where

$$D=[2m_\sigma(m_\sigma^2-4\mu^2)^{1/2}\,2M_c(4M_c^2-4\mu^2)^{1/2}+2m_\sigma^2(m_\sigma^2-4\mu^2)+2(m_\sigma^2-2\mu^2)(4M_c^2-m_\sigma^2)]/(4M_c^2-m_\sigma^2)\,,\qquad M_c\neq\tfrac{1}{2}m_\sigma\,.$$

$$I'_{\mu M}(0)=\frac{M^2\mu^2}{(M^2-\mu^2)^3}\ln\frac{M^2}{\mu^2}-\frac{1}{2}\,\frac{M^2+\mu^2}{(M^2-\mu^2)^2}\,. \tag{A4}$$

$$I'_{\mu\mu}(0;M_c)=-\frac{1}{6\mu^2}\,\frac{(4M_c^2-4\mu^2)^{3/2}}{(4M_c^2)^{3/2}}\,. \tag{A5}$$

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Asymptotic Behavior of Dual Amplitudes with Mandelstam Analyticity

D. Atkinson*
Physikalisches Institut der Universität Bonn, Bonn, Germany

and

A. P. Contogouris and R. Gaskell†
Physics Department, McGill University, Montréal, Canada

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The asymptotic ($|s|\to\infty$) behavior of a number of dual, crossing-symmetric amplitudes with nonzero Mandelstam double-spectral functions is studied. The difficulties in obtaining Regge behavior in the right-hand half of the s plane are analyzed in detail.

I. INTRODUCTION

Two years ago, Suzuki[1] proposed a model for a four-point function that resembles the Veneziano *Ansatz,* but which has complex trajectory functions. More recently, a number of models[2-5] have been invented that have a curved double-spectral-function boundary, which can be made to coincide exactly with the Mandelstam φ^3 boundary, and even, if one is prepared to add two terms together, with the correct φ^4 boundary. However, these models have been shown to have Regge asymptotic behavior only as $\mathrm{Re}\,s\to-\infty$, i.e., in the left half of the complex s plane. The main purpose of this paper is to examine the asymptotic behavior of these models in the right half of the s plane.

We find that, for a very restricted range of values of t (namely, $|t|<s_0$, where s_0 is the physical threshold), the models of Refs. 3–5 have Regge behavior in all directions of the complex s plane (except possibly along the positive real axis). However, for $|t|>s_0$, these models develop singularities that spoil the Regge behavior in a sector of the s plane around the real axis, the angle of which depends on t, and which can be large for perfectly physical values of t. For the model of Ref. 2 we find that, in addition to the above trouble, there are extra complications that destroy Regge behavior in a fixed sector $|\arg s|\leqslant 30°$ for all values of t.

In Secs. II and III, we consider models of the general form

$$A(s,t)=\int_0^1 dz\, z^{-\alpha(s,z)}(1-z)^{-\alpha(t,1-z)}, \tag{1.1}$$

where $\alpha(s,z)$ is a continuous function of z in $[0,1]$, with the property

$$\alpha(s,0)=\alpha(s)\equiv a+\lambda s+\Delta\alpha(s), \tag{1.2}$$

where

$$\Delta\alpha(s)=\frac{s}{\pi}\int_{s_0}^{\infty}\frac{ds'\,\mathrm{Im}\alpha(s')}{s'(s'-s)}, \tag{1.3}$$

while

$$\alpha(s,1)=a+\lambda s. \tag{1.4}$$

Technically, $\alpha(s,z)$ is said to be a homotopy of $\alpha(s)$, the trajectory function, onto its "linear part," i.e., $a+\lambda s$.

In Sec. II, we impose sufficient conditions on $\alpha(s,z)$, such that, for some $\beta(t)$,

$$A(s,t)\underset{|s|\to\infty}{\sim}\beta(t)(-s)^{\alpha(t)-1}, \tag{1.5}$$

for any fixed t [with $\mathrm{Re}\alpha(t)<1$ for simplicity], and in any direction in the left-hand half-plane of s, i.e., $\frac{1}{2}\pi<\varphi_s<\frac{3}{2}\pi$, where

$$\varphi_s\equiv\arg s. \tag{1.6}$$

In Sec. III, we consider the extra conditions that one must impose on $\alpha(s,z)$ in order to have the Regge behavior (1.5) also in the right-hand half-plane of s, with an infinitesimal wedge about the real axis deleted. At the end of Sec. III, then, we are armed with sufficient conditions for the behavior (1.5), for any fixed t, $\mathrm{Re}\alpha(t)<1$, and

$$|s|\to\infty,\quad \epsilon\le\varphi_s\le 2\pi-\epsilon. \tag{1.7}$$

We turn then to a consideration of the specific models, showing in Sec. IV that the model of Suzuki[1] satisfies the sufficient conditions of Sec. III, and therefore has Regge behavior in the right half s plane. In Sec. V we show that the model of Ref. 3 does not satisfy these sufficient conditions, unless t is in the small circle $|t|\le s_0$. For $|t|>s_0$, we show in detail that the amplitude does not have Regge behavior as $|s|\to\infty$, if

$$|\arg s|<\tfrac{1}{2}\pi-\tan^{-1}\left(\frac{\arg(t)}{\ln(|t|/s_0)}\right). \tag{1.8}$$

In the sector (1.8), the amplitude explodes exponentially as $|s|\to\infty$.

In Sec. VI, we show first that the model of Refs. 4–5 has exactly the same defect. We then show that the model of Ref. 2 is in an even more lamentable condition, in that there is not only a t-dependent forbidden sector of the s plane roughly like (1.8) (in detail a little different), but there is also trouble from a new source that causes non-Regge behavior in a fixed sector $|\arg s|\lesssim 30°$, irrespective of the value of t. Except for rather small values of $|t|$, the t-dependent sector will of course be greater than this, but in no case will there be Regge behavior inside this fixed sector. Finally, certain concluding remarks are presented in Sec. VII.

II. LEFT-HALF-PLANE BEHAVIOR

In this section, we consider the large-s behavior of (1.1), with

$$\tfrac{1}{2}\pi+\epsilon\le\varphi_s\le\tfrac{3}{2}\pi-\epsilon. \tag{2.1}$$

Since we shall require that

$$\frac{\alpha(s,z)}{\lambda s}\xrightarrow[|s|\to\infty]{}1 \tag{2.2}$$

uniformly with respect to z, $0\le z\le 1$, it follows that the phase of $\alpha(s,z)$ is asymptotically φ_s. Moreover, since $\mathrm{Re}\alpha(t,0)=\mathrm{Re}\alpha(t)<1$, we see that the integral (1.1) is well defined.

Make the change of variables

$$u=(1-z)|s| \tag{2.3}$$

so that

$$A(s,t)=\int_0^{|s|}\frac{du}{|s|}\left(1-\frac{u}{|s|}\right)^{-\alpha(s,1-u/|s|)}\left(\frac{u}{|s|}\right)^{-\alpha(t,u/|s|)}. \tag{2.4}$$

We would like to consider the limit $|s|\to\infty$, and conclude that (we employ the standard mathematical notation $A\sim B$, meaning $A/B\to 1$)

$$\alpha(s,1-u/|s|)\sim\lambda s,$$

so

$$\left(1-\frac{u}{|s|}\right)^{-\alpha(s,1-u/|s|)}\sim\exp(\lambda u e^{i\varphi_s}), \tag{2.5}$$

and that

$$\alpha(t,u/|s|)\sim\alpha(t,0)\equiv\alpha(t). \tag{2.6}$$

Hence

$$A(s,t)\sim|s|^{\alpha(t)-1}\int_0^{\infty}du\,u^{-\alpha(t)}\exp(\lambda u e^{i\varphi_s}). \tag{2.7}$$

This integral converges, since $\cos\varphi_s<0$. Set

$$w=\lambda u\exp[i(\varphi_s-\pi)] \tag{2.8}$$

and then rotate the w-integration contour by $\pi-\varphi_s$, thus bringing it down to the real, positive w axis. The result is

$$A(s,t)\sim(-\lambda s)^{\alpha(t)-1}\int_0^{\infty}dw\,w^{-\alpha(t)}e^{-w} =(-\lambda s)^{\alpha(t)-1}\Gamma(1-\alpha(t)). \tag{2.9}$$

The following points have to be checked in order to validate these manipulations:

(I) It will be observed that the asymptotic equalities (2.5) and (2.6) can be expected to be good only when $u \ll |s|$, so that they could not reasonably be maintained over the whole range of integration, $0 \leq u \leq |s|$. We shall split the integral (2.4) into two pieces:

$$\left(\int_0^{|s|^{1/2}} + \int_{|s|^{1/2}}^{|s|}\right) du \cdots .$$

For the first piece, $u/|s| \leq |s|^{-1/2} \ll 1$, and so (2.5) and (2.6) follow easily, being implied by the condition (2.2). The second piece of the integral will be shown to vanish faster than any inverse power of $|s|$, as $|s| \to \infty$, so that it can be neglected, in comparison with the first piece.

(II) The interchange in the order of the operations "$\lim_{|s|\to\infty}$" and the u integration in Eq. (2.1) must be justified. We shall show that, for suitable $\alpha(s, z)$,

$$\lim_{|s|\to\infty} \int_0^{|s|} du \cdots = \int_0^{\infty} du \lim_{|s|\to\infty} \cdots . \qquad (2.10)$$

(III) The rotation of the w contour in Eq. (2.9) must be justified.

We consider points I–III in order:

(I) Because of Eq. (2.2), and because $\cos\varphi_s < 0$, we can certainly find s_1 so large that

$$\mathrm{Re}[-\alpha(s, z)] > \tfrac{1}{2}\lambda |s| |\cos\varphi_s| \qquad (2.11)$$

for all $|s| \geq s_1$, and any z in $[0, 1]$. Hence, in particular, for $|s| \geq s_1$ and $|s|^{1/2} \leq u \leq |s|$, we have [we use the rigorous inequality $(1 - 1/x)^{ax} \leq e^{-a}$ for $a \geq 0$ and $x \geq 1$]

$$\left|\left(1 - \frac{u}{|s|}\right)^{-\alpha(s, 1-u/|s|)}\right| < (1 - |s|^{-1/2})^{\lambda |s| |\cos\varphi_s|/2} \leq \exp(-\tfrac{1}{2}\lambda |s|^{1/2} |\cos\varphi_s|) . \qquad (2.12)$$

Hence the piece of the integral (2.4) over the range $|s|^{1/2} \leq u \leq |s|$ vanishes faster than any inverse power of $|s|$, since the rest of the integral can contribute at most powers of $|s|$.

(II) For fixed t and φ_s, define $\sigma = |s|$, and

$$f(u, \sigma) = \sigma^{-\alpha(t)}\left(1 - \frac{u}{\sigma}\right)^{-\alpha(\sigma \exp(i\varphi_s), 1-u/\sigma)} \left(\frac{u}{\sigma}\right)^{-\alpha(t, u/\sigma)} , \qquad (2.13)$$

for $0 \leq u \leq \sigma$, and $f(u, \sigma) = 0$ for $u > \sigma$. Then Eq. (2.4) may be written

$$|s|^{-\alpha(t)+1} A(s, t) = \int_0^{\infty} du f(u, \sigma) . \qquad (2.14)$$

We want to find conditions such that

$$\lim_{\sigma\to\infty} \int_0^{\infty} du f(u, \sigma) = \int_0^{\infty} du \lim_{\sigma\to\infty} f(u, \sigma) . \qquad (2.15)$$

The following sufficient conditions may be enumerated:

(a) Both $\lim_{\sigma\to\infty} f(u, \sigma) \equiv f(u, \infty)$ and $\int_0^{\infty} du f(u, \infty)$ exist.

(b) The integral $\int_0^{\infty} du f(u, \sigma)$ converges uniformly with respect to σ.

(c) The function $f(u, \sigma)$ is a continuous function of σ, the continuity being uniform with respect to u.

Naturally, these conditions need only be satisfied in some fixed neighborhood of $\sigma = \infty$, say, $\sigma \geq s_1$.

It is easy to verify (a) in the present case, the convergence of the integral being assured at $u = \infty$, since $\cos\varphi_s < 0$, and at $u = 0$, since $\mathrm{Re}\alpha(t) < 1$. We have, in fact,

$$f(u, \sigma) \xrightarrow[u \text{ fixed}]{\sigma\to\infty} u^{-\alpha(t)} \exp(\lambda u e^{i\varphi_s}) ,$$

so we can find an s_1 so large that

$$|f(u, \sigma)| \leq 2 |u^{-\alpha(t)} \exp(\lambda u e^{i\varphi_s})| \qquad (2.16)$$

for all $\sigma \geq s_1$, $0 \leq u \leq \sigma^{1/2}$. Because of the bound (2.12), which holds for $\sigma^{1/2} \leq u \leq \sigma$, and the fact that $f(u, \sigma) = 0$ for $u > \sigma$, it follows that we can extend (2.16) to the whole range $0 \leq u < \infty$, and $\sigma \geq s_1$ (with a redefinition of s_1, if necessary). Since the right-hand side of the inequality (2.16) is independent of σ, and since its integral exists, the uniform (and absolute) convergence of the integral (b) follows immediately. The uniform continuity (c) can be readily assured by the requirement that $\alpha(s, z)/s$ be a continuous function of both s and z (separate continuity with respect to s and to z would not in general be enough), for $|s| \geq s_1$ and $0 \leq z \leq 1$.

(III) To validate the contour rotation, we observe that the integral in Eq. (2.7) has no singularities in the complex u plane, except for the points $u = 0$ and $u = \infty$. Accordingly, we write the integral in (2.7) as

$$\lim_{R\to\infty} \lim_{r\to 0} \int_r^R du \cdots \qquad (2.17)$$

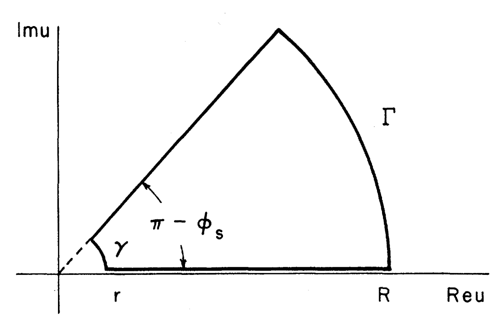

FIG. 1. Contour rotation for the integral (2.7).

and then we rotate the contour as shown in Fig. 1 (the circular sectors are necessary, in order to avoid the singularities at $u=0$ and $u=\infty$). It is easy to check that the contribution from Γ vanishes as $R\to\infty$, since $\cos\varphi_s<0$, and that that from γ vanishes as $r\to 0$, since $\mathrm{Re}\,\alpha(t)<1$. This concludes the validation of the contour rotation; and hence we have found sufficient conditions on $\alpha(s,z)$, such that the Regge behavior (2.9) is observed in the left-hand half of the s plane.

III. RIGHT-HALF-PLANE BEHAVIOR

We turn now to the case $|s|\to\infty$, $\mathrm{Re}\,s\not\to-\infty$. The main problem is that now the integral (1.1) diverges at $z=0$, and in fact it no longer represents the analytic continuation of $A(s,t)$. We shall first construct a new integral representation that is well defined in the sector

$$\epsilon\leqslant\varphi_s\leqslant\pi-\epsilon, \tag{3.1}$$

and which is equal to the right-hand side of (1.1) in the common domain of definition, namely,

$$\tfrac{1}{2}\pi+\epsilon\leqslant\varphi_s\leqslant\pi-\epsilon. \tag{3.2}$$

The new integral representation is then analyzed, as in the previous section, and we find that extra conditions have to be imposed on $\alpha(s,z)$, if the Regge behavior (2.9) is to be correct in the right half-plane of s. A similar treatment is possible for the lower half of the s plane, but this is not really necessary, since one has real analyticity in $s\times t$, and so the asymptotic behavior in the lower half-plane can be immediately inferred from that in the upper half-plane.

Make the transformation $x=-\ln z$ in the integral (2.1):

$$A(s,t)=\int_0^\infty dx\exp[-x+x\alpha(s,e^{-x})](1-e^{-x})^{-\alpha(t,1-e^{-x})}. \tag{3.3}$$

Following Suzuki,[1] we write this integral

$$\lim_{R\to\infty}\lim_{r\to 0}\int_r^R dx\cdots, \tag{3.4}$$

and then rotate the contour in the complex x plane by an angle $\theta<\frac{1}{2}\pi$ (just as in Fig. 1, except that the rotation angle is now θ, instead of $\pi-\varphi_s$). If no singularities are encountered, the integral becomes simply

$$\lim_{R\to\infty}\lim_{r\to 0}\left(\int_r^R e^{i\theta}dx'\cdots -\int_0^\theta iRe^{i\theta_1}d\theta_1+\cdots+\int_0^\theta ire^{i\theta_2}d\theta_2\cdots\right), \tag{3.5}$$

where we have set $x=x'e^{i\theta}$ in the first integral, $x=Re^{i\theta_1}$ in the second, and $x=re^{i\theta_2}$ in the third.

Consider, for the moment, only the sector (3.2). We will show that here the second and third integrals in Eq. (3.5) vanish in the respective limits $R\to\infty$, $r\to 0$, if certain extra conditions are satisfied by $\alpha(s,z)$. We can take the limits $R\to\infty$, $r\to 0$, inside the second and third integrals in Eq. (3.5), since these integrals are not improper in the sector (3.2). Consider the θ_1 integral first. We have already assumed that $\alpha(s,e^{-x})$ is an analytic function of x. We need in fact to suppose that there are no singularities in the sector $0\leqslant\arg(x)\leqslant\theta$, in order to justify the rotation (3.5), and eventually we would like to take $\theta=\frac{1}{2}\pi-\epsilon$. We need to impose another condition, namely, that

$$\alpha(s,e^{-x})\xrightarrow[x\to\infty]{}\alpha(s) \tag{3.6}$$

in all complex directions of the x plane such that $\arg(x)\leqslant\theta$. If $\alpha(s,z)$ is analytic at $z=0$, (3.6) will be implied by Eq. (1.2); but in many of the specific models, there is a singularity at this point, and then Eq. (3.6) must be verified separately. It follows then that

$$\alpha(s,\exp(-Re^{i\theta_1}))\underset{R\to\infty}{\sim}\alpha(s), \tag{3.7}$$

since $\theta_1\leqslant\theta\leqslant\frac{1}{2}\pi-\epsilon$. The θ_1 integral is hence asymptotically equal to

$$iR\int_0^\theta d\theta_1\exp\{e^{i\theta_1}[1+R\alpha(s)-R]\}. \tag{3.8}$$

Now in the sector (3.2), since $\alpha(s)\sim\lambda s$ as $|s|\to\infty$, the argument of the quantity between braces {} will be asymptotically $\theta_1+\varphi_s$, which means that the whole integral (3.8) will behave like a negative exponential of R, for s fixed and $|s|$ large enough and so it will vanish in the limit $R\to\infty$. Consider now the θ_2 integral in Eq. (3.5). The first factor in (3.3) tends to unity in this limit, while

$$[1-\exp(-re^{i\theta_2})]^{-\alpha(t,1-\exp(-re^{i\theta_2}))}\underset{r\to 0}{\sim}(re^{i\theta_2})^{-\alpha(t)}, \tag{3.9}$$

thanks to the condition (3.6). Hence the θ_2 integral will be asymptotically equal to

$$ir^{-\alpha(t)+1}\int_0^\theta d\theta_2\exp\{i\theta_2[1-\alpha(t)]\}, \tag{3.10}$$

which vanishes as $r\to 0$, since $\mathrm{Re}\,\alpha(t)<1$.

Finally, then, we have shown that, in the sector (3.2), and in the limit $r\to 0$, $R\to\infty$, only the x' integral in Eq. (3.5) remains. We have therefore (dropping the prime on x')

$$A(s,t)=e^{i\theta}\int_0^\infty dx\exp\{xe^{i\theta}[\alpha(s,\exp(-xe^{i\theta}))-1]\}\times[1-\exp(-xe^{i\theta})]^{-\alpha(t,1-\exp(-xe^{i\theta}))}. \quad (3.11)$$

This representation is equal to (3.3) in the sector (3.2), but converges throughout the sector (3.1), if we take $\theta=\frac{1}{2}\pi-\epsilon$. It is the required continuation into the first quadrant of the s plane. We note that it has already been assumed that there are no singularities in the right-hand half of the x plane.

The task now is to apply an asymptotic analysis, like that of Sec. II, to the representation (3.11). We need first to go back to the integration range $0\leq z\leq 1$ by the substitution $z=e^{-x}$; and then we change to $u=(1-z)|s|$, just as in Sec. II. The result is

$$A(s,t)=e^{i\theta}\int_0^{|s|}\frac{du}{|s|}\left(1-\frac{u}{|s|}\right)^{-1+Be^{i\theta}}\left[1-\left(1-\frac{u}{|s|}\right)^{e^{i\theta}}\right]^C, \quad (3.12)$$

where

$$B=1-\alpha(s,(1-u/|s|)^{e^{i\theta}}), \quad (3.13)$$

$$C=-\alpha(t,1-(1-u/|s|)^{e^{i\theta}}). \quad (3.14)$$

The required Regge behavior will follow if we can assert that, in the limit $|s|\to\infty$, in the sector (3.1), and for fixed u,

$$\alpha(s,(1-u/|s|)^{e^{i\theta}})\sim\lambda s, \quad (3.15)$$

so that

$$\left(1-\frac{u}{|s|}\right)^{-1+Be^{i\theta}}\sim\exp(\lambda ue^{i(\varphi_s+\theta)}). \quad (3.16)$$

This is a decreasing exponential in sector (3.1), since $\cos(\varphi_s+\theta)<0$. The last factor in (3.12) becomes asymptotically

$$\left(\frac{u}{|s|}e^{i\theta}\right)^{-\alpha(t)}, \quad (3.17)$$

so that, finally, with the substitution

$$w=\lambda u\exp[i(\varphi_s+\theta-\pi)], \quad (3.18)$$

and then a rotation of the w contour by $\pi-\varphi_s-\theta$, we recover precisely the expression (2.9). The justification for these steps is just as in Sec. II.

In concluding this section, let us bring together sufficient conditions on $\alpha(s,z)$ such that the above steps can be validated, as in Sec. II, but now for sector (3.1).

(a) We would like $\alpha(s,z)$ to be an analytic function of z. Some singularities can be tolerated, however, since it is enough if

$$\exp[x\alpha(s,e^{-x})] \quad (3.19)$$

and

$$(1-e^{-x})^{-\alpha(t,1-e^{-x})} \quad (3.20)$$

are analytic in x for

$$-\tfrac{1}{2}\pi+\epsilon\leq\arg(x)\leq\tfrac{1}{2}\pi-\epsilon, \quad (3.21)$$

except possibly for $x=0$ and $x=\infty$, where singularities may be permitted.

(b) $\alpha(s,z)/s$ should be a continuous function of both s and z, in complex as well as real directions. Although $\alpha(s,z)$ need not be analytic at $z=0$ and $z=1$, we must at least have continuity of $\alpha(s,e^{-x})$ at $x=0$ and $x=\infty$, in the sector (3.21). The continuity with respect to s may be limited to some neighborhood of infinity, say, $|s|\geq s_1$, but must include $s=\infty$. In particular,

$$\alpha(s,z)/s\xrightarrow[|s|\to\infty]{}\lambda. \quad (3.22)$$

IV. SUZUKI *ANSATZ*

We come now to our discussion of specific, ancestor-free models that have been proposed. One of the earliest models in the canonical form (1.1)–(1.4) is that of Suzuki.[1] His choice is

$$\alpha(s,z)=\alpha(s)-f(z)\Delta\alpha(s), \quad (4.1)$$

where $f(z)$ is a van der Corput neutralizer function, defined by

$$f(z)=g(z)/g(1), \quad (4.2)$$

with

$$g(z)=\int_0^z dy(-\ln y)^{\ln y}[-\ln(1-y)]^{\ln(1-y)}. \quad (4.3)$$

The neutralizer has the property

$$f(0)=0, \qquad f(1)=1 \quad (4.4a)$$

so that Eqs. (1.2) and (1.4) are satisfied. In addition,

$$\left(\frac{d}{dz}\right)^n f(z)=0 \text{ for } z=0 \text{ and } z=1; \quad n=1,2,3,\ldots. \quad (4.4b)$$

The neutralizer has branch points at $z=0$ and $z=1$, but is otherwise free of singularities, and so $\alpha(s,z)$ also has these properties.

If $f(z)$ is uniformized, it is found to have essential singularities at the images of $z=0$ and $z=1$. In fact, the variable $x=-\ln z$ of Eq. (3.3) goes part of the way toward such a uniformization. We have

$$g(e^{-x})=\int_x^\infty e^{-\xi}d\xi\,\xi^{-\xi}[-\ln(1-e^{-\xi})]^{\ln(1-e^{-\xi})}. \quad (4.5)$$

The important factor here is $\xi^{-\xi}$. Since we have to rotate the x contour by an angle $\theta\leq\frac{1}{2}\pi-\epsilon$, for the work in Sec. III, we shall have to deal with the quantity

$$(|\xi|e^{i\theta})^{-|\xi|e^{i\theta}}, \tag{4.6}$$

the modulus of which is

$$\exp\{-|\xi|(\ln|\xi|)\cos\theta+|\xi|\theta\sin\theta\}. \tag{4.7}$$

This vanishes as $|\xi|\to\infty$, for any $|\theta|\leq\frac{1}{2}\pi-\epsilon$, so that we have the continuity at $x=\infty$ in the sector (3.21), as required. Continuity at $x=0$ is easy to check, since it does not involve leaving the first Riemann sheet of the z plane. Finally, we have the necessary continuity in s, and in particular the limit (3.22), if we require $\Delta\alpha(s)$ to be continuous, and to satisfy in fact

$$\Delta\alpha(s)/s \underset{s\to\infty}{\longrightarrow} 0 \tag{4.8}$$

in complex, as well as in real, directions. The above analysis was essentially contained already in the paper of Suzuki; we have merely spelled it out in a little greater detail.

Franzen,[6] and independently Adjei *et al.*,[7] have proposed a model which also uses the van der Corput neutralizer (4.2)–(4.3). This model is

$$\int_0^1 dz\, z^{-a-\lambda s}(1-z)^{-a-\lambda t}f(z), \tag{4.9}$$

which is not in the canonical form. The resonances are δ functions, and in fact (4.9) is not supposed to be a complete amplitude, but only a part of it. Nevertheless, it is of interest to consider the asymptotics because they are indicative of various models with neutralizers. Away from the positive real s axis there is no trouble, since we have seen above that $f(e^{-x})$ is well behaved as $|x|\to\infty$, $\arg x\leq\frac{1}{2}\pi-\epsilon$. This confirms the asymptotic analysis of Franzen, who used a slightly different method. Nevertheless, the function (4.9) has some strange features. First, there are no singularities in the finite s plane, since the neutralizer ensures convergence of the integral at $z=0$, for all finite s. There is an essential singularity at $s=\infty$, and so there must be bad, non-Regge behavior in some neighborhood of the positive real axis (remember that the general proof only works for $\epsilon\leq\varphi_s\leq 2\pi-\epsilon$). One can get an idea of how bad the behavior must be by simplifying (4.9) a little. Suppose we set $t=-a/\lambda$ to remove one term, and replace $f(z)$ by $\hat{g}(z)/\hat{g}(1)$ with

$$\hat{g}(z)=\int_0^z dy(-\ln y)^{\ln y}, \tag{4.10}$$

since we are only interested in the integrand near $z=0$. In terms of the variable $x=-\ln z$, we have

$$\left\{[\hat{g}(1)]^{-1}\int_0^\infty dx\exp[-x(\ln x-a-\lambda s)]-1\right\}\Big/(1+a+\lambda s), \tag{4.11}$$

where a partial integration has been performed.

The behavior of this expression for large s can be estimated by the method of steepest descents, the result being

$$\frac{(2\pi)^{1/2}}{\lambda s\hat{g}(1)}\exp\left\{\frac{\lambda s+a-3}{2}+e^{\lambda s+a-3}\right\}, \tag{4.12}$$

which is rather frightening, to say the least.

V. A REPRESENTATION WITH CURVED SPECTRAL BOUNDARY

We turn to a recent model[3] that has the standard form (1.1), with

$$\alpha(s,z)=a+\lambda s+\frac{s}{\pi}\int_{s_0/(1-z)}^\infty ds'\,\frac{\mathrm{Im}\,\alpha(s')}{s'(s'-s)}. \tag{5.1}$$

The corresponding amplitude has a nonvanishing double-spectral function, with boundary

$$(s-s_0)(t-s_0)=s_0^{\,2}. \tag{5.2}$$

If one takes $s_0=4\mu^2$, this is inside the boundary for a φ^3 scalar theory, namely

$$(s-4\mu^2)(t-4\mu^2)=4\mu^4.$$

If instead one takes (5.1) for $\alpha(s,z)$, and for $\alpha(t,1-z)$ the same expression, but with t_0 in place of s_0 (and $1-z$ in place of z), and sets $s_0=4\mu^2$, $t_0=16\mu^2$, one finds the boundary

$$(s-4\mu^2)(t-16\mu^2)=64\mu^4,$$

which is exactly correct for the s-channel elastic wing of the spectral function in a φ^4 theory. To obtain the t-channel wing, and to restore s-t crossing, one would add a second term to the amplitude with $s_0=16\mu^2$ and $t_0=4\mu^2$ [this procedure would correspond to an interference model (rather than a dual model) and is mentioned simply as an alternative possibility]. We shall content ourselves however with a discussion of the simplest case, in which $s_0=t_0$.

It is not difficult to see, by the method of Sec. II, that Regge behavior is assured for $|s|\to\infty$, $\frac{1}{2}\pi+\epsilon\leq\varphi_s\leq\frac{3}{2}\pi-\epsilon$, and fixed t, $\mathrm{Re}\,\alpha(t)<1$, if $\mathrm{Im}\,\alpha(t)$ is a continuous function, such that

$$\frac{(\ln s)^{1+\epsilon}}{s}\mathrm{Im}\,\alpha(s)\underset{s\to\infty}{\longrightarrow}0. \tag{5.3}$$

This indeed is rather more than is necessary. However, to investigate the right-hand half of the s plane, as in Sec. III, we have to make the rotation in the x plane, which means that the s' contour in Eq. (5.1) will become complex. Thus we must require $\mathrm{Im}\,\alpha(s)$ to have some analyticity. We shall suppose that $\mathrm{Im}\,\alpha(s'+i\epsilon)$ is the boundary value on $s_0\leq s'<\infty$ of an analytic function that has no singularities in the right half-plane of s. The integrand of (1.1) will have, aside from the usual singularities at $z=0$ and $z=1$, a branch point at

$$z=z_s\equiv 1-s_0/s \tag{5.4}$$

arising from $\alpha(s,z)$, and one at

$$z=z_t\equiv t_0/t \tag{5.5}$$

arising from $\alpha(t, 1-z)$. We consider in turn the effects of these singularities on the x-plane rotation of Sec. III.

In terms of the variable $x=-\ln z$, the branch point (5.4) is mapped into the sequence of branch points at

$$x=x_s^{(n)}\equiv -\ln|z_s|-i(\varphi_{z_s}+2\pi n), \tag{5.6}$$

$n=0, \pm1, \pm2, \ldots$, where $\varphi_{z_s}=\arg z_s$. For $|s|$ large, and $\epsilon\leq\varphi_s\leq\frac{1}{2}\pi$, φ_{z_s} is positive, but small, and $|z_s|\to 1$ as $s\to\infty$. Hence these branch points, by themselves, would allow one to rotate the x contour almost to the imaginary axis. Precisely, given any $\epsilon>0$, one can find s_1 such that the sector $0\leq\arg x\leq\frac{1}{2}\pi-\epsilon$ is free of the singularities $x_s^{(n)}$, for all s such that $|s|\geq s_1$. The branch points $x_s^{(n)}$ are shown in Fig. 2, for a typical value of s ($|s|$ large).

The branch point (5.5) can cause trouble, however. This is mapped into the sequence of branch points at

$$x=x_t^{(k)}\equiv\ln\frac{t}{t_0}+2i\pi k, \tag{5.7}$$

$k=0, \pm1, \pm2, \ldots$. For $|t|\leq t_0$, $\mathrm{Re}\,x_t^{(k)}\leq 0$, and so there is no problem; but for $|t|>t_0$, the branch points (5.7) are in the right-hand half of the x plane, and interfere with the rotation of the x contour (see Fig. 2). However, we can still rotate by an angle $\theta^{(0)}$ as before (see Fig. 3), with

$$\tan\theta^{(0)}=\varphi_t\Big/\ln\frac{|t|}{t_0}, \tag{5.8}$$

where $\varphi_t=\arg t$, which is enough to guarantee Regge

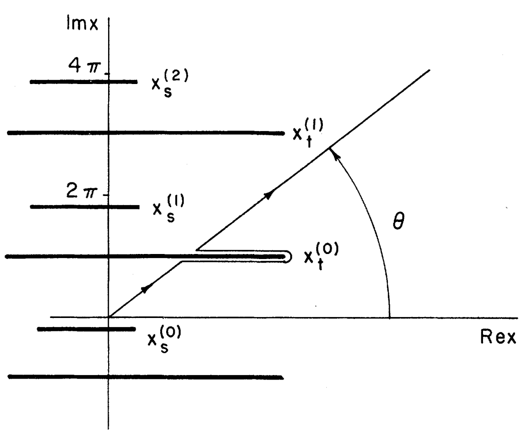

FIG. 2. The branch points $x_s^{(n)}$ and $x_t^{(n)}$ and the x-contour rotation.

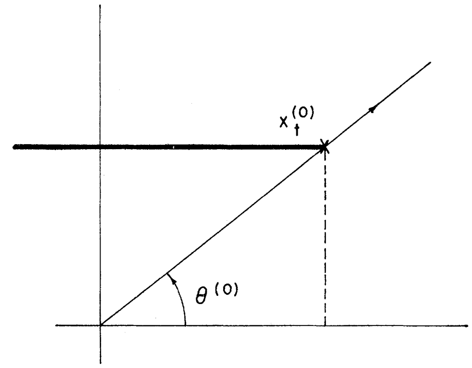

FIG. 3. Characterization of $\theta^{(0)}$.

behavior, as in Sec. III, for the sector

$$\pi\geq\varphi_s>\tfrac{1}{2}\pi-\theta^{(0)} \tag{5.9}$$

in the s plane. To obtain the analytic continuation into the sector

$$\tfrac{1}{2}\pi-\theta^{(0)}\geq\varphi_s\geq\epsilon, \tag{5.10}$$

we have to rotate by an angle $\theta>\theta^{(0)}$, as in Fig. 2, and this means that we must wrap the contour around the branch cut. We now show that the asymptotic analysis of Sec. III does *not* apply, and that in fact one does not have Regge behavior in the sector (5.10).

Let us take the angle of rotation of the contour, θ, such that

$$\theta>\tfrac{1}{2}\pi-\varphi_s>\theta^{(0)}, \tag{5.11}$$

as in Fig. 4. Then

$$\varphi_s+\theta>\tfrac{1}{2}\pi,\quad\text{so}\quad\cos(\varphi_s+\theta)<0 \tag{5.12}$$

and

$$\varphi_s+\theta^{(0)}<\tfrac{1}{2}\pi,\quad\text{so}\quad\cos(\varphi_s+\theta^{(0)})>0\,. \tag{5.13}$$

We will break up the integral of Fig. 4 into three

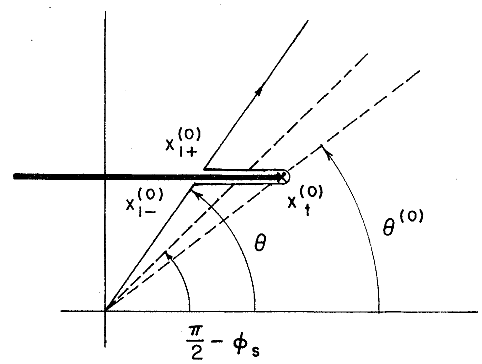

FIG. 4. Integration contour for Eq. (5.14).

pieces:

$$A=\left\{\int_{x_{1+}^{(0)}}^{\infty e^{i\theta}}+\int_0^{x_{1-}^{(0)}}+\oint_{\text{cut}}\right\}dx \times\exp[-x+x\alpha(s,e^{-x})]g(t,x) \equiv A_1+A_2+A_3, \quad (5.14)$$

where

$$g(t,x)=[1-e^{-x}]^{-\alpha(t,1-e^{-x})} \quad (5.15)$$

[see Eq. (3.3)]. We will show that

(a) $A_1\to 0$ faster than any inverse power, as $|s|\to\infty$;
(b) A_2 has Regge behavior;
(c) A_3 grows faster than any power, as $|s|\to\infty$, thus spoiling the asymptotic behavior.

In the integrals A_1 and A_2, set $x'=xe^{-i\theta}$, $z'=e^{-x'}$, and finally $u=(1-z')|s|$, as in Sec. III. The result is

$$A_1+A_2=e^{i\theta}\left\{\int_{p|s|}^{|s|}+\int_0^{p|s|}\right\}\frac{du}{|s|} \times\left\{1-\frac{u}{|s|}\right\}^{-1+Be^{i\theta}}\left\{1-\left(1-\frac{u}{|s|}\right)^{e^{i\theta}}\right\}^{C}, \quad (5.16)$$

where $p=(1-e^{-|x_1^{(0)}|})$, so $0<p<1$. The only difference between this equation and Eq. (3.12) of Sec. III is that here

$$C(s,t,u) \quad (5.17)$$

has a discontinuity in u at the point $u=p|s|$, corresponding to the jump over the cut of Fig. 4. This, however, makes no difference to the asymptotic analysis of A_1+A_2.

(a) The asymptotic equality (3.16) is still true, and so we can certainly find an s_1 so large that

$$\left|\left\{1-\frac{u}{|s|}\right\}^{-1+Be^{i\theta}}\right|\leq\exp[-\tfrac{1}{2}\lambda u|\cos(\varphi_s+\theta)|] \quad (5.18)$$

for all $|s|\geq s_1$, and so

$$|A_1|\leq|s|^{-1}\exp[-\tfrac{1}{2}\lambda p|s\cos(\varphi_s+\theta)|] \times\int_{p|s|}^{|s|}du\left|\left\{1-\left(1-\frac{u}{|s|}\right)^{e^{i\theta}}\right\}^{C}\right|, \quad (5.19)$$

which decreases faster than any power of $|s|^{-1}$, since the integral can contribute at most powers of $|s|$, just as in the discussion following Eq. (2.11).

(b) We can divide the integral A_2 into two pieces (for large $|s|$):

$$A_2=\left\{\int_0^{|s|^{1/2}}+\int_{|s|^{1/2}}^{p|s|}\right\}\cdots \quad (5.20)$$

and these integrals are treated just as in Sec. II and III. The first piece gives Regge behavior, and the second piece vanishes faster than any power of $|s|^{-1}$ Thus we have finally

$$A_2\sim(-\lambda s)^{\alpha(t)-1}\Gamma(1-\alpha(t)). \quad (5.21)$$

We note that the discontinuity in C has made no difference to this result, because it occurs at the point $u=p|s|$, which is not in an asymptotically significant part of the integrand. We have treated A_1+A_2 carefully, so that we can be quite sure that they cannot cancel the bad behavior of A_3, which we now elucidate:

(c) For the finite integral A_3, we stay in the x plane, and can take the limit $|s|\to\infty$ under the integral. The asymptotic equality (3.16) becomes

$$\exp[-x+x\alpha(s,e^{-x})]\sim\exp(\lambda sx) \quad (5.22)$$

so we have asymptotically

$$A_3\sim\int_{x_1^{(0)}}^{x_t^{(0)}}dx\exp(\lambda sx)\{g(t,x_-)-g(t,x_+)\}. \quad (5.23)$$

Unfortunately, although we see that

$$|\exp(\lambda sx_1^{(0)})|=\exp[-\lambda|sx_1^{(0)}\cos(\varphi_s+\theta)|] \quad (5.24)$$

decreases exponentially, in view of Eq. (5.12), we have

$$|\exp(\lambda sx_t^{(0)})|=\exp[+\lambda|sx_t^{(0)}\cos(\varphi_s+\theta^{(0)})|], \quad (5.25)$$

which explodes, in view of Eq. (5.13).

To establish this divergent behavior beyond all shadow of doubt, note that $x_1^{(0)}$, $x_t^{(0)}$, and $\{g(t,x_-)-g(t,x_+)\}$ are all independent of $|s|$. The point $x_t^{(0)}$ is a branch point of $g(t,x)$, so we cannot make a Taylor expansion about that point. We shall suppose that

$$\{g(t,x_-)-g(t,x_+)\}\equiv\gamma(t,x)\exp\{i\,\delta(t,x)\} \quad (5.26)$$

is not zero at $x=x_t^{(0)}$, and that there is some neighborhood of this point free from zeros (the following argument can be generalized to the case where $\gamma(t,x)$ tends to zero as $x\to x_t^{(0)}$ less quickly than $\exp[-|x-x_t^{(0)}|^{-1}]$). Consider the real part of a little piece of the integral (5.23) very near the end point:

$$\int_{x_t^{(0)}-\kappa}^{x_t^{(0)}}dx\;\gamma\exp\{\lambda|sx|\cos(\varphi_s+\varphi_x)\} \times\cos\{\lambda|sx|\sin(\varphi_s+\varphi_x)+\delta\}, \quad (5.27)$$

where

$$\varphi_x=\arg(x)\text{ and }\kappa=\frac{\eta}{|s|}\text{Re}\,x_t^{(0)}.$$

We can choose η small, but independent of $|s|$, such that the sign of this integrand does not change, so the integral will be, by the mean-value theorem,

$$\kappa\gamma(t, \bar{x}) \exp\{\lambda |s\bar{x}| \cos(\varphi_s + \bar{\varphi}_x)\} \times \cos\{\lambda |s\bar{x}| \sin(\varphi_s + \bar{\varphi}_x) + \delta(t, \bar{x})\}, \quad (5.28)$$

where $\bar{x}$ lies on the chord joining $x_t^{(0)}$ and $x_t^{(0)} - \kappa$. This contribution diverges exponentially, and cannot be canceled by the real part of the rest of the integral (5.23), since this has a weaker behavior.

In the next section we discuss the Cohen-Tannoudji–Henyey–Kane–Zakrzewski[2] (CHKZ) and Bugrij-Jenkovsky-Kobylinsky-Schmidt[4,5] (BJKS) *Ansätze*.

VI. THE CHKZ AND BJKS *ANSÄTZE*

The following model, also possessing a double-spectral function, was suggested by CHKZ[2]:

$$A(s, t) = \int_0^1 dz\, z^{-\alpha[s(1-z)]}(1-z)^{-\alpha[tz]} \times f[s(1-z)]f[tz], \quad (6.1)$$

where $\alpha[y]$ is a trajectory function satisfying (1.2)–(1.3), and $f[y]$ is a suitable function that decreases faster than any inverse power as $|y| \to \infty$, in all directions. One could take

$$f(y) = \exp[-\beta(s_0 - y)^{1/4}]. \quad (6.2)$$

This *Ansatz* could be cast (somewhat artificially) into the standard form (1.1) by identifying

$$\alpha(s, z) = \alpha[s(1-z)] - \ln f[s(1-z)]/\ln z. \quad (6.3)$$

However, one should note that $\alpha(s, 1)$ is not equal to $a + \lambda s$, nor does $A(s, t)$ reduce to the B function when $\mathrm{Im}\,\alpha(s) = 0$, as is the case for the standard form (1.1)–(1.4). Although (6.1) has Regge behavior for $\mathrm{Re}\,s \to -\infty$, this comes about in rather a different way from that studied in Sec. II, the presence of the function $f[y]$ being in fact essential.

An improvement of the CHKZ *Ansatz* was suggested by Bugrij, Jenkovsky and Kobylinsky,[4] and independently by Schmidt[5] (BJKS). This model has the standard form (1.1), with the choice

$$\alpha(s, z) = a + \lambda s + \Delta\alpha[s(1-z)], \quad (6.4)$$

which evidently does satisfy Eq. (1.4). Note that there is no function $f[y]$ in this case. The model is much closer in spirit to the *Ansatz* of Ref. 3; and we shall demonstrate briefly that is suffers from the same defect, namely the existence of a sector of non-Regge behavior about the positive real s axis, the half-angle of which depends on t in the same way as in Sec. V. Having done this, we will then turn to the CHKZ model (6.1) and show that this has two sources of non-Regge behavior, one of which is the same as in the BJKS case, and the other of which is specific to the CHKZ *Ansatz*.

The integrand of (1.1), with the BJKS form (6.4), has precisely the same branch points, as a function of z, as has the model of Ref. 3, namely at $z = 0, 1, z_s, z_t$ [see Eqs. (5.4) and (5.5)]. As in Sec. V, we can show that the branch point $z = z_s$ causes no trouble, but that z_t does cause difficulties in the sector (5.10), and one has to split the x plane integral into three pieces, as in Eq. (5.14). As before, one shows that $A_1 + A_2$ has Regge behavior, and that A_3 explodes faster than any power of $|s|$. The whole proof parallels that of Sec. V, the only difference being that Eq. (6.4) replaces Eq. (5.1), and that it is no longer necessary to suppose $\mathrm{Im}\,\alpha(s' + i\epsilon)$ to be the boundary value of an analytic function. It is enough for $\mathrm{Im}\,\alpha(s')/s'$ to be absolutely integrable on $s_0 \le s' < \infty$, and $\Delta\alpha(s)/s \to_{|s|\to\infty} 0$ [since one never considers $\varphi_s = 0$, the integral (1.3) is never singular, and so one does not need to require the Hölder continuity of $\mathrm{Im}\,\alpha(s')$].

We turn now to the CHKZ model (6.1). Here there is the difficulty that, if we use the variable $x = -\ln z$, as in Sec. III, the term $z^{-\alpha[s(1-z)]}$ in Eq. (6.1) becomes

$$\exp\{x[a + \lambda s(1 - e^{-x}) + \Delta\alpha(s(1 - e^{-x}))]\}. \quad (6.5)$$

For large $|s|$ and fixed x, this is asymptotically equal to

$$\exp[\lambda s x(1 - e^{-x})]. \quad (6.6)$$

If one makes the rotation in the x plane by the angle $\theta > \frac{1}{2}\pi - \varphi_s$, the expression (6.6) will certainly be indistinguishable from

$$\exp\{\lambda s x\} = \exp\{\lambda |sx| e^{i(\theta + \varphi_s)}\} \quad (6.7)$$

in the limit as $|x| \to \infty$, so that the integral will converge at $x = \infty$, since $\cos(\theta + \varphi_s) < 0$. However, the presence of the term e^{-x} in (6.6) means that, for some finite x, the phase of $sx(1 - e^{-x})$ could become less than $\frac{1}{2}\pi$, leading to non-Regge behavior from the corresponding parts of the x integral.

The neatest way to show this is to change the integration variable to $y = x(1 - e^{-x})$, or equivalently

$$y = -(1 - z)\ln z, \quad (6.8)$$

so that one obtains

$$A(s, t) = \int_0^\infty dy\, e^{\lambda s y} \frac{z}{z - 1 + z\ln z} z^{-a - \Delta\alpha[s(1-z)]} \times f[s(1-z)]h(t, y), \quad (6.9)$$

where

$$h(t, y) = (1 - z)^{-\alpha(tz)} f(tz), \quad (6.10)$$

and where z is to be considered as a function of y, through Eq. (6.8) (which has as inverse). The rotation through an angle θ will now be done in the y plane instead of the x plane. The term

$$\frac{z}{z-1+z\ln z}=\frac{dz}{dy} \tag{6.11}$$

is the Jacobian of the transformation $z\to y$, and will produce extra branch points at

$$y=y_n\equiv-(1-z_n)\ln z_n, \tag{6.12}$$

for $n=0, \pm1, \pm2, \ldots$, where z_n are solutions of

$$z_n-1+z_n\ln z_n=0\,. \tag{6.13}$$

Details of a numerical solution of Eqs. (6.12) and (6.13) are given in the Appendix. One has, in particular, $y_0=0$, which causes no trouble, but

$$y_{\pm1}\approx 2.6\pm 4.4i\,. \tag{6.14}$$

These fixed branch points mean that the y contour can be rotated only by $\theta\leqslant 60°$, so that one does not have Regge behavior for

$$\varphi_s<30° \tag{6.15}$$

for *any* value of t. Some of the branch points y_n are shown in Fig. 5, where it will be seen that $y_{\pm2}, y_{\pm3}, \ldots$ lie outside the sector defined by the origin and $y_{\pm1}$, and so cause no further trouble. The fixed, forbidden sector (6.15) is the price one pays for having the extra term e^{-x} in Eq. (6.6). It does not arise in the BJKS case, because here e^{-x} only occurs in $\Delta\alpha[s(1-e^{-x})]$ which is negligible in the limit $|s|\to\infty$, as compared with λs.

In addition to the fixed branch points, y_n, the integrand in Eq. (6.9) has branch points at $z=z_s$ and z_t [Eqs. (5.4) and (5.5)] which map into

$$y=y_s^{(n)}\equiv-(1-z_s)[\ln z_s+2\pi i n], \tag{6.16}$$

$n=0, \pm1, \pm2, \ldots$, and

$$y=y_t^{(n)}\equiv\left(1-\frac{t_0}{t}\right)\left[\ln\frac{t}{t_0}+2\pi i k\right], \tag{6.17}$$

$k=0, \pm1, \pm2, \ldots$. It is easy to see that the $y_s^{(n)}$ cause no trouble, as in Sec. V. We shall show now that the branch points $y_t^{(k)}$ give rise to a t-dependent sector in the s plane, in which Regge behavior is not observed, much as in Sec. V. There is some interest in examining this case in detail, since the branch points $y_t^{(k)}$ are not at the same positions as the $x_t^{(k)}$, and also the Regge behavior (when it obtains) comes about in a different manner.

For $|t|\gg t_0$, the $y_t^{(k)}$ are almost at the same position as the $x_t^{(k)}$, and so the critical angle $\theta^{(0)}$, defining the maximum permitted angle of rotation in the y plane before a branch point is encountered, will be approximately as in Eq. (5.8) and Fig. 3. One is not interested in values of $\theta^{(0)}$ in excess of 60°, because in such cases the rotation is stopped first by the fixed branch point y_1 [Eq. (6.14)]. For t real and negative, for example, $\theta^{(0)}=60°$ for $t\approx-4t_0$. For $|t|$ of the order of t_0, the $y_t^{(k)}$ are not even approximately equal to $x_t^{(k)}$. In fact, for $|t|<t_0$, when the $x_t^{(k)}$ are all decently in the left half-plane, there are some $y_t^{(k)}$ still in the right half-plane. The situation is quite complicated, since

$$\mathrm{Re}\,y_t^{(k)}=-\left[1-\frac{t_0}{|t|}\cos\varphi_t\right]\ln\frac{t_0}{|t|}-\frac{t_0}{|t|}\sin\varphi_t\,(\varphi_t+2\pi k), \tag{6.18a}$$

$$\mathrm{Im}\,y_t^{(k)}=\left[1-\frac{t_0}{|t|}\cos\varphi_t\right](\varphi_t+2\pi k)-\frac{t_0}{|t|}\sin\varphi_t\,\ln\frac{t_0}{|t|}\,. \tag{6.18b}$$

If $|t|\ll t_0$, $0\leqslant\varphi_t\leqslant\pi$, $\cos\varphi_t<|t|/t_0$, then $\mathrm{Re}\,y_t^{(k)}<0$,

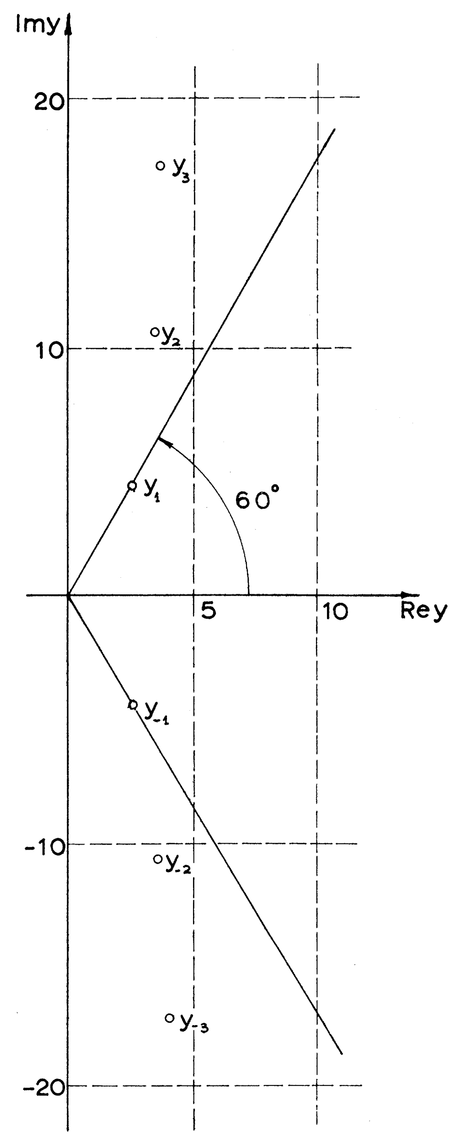

FIG. 5. Solutions of Eqs. (6.12) and (6.13).

unless k is very large and negative, but in that case $\mathrm{Im}\, y_t^{(k)} < 0$, so there is no trouble in either case. However, for smaller values of φ_t, or larger values of $|t|$ (but still less than t_0), some of the $y_t^{(k)}$ encroach upon the first quadrant of the y plane, as the reader may verify by considering Eq. (6.18) in detail. This source of non-Regge behavior for the corresponding sector of the s plane is in addition to the fixed sector of angle 30°, which applies for all values of t.

Finally, let us examine briefly the asymptotic behavior of the CHKZ amplitude, to check that a singularity in the first quadrant of the y plane does indeed mean non-Regge behavior in a sector of the s plane. We integrate along a contour as in Fig. 4 (but in the y plane instead of the x plane), and we split up the integration into three pieces A_1, A_2, A_3, as in Eq. (5.14). In the integrals A_1 and A_2, we set $y' = ye^{-i\theta}$ to obtain a real integration contour:

$$A_1 + A_2 = e^{i\theta}\left\{\int_{|y_1^0|}^{\infty} + \int_0^{|y_1^0|}\right\} dy' \times \exp[\lambda s y' e^{i\theta}]\, \frac{z}{z-1+z\ln z} \times z^{-a-\Delta\alpha[s(1-z)]} f[s(1-z)] h(t, y'e^{i\theta}). \tag{6.19}$$

It may be shown that A_1 vanishes faster than any inverse power of $|s|$. Indeed, if we divide the A_2 integration domain into the two pieces $0 \le y' \le |s|^{-3/2}$ and $|s|^{-3/2} \le y' \le |y_1^0|$, the contribution from the latter piece will also vanish faster than any inverse power of $|s|$, because of the factor $f[s(1-z)]$. For $0 \le y' \le |s|^{-3/2}$, and $|s|$ large, it follows that $y \approx 0$ and so from Eq. (6.8) we see that

$$y = (1-z)^2 + O[(1-z)^3]. \tag{6.20}$$

Asymptotically, in the above interval,

$$\exp[\lambda s y' e^{i\theta}] \to 1, \tag{6.21}$$

$$\frac{z}{z-1+z\ln z} z^{-a-\Delta\alpha[s(1-z)]} \sim (y'e^{i\theta})^{-1/2}, \tag{6.22}$$

$$f[s(1-z)] \sim f[s(y'e^{i\theta})^{1/2}], \tag{6.23}$$

$$h(t, y'e^{i\theta}) \sim f[t][y'e^{i\theta}]^{-\alpha(t)/2}, \tag{6.24}$$

so that finally we obtain

$$A_2 \sim e^{i[1-\alpha(t)]\theta/2} f(t) \int_0^{|s|^{-3/2}} dy' (y')^{-[1+\alpha(t)]/2} \times \exp\{-\beta[s_0 - s(y'e^{i\theta})^{1/2}]^{1/4}\}. \tag{6.25}$$

It is not difficult to show that this is asymptotically equal to

$$s^{\alpha(t)-1}\left\{2f[t]\int_0^{\infty} dw\, w^{-\alpha(t)} \exp[-\beta(s_0 - w)^{1/4}]\right\}, \tag{6.26}$$

which certainly has the Regge form, although the detailed way in which this was achieved is quite different from our above analysis of the BJKS *Ansatz*.

For the finite integral A_3, we have asymptotically

$$A_3 \sim \int_{y_1^0}^{y_t^0} dy \exp(\lambda s y)\{\bar{g}(t, y_-) - \bar{g}(t, y_+)\}, \tag{6.27}$$

where

$$\bar{g}(t, y) = \frac{z^{1-a-\Delta\alpha[s(1-z)]}}{z-1+z\ln z} f[s(1-z)] h(t, y). \tag{6.28}$$

The analysis goes now just as in Sec. V, following Eq. (5.22). We take $\theta > \frac{1}{2}\pi - \varphi_s > \theta^{(0)}$, and show that the integral (6.27) certainly explodes exponentially, since $\cos(\varphi_s + \theta^{(0)}) > 0$. Note that the term $f[s(1-z)]$ supplies a decreasing factor $f[s(1-z_t^{(0)})]$ which is, however, quite insufficient to cancel $\exp[\lambda s y_t^{(0)}]$, since the function $f[w]$ cannot decrease faster than $\exp[-\beta w^{1/2-\epsilon}]$ as $|w| \to \infty$, because it must decrease in all complex directions.

VII. CONCLUDING REMARKS

We have seen that there are serious difficulties associated with a number of crossing-symmetric representations that have a curved double-spectral boundary in the s-t plane. Nevertheless, the possibility of constructing such a representation with Regge behavior for all t (or at least for a range of t considerably greater than $|t| < s_0$) must be regarded as still open.

A recent suggestion of Cohen-Tannoudji, Henyey, and Lacaze[8] has our standard form (1.1) with (in our notation)

$$\alpha(s, z) = a + \lambda s + \Delta\alpha[(s - \tfrac{1}{2}s_0) f(1-z)], \tag{7.1}$$

where $f(z)$ is a van der Corput neutralizer, satisfying Eqs. (4.4) and (4.5). In Ref. 8, the following suggestion is made for $f(z)$:

$$f(z) = g(z)/g(1), \tag{7.2}$$

with

$$g(z) = \int_0^z dy \exp\left[-\frac{1}{y(1-y)}\right]. \tag{7.3}$$

However, it is easy to see that this choice does not allow any rotation at all in the x plane, and that there is no Regge behavior in the whole of the right-hand half of the s plane. The situation can be ameliorated by the Suzuki choice for $g(z)$ [see Eq. (4.3)], instead of Eq. (7.3). However, there are still singularities in the right half of the x plane for some values of t, and so there will again be a t-dependent sector of non-Regge behavior in the s plane.[9]

A different possibility is to choose an $\alpha(s, z)$ that only has z-plane singularities at $z=0$ and $z=1$, for all s. The Suzuki *Ansatz* is a good example of this type of model. The van der Corput neutralizer complicates the model somewhat: If one is prepared to tolerate second-sheet singularities other than simple poles, then considerable simplification is possible, for example,

$$\alpha(s, z) = \alpha(s) - z\Delta\alpha(s)\,. \tag{7.4}$$

Unfortunately, models of this kind have a square-cornered double-spectral function, but it might be reasonable to take $s_0 \geq 20\mu^2$ in Eq. (1.3) (in the $\pi\pi$ case), so that the "inelastic kernel", $A(s, t)$, has a double-spectral function with support $s \geq s_0$, $t \geq s_0$ which is inside the Mandelstam spectral region, and then to leave the generation of the correct boundary to a perturbative iteration of $A(s, t)$ through the unitarity equation.

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APPENDIX

We examine graphically the complex numbers

$$y_n = -(1 - z_n)\ln z_n\,, \tag{A1}$$

where the z_n are solutions of

$$z_n - 1 + z_n \ln z_n = 0\,. \tag{A2}$$

Set

$$z_n = e^{p+iq-1}\,, \tag{A3}$$

with p and q real. Then (A2) becomes

$$p = e^{1-p}\cos q\,, \tag{A4}$$

$$q = -e^{1-p}\sin q\,. \tag{A5}$$

We have to find solutions of these two equations. It makes a prettier picture to combine (A4) and (A5) to give

$$p^2 + q^2 = e^{2(1-p)}\,. \tag{A6}$$

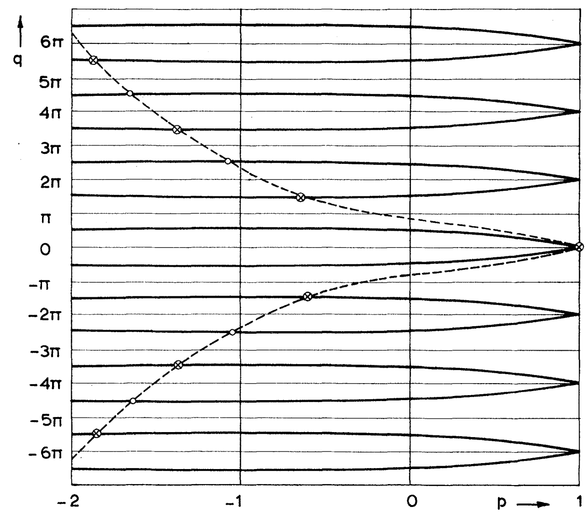

FIG. 6. The curves $p = e^{1-p}\cos q$ (solid) and $p^2 + q^2 = e^{2(1-p)}$ (dashed).

In Fig. 6 we sketch Eqs. (A4) and (A6) in the (p, q) plane. Solutions are represented by the points where the solid curves (A4) cross the dashed curve (A6). Clearly any solution of (A4) and (A5) is also a solution of (A4) and (A6). It is easy to see that only the points marked ⊗ are solutions of (A4) and (A5), the extra points, marked ○, correspond to a positive sign in Eq. (A5), and are of no interest.

The first few solutions are as follows:

n	0	±1	±2	±3
p	1	−0.58	−1.40	−1.87
q	0	±4.6	±11.0	±17.3
y_n	0	$2.6 \pm 4.4i$	$3.3 \pm 10.6i$	$3.8 \pm 17.3i$

These points are plotted in Fig. 5, where it will be seen that the sector defined by the origin and $y_{\pm 1}$, of half-angle 60°, excludes the higher solutions y_n, $n = \pm 2, \pm 3, \ldots$.

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*Present address: Institute for Theoretical Physics, Post Office Box 800, Groningen, The Netherlands.

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Hadron Structure and the Parton-Parton Interaction in a Partial Bootstrap Model*

Fred Cooper and Edmond Schonberg
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033
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We present a model where hadrons are infinitely composite objects made up of pointlike constituents (partons). The requirement that the hadron wave function be power-behaved determines the nature of the parton-parton interaction. In particular σ(parton-parton) $\sim$ const at high energies. If this is implemented by vector-gluon exchange between partons, the following simple picture of electromagnetic form factors and large-angle hadron-hadron scattering arises: (a) The electromagnetic form factor $F(t)$ has the same power behavior as the hadron wave function. (b) The large-angle scattering cross section is proportional to $|F(t)|^4$. (c) The effective interaction of the vector gluon with the composite hadron is described by the same form factor $F(t)$ as the electromagnetic interaction of the hadron.

I. INTRODUCTION

The structure of hadrons has been explored along three complementary lines:

(a) Elastic e-p scattering has shown that the electromagnetic form factor of the proton falls off as $(q^2)^{-2}$ or faster, for large q^2. This is seen to be strong evidence for the composite nature of the proton, and various bootstrap models have been proposed to describe it.

(b) Inelastic e-p scattering has shown that the hadronic structure functions W_1 and νW_2 are functions of the dimensionless variable $\omega = \nu/q^2$ alone, for $\nu, q^2 \to \infty$. The simplest model that reproduces this scaling is the parton picture, which describes the proton as "made up" in some fashion of bare pointlike constituents which interact locally with the electromagnetic field.[1] Furthermore, the fact that $W_1 \to \omega\nu W_2$ as $\nu/q^2 \to \infty$ indicates that partons, if they exist, have spin $\frac{1}{2}$. Finally, with the use of sidewise dispersion relations one can relate the wave-function renormalization constant Z_3 to an integral over the structure functions W_1 and νW_2 and obtain thus rigorous bounds for Z_3. Present data are consistent with $Z_3 = 0$, which is the field-theoretical criterion for compositeness.[2]

(c) Elastic p-p scattering at large angles is reasonably well described by[3,4]

$$\frac{d\sigma}{dt} \sim |F_{\rm em}(t)|^4 f(s, t),$$

where $f(s, t)$ is some slow-varying function for which various forms have been suggested. This seems to indicate that hadronic charge and electric charge have the same spatial distribution in the hadron. It also indicates that each hadron "probes" the wave function of the other in the same fashion that a photon does.[3] This points in the direction of a vector interaction between the constituents of the hadrons. Results in current algebras and the analysis of light-cone singularities also suggest that partons (or quarks) interact among themselves via vector "gluons."

In this paper we would like to propose a simple model to obtain a unified picture of (a), (b), and (c). We use an off-shell partial-bootstrap approach to describe the proton as an infinitely composite object, i.e., containing an infinite number of pointlike partons. We assume that the force that binds the parton to the hadron is the parton-parton force itself. The bootstrap is partial in the sense that the parton-parton interaction itself is considered as primitive. Only the composite hadron structure is bootstrapped. The material is organized as follows: In Sec. II we develop our partial-bootstrap model for the hadron-hadron-parton vertex