Weakly Broken Subsymmetries of Hadrons

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The relation between explicit and spontaneous breaking of chiral SU(3) × SU(3) symmetry is analyzed. Because it is assumed that the transformations of the asymptotic fields, even if nonlinear, are the same up to the linear term as they are in the limit of no explicit symmetry breaking, it is possible to solve the equations of the Glashow-Weinberg model. Using only the pseudoscalar-meson masses and the ratio of the kaon and pion leptonic decay constants as input, the remaining parameters of the model are computable. Numerical results are obtained for M_κ^2, $Z_K^{1/2}/Z_\pi^{1/2}$, $\epsilon_8/\sqrt{2}\epsilon_0$, and $\langle 0|u_8|0\rangle/\sqrt{2}\,\langle 0|u_0|0\rangle$. Figures display the dependence of M_κ^2 on the ratio F_K/F_π. Two noteworthy features of the analysis are: (1) The Glashow-Weinberg model admits two types of solutions simultaneously, one characterized by chiral SU(2) × SU(2) as a weakly broken subsymmetry [i.e., by the hadron Hamiltonian's being approximately invariant under chiral SU(2) × SU(2)], which is the view of Gell-Mann, Oakes, and Renner, and the other by SU(3) as a weakly broken subsymmetry, which is the view of Brandt and Preparata. (2) The limit of no explicit symmetry breaking is realized with the entire spinless chiral multiplet degenerate in mass, and massless if either the full symmetry or any subsymmetry is spontaneously broken. It is made plausible that recent studies of the limit of no explicit symmetry breaking which find an octet of pseudoscalar Goldstone bosons and an SU(3)-invariant vacuum do so because their assumptions or approximations guarantee that they must.

I. INTRODUCTORY REMARKS AND SUMMARY OF RESULTS

That the hadron Hamiltonian should consist of one part H_0 which is invariant under chiral SU(3) × SU(3) and another part H' which transforms as a $(3, 3^*)+(3^*, 3)$ representation of the group was suggested by Gell-Mann.[1] Explicitly, the hypothesis was

$$H = H_0 + H' , \tag{1a}$$

with

$$H' = \int d\vec{x}[\epsilon_0 u_0(x) + \epsilon_8 u_8(x)] , \tag{1b}$$

where the $u_i(x)$ belong to a nonet of scalar densities, which together with a nonet of pseudoscalar densities $v_j(x)$ make up a $(3, 3^*)+(3^*, 3)$ representation of chiral SU(3) × SU(3).

This idea has been reconsidered by Glashow and Weinberg[2] (GW), who make an additional assumption about smoothness of certain vertex functions and find inequalities to be satisfied by the κ-meson mass, a condition on the K_{l3} form factor $f_+(t)$ at $t = 0$, and a formula which relates the masses of the pseudoscalar nonet and of the κ meson and which reduces to the Gell-Mann–Okubo mass formula when the vacuum is SU(3)-invariant. In these equations appear the leptonic decay constants of the pseudoscalar mesons and κ meson, needed ratios of which cannot be determined without introducing additional assumptions (GW use spectral-function sum rules).

The $(3, 3^*)+(3^*, 3)$ model has also been reconsidered by Gell-Mann, Oakes, and Renner[3] (GOR), who neglect the effect of η-η' mixing and of scalar mesons and who assume pole dominance of the axial-vector current divergences by π, K, and η mesons and approximate SU(3) symmetry for matrix elements of the scalar densities between one-pseudoscalar-meson states. In this context, GOR find that the hadron Hamiltonian is approximately chiral-SU(2) × SU(2)-invariant and the vacuum approximately SU(3)-invariant: Explicitly,

$$a \simeq -0.89 \tag{2a}$$

and

$$b \simeq 0 , \tag{2b}$$

where we have used the definitions

$$a \equiv \epsilon_8/\sqrt{2}\,\epsilon_0 \tag{3a}$$

and

$$b \equiv \lambda_8/\sqrt{2}\,\lambda_0 , \tag{3b}$$

with

$$\lambda_i \equiv \langle 0|u_i(x)|0\rangle = \delta_{i0}\lambda_0 + \delta_{i8}\lambda_8 . \tag{4}$$

[Strictly speaking, GOR find

$$\lambda_8 \simeq 0 \tag{5}$$

rather than Eq. (2b), which is then a consequence as long as $\lambda_0 \neq 0$.] For the leptonic decay constants defined by

$$\langle 0|A_\mu^B(0)|P_i(\vec{k})\rangle \equiv ik_\mu F_{Bi}/[(2\pi)^3 2\omega_{\vec{k}}]^{1/2}, \qquad (6a)$$

where

$$F_{Bi}=F_B\delta_{Bi} \quad \text{for } B=1,2,\ldots,7 \qquad (6b)$$

(large Latin indices run from 1 to 8, small Latin indices from 0 to 8), GOR obtain

$$F_\pi \simeq F_K \simeq F_\eta ; \qquad (7)$$

and for the renormalization constants defined by

$$\langle 0|v_i(0)|P_j(\vec{k})\rangle \equiv Z_{ij}{}^{1/2}/[(2\pi)^3 2\omega_{\vec{k}}]^{1/2}, \qquad (8a)$$

where

$$Z_{ij}{}^{1/2}=Z_i{}^{1/2}\delta_{ij} \quad \text{for } i=1,2,\ldots,7 \qquad (8b)$$

they obtain

$$Z_{08}{}^{1/2}\simeq 0 \qquad (9a)$$

and

$$Z_\pi{}^{1/2}\simeq Z_K{}^{1/2}\simeq Z_\eta{}^{1/2}. \qquad (9b)$$

Or for the ratios

$$R\equiv F_K/F_\pi \qquad (10a)$$

and

$$\zeta\equiv Z_K{}^{1/2}/Z_\pi{}^{1/2}, \qquad (10b)$$

their results are

$$R\simeq 1 \qquad (11a)$$

and

$$\zeta\simeq 1. \qquad (11b)$$

The result given in Eq. (2a) is correlated by GOR with the relative smallness of the mass of the physical pion, which, if the parameter a were exactly equal to -1 [corresponding to no explicit breaking of chiral $SU(2)\times SU(2)$], would be a massless, Goldstone boson. The result given in Eq. (2b) is close to the limiting value $b=0$, for which the vacuum would be invariant under SU(3).

Recently, in a different scheme which also uses the $(3,3^*)+(3^*,3)$ model for broken chiral $SU(3)\times SU(3)$ symmetry, Brandt and Preparata[4] (BP) have instead found

$$a\simeq -0.17, \qquad (12a)$$

$$R\simeq 1.2, \qquad (12b)$$

and

$$\zeta\simeq 0.92 m_K{}^2/m_\pi{}^2\simeq 12. \qquad (12c)$$

The result in Eq. (12a) is close to the limiting value $a=0$, which corresponds to no explicit breaking of SU(3) symmetry. The possibility of $a\simeq 0$ was, of course, the original suggestion of Gell-Mann,[1] but is controverted in the GOR reformulation.

The disparity between the GOR and BP versions of Gell-Mann's model,

$$a_{\rm GOR}\simeq -0.89, \quad \zeta_{\rm GOR}\simeq 1, \quad b_{\rm GOR}\simeq 0, \qquad (13)$$

vs

$$a_{\rm BP}\simeq -0.17, \quad \zeta_{\rm BP}\simeq 0.92 m_K{}^2/m_\pi{}^2, \quad b_{\rm BP}\simeq ?, \qquad (14)$$

is due to their proximity to the limiting situations of $b=0$ vs $a=0$, SU(3)-invariant vacuum vs SU(3)-invariant Hamiltonian. Both GOR-like and BP-like results may be obtained as solutions of the GW model. We find this to be so in our analysis.

Already Auvil and Deshpande[5] have shown that the GW model admits, under the conditions of no η-η' mixing ($Z_{08}{}^{1/2}=0$) and of Gell-Mann–Okubo mass splitting for the 0^- octet, as possible limiting solutions

$$b=0, \quad a=-0.89, \quad \zeta=1, \quad R=1, \qquad (15)$$

and

$$a=0, \quad b=-0.89, \quad \zeta=m_K{}^2/m_\pi{}^2, \quad R=1. \qquad (16)$$

(We note that these values for ζ and R are not the only possibilities, and that we do not agree completely with their conclusions concerning the mass and leptonic decay constant of the κ meson.) The GOR results almost coincide with (15), and the BP results at least tend toward (16).

We consider herein what may be called a GW-like theory. By a GW theory we mean a theory of broken chiral symmetry which has three essential components: (1) Gell-Mann's hypothesis of $(3,3^*)+(3^*,3)$ transformation properties for the symmetry-breaking part of the hadron Hamiltonian, (2) the possibility of spontaneous breaking of both $SU(3)\times SU(3)$ and SU(3) symmetries, and (3) both 0^- and 0^+ mesons. The GOR and BP theories do not satisfy all these criteria: The scalar mesons and the SU(3) asymmetry of the vacuum have been either eliminated (GOR) or ignored (BP).

In GW theories, in order to get results for a, b, masses, etc., hitherto there have been two ways to get needed relations among the leptonic decay constants: Either spectral-function sum rules are used to determine R and the other relevant ratios of leptonic decay constants,[2,6] or the renormalization constants are constrained to be all equal due to various smoothness requirements[5,7] or all equal to unity in a tree-diagram approximation.[8,9] Whenever the renormalization constants are so constrained, certain of the GW equations reduce to useful relations among the leptonic decay constants. However, as may be expected from Eqs. (15) and as we shall see later, the constraint $\zeta=1$, or even $\zeta\simeq 1$, automatically prejudices a GW theory to favor a GOR world in which b is small and $a\simeq -1$.

Therefore, the fact that Dutt, Eliezer, and Nanda,[7] Carruthers and Haymaker,[8] and Olshansky[9] find that the limit of no explicit symmetry breaking, $\epsilon_0 \to 0$ and $\epsilon_8 \to 0$, is realized in the GOR manner is perhaps not really a prediction of their analyses, but rather a self-consistent result. [Actually Olshansky also finds another possibility, $\lambda_0 = \lambda_8 = 0$ and all $0^\pm$ masses degenerate (the normal solution), with the choice between the GOR solution and the normal solution to be settled ultimately by the value of the π-N σ commutator.] It would be very interesting to see, if possible, the results of an analysis as in Refs. 8 and 9 with the renormalization constants unconstrained.

Our theory is GW-like in that it contains the three essential components, except that the first component is modified, as will be discussed below. Because we are willing to make a commitment as to how the asymptotic (in- or out-) fields transform, we can obtain not only the GW equations, but also relations among the leptonic decay constants, so that we are able to solve the GW equations for arbitrary values of R, with no use of spectral-function sum rules and no prior constraint on ξ.

Although we do not believe that the hadron Hamiltonian transforms strictly as a $(3,3^*)+(3^*,3)$ representation,[10,11] we nevertheless determine the consequences of assuming that, at least for vacuum-to-one-spinless-meson and vacuum-to-vacuum matrix elements of the commutators $[G_A,H]$ and $[G_A,[G_A,H]]$, respectively [where G_A is a generator of chiral SU(3)×SU(3)], we can take H' to *effectively* belong to a $(3,3^*)+(3^*,3)$ representation [i.e., we use Eqs. (1) only under these circumstances].

We limit our attention to the nonets of $0^\pm$ mesons, and we assume that, even when the hadron Hamiltonian is asymmetric, we can take the transformations of the asymptotic fields of the $0^\pm$ mesons to be the same, up to the terms linear in the fields, as they are[12] when H is invariant. If H were invariant, then, of course, the transformations of asymptotic fields are necessarily purely linear – since the Hamiltonian when expressed in terms of asymptotic fields (which are free) is bilinear, no local nonlinear transformations of the fields can leave it invariant. When H is not required to be invariant, we can no longer expect the transformations of asymptotic fields to be at most linear (or even purely local), but, for simplicity, we assume that the coefficients of the leading local terms are not appreciably modified.[13]

The spontaneous breakdown of the symmetry is introduced[12] via the appearance, in the commutators of the generators with the asymptotic fields, of c numbers C_0 and C_8 added to the $I=Y=0$ scalar asymptotic fields $S_0(x)$ and $S_8(x)$: $C_0=C_8=0$ is required for a chiral-SU(3)×SU(3)-invariant vacuum, $C_8=0$ for an SU(3)-invariant vacuum, $C_8=2\sqrt{2}C_0$ for a chimeral-SU(3)-invariant[14] vacuum, and $C_8=-\sqrt{2}C_0$ for a chiral-SU(2)×SU(2)-invariant vacuum. These c numbers are expressible[12] in terms of the leptonic decay constants of the pseudoscalar nonet and of the κ meson, so that, for example, $C_8=0$ implies $F_\kappa=0$ and $F_\pi=F_K$, etc. Since there are only two c numbers, there can only be two independent leptonic decay constants among the five that appear; and it is just the consequent relations among the five leptonic decay constants which enable us to go beyond the results of the usual GW theories, without constraining the renormalization constants.

The effects of the existence of the c numbers *in the limit of no explicit breaking of chiral SU(3) ×SU(3) symmetry,* in which case the nonlinear and/or nonlocal terms in the commutators of the generators with the asymptotic fields [the unwritten terms indicated by the dots in Eqs. (22)] vanish, can be determined by using the explicit expression for H in terms of asymptotic fields to determine the implications of $0=\dot{G}_A=-i[G_A,H]$ (and also by examining $\langle 0|[G_A,\phi_i]|0\rangle$,[15] where ϕ_i is an asymptotic field). In the case of no mixing, for the SU(3) generator T_A, we have

$$\dot{T}_A = f_{Aij}\int d\vec{x}(M_i^2 C_j S_i + m_i^2 P_i P_j + M_i^2 S_i S_j)\,,$$

while for the chiral generator X_A, we have

$$\dot{X}_A = d_{Aij}\int d\vec{x}[m_i^2 C_j P_i + (m_i^2 - M_j^2)P_i S_j]\,.$$

Requiring the coefficients of the bilinear terms in $\dot{T}_A$ to vanish implies mass degeneracy within the 0^- and 0^+ octets: $\dot{T}_\kappa=0$ implies $m_\pi^2=m_K^2$, $m_K^2=m_\eta^2$, $M_\delta^2=M_\kappa^2$, and $M_\kappa^2=M_\sigma^2$; requiring the bilinear coefficients in $\dot{X}_A$ to vanish implies mass degeneracy within the entire $0^\pm$ chiral multiplet: $\dot{X}_\pi=0$ implies $m_\pi^2=M_\sigma^2$, $m_\pi^2=M_{\sigma'}^2$, $m_K^2=M_\kappa^2$, $m_\eta^2=M_\delta^2$, and $m_{\eta'}^2=M_\delta^2$; $\dot{X}_K=0$ implies $m_\pi^2=M_\kappa^2$, $m_K^2=M_\delta^2$, $m_K^2=M_\sigma^2$, $m_K^2=M_{\sigma'}^2$, $m_\eta^2=M_\kappa^2$, and $m_{\eta'}^2=M_\kappa^2$; and $\dot{X}_8=0$ implies $m_\pi^2=M_\delta^2$, $m_K^2=M_\kappa^2$, $m_\eta^2=M_\sigma^2$, $m_\eta^2=M_{\sigma'}^2$, and $m_{\eta'}^2=M_\sigma^2$. These are, admittedly, unsurprising results, since, *when H is invariant,* we assign the $0^\pm$ asymptotic fields to a linear representation. Since the c numbers are expressible in terms of the leptonic decay constants, requiring the coefficients of the linear terms in $\dot{G}_A$ to vanish implies that either the $0^\pm$ mesons are massive and their leptonic decay constants are zero (the normal case) or the $0^\pm$ mesons are massless and their leptonic decay con-

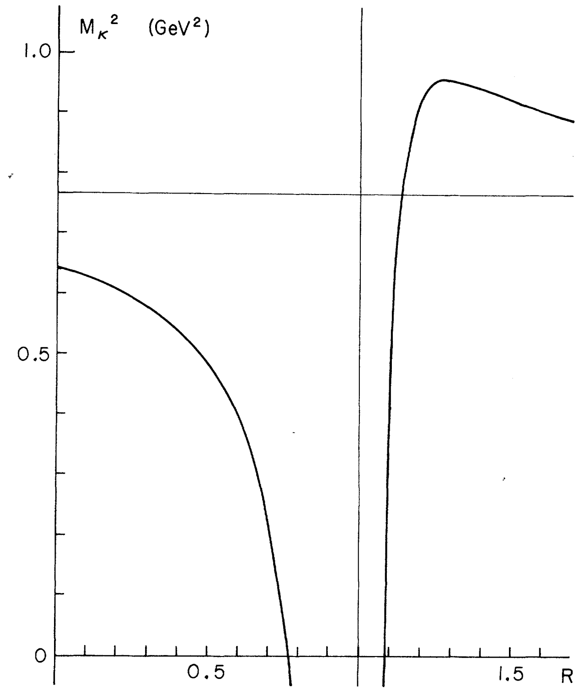

FIG. 1. Plot of $M_K{}^2$ vs R for case I, mixing angle $\theta = 0$.

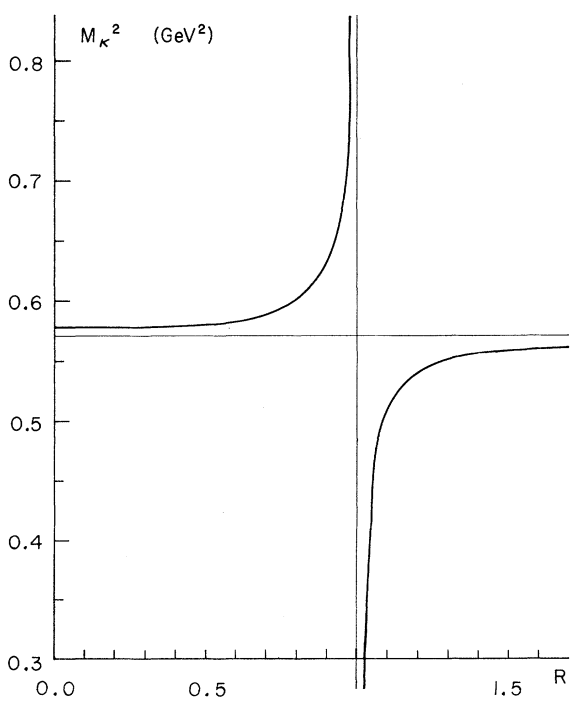

FIG. 2. Plot of $M_K{}^2$ vs R for case II, mixing angle $\theta = 10.6°$.

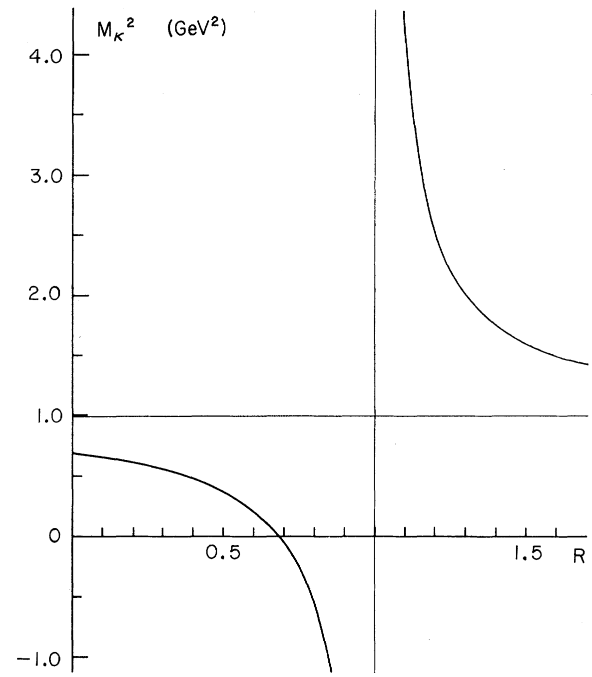

FIG. 3. Plot of $M_K{}^2$ vs R for case III, mixing angle $\theta = -10.6°$.

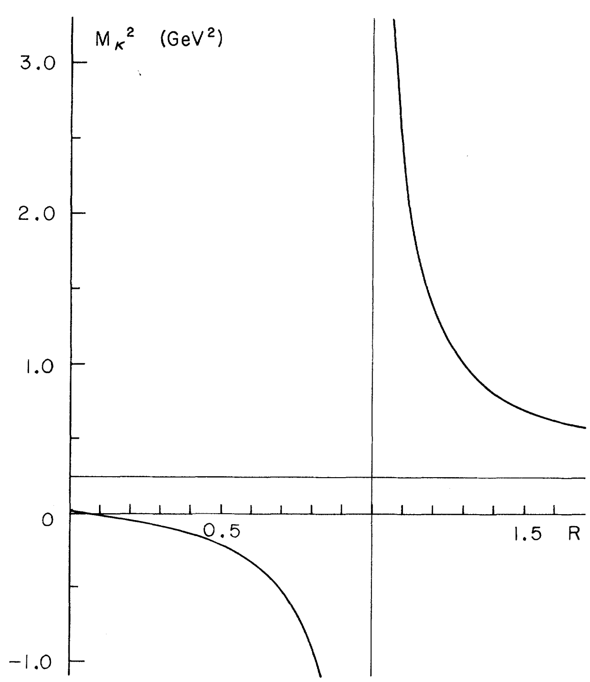

FIG. 4. Plot of $M_K{}^2$ vs R for case IV, $\zeta_1 = 1$ and $\zeta_2 = m_K{}^2/m_\pi{}^2$.

TABLE I. Results for case I, mixing angle $\theta=0$.

R	M_κ^2 (GeV^2)	ζ_1	ζ_2	$a_1=b_2$	$b_1=a_2$	$C_8/\sqrt{2}C_0$
1.18	0.898	0.932	14.0	−0.912	−0.0622	−0.107
1.23	0.951	0.949	13.7	−0.914	−0.100	−0.133
1.28	0.961	0.975	13.4	−0.915	−0.142	−0.157
1.34	0.956	1.02	12.8	−0.915	−0.194	−0.185

TABLE III. Results for case III, mixing angle $\theta=-10.6°$.

R	M_κ^2 (GeV^2)	ζ_1	ζ_2	$a_1=b_2$	$b_1=a_2$	$C_8/\sqrt{2}C_0$
1.18	2.70	1.17	11.1	−0.890	−0.204	−0.107
1.23	2.33	1.27	10.3	−0.886	−0.272	−0.133
1.28	2.09	1.41	9.28	−0.879	−0.348	−0.157
1.34	1.90	1.65	7.90	−0.865	−0.448	−0.185

stants need not be zero or equal (the spontaneous breakdown case). Note that if *any* subsymmetry is spontaneously broken (i.e., the relevant leptonic decay constant is different from zero), *all* eighteen $0^\pm$ mesons are massless due to the degeneracy. Our view of the limit of no explicit symmetry breaking differs from the popular view due to GOR,[3] who have only π, K, and η mesons massless, with $F_\pi = F_K \neq 0$ and $F_\kappa = 0$. But, as we have mentioned before, they are working in a theory where η' and 0^+ mesons and SU(3) vacuum asymmetry are neglected.

Upon deriving the GW equations and our relations among the leptonic decay constants, we are able to obtain equations for M_κ^2, for ζ, for a, for b, and for $C_8/\sqrt{2}C_0$, which require as basic input data only the π, K, η, and η' masses and the F_K/F_π ratio R. For the κ-mass equation, we consider four cases: (I) no mixing; (II) positive mixing angle; (III) negative mixing angle; and (IV) $\zeta=1$, or $\zeta=m_K^2/m_\pi^2$, constraints which yield a well-known equation which expresses M_κ^2 simply in terms of m_K^2, m_π^2, and R. We exhibit plots (see Figs. 1–4) of M_κ^2 as a function of R, showing the existence of certain upper and lower bounds for M_κ^2 and showing that certain values of R correspond to negative M_κ^2 and are consequently forbidden, and also numerical results (see Tables I–IV), using a few interesting values of R, for M_κ^2, ζ, a, b, and $C_8/\sqrt{2}C_0$.

Since a general GW theory with no constraints on ζ yields a quadratic equation for ζ,[16] we obtain two solutions: (1) a solution similar to that favored by GOR: $\zeta_1 = O(1)$; and (2) a solution similar to that favored by BP: $\zeta_2 = O(m_K^2/m_\pi^2)$. Corresponding to these solutions, we get, e.g., with no mixing and with $R=1.28$, the results $a_1 \simeq -0.91$, $b_1 \simeq -0.14$ and $a_2 \simeq -0.14$, $b_2 \simeq -0.91$; we also get $C_8/\sqrt{2}C_0 \simeq -0.16$, this result being independent of the value of ζ. The fact that $a_1=b_2$ and $a_2=b_1$ is no accident – it can be shown that there is a single equation relating a and b with only m_π, m_K, m_η, and $m_{\eta'}$ as parameters, which is symmetric under the interchange $a \leftrightarrow b$. This same symmetry in the a-b plane is manifested in the Mathur-Okubo plot[14] of the allowed domains for broken chiral symmetry in the GW model.

We can determine under what circumstances the simple case IV ($\zeta=1$ or $\zeta=m_K^2/m_\pi^2$) could serve as an approximation to cases I, II, or III (mixing angle $\theta=0$, $\theta>0$, or $\theta<0$). We find that, for reasonable values of R, case IV can approximate only case I (and then only over a very short range); specifically, we obtain

$$\text{for } R \simeq 1.32\,, \tag{17a}$$

$$(M_\kappa^2)_{\text{IV}} \simeq (M_\kappa^2)_{\text{I}} \simeq 0.96 \text{ GeV}^2\,, \tag{17b}$$

$$\zeta_1 \simeq 1.00 \text{ and } \zeta_2 \simeq 13.05 = m_K^2/m_\pi^2\,, \tag{17c}$$

$$a_1 = b_2 \simeq -0.91 \text{ and } b_1 = a_2 \simeq -0.18\,, \tag{17d}$$

$$C_8/\sqrt{2}C_0 \simeq -0.18\,. \tag{17e}$$

However, the mass required, $M_\kappa \simeq 980$ MeV, lies between the values, ~890 MeV and ~1200 MeV, which have been suggested.[17]

Although our preliminary results, Eqs. (29), (33), and (35), were also obtained by Auvil and Deshpande,[5] apart from their limiting GOR-like and BP-like solutions, our Eqs. (15) and (16), their further work is restricted by the condition $\zeta=1$.

II. THEORY

Since the hadron Hamiltonian, when expressed in terms of the asymptotic fields, must describe free hadrons, for the $0^\pm$ mesons we must have

$$H = (\text{kinetic terms}) + H'\,, \tag{18}$$

TABLE II. Results for case II, mixing angle $\theta=10.6°$.

R	M_κ^2 (GeV^2)	ζ_1	ζ_2	$a_1=b_2$	$b_1=a_2$	$C_8/\sqrt{2}C_0$
1.18	0.540	0.896	14.6	−0.915	−0.0368	−0.107
1.23	0.547	0.886	14.7	−0.919	−0.0564	−0.133
1.28	0.551	0.881	14.8	−0.923	−0.0785	−0.157
1.34	0.555	0.881	14.8	−0.926	−0.107	−0.185

TABLE IV. Results for case IV, $\zeta_1=1$ and $\zeta_2=m_K^2/m_\pi^2$.

R	M_κ^2 (GeV^2)	ζ_1	ζ_2	$a_1=b_2$	$b_1=a_2$	$C_8/\sqrt{2}C_0$
1.18	1.51	1	13.05	−0.906	−0.107	−0.107
1.23	1.23	1	13.05	−0.909	−0.133	−0.133
1.28	1.06	1	13.05	−0.913	−0.157	−0.157
1.34	0.914	1	13.05	−0.916	−0.185	−0.185

with

$$H' = \int d\vec{x}\, \tfrac{1}{2}(m_\pi^2\pi^2 + m_K^2K^2 + m_\eta^2\eta^2 + m_{\eta'}^2\eta'^2 + M_\delta^2\delta^2 + M_\kappa^2\kappa^2 + M_\sigma^2\sigma^2 + M_{\sigma'}^2\sigma'^2)\,. \tag{19}$$

The hadron Hamiltonian as given by Eqs. (18) and (19) is exactly equal to the expression for the hadron Hamiltonian in terms of interpolating fields.

In terms of the SU(3) bases, we have

$$H' = \int d\vec{x}\, \tfrac{1}{2}(m_i^2P_i^2 + \Delta P_0P_8 + M_j^2S_j^2 + \Delta_S S_0S_8)\,, \tag{20}$$

where there is a sum on i and j from 0 to 8 and where we have introduced singlet-octet mixing:

$$P_8 = \eta\cos\theta + \eta'\sin\theta\,,$$

$$P_0 = -\eta\sin\theta + \eta'\cos\theta\,,$$

$$m_8^2 = m_\eta^2\cos^2\theta + m_{\eta'}^2\sin^2\theta\,,$$

$$m_0^2 = m_\eta^2\sin^2\theta + m_{\eta'}^2\cos^2\theta\,,$$

and

$$\Delta = (m_{\eta'}^2 - m_\eta^2)\sin 2\theta\,,$$

with analogous equations for the scalar nonet. The question of whether or not we should use the Gell-Mann–Okubo mass formula in this theory will be discussed shortly. If we do not impose it, then not all the parameters introduced by mixing can be specified by knowing only the masses of π, K, η, and η' mesons; hence, we neglect mixing and put $\theta = 0$ (our case I). If we do impose it, then all the parameters introduced by mixing can be specified as usual,

$$m_8^2 = \tfrac{1}{3}(4m_K^2 - m_\pi^2)\,,$$

$$m_0^2 = m_\eta^2 + m_{\eta'}^2 - m_8^2\,,$$

and

$$\Delta = \pm 2[(m_{\eta'}^2 - m_8^2)(m_8^2 - m_\eta^2)]^{1/2}\,,$$

where the plus or minus sign corresponds to positive or negative mixing angle θ (cases II and III).

If Gell-Mann's hypothesis[1] is correct, then we should be able to equate the right-hand sides of Eqs. (1b) and (20). Actually the right-hand side of Eq. (20) contains a small amount of SU(3)×SU(3) singlet, but that is of no consequence since H' is only used in commutators with the generators. We put

$$\epsilon_0u_0 + \epsilon_8u_8 = \tfrac{1}{2}(m_i^2P_i^2 + \Delta P_0P_8 + M_j^2S_j^2 + \Delta_S S_0S_8)\,, \tag{21}$$

with the stipulation that this "equality" only be used as described in Sec. I.

The scalar and pseudoscalar densities in the theory are assigned to the $(3,3^*)+(3^*,3)$ representation. For the $0^\pm$ asymptotic fields, we make the assumption that

$$[T_A, S_i] = if_{Aij}(S_j + C_j) + \cdots\,, \tag{22a}$$

$$[T_A, P_i] = if_{Aij}P_j + \cdots\,, \tag{22b}$$

$$[X_A, S_i] = -id_{Aij}P_j + \cdots\,, \tag{22c}$$

and

$$[X_A, P_i] = id_{Aij}(S_j + C_j) + \cdots\,, \tag{22d}$$

where

$$C_j = \delta_{j0}C_0 + \delta_{j8}C_8 \tag{23}$$

is a c number. Since the Hamiltonian is not invariant, we admit the possibility of nonlinear terms (indicated by the dots). However, for the sake of simplicity, we tentatively retain the same coefficients for the leading terms as would be necessary were the Hamiltonian invariant.

If we believed Eq. (21) to be a strict equality, then we could use Eqs. (22) to compute $\langle K^-|[T_{K^+}, \dot{T}_{K^+}]|K^+\rangle$. This would yield the Gell-Mann–Okubo mass formula. However, we do not know *a priori* that the hadron Hamiltonian, Eqs. (18)–(20), contains no SU(3) representations higher than the octet. Hence we allow for the possibility that the Gell-Mann–Okubo mass formula may (cases II and III) or may not (case I) hold. For case IV, it makes no difference.

The c numbers are necessarily present in the above commutators, Eqs. (22), because, for example,

$$\langle 0|[X_\pi(x_0), \pi(x)]|0\rangle = [\langle 0|A_0^\pi(0)|\pi(\vec{0})\rangle - \langle\pi(\vec{0})|A_0^\pi(0)|0\rangle][(2\pi)^3/2m_\pi]^{1/2} = iF_\pi\,. \tag{24}$$

Thus, from Eqs. (22a) and (22d) we obtain

$$F_\kappa = \tfrac{1}{2}\sqrt{3}\,C_8 \tag{25a}$$

and

$$F_{Ai} = d_{Aij}C_j\,, \tag{25b}$$

so that, with Eq. (23),

$$F_\pi = (\sqrt{2}\,C_0 + C_8)/\sqrt{3}\,, \tag{26a}$$

$$F_K = (\sqrt{2}\,C_0 - \tfrac{1}{2}C_8)/\sqrt{3}\,, \tag{26b}$$

$$C_8/\sqrt{2}\,C_0 = -2(R-1)/(2R+1)\,, \tag{27}$$

$$F_\pi = F_K + F_\kappa\,, \tag{28a}$$

$$F_{88} = \tfrac{1}{3}(4F_K - F_\pi)\,, \tag{28b}$$

and

$$F_{80} = \tfrac{2}{3}\sqrt{2}\,(F_\pi - F_K)\,. \tag{28c}$$

From Eq. (27) it is clear that the limits of invariance of the vacuum under SU(3), $R\to 1$, under chimeral SU(3), $R\to 0$, and under chiral SU(2) $\times$SU(2), $R\to\infty$, are realized when $C_8/\sqrt{2}\,C_0$ is equal to 0, 2, and –1, respectively. Equations (28) are just what enable us to go beyond the usual GW theories.

If we use Eq. (21) and the transformations of both the $0^\pm$ densities and the $0^\pm$ asymptotic fields to evaluate $\langle 0|[G_A,H]|1\rangle$, we get

$$\epsilon_\kappa Z_\kappa^{\;1/2}=M_\kappa^{\;2}F_\kappa\,, \tag{29a}$$

$$\epsilon_\pi Z_\pi^{\;1/2}=m_\pi^{\;2}F_\pi\,, \tag{29b}$$

$$\epsilon_K Z_K^{\;1/2}=m_K^{\;2}F_K\,, \tag{29c}$$

$$\epsilon_{88}Z_{88}^{\;1/2}+\epsilon_{80}Z_{08}^{\;1/2}=m_8^{\;2}F_{88}+\tfrac{1}{2}\Delta F_{80}\,, \tag{29d}$$

and

$$\epsilon_{88}Z_{80}^{\;1/2}+\epsilon_{80}Z_{00}^{\;1/2}=m_0^{\;2}F_{80}+\tfrac{1}{2}\Delta F_{88}\,, \tag{29e}$$

where we have used the definitions

$$\epsilon_{ij}\equiv d_{ijk}\epsilon_k\,, \tag{30a}$$

$$\epsilon_\pi\equiv\epsilon_{11}=\epsilon_{22}=\epsilon_{33}\,, \tag{30b}$$

$$\epsilon_K\equiv\epsilon_{44}=\epsilon_{55}=\epsilon_{66}=\epsilon_{77}\,, \tag{30c}$$

$$\epsilon_\kappa\equiv\tfrac{1}{2}\sqrt{3}\,\epsilon_8\,, \tag{30d}$$

and

$$Z_\kappa^{\;1/2}\equiv\langle 0|u_\kappa(0)|\kappa(\vec{k})\rangle[(2\pi)^3 2\omega_{\vec{k}}]^{1/2}\,. \tag{31}$$

If we expand[18] the scalar and pseudoscalar densities $u_i(x)$ and $v_i(x)$ in terms of asymptotic fields,

$$\begin{aligned}u_i(x)=\lambda_i&+Z_{ij}^{(S)\,1/2}S_j(x)\\&+\int dydz:P_j(y)P_k(z):F_{ijk}(xyz)\\&+\int dydz:S_j(y)S_k(z):G_{ijk}(xyz)+\cdots\end{aligned} \tag{32a}$$

and

$$\begin{aligned}v_i(x)=Z_{ij}^{\;1/2}P_j(x)&\\&+\int dydz:S_j(y)P_k(z):H_{ijk}(xyz)+\cdots\,,\end{aligned} \tag{32b}$$

and evaluate, again using the transformation properties of both the $0^\pm$ densities and the $0^\pm$ asymptotic fields, the matrix elements $\langle 0|[T,u]|0\rangle$ and $\langle 0|[X,v]|0\rangle$, we find

$$\lambda_\kappa=F_\kappa Z_\kappa^{\;1/2}\,, \tag{33a}$$

$$\lambda_\pi=F_\pi Z_\pi^{\;1/2}\,, \tag{33b}$$

$$\lambda_K=F_K Z_K^{\;1/2}\,, \tag{33c}$$

$$\lambda_{88}=F_{88}Z_{88}^{\;1/2}+F_{80}Z_{80}^{\;1/2}\,, \tag{33d}$$

and

$$\lambda_{80}=F_{80}Z_{00}^{\;1/2}+F_{88}Z_{08}^{\;1/2}\,, \tag{33e}$$

where we have used the definitions

$$\lambda_{ij}\equiv d_{ijk}\lambda_k\,, \tag{34a}$$

$$\lambda_\pi\equiv\lambda_{11}=\lambda_{22}=\lambda_{33}\,, \tag{34b}$$

$$\lambda_K\equiv\lambda_{44}=\lambda_{55}=\lambda_{66}=\lambda_{77}\,, \tag{34c}$$

and

$$\lambda_\kappa\equiv\tfrac{1}{2}\sqrt{3}\,\lambda_8\,. \tag{34d}$$

Although Eqs. (29) are, within the context of our assumptions, exact, in Eqs. (33) we have neglected effects coming from commutators of the generators with the bilinear and higher terms in the expansions for u and v. The terms we have neglected can be shown to arise from multiparticle intermediate states. How Eqs. (33) can be derived by considering only single-particle intermediate states has been demonstrated by Auvil and Deshpande.[5] That our approximation is sufficient for our purposes will become clear shortly. In a theory with the renormalization constants constrained to be all diagonal and to be all equal or all equal to unity, Eqs. (33) simplify and imply the very useful Eqs. (28). In the limit of no explicit symmetry breaking, Eqs. (33) are just a straightforward application of a general formula derived in Ref. 15, and are exact.

Eliminating the renormalization constants by simultaneous use of Eqs. (29) and (33) yields

$$\epsilon_\kappa\lambda_\kappa=M_\kappa^{\;2}F_\kappa^{\;2}\,, \tag{35a}$$

$$\epsilon_\pi\lambda_\pi=m_\pi^{\;2}F_\pi^{\;2}\,, \tag{35b}$$

$$\epsilon_K\lambda_K=m_K^{\;2}F_K^{\;2}\,, \tag{35c}$$

and

$$\epsilon_{88}\lambda_{88}+\epsilon_{80}\lambda_{80}=m_8^{\;2}F_{88}^{\;2}+m_0^{\;2}F_{80}^{\;2}+\Delta F_{88}F_{80}\,. \tag{35d}$$

Even in a theory with the renormalization constants constrained, these relations still hold, of course. It is interesting to note that these same equations (35) could have been derived directly by evaluating, using Eq. (21) and the transformation properties of both the $0^\pm$ densities and the $0^\pm$ asymptotic fields, the matrix elements $\langle 0|[T_\kappa,\dot{T}_\kappa]|0\rangle$, $\langle 0|[X_\pi,\dot{X}_\pi]|0\rangle$, $\langle 0|[X_K,\dot{X}_K]|0\rangle$, and $\langle 0|[X_8,\dot{X}_8]|0\rangle$. Thus, the assumptions [Eqs. (21) and (22)] used in such a derivation and in the derivation of Eqs. (29) are clearly compatible with the assumption [Eqs. (22)] and one-particle-pole approximation used in the derivation of Eqs. (33).

By means of Eqs. (30) and (34) it is possible to show that Eqs. (35) can be combined into[19]

$$4(M_\kappa{}^2F_\kappa{}^2+m_K{}^2F_K{}^2) = 3(m_8{}^2F_{88}{}^2+m_0{}^2F_{80}{}^2+\Delta F_{88}F_{80})+m_\pi{}^2F_\pi{}^2\,. \tag{36}$$

Suppose we do not impose the Gell-Mann–Okubo mass formula and we take $\theta=0$ (our case I); if we put in the experimental 0^- masses and then, recalling Eqs. (28), study the limit $C_8\to 0$ and $|C_0|<\infty$ (i.e., $R\to 1$), which is the limit of an SU(3)-invariant vacuum, we find $M_\kappa{}^2\to -\infty$. On the other hand, suppose we do impose the Gell-Mann–Okubo mass formula and we consider $\theta\neq 0$ (our cases II and III); if we repeat the procedure, we find that Eq. (36) just degenerates into the Gell-Mann–Okubo mass formula while $M_\kappa{}^2$ drops out and is thus unspecified.

The explicit functional forms we find from Eq. (36) using Eqs. (28) and the experimental data[20] for the 0^- masses are: for $\theta=0$ (case I),

$$M_\kappa{}^2=0.767+\frac{0.110}{R-1}-\frac{0.0158}{(R-1)^2}\,; \tag{37a}$$

for $\theta>0$ (case II),

$$M_\kappa{}^2=0.571-\frac{0.00583}{R-1}\,; \tag{37b}$$

for $\theta<0$ (case III),

$$M_\kappa{}^2=0.990+\frac{0.308}{R-1}\,. \tag{37c}$$

In Figs. 1, 2, and 3 we have plotted these expressions for $M_\kappa{}^2$ as a function of R. At $R=1$, the curves exhibit the behavior described above. We can, if we wish, establish certain bounds on R since those values corresponding to $M_\kappa{}^2<0$ must be unphysical. Since the experimental value of $M_\kappa{}^2$ is still rather uncertain,[17] we attempt no prediction for R. However, we can rule out case II if we regard $R<1$ as unlikely.

Since Eqs. (30) imply $\epsilon_\pi=\epsilon_K+\epsilon_\kappa$, we conclude from Eqs. (29) that[2]

$$m_\pi{}^2F_\pi Z_\pi{}^{-1/2}=m_K{}^2F_KZ_K{}^{-1/2}+M_\kappa{}^2F_\kappa Z_\kappa{}^{-1/2}\,; \tag{38}$$

since Eqs. (34) imply $\lambda_\pi=\lambda_K+\lambda_\kappa$, we conclude from Eqs. (33) that[2]

$$F_\pi Z_\pi{}^{1/2}=F_KZ_K{}^{1/2}+F_\kappa Z_\kappa{}^{1/2}\,. \tag{39}$$

These equations are analogous in form to our earlier result

$$F_\pi=F_K+F_\kappa\,. \tag{28a}$$

If we put $Z_\pi{}^{1/2}=Z_K{}^{1/2}=Z_\kappa{}^{1/2}$ (i.e., $\zeta=1$), then Eq. (39) reduces to Eq. (28a), and Eq. (38) can be used to compute $M_\kappa{}^2$; if we put the various $Z^{1/2}$'s proportional to the corresponding masses squared (i.e., $\zeta=m_K{}^2/m_\pi{}^2$ and $Z_\kappa{}^{1/2}/Z_\pi{}^{1/2}=M_\kappa{}^2/m_\pi{}^2$), then Eq. (38) reduces to Eq. (28a), and Eq. (39) can be used to compute $M_\kappa{}^2$. Thus, for either $\zeta=1$ or $\zeta=m_K{}^2/m_\pi{}^2$, Eqs. (38) and (39) reduce to Eq. (28a) and to

$$M_\kappa{}^2=\frac{Rm_K{}^2-m_\pi{}^2}{R-1} =0.246+\frac{0.227}{R-1}\,. \tag{40}$$

In Fig. 4, we have $M_\kappa{}^2$ as a function of R for this case IV. At $R=1$, $M_\kappa{}^2$ is, as in cases II and III, unspecified.

It is more interesting, however, to eliminate $Z_\kappa{}^{1/2}$ and F_κ by combining Eqs. (38), (39), and (28a) to get a single quadratic equation for ζ in terms of $M_\kappa{}^2$, $m_K{}^2$, $m_\pi{}^2$, and R. The solutions are

$$\zeta=(m_\pi{}^2+m_K{}^2R^2+M_\kappa{}^2(1-R)^2\pm\{[m_\pi{}^2+m_K{}^2R^2+M_\kappa{}^2(1-R)^2]^2-4m_\pi{}^2m_K{}^2R^2\}^{1/2})/2m_\pi{}^2R\,. \tag{41}$$

We will denote the solution with the negative square root as ζ_1 and that with the positive as ζ_2. If we use the κ-mass equations, Eqs. (36) and (40), to determine $\lim_{R\to 1}M_\kappa{}^2(1-R)^2$, we find: For case I (no Gell-Mann–Okubo mass formula) and $R=1$ that $\zeta_1\simeq 1$ and $\zeta_2\simeq m_K{}^2/m_\pi{}^2$; for cases II and III (with Gell-Mann–Okubo mass formula) and $R=1$ that $\zeta_1=1$ and $\zeta_2=m_K{}^2/m_\pi{}^2$; and for case IV (without need for Gell-Mann–Okubo mass formula) and $R=1$ that $\zeta_1=1$ and $\zeta_2=m_K{}^2/m_\pi{}^2$. The results for case IV are, of course, just what we assumed in order to get Eq. (40).

Using Eqs. (29b), (29c), and (30a)–(30c), we get

$$a=-2(m_K{}^2R-m_\pi{}^2\zeta)/(2m_K{}^2R+m_\pi{}^2\zeta)\,; \tag{42}$$

using Eqs. (33b), (33c), and (34a)–(34c), we get

$$b=-2(R\zeta-1)/(2R\zeta+1)\,. \tag{43}$$

For case I and $R=1$, we have $a_1=b_2\simeq -0.9$ and $a_2=b_1\simeq -0.05$; for cases II, III, and IV and $R=1$, we have $a_1=b_2=-0.89$ and $a_2=b_1=0$. The GOR results (without mixing, but with the Gell-Mann–Okubo mass formula) seem to coincide with the solution (ζ_1, a_1, b_1) for cases II, III, and IV at $R=1$.

It is easy to verify, by use of Eq. (41) that $a_1=b_2$ and $a_2=b_1$ for any R. Alternatively, if one exploits Eqs. (35b)–(35d) in such a way as to eliminate R, one gets an equation (with only the 0^- masses as parameters) which is manifestly symmetric under the interchange $a\longleftrightarrow b$ and which for each b has a possibility of as many as two real solutions for a and vice versa. Therefore, $a_1=b_2$ and $a_2=b_1$ is not unexpected.

In Tables I–IV, we give the results for ξ, a, b, and $C_8/\sqrt{2}C_0$ for a few interesting values of R in the cases I–IV. It is obvious that, at least for $1.34 \geq R \geq 1.18$, solution 1 is more nearly GOR-like [weakly broken chiral SU(2)×SU(2) symmetry] while solution 2 is more nearly BP-like [weakly broken SU(3) symmetry].

By simultaneously solving Eqs. (40) and (37), we can find for which values of R case IV coincides with cases I–III. Cases IV and I coincide at $R \simeq 0.91$ ($M_K^2 < 0$) and at $R \simeq 1.32$ (further details were given in Sec. I); however, since near $R = 1.32$ Fig. 1 exhibits much more curvature than Fig. 4, it is clear that case IV can serve as an approximation to case I only over a very small range of R. Case IV and II coincide at $R \simeq 1.72$ (not interesting) and at $R = 1$. Cases IV and III coincide at $R \simeq 0.89$ ($M_K^2 < 0$) and at $R = 1$.

III. FINAL REMARKS

The results which follow when our Eqs. (28) hold, are obtainable because we have made a conjecture, Eqs. (22), as to how the asymptotic $0^{\pm}$ fields might transform to linear order. We have imposed no constraint on the renormalization constants.

It is now manifest how a GW-like theory of broken chiral SU(3)×SU(3) symmetry may admit either[3] chiral SU(2)×SU(2) or[4] SU(3) as a weakly broken subsymmetry. It should be clear that any argument[7–9] which appears to support the GOR realization of the limit of no explicit symmetry breaking, but has constrained the renormalization constants to be all equal or all equal to unity, has only obtained a self-consistent result.

It would be interesting to do, if possible, an analysis as in Refs. 8 or 9 without requiring the renormalization constants to be all equal, or at least to see what happens if all the renormalization constants are proportional to their corresponding masses squared (as in the BP scheme). In any event, it would be useful to see if contact with experimental results can be made without prior commitment to a specific viewpoint (i.e., neither necessarily a GOR theory nor necessarily a BP theory), in order to determine whether or not experiment is able to choose between a GOR world or a BP world in the context of such an impartial theory.

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