White,[11] although it is more general because we are not restricted to the nonsense pole. We agree with Ref. 11 that the inclusive vertex vanishes linearly at $t=0$, but we disagree with the arguments put forth there.

II. THE TWO TRIPLE-POMERANCHUKON COUPLINGS IN A MODEL

Gordon[5] and Mueller and Trueman[6] have calculated the discontinuity in M^2 of the amplitude shown in Fig. 3. In this section we consider the contribution of Fig. 4 to the forward reaction $p+p_1 \to p+p_1$ for $s=(p+p_1)^2 \gg m^2$. By casting the amplitude into the form given by Gribov's calculus, we can read off the contribution to Gribov's triple-Pomeranchukon vertex made by the nonplanar connection in the center of Fig. 4. This same connection appears in the inclusive process of Fig. 3, so a comparison can be made in the model. We follow the plan of Ref. 8, which outlines most of our results.

We first consider the upper piece of Fig. 4, as shown in Fig. 5. This diagram has been considered in Ref. 8, but with the lower legs on the mass shell. In writing the amplitude we use Sudakov variables,[12] and for the moment we generalize away from the forward direction:

$$
\begin{aligned}
q &= \alpha_1 p_1' + \beta_1 k + q_\perp ,\\
k_1 &= \alpha_2 p_1' + \beta_2 k + k_{1\perp} ,\\
q' &= \alpha_1' p_1' + \beta_1' k + q_\perp' ,\\
\Delta &= p_1' - p_1 \sim \frac{1}{s}(k'^2 - k^2 - \Delta^2) p_1' - \frac{\Delta^2}{s} k + \Delta_\perp ,\\
k'^2 &= (p_1' - p_1 - k)^2 , \quad s = (p_1' - k)^2 .
\end{aligned}
\tag{2.1}
$$

Here the vectors $q_\perp$, etc., are two-dimensional spacelike vectors orthogonal to p_1' and k, and we have retained the leading terms in s^{-1} in writing Δ. For large s the momentum transfer squared is $\Delta^2 \sim \Delta_\perp{}^2$. The propagators for the lower half of Fig. 5 are

$$
\begin{aligned}
& q^2 - m^2 + i\epsilon = (\alpha_1 + \beta_1)(\alpha_1 m^2 + \beta_1 k^2) - \alpha_1\beta_1 s + q_\perp{}^2 - m^2 + i\epsilon ,\\
& (q-k_1)^2 - m^2 + i\epsilon = (\alpha_1 - \alpha_2 + \beta_1 - \beta_2)(\alpha_1 m^2 - \alpha_2 m^2 + \beta_1 k^2 - \beta_2 k^2) - (\alpha_1 - \alpha_2)(\beta_1 - \beta_2) s + (q_\perp - k_{1\perp})^2 - m^2 + i\epsilon ,\\
& (q-k)^2 - m^2 + i\epsilon = (\alpha_1 + \beta_1 - 1)(\alpha_1 m^2 + \beta_1 k^2 - k^2) - \alpha_1(\beta_1 - 1) s + q_\perp{}^2 - m^2 + i\epsilon ,\\
& (p_1' - p_1 - k + q - k_1)^2 - m^2 + i\epsilon \\
& \quad = \left[\frac{1}{s}(k'^2 - k^2 - 2\Delta^2) + \alpha_1 - \alpha_2 + \beta_1 - \beta_2 - 1\right]\left[\frac{m^2}{s}(k'^2 - k^2 - \Delta^2) + \alpha_1 m^2 - \alpha_2 m^2 - \frac{k^2\Delta^2}{s} - k^2 + \beta_1 k^2 - \beta_2 k^2\right] \\
& \quad\quad - \left[\frac{1}{s}(k'^2 - k^2 - \Delta^2) + \alpha_1 - \alpha_2\right]\left[-\frac{\Delta^2}{s} - 1 + \beta_1 - \beta_2\right] s + (\Delta_\perp + q_\perp - k_{1\perp})^2 - m^2 + i\epsilon .
\end{aligned}
\tag{2.2}
$$

We expect the Regge residues to damp rapidly when internal masses are large,[4,8] so when s is large our amplitude will be dominated by values of α_1 and α_2 of order m^2/s, and by β_1 and β_2 of order 1. Examination of the analogous propagators for the upper half of Fig. 5 shows that β_1' and β_2 must be of order m^2/s, while α_1' and α_2 can be of order 1. Taken together, these arguments allow us to drop β_2 relative to β_1 in the lower propagators, and α_2 relative to α_1' in the upper propagators. If we drop other terms proportional to s^{-1} in Eqs. (2.2), the lower propagators become

$$
\begin{aligned}
& q^2 - m^2 + i\epsilon \sim -\alpha_1\beta_1 s + \beta_1{}^2 k^2 + q_\perp{}^2 - m^2 + i\epsilon ,\\
& (q-k_1)^2 - m^2 + i\epsilon \sim -(\alpha_1 - \alpha_2)\beta_1 s + \beta_1{}^2 k^2 + (q_\perp - k_{1\perp})^2 - m^2 + i\epsilon ,\\
& (q-k)^2 - m^2 + i\epsilon \sim -\alpha_1(\beta_1 - 1)s + (1-\beta_1)^2 k^2 + q_\perp{}^2 - m^2 + i\epsilon ,\\
& (p_1' - p_1 - k + q - k_1)^2 - m^2 + i\epsilon \sim -(\alpha_1 - \alpha_2)(\beta_1 - 1)s + (1-\beta_1)^2 k^2 + (1-\beta_1)(k'^2 - k^2 - \Delta^2) + (\Delta_\perp + q_\perp - k_{1\perp})^2 - m^2 + i\epsilon .
\end{aligned}
\tag{2.3}
$$

Examination of the upper propagators shows that at large s they can be obtained from the lower ones by the substitutions $\alpha_1 \to \beta_1'$, $\beta_1 \to \alpha_1'$, $\alpha_2 \to \beta_2$, $q_\perp \to q_\perp'$, $k^2 \to m^2$, $k'^2 \to m^2$.

Each Pomeranchuk pole in Fig. 5 is represented by the factorized form

$$-g(m_1^2, m_2^2) g(m_3^2, m_4^2) \frac{e^{-i\pi\alpha(t)}}{\sin\pi\alpha(t)} [(s+i\epsilon)^{\alpha(t)} + \delta(-s+i\epsilon)^{\alpha(t)}],$$

where δ is the signature. ($\delta = 1$ for Pomeranchukons.) The energies and momentum transfers we need are

$$\begin{aligned} &-2q\cdot q' \sim \alpha_1'\beta_1 s, \quad k_1^2 \sim k_{1\perp}^2, \\ &2(q-k)\cdot(p_1'-q') \sim (1-\alpha_1')(1-\beta_1)s, \quad (p_1'-p_1-k)^2 \sim (\Delta_\perp - k_{1\perp})^2. \end{aligned} \tag{2.4}$$

Assembling our results, we see that the amplitude for Fig. 5 factorizes in its dependence on α_1, β_1, α_2, $q_\perp$ and α_1', β_1', β_2, $q_\perp'$. It takes the form

$$\begin{aligned} F_{\rm I} \sim \frac{i}{|s|} \int \frac{d^2k_{1\perp}}{(2\pi)^2} s^{\alpha(k_{1\perp}^2)+\alpha((\Delta_\perp-k_{1\perp})^2)} \left[\frac{e^{-i\pi\alpha(k_{1\perp}^2)}+\delta}{\sin\pi\alpha(k_{1\perp}^2)}\right] \left[\frac{e^{-i\pi\alpha((\Delta_\perp-k_{1\perp})^2)}+\delta}{\sin\pi\alpha((\Delta_\perp-k_{1\perp})^2)}\right] N_{\alpha(k_{1\perp}^2),\alpha((\Delta_\perp-k_{1\perp})^2)}(k^2, k'^2, \Delta_\perp, k_{1\perp}) \\ \times N_{\alpha(k_{1\perp}^2),\alpha((\Delta_\perp-k_{1\perp})^2)}(m^2, m^2, \Delta_\perp, k_{1\perp}). \end{aligned} \tag{2.5}$$

Here

$$\begin{aligned} &N_{l_1,l_2}(k^2, k'^2, \Delta_\perp, k_{1\perp}) \\ &= \frac{m^4}{2^{3/2}(2\pi)^5} \int d^2q_\perp \int_0^1 d\beta_1 \beta_1^{l_1}(1-\beta_1)^{l_2} \int_{-\infty}^{\infty} d\bar\alpha \int_{-\infty}^{\infty} d\bar\alpha_1 \\ &\times \frac{g(m^2\bar\alpha_1\beta_1 + \beta_1^2k^2 + q_\perp^2, m^2(\bar\alpha_1-\bar\alpha)\beta_1 + \beta_1^2k^2 + (k_{1\perp}-q_\perp)^2)}{[m^2\bar\alpha_1\beta_1 + \beta_1^2k^2 + q_\perp^2 - m^2 + i\epsilon][m^2(\bar\alpha_1-\bar\alpha)\beta_1 + \beta_1^2k^2 + (k_{1\perp}-q_\perp)^2 - m^2 + i\epsilon]} \\ &\times \frac{g(m^2\bar\alpha_1(\beta_1-1) + (1-\beta_1)^2k^2 + q_\perp^2, m^2(\bar\alpha_1-\bar\alpha)(\beta_1-1) + (1-\beta_1)^2k^2 + (1-\beta_1)(k'^2-k^2-\Delta^2) + (\Delta_\perp+q_\perp-k_{1\perp})^2)}{[m^2\bar\alpha_1(\beta_1-1) + (1-\beta_1)^2k^2 + q_\perp^2 - m^2 + i\epsilon][m^2(\bar\alpha_1-\bar\alpha)(\beta_1-1) + (1-\beta_1)^2k^2 + (1-\beta_1)(k'^2-k^2-\Delta^2) + (\Delta_\perp+q_\perp-k_{1\perp})^2 - m^2 + i\epsilon]}. \end{aligned} \tag{2.6}$$

We have used the scaled integration variables $\alpha_2 = -m^2\bar\alpha/s$, $\alpha_1 = -m^2\bar\alpha_1/s$. The restriction on the range of β_1 is obtained by noting that the Regge residues are analytic functions of the masses in the upper half plane. Therefore, the integral over $\bar\alpha$ vanishes unless β_1 and $1-\beta_1$ have the same sign.

The factor $|s|^{-1}$ in front of the integral of Eq. (2.5) comes from the Jacobian of the transformation to Sudakov variables. It does not change sign under $s \to -s$, so $F_{\rm I}$ is even under $s \to -s + i\epsilon$. We write $F_{\rm I}$ as a Sommerfeld-Watson integral over the even-signature partial-wave amplitude:

$$F_{\rm I} = -\frac{1}{2\pi i} \int dj \left[\frac{e^{-i\pi j}+1}{\sin\pi j}\right] f_{{\rm I}j}(\Delta_\perp^2) s^j. \tag{2.7}$$

It is easy to check that the partial-wave amplitude is

$$\begin{aligned} f_{{\rm I}j}(\Delta_\perp^2) = \int \frac{d^2k_{1\perp}}{(2\pi)^2} \frac{\gamma_{\alpha(k_{1\perp}^2),\alpha((\Delta_\perp-k_{1\perp})^2)}}{j+1-\alpha(k_{1\perp}^2)-\alpha((\Delta_\perp-k_{1\perp})^2)} \\ \times N_{\alpha(k_{1\perp}^2),\alpha((\Delta_\perp-k_{1\perp})^2)}(k^2, k'^2, \Delta_\perp, k_{1\perp}) N_{\alpha(k_{1\perp}^2),\alpha((\Delta_\perp-k_{1\perp})^2)}(m^2, m^2, \Delta_\perp, k_{1\perp}), \end{aligned} \tag{2.8}$$

where

$$\gamma_{l_1,l_2} = \frac{\cos\frac{1}{2}\pi(l_1+l_2+1-\delta)}{\sin\frac{1}{2}\pi(l_1+\frac{1}{2}-\frac{1}{2}\delta)\sin\frac{1}{2}\pi(l_2+\frac{1}{2}-\frac{1}{2}\delta)}. \tag{2.9}$$

Equation (2.8) exhibits a cut in the angular momentum plane. The numerator of Eq. (2.9) guarantees that there is no anomalous cut in the t plane at even integer j. The numerator also introduces the minus sign that makes the Mandelstam Pomeranchukon cuts have opposite sign from the Amati-Fubini-Stanghellini cuts.

It is necessary to attach $F_{\rm I}$ to the lower Pomeranchukon pole in order to construct the amplitude of Fig. 4. The problem of connecting two Regge amplitudes in series was considered by Gribov, and we agree with his conclusion.[8] However, the limits on the integral of Eq. (35) in Ref. 8 seem unmotivated to us, so we give an independent derivation. The process considered is shown in Fig. 6, and now for simplicity we specialize to forward scattering. We use the Sudakov variables

$$k=\alpha p_1+\beta p+k_\perp\,, \quad s=(p+p_1)^2\,,$$
$$k^2-m^2+i\epsilon=(\alpha-\beta)^2m^2+\alpha\beta s+k_\perp{}^2-m^2+i\epsilon\,,$$
$$2k\cdot p_1\sim\beta s\,, \quad -2k\cdot p\sim-\alpha s\,. \tag{2.10}$$

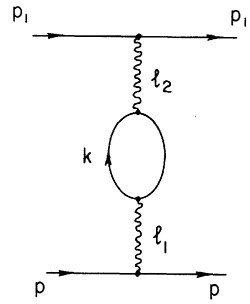

FIG. 6. Regge amplitudes in series.

The amplitude is

$$F_{\text{II}}\sim\frac{-i|s|}{2(2\pi)^4}\frac{e^{-i\pi(l_1+l_2)}}{(\sin\pi l_1)(\sin\pi l_2)}\int d^2k_\perp\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta\frac{g_1(k^2)g_2(k^2)}{(k^2-m^2+i\epsilon)^2}[(-\alpha s+i\epsilon)^{l_1}+\delta_1(\alpha s+i\epsilon)^{l_1}]$$
$$\times[(\beta s+i\epsilon)^{l_2}+\delta_2(-\beta s+i\epsilon)^{l_2}]\,. \tag{2.11}$$

Here l_1 and l_2 are angular momenta which are fixed for pole trajectories, and get integrated over for cut trajectories. In Eq. (2.11) there is no simple scaling that picks out the dominant region of integration, so we make no approximations in the propagator $(k^2-m^2+i\epsilon)^{-1}$ other than dropping the term $-2\alpha\beta m^2$, which is uniformly dominated by $s\alpha\beta$. We introduce a spectral representation for the dependence of the Regge residues on the internal mass[4]:

$$\frac{g_1(k^2)g_2(k^2)}{(k^2-m^2+i\epsilon)^2}=\int_{\mu_0{}^2}^{\infty}\frac{d\mu^2\rho(\mu^2)}{(k^2-\mu^2+i\epsilon)^2}\,. \tag{2.12}$$

When the integrals over α and β are split into the ranges $-\infty$ to 0, and 0 to ∞, F_{II} can be reorganized to read

$$F_{\text{II}}=\frac{-i|s|s^{l_1+l_2}\delta}{(2\pi)^4}\left(\frac{1+\delta_1\delta_2}{2}\right)\left(\frac{e^{-i\pi l_1}+\delta}{\sin\pi l_1}\right)\left(\frac{e^{-i\pi l_2}+\delta}{\sin\pi l_2}\right)\int d^2k_\perp\int_{\mu_0{}^2}^{\infty}d\mu^2\rho(\mu^2)[I(s)+\delta I(-s+i\epsilon)]\,, \tag{2.13}$$

where[13]

$$I(s)=\int_0^\infty d\alpha\,\alpha^{l_1}\int_0^\infty d\beta\,\beta^{l_2}G(\alpha,\beta)\,,$$
$$G(\alpha,\beta)=[s\alpha\beta+m^2(\alpha^2+\beta^2)+k_\perp{}^2-\mu^2+i\epsilon]^{-2}\,. \tag{2.14}$$

Note that F_{II} vanishes unless the signature of the two amplitudes being joined is the same; we denote the common signature by δ. In Eq. (2.13) we have used the relation

$$I(-s+i\epsilon)=\int_0^\infty d\alpha\,\alpha^{l_1}\int_0^\infty d\beta\,\beta^{l_2}G(-\alpha,\beta)\,. \tag{2.15}$$

In the Appendix we evaluate I for large s, and find for F_{II}

$$F_{\text{II}}\sim\frac{2\Gamma(1-\frac{1}{2}l_1-\frac{1}{2}l_2)}{(2\pi)^4(l_2-l_1)}\left\{e^{-i\pi l_1}\left(\frac{e^{-i\pi l_2}+\delta}{\sin\pi l_2}\right)\Gamma(1+l_1)\Gamma(1+\tfrac{1}{2}l_2-\tfrac{1}{2}l_1)(m)^{l_1-l_2}s^{l_2}-(l_1\leftrightarrow l_2)\right\}$$
$$\times\int d^2k_\perp\int_{\mu_0{}^2}^{\infty}d\mu^2\rho(\mu^2)(k_\perp{}^2-\mu^2+i\epsilon)^{l_1/2+l_2/2-1}\,. \tag{2.16}$$

This expression is valid for l_1 and l_2 such that all Γ functions have positive arguments; the point $l_1=1$, $l_2=1$ is near the middle of the region. We use the representation

$$(k_\perp{}^2-\mu^2+i\epsilon)^{l_1/2+l_2/2-1}=-\frac{2e^{(i\pi/2)(l_1+l_2)}}{\pi(l_1+l_2)}\sin\tfrac{1}{2}\pi(l_1+l_2)\int_0^\infty\frac{dy\,y^{l_1/2+l_2/2}}{(-y+\lambda+i\epsilon)^2}\qquad(\lambda=k_\perp{}^2-\mu^2) \tag{2.17}$$

to write

$$F_{\text{II}} \sim \frac{-2}{(2\pi)^4 (l_2 - l_1)\Gamma(1+\frac{1}{2}l_1+\frac{1}{2}l_2)} \left\{ e^{(i\pi/2)(l_2-l_1)} \left(\frac{e^{-i\pi l_2}+\delta}{\sin\pi l_2} \right) \Gamma(1+l_1)\Gamma(1+\tfrac{1}{2}l_2-\tfrac{1}{2}l_1)(m)^{l_1-l_2} s^{l_2} - (l_1 \leftrightarrow l_2) \right\}$$
$$\times \int d^2k_\perp \int_0^\infty dy\, y^{(l_1+l_2)/2} \frac{g_1(-y+k_\perp^2)\, g_2(-y+k_\perp^2)}{(y-k_\perp^2+m^2)^2}. \tag{2.18}$$

The partial-wave amplitude that produces F_{II} is

$$f_{\text{II}j} = \frac{1}{j-l_1} r^j_{l_1+l_2} \frac{1}{j-l_2},$$
$$r^j_l = \frac{(m)^{l-2j} e^{-i\pi(l/2-j)} \Gamma(1+l-j)\Gamma(1-\frac{1}{2}l+j)}{\pi(2\pi)^3 \Gamma(1+\frac{1}{2}l)} \int d^2k_\perp \int_0^\infty dy\, y^{l/2} \frac{g_1(-y+k_\perp^2)\, g_2(-y+k_\perp^2)}{(y-k_\perp^2+m^2)^2}. \tag{2.19}$$

In the limit $l_1 \approx l_2$, we may set $\frac{1}{2}(l_1+l_2)=j$. Then, apart from a constant related to the normalization to Eq. (2.7), we recover Gribov's result:

$$r_j = \frac{1}{\pi(2\pi)^3} \int d^2k_\perp \int_0^\infty dy\, y^j \frac{g_1(-y+k_\perp^2)\, g_2(-y+k_\perp^2)}{(y-k_\perp^2+m^2)^2}. \tag{2.20}$$

It is worth noting that in Eq. (2.11) *both* Regge amplitudes have large energies, in contrast to Gribov's implication. [This is why Gribov's Eq. (35) is puzzling.] This is all to the good because it would make little sense to insert Regge limits of amplitudes if major contributions to the Feynman integral came from low-energy regions. We can check this point by computing the energy of the upper amplitude when averaged over α and β. With the aid of results in the Appendix we find

$$\langle \beta s \rangle = \frac{s}{I(s)} \int_0^\infty d\alpha\, \alpha^{l_1} \int_0^\infty d\beta\, \beta^{l_2+1} G(\alpha,\beta)$$
$$= \left(\frac{k_\perp^2 - \mu^2 + i\epsilon}{m^2} \right)^{1/2} \begin{cases} \text{const} \times s & (l_2 > l_1) \\ \text{const} \times s/\ln s & (l_2 = l_1) \\ \text{const} \times s^{l_2-l_1+1} & (l_1 - 1 < l_2 < l_1). \end{cases} \tag{2.21}$$

We see that both energies will be large for $|l_1 - l_2| < 1$, and Eq. (2.19) is useful for coupling trajectories up to one unit of spin from each other.

We now assemble Fig. 4 by replacing $g_2(k^2)$ in Eq. (2.11) by the operator

$$\int \frac{d^2k_{1\perp}}{(2\pi)^2} \gamma_{\alpha(k_{1\perp}^2),\,\alpha(k_{1\perp}^2)} N_{\alpha(k_{1\perp}^2),\,\alpha(k_{1\perp}^2)}(k^2,k^2,0,k_{1\perp}) N_{\alpha(k_{1\perp}^2),\,\alpha(k_{1\perp}^2)}(m^2,m^2,0,k_{1\perp}) \int dl_2\, \delta(l_2+1-2\alpha(k_{1\perp}^2)). \tag{2.22}$$

We also replace $g_1(k^2)$ by $g(k^2)g(m^2)$, the factorized Pomeranchukon pole residue. These manipulations can be performed directly on Eq. (2.19) to determine the partial-wave amplitude for Fig. 4:

$$f_j = g(m^2) \frac{1}{j-\alpha(0)} \int \frac{d^2k_{1\perp}}{(2\pi)^2} \frac{\gamma_{\alpha(k_{1\perp}^2),\,\alpha(k_{1\perp}^2)}\, r^j_{\alpha(k_{1\perp}^2),\,\alpha(k_{1\perp}^2)}(k_{1\perp})}{j+1-2\alpha(k_{1\perp}^2)} N_{\alpha(k_{1\perp}^2),\,\alpha(k_{1\perp}^2)}(m^2,m^2,0,k_{1\perp}), \tag{2.23}$$

where the Gribov triple-Pomeranchukon vertex is

$$r^j_{\alpha(k_{1\perp}^2),\,\alpha(k_{1\perp}^2)}(k_{1\perp}) = (-im)^{\alpha(0)+2\alpha(k_{1\perp}^2)-1-2j} \frac{\Gamma(\alpha(0)+2\alpha(k_{1\perp}^2)-j)}{\Gamma(\frac{1}{2}\alpha(0)+\alpha(k_{1\perp}^2)+\frac{1}{2})} \frac{\Gamma(j+\frac{3}{2}-\frac{1}{2}\alpha(0)-\alpha(k_{1\perp}^2))}{\pi(2\pi)^3}$$
$$\times \int d^2k_\perp \int_0^\infty dy \frac{y^{\alpha(0)/2+\alpha(k_{1\perp}^2)-1/2}\, g(-y+k_\perp^2)}{(y-k_\perp^2+m^2)^2} N_{\alpha(k_{1\perp}^2),\,\alpha(k_{1\perp}^2)}(-y+k_\perp^2, -y+k_\perp^2, 0, k_{1\perp}). \tag{2.24}$$

This agrees with Gribov's expression in the limit $j \to \frac{1}{2}\alpha(0)+\alpha(k_{1\perp}^2)-\frac{1}{2}$. Equation (2.24) is an improvement on Gribov's definition in the sense that Eq. (2.19) is an improvement over Eq. (2.20).

We evaluate r by substituting N from Eq. (2.6), putting $l_1 - l_2 = k^2 - k'^2 = \Delta = 0$. First we make the substitution $\bar{\alpha} \to \bar{\alpha} + 2k_{1\perp}\cdot k_\perp/m^2$ in Eq. (2.6), and note that the resulting integrand is invariant under the replacement $\beta_1 \to 1-\beta_1$, $\bar{\alpha}_1 \to -\bar{\alpha}_1$, $\bar{\alpha} \to -\bar{\alpha}$, and $k_\perp \to -k_\perp$. This invariance permits us to restrict $\bar{\alpha}$ to the range $-\infty < \bar{\alpha} < 0$ because the integral in Eq. (2.24) is symmetric in $k_\perp$. Next make the further substitutions

$$\bar{\alpha} \to -\frac{M^2}{m^2}\beta, \quad \beta_1 \to \beta_1/\beta, \quad y \to -s\alpha\beta, \quad q_\perp \to q_\perp - \beta_1 k_\perp/\beta,$$
$$\bar{\alpha}_1 \to \frac{1}{m^2}[\alpha_1\beta s + 2q_\perp \cdot k_\perp - 2\beta_1 k_\perp{}^2/\beta - s\alpha\beta_1]. \tag{2.25}$$

From Eq. (2.24) we obtain the relation

$$-\frac{i\pi}{2^{1/2}}\left(\frac{s}{M}\right)^2 r^{j}_{\alpha(k_{1\perp}{}^2),\, \alpha(k_{1\perp}{}^2)}(k_{1\perp}) \frac{s^{\alpha(0)/2+\alpha(k_{1\perp}{}^2)-3/2}\Gamma(\frac{1}{2}\alpha(0)+\alpha(k_{1\perp}{}^2)+\frac{1}{2})}{\Gamma(j+\frac{3}{2}-\frac{1}{2}\alpha(0)-\alpha(k_{1\perp}{}^2))\Gamma(\alpha(0)+2\alpha(k_{1\perp}{}^2)-j)}(-im)^{2j+1-\alpha(0)-2\alpha(k_{1\perp}{}^2)}$$
$$= \frac{-2i}{4(2\pi)^8}\int_{-\infty}^{0} d\alpha \int_{-\infty}^{\infty} d\alpha_1 \int_0^\infty d\beta \left(\frac{-\beta}{\alpha}\right)^{\alpha(0)/2+1/2-\alpha(k_{1\perp}{}^2)} \int_0^\beta d\beta_1 \int d^2q_\perp d^2k_\perp (-\alpha s)^{\alpha(0)}(\beta, s)^{\alpha(k_{1\perp}{}^2)}[(\beta-\beta_1)s]^{\alpha(k_{1\perp}{}^2)}$$
$$\times \frac{g(\alpha\beta s + k_\perp{}^2)}{[\alpha\beta s + k_\perp{}^2 - m^2 + i\epsilon]^2} \frac{g(\alpha_1\beta_1 s + q_\perp{}^2,\, \alpha_1\beta_1 s + M^2\beta_1 + (q_\perp - k_{1\perp})^2)}{[\alpha_1\beta_1 s + q_\perp{}^2 - m^2 + i\epsilon][\alpha_1\beta_1 s + M^2\beta_1 + (q_\perp - k_{1\perp})^2 - m^2 + i\epsilon]}$$
$$\times \frac{g((\alpha_1-\alpha)(\beta_1-\beta)s + (q_\perp - k_\perp)^2,\, (\alpha_1-\alpha)(\beta_1-\beta)s + M^2(\beta_1-\beta) + (q_\perp - k_\perp - k_{1\perp})^2)}{[(\alpha_1-\alpha)(\beta_1-\beta)s + (q_\perp - k_\perp)^2 - m^2 + i\epsilon][(\alpha_1-\alpha)(\beta_1-\beta)s + M^2(\beta_1-\beta) + (q_\perp - k_\perp - k_{1\perp})^2 - m^2 + i\epsilon]}. \tag{2.26}$$

The right-hand side of this equation agrees with $F_1(s, M^2, t)$ of Sec. III, Eq. (7) in Ref. 6, provided $\alpha(0) = 2\alpha(k_{1\perp}{}^2) - 1$.[14] For $\alpha(0) = 1$, this condition is satisfied at $t = 0$. Corresponding to this, the inclusive amplitude has the asymptotic form

$$F(s, M^2, 0) \sim -\frac{i\pi}{2^{1/2}}\left(\frac{s}{M}\right)^2 r^1_{1,1}(0) \tag{2.27}$$

at $t = 0$. Its discontinuity (here its imaginary part) is proportional to Gribov's vertex.

The left-hand side of Eq. (2.26) exhibits various explicit singularities in j when j is continued away from $\alpha(0) = 2\alpha(k_{1\perp}{}^2) - 1$. These singularities are also present in r since the right-hand side of Eq. (2.26) is independent of j, and they are present only because of the definition we have chosen for r. It is desirable to use Eq. (2.26) only "on the nonsense pole" $j = \alpha(0) = 2\alpha(k_{1\perp}{}^2) - 1$. In the next section it will be apparent that the inclusive vertex is itself the natural continuation of Gribov's vertex off the nonsense pole.

III. PARTIAL-WAVE AMPLITUDES IN THE $B\bar{B}$ CHANNEL

In Sec. II we found that the Gribov and inclusive vertices can be identified at $t = 0$, but in the model we used, both objects were finite. In order to see that the inclusive vertex vanishes, and how it vanishes, we must use a formalism that includes the effects of pole-cut collisions. One such formalism is that of Reggeon unitary partial-wave amplitudes in the $B\bar{B}$ channel. By using it we can directly compare our argument with that of Ref. 11.

We examine the partial-wave amplitude for the process depicted in Fig. 1, in the channel $B + \bar{B} \to (\bar{A} + C) + (\bar{C} + A)$. The energy squared will be denoted $t_0 = (p_B + p_{\bar{B}})^2$, and we generalize to $t_0 \neq 0$. The particles in parentheses are organized into pairs of angular momentum, helicity, and energy l_i, m_i, and t_i $(i = 1, 2)$, and j is the channel angular momentum. As discussed in Refs. 3 and 11, the full amplitude in the limit of Eq. (1.1) is controlled by the residue of the partial-wave amplitude at Regge and helicity poles. These singularities are at $j = \alpha(t_0)$, $l_i = m_i = \alpha(t_i)$, and for $t_0 = 0$, $t_1 = t_2 = t$.

Gribov, Pomeranchuk, and Ter-Martirosyan[7] have written the residue at the singularities $\alpha_i = m_i = \alpha(t_i)$ in a form displaying the nonsense pole explicitly[15]:

$$T^{j}_{\alpha_1\alpha_2}(t_0) = \frac{N^{j}_{\alpha_1\alpha_2}(t_0)}{j + 1 - \alpha_1 - \alpha_2} + R^{j}_{\alpha_1\alpha_2}(t_0). \tag{3.1}$$

Here $\alpha_i \equiv \alpha(t_i)$. The function N, which is called the Reggeon production amplitude in Ref. 7, still contains the Pomeranchukon pole at $j = \alpha(t_0)$, and the two-Pomeranchukon cut as well. (The multi-Pomeranchukon cuts are thought to be weak and will be ignored in the following.[7,10]) Gribov *et al.*[7] have shown that near $t_1 = t_2 = \frac{1}{4}t_0$, the Reggeon unitarity equations require that N have the form

$$N^{j}_{\alpha_1\alpha_2}(t_0) = \frac{A^{1/2}(t_0, j)}{B(t_0, j) + [1/\alpha'(\frac{1}{4}t_0)]\ln[j + 1 - 2\alpha(\frac{1}{4}t_0)]}, \tag{3.2}$$

where A and B are meromorphic in the neighborhood of the two-Pomeranchukon cut.

Equations (3.1) and (3.2) provide the framework for understanding the behavior of the inclusive cross section as a function of $t_1 = t_2 = t$. First we take the residue of the Regge pole at $j = \alpha(t_0)$ in Eq. (3.1). Note that with $j = \alpha(t_0)$ the nonsense pole provides a factor that cancels the sine function introduced by taking the discontinuity in M^2. The presence of this sine means that the contribution of R to inclusive cross sections vanishes linearly at $t=0$ for linear trajectories. Aside from this linear term, the inclusive cross section is proportional to $n_{\alpha(0),\alpha(t),\alpha(t)}$, the residue of $N^j_{\alpha(t)\alpha(t)}(t_0)$ at $j = \alpha(t_0)$ for $t_0 = 0$.

The residue $n_{\alpha(0),\alpha(t),\alpha(t)}$ can be calculated from Eq. (3.2) only at $t=0$ because of the restriction $t_1 = t_2 = \frac{1}{4}t_0$; there it must vanish. Nevertheless, it is instructive to overlook this restriction on the applicability of Eq. (3.2), and to postulate that the Pomeranchukon pole at $j = \alpha(t_0)$ occurs as a pole of $A^{1/2}$. If the pole occurs in this way, the logarithm in the denominator provides an automatic zero in $n_{\alpha(t_0),\alpha(t_0/4),\alpha(t_0/4)}$ at $t_0 = 0$ due to the Pomeranchukon pole-cut collision there. Further, since the pole-cut collision occurs at $t_0 = 0$ regardless of t_1 and t_2 it is natural to suppose that $n_{\alpha(0),\alpha(t_1),\alpha(t_2)}$ vanishes identically. Then the inclusive reaction coupling is due solely to R in Eq. (3.1), and vanishes linearly at $t=0$. Just this argument has been put forward in Ref. 11.

Unfortunately for the foregoing argument, the pole at $j = \alpha(t_0)$ must be due to a zero in the denominator of Eq. (3.2), and is not a pole in $A^{1/2}$. This is immediately evident from expressions for the Reggeon-Reggeon and $B\bar{B}$ scattering amplitudes, which are developed in Ref. 7 in the same approximation as Eq. (3.2). The Reggeon-Reggeon scattering amplitude is

$$T^j_{\alpha_1'\alpha_2',\alpha_1\alpha_2}(t_0) = \frac{M^j_{\alpha_1'\alpha_2',\alpha_1\alpha_2}(t_0)}{(j+1-\alpha_1'-\alpha_2')(j+1-\alpha_1-\alpha_2)} + \cdots, \tag{3.3}$$

$$M^j_{\alpha_1'\alpha_2',\alpha_1\alpha_2}(t_0) = \frac{1}{B(t_0,j) + [1/\alpha'(\frac{1}{4}t_0)]\ln[j+1-2\alpha(\frac{1}{4}t_0)]}.$$

T is the residue of Regge poles in pairs of particles in the 4 particle → 4 particle amplitude, and in M we have put $\alpha_i = \alpha_i' = \alpha(\frac{1}{4}t_0)$. The amplitude for $B+\bar{B} \to B+\bar{B}$ is

$$F(t_0,j) = \frac{A(t_0,j)}{B(t_0,j) + [1/\alpha'(\frac{1}{4}t_0)]\ln[j+1-2\alpha(\frac{1}{4}t_0)]} + D(t_0,j). \tag{3.4}$$

We note from Eq. (3.3) that the Regge pole at $j = \alpha(t_0)$ is absent unless it is a zero of the denominator. However, if M had no Regge pole at $j = \alpha(t_0)$, the residue of the missing pole, which is $[n_{\alpha(t_0),\alpha(t_0/4),\alpha(t_0/4)}]^2$, would have to vanish identically in t_0, and not just at $t_0 = 0$, where pole and cut collide. No reason for this to happen has ever been put forward. Even worse, $F(t_0,j)$ has a second-order pole at $j = \alpha(t_0)$ if the singularity is in A. An illegal second-order pole cannot be canceled by D because D does not have the two-Pomeranchukon cut.

Once the Pomeranchukon pole is seen to occur in the denominator of Eqs. (3.2)–(3.4), the pole-cut collision produces a branch point in $\alpha(t_0)$, and the zero in Gribov's vertex can no longer be traced to an obvious factor like the logarithm. One is forced to the less conclusive position of Refs. 9 and 10, which is that no satisfactory theory can be constructed with a nonzero vertex at $t_0 = 0$. For example, in Ref. 10 a satisfactory S matrix can be constructed only when the functions A and B share a second-order Castillejo-Dalitz-Dyson (CDD) pole that moves through $j=1$ and $t_0 = 0$. The CDD pole is the source of the vanishing of $n_{\alpha(0),\alpha(0),\alpha(0)}$, and it is forced to be there by the Pomeranchukon pole-cut collision; but it is by no means as obvious as the logarithm in Eqs. (3.2)–(3.4).

In order to validate the arguments just given, and to determine the behavior of $n_{\alpha(0),\alpha(t),\alpha(t)}$ near $t=0$, we now remove the restriction $t_1 = t_2 = \frac{1}{4}t_0$ from our formulas. To do this we must go beyond the approximations used in Ref. 7. The first question that arises when t_1 and t_2 are free is what singularities might be present in these variables. Bound-state (Regge) and threshold singularities have been removed in Ref. 7; beyond this little is known except that the amplitudes are likely to be rife with anomalous and complex singularities. We make the relatively modest assumption that the amplitudes N and M are analytic at $t_1 = \frac{1}{4}t_0$ and $t_2 = \frac{1}{4}t_0$, so that a power-series development is possible. Generalizing the *Ansätze* of p. B197, Ref. 7, we set

$$\begin{aligned} N^j_{\alpha_1\alpha_2}(t_0) &= [c_j(t_0) + (t_1+t_2-\tfrac{1}{2}t_0)b_j(t_0) + \cdots][p(t_0,t_1,t_2)]^{-1}, \\ M^j_{\alpha_1'\alpha_2',\alpha_1\alpha_2}(t_0) &= [d_j(t_0) + (t_1'+t_2'-\tfrac{1}{2}t_0)e_j(t_0) + e_j(t_0)(t_1+t_2-\tfrac{1}{2}t_0) + (t_1'+t_2'-\tfrac{1}{2}t_0)f_j(t_0)(t_1+t_2-\tfrac{1}{2}t_0) + \cdots] \\ &\quad \times [p(t_0,t_1',t_2')p(t_0,t_1,t_2)]^{-1}. \end{aligned} \tag{3.5}$$

The factors p^{-1} reflect an orbital angular momentum of minus one for the Pomeranchukons. We substitute these expressions into the Regge-pole unitarity equations (50), (59), and (60) of Ref. 7. We retain only the terms exhibited in Eq. (3.5), and the discontinuities across the two-Pomeranchukon cut are

$$\mathrm{Disc}_j\, d_j = -\frac{\pi}{\alpha'}\left[d_j d_j^\dagger + \lambda e_j d_j^\dagger + \lambda d_j e_j^\dagger + \lambda^2 e_j e_j^\dagger \right], \tag{3.6a}$$

$$\mathrm{Disc}_j\, e_j = -\frac{\pi}{\alpha'}\left[e_j d_j^\dagger + \lambda e_j e_j^\dagger + \lambda f_j d_j^\dagger + \lambda^2 f_j e_j^\dagger \right], \tag{3.6b}$$

$$\mathrm{Disc}_j f_j = -\frac{\pi}{\alpha'}\left[e_j e_j^\dagger + \lambda e_j f_j^\dagger + \lambda f_j e_j^\dagger + \lambda^2 f_j f_j^\dagger \right], \tag{3.6c}$$

$$\mathrm{Disc}_j\, c_j = -\frac{\pi}{\alpha'}\left[c_j d_j^\dagger + \lambda c_j e_j^\dagger + \lambda b_j d_j^\dagger + \lambda^2 b_j e_j^\dagger \right], \tag{3.6d}$$

$$\mathrm{Disc}_j\, b_j = -\frac{\pi}{\alpha'}\left[c_j e_j^\dagger + \lambda c_j f_j^\dagger + \lambda b_j e_j^\dagger + \lambda^2 b_j f_j^\dagger \right], \tag{3.6e}$$

$$\mathrm{Disc}_j F = -\frac{\pi}{\alpha'}\left[c_j c_j^\dagger + \lambda c_j b_j^\dagger + \lambda b_j c_j^\dagger + \lambda^2 b_j b_j^\dagger \right]. \tag{3.6f}$$

In evaluating the phase-space integrals we have assumed linear trajectories for simplicity, and

$$\lambda = \frac{j - j_c(t_0)}{\alpha'} = \frac{j+1-2\alpha(\frac{1}{4}t_0)}{\alpha'} = \frac{j-1-\frac{1}{2}\alpha' t}{\alpha'}. \tag{3.7}$$

Equations (3.6a)–(3.6c) can be solved by diagonalizing the quadratic forms on the right-hand sides. The diagonalizing matrix is analytic in λ in a neighborhood of $\lambda = 0$, so no new singularities are introduced. The diagonalized unitarity equations are readily solved by the techniques of Ref. 7. Having constructed M, we can apply the same procedure to Eqs. (3.6d) and (3.6e) to construct the Reggeon production amplitude:

$$N^j_{\alpha_1'\alpha_2}(t_0) = \frac{A^{1/2}\left[1+\lambda C + (t_1+t_2-\frac{1}{2}t_0)(\lambda - C)\right]}{B + [1/\alpha']\ln\lambda} + (t_1+t_2-\tfrac{1}{2}t_0-\lambda)D. \tag{3.8}$$

C and D, like A and B, are arbitrary meromorphic functions of t_0 and j in the neighborhood of $\lambda = 0$.

By examining Eq. (3.8) and companion expressions for M and F, it is readily seen that the Pomeranchukon pole must appear as a zero of the denominator of the first term of Eq. (3.8). For this to happen, A and B must continue to have second-order poles that are functions of t_0 and j, and which pass through $j=1$ at $t_0=0$. In addition, it is consistent for C to carry the same CDD pole, but as a first-order pole. If this happens $n_{\alpha(0),\alpha(t),\alpha(t)}$ does not vanish identically, but only linearly in t. This leads to a linear behavior of the inclusive cross section near $t=0$, but for a different reason than the one given in Ref. 11.

Looking back over the calculation, the linear behavior of the triple-Pomeranchukon residue can be traced to the power-series development in Eq. (3.5).

APPENDIX

Here we establish the asymptotic behavior

$$I(s) = \int_0^\infty d\alpha\, \alpha^{l_1} \int_0^\infty d\beta\, \beta^{l_2}\left[\alpha\beta s + m^2\alpha^2 + m^2\beta^2 + \lambda + i\epsilon\right]^{-2}$$
$$= \frac{\Gamma(1-\frac{1}{2}l_1-\frac{1}{2}l_2)(\lambda+i\epsilon)^{l_1/2+l_2/2-1}}{l_2-l_1}\left[\Gamma(1+l_1)\Gamma(1+\tfrac{1}{2}l_2-\tfrac{1}{2}l_1)m^{l_1-l_2}s^{-l_1-1} - (l_1 \leftrightarrow l_2)\right] + O(s^{-l_1/2-l_2/2-2}). \tag{A1}$$

Evaluate $I(s)$ using the variables $x=\alpha\beta$, $y=\alpha/\beta$. The integral over x converges for $-2<l_1+l_2<2$, and can be evaluated by contour integration. This leaves

$$I = \frac{\pi(l_1+l_2)(\lambda+i\epsilon)^{l_1/2+l_2/2-1}}{4\sin\frac{1}{2}\pi(l_1+l_2)}J,$$

$$J = \int_0^\infty dy\, y^{l_1/2-l_2/2-1}(s+m^2y+m^2/y)^{-l_1/2-l_2/2-1}. \tag{A2}$$

J converges for $-1 < l_1$, $-1 < l_2$, so I exists in the triangle defined by these two boundaries and $l_1 + l_2 < 2$. J can be written as integrals on (0, 1) and (1, ∞); the latter can be converted to (0, 1) by the substitution $y \to y^{-1}$. Thus

$$J = J(l_1, l_2) + J(l_2, l_1),$$
$$J(l_1, l_2) = \int_0^1 dy\, y^{l_1}(ys + m^2 + y^2m^2)^{-l_1/2 - l_2/2 - 1}. \tag{A3}$$

The terms m^2 and y^2m^2 are important only for $y \lesssim m^2/s$. Therefore, we may drop y^2m^2 relative to m^2. Introduce the scaling $y = m^2z/s$:

$$J(l_1, l_2) \sim s^{-l_1 - 1} m^{l_1 - l_2} \int_0^{s/m^2} dz\, z^{l_1/2 - l_2/2 - 1}(1 + 1/z)^{-l_1/2 - l_2/2 - 1}. \tag{A4}$$

Suppose $l_1 < l_2$. The integral may be extended to infinity and related to a beta function. In the extra piece, which must be subtracted off, one can expand the last factor in Eq. (A4) in powers of $1/z$. The result is

$$J(l_1, l_2) \sim s^{-l_1 - 1} m^{l_1 - l_2} \left[\frac{\Gamma(1 + l_1)\Gamma(\frac{1}{2}l_2 - \frac{1}{2}l_1)}{\Gamma(1 + \frac{1}{2}l_1 + \frac{1}{2}l_2)} + \frac{2(s/m^2)^{l_1/2 - l_2/2}}{l_1 - l_2} - \frac{(2 + l_1 + l_2)}{l_1 - l_2 - 2}(s/m^2)^{l_1/2 - l_2/2 - 1} + \cdots \right]. \tag{A5}$$

$J(l_1, l_2)$ is analytic at $l_1 = l_2$, so Eq. (A5) remains correct for $l_2 < l_1$ by analytic continuation. Upon adding $J(l_1, l_2)$ and $J(l_2, l_1)$, the middle term in Eq. (A5) cancels, and we are left with Eq. (A1).

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[13] The terms $m^2(\alpha^2 + \beta^2)$ in G are necessary for $I(s)$ to exist. It can be shown that such terms are present even when the problem is formulated in terms of lightlike combinations of p and p_1, as in Ref. 8.
[14] $t = p_{2\perp}{}^2$ in Ref. 6. The factor $(2\pi)^{-4}$ in Ref. 6 has been changed to $(2\pi)^{-8}$ because there are two loop momenta in the nonplanar coupling.
[15] The point $j = \alpha_1 + \alpha_2 - 1$ is a sense-nonsense point, so the conventional partial-wave amplitude has a square-root singularity there. We have followed the convention of Ref. 11 and introduced a compensating square root to give T a pole. When considering full amplitudes, this extra square root is naturally supplied by the group-representation functions. Finally, R can obviously be combined with N; we have separated it here to make it easy to describe contributions to T which do not have the nonsense pole.